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Abstract

The paper contains constructions of Hilbert systems for the action of the circle group $T$ using subgroups of implementable Bogoljubov unitaries w.r.t. Fock representations of the Fermion algebra for suitable data of the selfdual framework: $\mathcal{H}$ is the reference Hilbert space, $\Gamma$ the conjugation and $P$ a basis projection on $\mathcal{H}$. According to a general result for Hilbert systems of this type, the group $C(\text{spec } \mathcal{Z} \rightarrow T)$ of $T$-valued functions on $\text{spec } \mathcal{Z}$ is isomorphic to the stabilizer of $\mathcal{A}$. In particular, examples are presented where the center $\mathcal{Z}$ of the fixed point algebra $\mathcal{A}$ can be calculated explicitly.\(^1\)

1 Introduction

It is well-known that the DHR-selection criterion in superselection theory leads to a (global) symmetry group $\hat{\mathcal{G}}$ of the theory, acting on the so-called field algebra. This group $\hat{\mathcal{G}}$ is compact. These $C^*$-systems consisting of the field algebra, together with the automorphism group $\hat{\mathcal{G}}$, are called Hilbert systems (for a precise definition see below) and they can be constructed as crossed products with the dual of $\mathcal{G}$. Decisive for this construction is the property that the relative commutant of the fixed point algebra $\mathcal{A}$ under the $\hat{\mathcal{G}}$ action is trivial and this requires the triviality of the center of $\mathcal{A}$ (see [DR1]).

Later it turned out that Hilbert systems with nontrivial center $\mathcal{Z}$ of $\mathcal{A}$ may be also of interest. For example, in such systems there is an intrinsic link between the group $\hat{\mathcal{G}}$ and a subgroup of the continuous $\hat{\mathcal{G}}$-valued functions on $\text{spec } \mathcal{Z}$, denoted by $C(\text{spec } \mathcal{Z} \rightarrow \hat{\mathcal{G}})$, given by its isomorphism with $\text{stab } \mathcal{A}$. This is due to the fact that the property of the model to be Galois closed breaks down (see [BL1]).

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If $\text{spec } Z$ can be considered as a “base space” and $\mathcal{G}$ as a “global gauge group” then this means that $\text{stab } \mathcal{A}$ appears as a “local gauge group”.

We adopt here a pure mathematical point of view. Nevertheless, the property just mentioned suggests to construct models such that these interpretations may be justified.

We concede that only the last example of the present paper points into this direction. The interpretation of $\mathcal{Z}$ in the other examples is rather that of a space of labels for multiplicity resp. degeneracy.

We restrict the treatment to abelian groups $\mathcal{G}$ because in this case the extension problem is solved in general terms (see, for example, [Bl1],[Bl2]). The case $\mathcal{G} = T$ is of special interest.

This paper presents constructions for the action of $T$ which are closely related to fermion quantization in general, and in particular to the implementation on the corresponding Fock spaces of the automorphisms of the CAR algebra defined by loop groups. Techniques and ideas from [CR1] and other related papers are used and partly simplified, for example by use of the selfdual framework.

First a construction is given on the reference Hilbert space $\mathcal{H}$ without use of quantization procedures which in turn yields a further example with nonnegative spectrum by quantising in the Fock space using the “positive energy” criterion. The third construction is a general procedure within the selfdual framework of fermion quantization (see [A1]). This procedure is then applied to an example where the implementation of the automorphisms defined by loop groups is used. In this example the center is calculated explicitly. According to the general observation mentioned above this means simply that in this model the group $C(\text{spec } Z \to T)$ is nothing other than the stabilizer of the fixed point algebra $\mathcal{A}$.

Finally we present an example, using a simple tensor product construction, where $\text{spec } Z = S^1$ is satisfied. This example shows that, at least in special cases, the interpretation of $\text{stab } \mathcal{A}$ as a “local gauge group” seems to be reasonable.

**DEFINITION.** A Hilbert system for a compact abelian group $\mathcal{G}$ is a $C^*$-system $\{\mathcal{F}, a_\mathcal{G}\}, a_\mathcal{G} \subset \text{aut } \mathcal{F}$, $a_\mathcal{G}$ continuous w.r.t. the topology of pointwise norm convergence, with $\text{spec } a_\mathcal{G} = \hat{\mathcal{G}}$ such that each spectral subspace contains a unitary.

As usual, the stabilizer $\text{stab } \mathcal{A}$ for a unital $C^*$-subalgebra $\mathcal{A} \subset \mathcal{F}$ is defined by $\text{stab } \mathcal{A} := \{\beta \in \text{aut } \mathcal{F} : \beta(A) = A \text{ for all } A \in \mathcal{A}\}$.

The terminology Hilbert system can be traced back to Doplicher and Roberts [DR2] where - in the case just mentioned - the spectrum of $a_\mathcal{G}$ is called the Hilbert spectrum.
2 Selected properties of Hilbert $C^*$-systems

For convenience we recall some properties of Hilbert systems for the action $T$ which are useful for the following constructions (cf. [BL1]).

Let $\{\mathcal{F}, \phi_T\}$ be a Hilbert system, $\Pi_n$, $n \in \mathbb{Z}$ its spectral projections and $\mathcal{F}_n := \Pi_n \mathcal{F}$ the corresponding spectral subspaces. $\mathcal{F}_0 := A$ is the fixed point algebra and $\mathcal{Z}$ its center.

If $V \in \mathcal{F}_1$ is a unitary then $V^n \in \mathcal{F}_n$, $\mathcal{F}_n = AV^n$ and

$$\mathcal{F} = \overline{\text{cl}} \{ \sum_{n \in \mathbb{Z}} A_n V^n, A_n \in A, \text{ finite sum} \}.$$ 

Put $\kappa := \text{Ad} V A$ then $\kappa \in \text{aut} A$. If $\tilde{V} := UV$ with $U \in \mathcal{U}(A)$ then $\text{Ad} V A = \text{Ad} U \circ \kappa =: \tilde{\kappa}$, i.e. $\kappa$ and $\tilde{\kappa}$ are unitarily equivalent. If $\kappa = \tilde{\kappa}$, i.e. $UVAV^{-1} U^{-1} = VAV^{-1}$ for all $A \in A$ then $UBU^{-1} = B$ for all $B \in A$ and $U \in \mathcal{Z}$ follows. This means: $\kappa$ determines uniquely only $V \mathcal{Z} = \mathcal{Z} V$. The latter equation is true because of $\kappa(\mathcal{Z}) = \mathcal{Z}$. For $\alpha, \beta \in \text{aut} A$ the space of intertwiners is defined by

$$(\alpha, \beta) := \{ X \in A : X \alpha(A) = \beta(A) X \text{ for all } A \in A \}.$$ 

Then one obtains: $(\kappa^n, \iota) = \{ 0 \}$ for all $n \neq 0$ or $\kappa^n$ and $\kappa^m$ are mutually disjoint for $m \neq n$ iff $A^l \cap \mathcal{F} = \mathcal{Z}$ (see [BL1]). In this case the stabilizer turns out to be isomorphic to $\mathcal{U}(\mathcal{Z})$ where the isomorphism is given by

$$\text{stab} A \ni \beta \rightarrow Z(\beta) \in \mathcal{U}(\mathcal{Z}) : \beta(V) = VZ(\beta).$$

Note that $\mathcal{U}(\mathcal{Z}) \cong C(\text{spec } \mathcal{Z} \rightarrow T)$. Moreover, each element $\beta \in \text{stab} A$ commutes with the starting group $\phi_T$, i.e. $\beta \circ \alpha_\zeta = \alpha_\zeta \circ \beta$ for all $\zeta \in T$ and, if $V \in \mathcal{F}_1$ then $\beta(V) \in \mathcal{F}_1$ and $\text{Ad} \beta(V) | A = \kappa$, i.e. the automorphism of $A$ generated by $V$ is invariant w.r.t. application of $\beta$. These properties of $\text{stab} A$ point to possible interpretation in suitable models.

Concerning the center $\mathcal{Z}$ note the following observation (see [BL1]): If $\omega_0$ is a state of $A$ such that the corresponding GNS-representation $\pi_0$ is faithful and $\pi$ is the GNS-representation of $\mathcal{F}$ w.r.t. $\omega(F) := \omega_0(\Pi, F)$ and $U(T)$ the (unique) implementer of $\phi_T$ then $\pi(A) \cap U(T)' = C1$ and $\pi(A)' \subseteq U(T)'$ follow. Now, if $\mathcal{Z} \subseteq C1$ then

$$\pi(A)' \subseteq U(T)'$$

follows, i.e. the breakdown of the Galois closedness implies the ”gap” described by this proper inclusion (in the case $\mathcal{Z} = C1$ one has equality). This means the group $U(T)$ does not determine $\pi(A)$ completely. On the other hand the representation $\pi_0$ is not irreducible which points to a ”degeneracy of the vacuum” in corresponding models.
3 Constructions using the regular representation of $T$ and the bilateral shift

3.1 A direct approach

The unit circle is denoted by $S^1$ (as a topological space) and by $T$ (as the 1-torus group). By $e_n, n \in Z$, we denote the canonical orthonormal basis $e_n(\xi) := \xi^n$ in $L^2(S^1)$ and by $P_n := (e_n, \cdot)e_n$ the corresponding 1-dimensional projection. Let $K$ be a Hilbert space. We put $H_0 := L^2(S^1, K)$. The regular representation of $T$ on $H$ is denoted by $U_\zeta, \zeta \in T$, where

$$(U_\zeta f)(\xi) := f(\zeta \xi), \ f \in H_0.$$ 

Putting $E_n := P_n \otimes 1_K$ we have $U_\zeta = \sum_{n \in Z} \zeta^n E_n$, i.e. $E_n$ is the isotypical projection w.r.t. the label $n$, and with $U_T = \{U_\zeta | \zeta \in T\}$ we have spec $U_T = Z$. The bilateral shift on $L^2(S^1)$ w.r.t. the canonical basis is denoted by $V_0 : V_0 e_n := e_{n+1}, n \in Z$. Then $V := V_0 \otimes 1_K$ satisfies

$$U_\zeta VU_\zeta^{-1} = \zeta V, \ \zeta \in T,$$

(1)

because

$$U_\zeta VU_\zeta^{-1} e_n \otimes k = \zeta^{-n} U_\zeta V e_n \otimes k = \zeta^{-n} U_\zeta e_{n+1} \otimes k = \zeta^{-n} \zeta^{n+1} e_{n+1} \otimes k = \zeta V e_n \otimes k.$$

Using $U_T$ and $V$ one can construct Hilbert systems w.r.t. $T$ as follows. Let $C^*(U_T, V)$ be the $C^*$-algebra generated by $U_T, V$.

PROPOSITION 1. Let $\mathcal{F}$ be a $C^*$-algebra with

$$C^*(U_T, V) \subseteq \mathcal{F} \subseteq L(H_0).$$

Then:

(I) $\alpha_\zeta := \text{Ad } U_\zeta | \mathcal{F} \in \text{aut } \mathcal{F}, \zeta \in T$, and $\zeta \rightarrow \alpha_\zeta$ is continuous w.r.t. pointwise norm convergence.

(II) $\{\mathcal{F}, \alpha_T\}$ is a Hilbert system and the spectral subspaces $\mathcal{F}_n$ are given by

$$\mathcal{F}_n = A^n, \ n \in Z,$$

where $A$ denotes the fixed point algebra $A := \mathcal{F}_0 = U_T' \cap \mathcal{F}$.

(III) The center $Z$ of $A$ satisfies

$$Z = A' \cap \mathcal{F},$$

(3)

i.e. it coincides with the relative commutant of $A$. Moreover $C^*(U_T) \subseteq Z$.

Proof. These facts are elementary consequences of the definitions. Note that $Z = A \cap A' = \mathcal{F} \cap U_T' \cap A' = \mathcal{F} \cap (U_T \cup A)' = \mathcal{F} \cap A'$ because $U_T \subset U_T'$, hence $U_T \subset A$ follows. Note further $A \subseteq U_T'$ hence $U_T \subset U_T' \subseteq A'$ and $U_T \subset Z$ follows. $\square$
In particular, we consider the extremal special cases of the inclusion (2). First let \( F := \mathcal{L}(H_0) \). Then \( A = U_T^\nu \) , i.e. \( A \in \mathcal{A} \) iff \( AE_n = E_n A \) for all \( n \in \mathbb{Z} \) which means that there is a one-to-one correspondence \( A \leftrightarrow \{ B_n \}_{n \in \mathbb{Z}}, B_n \in \mathcal{L}(K), \sup_n \| B_n \| < \infty \), i.e. \( \mathcal{A} \) corresponds to the so-called algebra of all diagonalizable operators. Moreover, \( \mathcal{A}' = U_T'^\nu \) and \( \mathcal{Z} = U_T' \cap U_T'^\nu = \{ \sum_{n \in \mathbb{Z}} \alpha_n E_n, \sup_n |\alpha_n| < \infty \} \), i.e. \( \mathcal{Z} \) is isomorphic to the algebra of all bounded complex-valued functions on \( \mathcal{Z} \). Then \( \text{spec} \mathcal{Z} \) coincides with the Stone-Cech-compactification of \( \mathcal{Z} \). Second let \( F := C^*(U_T, V) \). Then \( F = \text{clo}\|\{ \sum_{n \in \mathbb{Z}} C_n V^n \text{, finite sum, } C_n \text{ polynomial w.r.t. } U_T \}\) because \( U_V V = \zeta V U_V \). Hence \( \mathcal{A} = C^*(U_T) \) follows, i.e. \( \mathcal{A} = \mathcal{Z} \).

### 3.2 Quantized version with nonnegative spectrum

Applying quasifree fermion quantization to the foregoing example one gets Hilbert systems where the representation (of T), which defines via “Ad” the automorphism group, has nonnegative spectrum. Recall that the generator \( H \) of \( U_{it} = e^{itH}, H = \sum_n E_n \) may be used to define by implementation, the free fermion Hamiltonian or, in fermionic conformal field theory models, a generator of the Virasoro algebra. In the following we assume \( \text{dim } K < \infty \). We put

\[
E_{\geq 0} := \sum_{n \geq 0} E_n, \quad E_{< 0} := 1 - E_{\geq 0}.
\]

Now let \( \gamma \) be an arbitrary conjugation on \( H_0 \), i.e. \( \gamma \) is anti-unitary and \( \gamma^2 = 1 \). Put \( H := H_0 \oplus H_0 \). Then \( \Gamma \) defined by \( \Gamma(f, g) := (\gamma g, \gamma f) \) is a conjugation on \( H \). Adopting the terminology of the selfdual framework for the fermion algebra \( CAR(H, \Gamma) \) all operators \( \phi(U) \) defined by

\[
\phi(U) (f, g) := (Uf, \gamma Ugf), \quad (f, g) \in H, \quad U \in \mathcal{U}(H_0), \quad (4)
\]

are Bogoljubov unitaries on \( H \) defining Bogoljubov automorphisms \( \alpha_U \) of \( CAR(H, \Gamma) \) and the projection \( \Pi \) defined by

\[
\Pi(f, g) := (E_{\geq 0} f, \gamma E_{< 0} gf), \quad (f, g) \in H,
\]

is a basis projection, i.e. it defines a Fock representation \( \pi \) of \( CAR(H, \Gamma) \) on the Fock space \( \mathcal{F}_\Pi(H, \Gamma) \). According to the well-known implementation criterion the operators (4) are implementable by unitaries \( \Phi(U) \) on the Fock space w.r.t. \( \Pi \), this means

\[
\pi \circ \alpha_U = \text{Ad } \Phi(U) \circ \pi,
\]

iff \( E_{\geq 0} UE_{< 0} \) and \( E_{< 0} UE_{\geq 0} \) are Hilbert-Schmidt operators, i.e. are from \( \mathcal{L}_2(H_0) \) (note that these are two independent conditions). The implementer \( \Phi(U) \) is unique mod T1 in general.

Note that \( \phi(V) \) is implementable. Choose, for example, \( E_{\geq 0} V E_{< 0} \). The condition \( E_{\geq 0} V E_{< 0} \in \mathcal{L}_2(H_0) \) is equivalent to \( VE_{\geq 0} = E_{\geq 0} V \in \mathcal{L}_2(H_0) \). That is, according to \( VE_n = E_n V E_n \) or \( V = \sum_n E_n V E_n \) we obtain

\[
\sum_{n \geq 0} E_n \sum_m E_{m+1} V E_m - \sum_m E_{m+1} V E_m \sum_{n \geq 0} E_n =
\]

5
\[
\sum_{m \geq 1} E_{m+1} V E_m - \sum_{m \geq 0} E_{m+1} V E_m = E_0 V E_{-1}
\]
which is finite-dimensional.

Furthermore, \( \phi(U_\zeta) \) is implementable without phase ambiguity because it commutes with \( \Pi \). Then we can state

**Lemma 1.** The implemented strongly continuous unitary group \( \Phi(U_T) \) on \( \mathcal{F}_\Pi(\mathcal{H}, \Gamma) \) has nonnegative spectrum, \( \text{spec} \Phi(U_T) \geq 0 \).

**Proof.** One has \( \Pi \mathcal{H} = E_{\geq 0} \mathcal{H}_0 \oplus \gamma E_{<0} \mathcal{H}_0 \). Obviously, \( \text{spec} U_T \mid E_{\geq 0} \mathcal{H}_0 \geq 0 \). But also \( \text{spec} \gamma U_T \gamma \mid E_{<0} \mathcal{H}_0 \geq 0 \), because \( \gamma U_\zeta \gamma = \sum_n \zeta^{-n} \gamma E_n \gamma \) and a label \( n < 0 \) gives a positive spectral value \(-n\). \( \square \)

The relation (1) implies the corresponding relation for the implementers:

\[
\Phi(U_\zeta) \Phi(V) \Phi(U_\zeta)^{-1} = \zeta \Phi(V), \quad \zeta \in T.
\]  

(5)

This leads to the following Hilbert systems:

**Proposition 2.** Let \( \mathcal{F} \) be a \( C^* \)-algebra on the Fock space \( \mathcal{F}_\Pi(\mathcal{H}, \Gamma) \) with

\[
C^*(\Phi(U_T), \Phi(V)) \subseteq \mathcal{F} \subseteq C^*(\Phi(U), U \in \mathcal{U}(\mathcal{H}), \phi(U) \text{ implementable}).
\]

Then:

(i) \( \alpha_\zeta := \text{Ad} \Phi(U_\zeta) \mathcal{F} \in \text{aut} \mathcal{F}, \zeta \in T, \text{ and } \zeta \to \alpha_\zeta \) is continuous w.r.t. the pointwise norm convergence.

(ii) \( \{ \mathcal{F}, \alpha_T \} \) is a Hilbert system and the spectral subspaces \( \mathcal{F}_n, n \in \mathbb{Z}, \) are given by

\[
\mathcal{F}_n = A \Phi(V)^n, \quad n \in \mathbb{Z},
\]

where \( A \) denotes the fixed point algebra \( A := \mathcal{F}_0 = \mathcal{F} \cap \Phi(U_T)' \).

(iii) The center \( \mathcal{Z} \) of \( A \) satisfies

\[
\mathcal{Z} = A' \cap \mathcal{F}
\]

and one has

\[
C^*(\Phi(U_T)) \subseteq \mathcal{Z}.
\]

**Proof.** Similar to Proposition 1. \( \square \)

According to this result the question arises: find conditions for proper extensions \( \mathcal{F} \supset C^*(\Phi(U_T), \Phi(V)) \) such that \( \mathcal{Z} = C^*(\Phi(U_T)) \).

4 Constructions on the quantized level for the identical representation of \( T \)

4.1 A general procedure

The aim of this section is to present an approach for the construction of Hilbert systems for the action of \( T \) such that the center of the fixed point algebra can
be calculated explicitly.

The starting point for this approach is the selfdual framework of the fermion algebra $CAR(\mathcal{H}, \Gamma)$ and the essential tools are Fock representations, implementation theorems and the CAR-CCR correspondence given by the so-called Schwinger term.

### 4.1.1 Preliminary remarks

We start with an infinite-dimensional Hilbert space $\mathcal{H}_0$ and an arbitrary conjugation $\gamma$ on $\mathcal{H}_0$. We put $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_0$ and define the conjugation $\Gamma$ on $\mathcal{H}$ as before by $\Gamma(f, g) := (\gamma g, \gamma f), (f, g) \in \mathcal{H}$. For unitaries $U \in \mathcal{U}(\mathcal{H}_0)$ the unitary $\phi(U)$ defined by

$$\phi(U)(f, g) := (Uf, \gamma U\gamma g), \quad (f, g) \in \mathcal{H}$$

is a Bogoliubov unitary w.r.t. $\Gamma$ on $\mathcal{H}$. Further we choose an arbitrary orthoprojection $P$ on $\mathcal{H}_0$, $0 < P < 1$, with $\dim P = \dim P^\perp = \infty$, where $P^\perp := 1 - P$. Then the projection $\Pi$ on $\mathcal{H}$ defined by

$$\Pi(f, g) := (Pf, \gamma P^\perp \gamma g), \quad (f, g) \in \mathcal{H}$$

is a basis projection w.r.t. $\Gamma$ and $\mathcal{H}$. To $\Pi$ there corresponds a unique Fock representation $\pi$ of $CAR(\mathcal{H}, \Gamma)$ on the corresponding Fock space, denoted by $\mathcal{F}_\Pi(\mathcal{H}, \Gamma)$. Bogoliubov unitaries $V$ on $\mathcal{H}$ define Bogoliubov automorphisms $\alpha_V \in \text{aut} CAR(\mathcal{H}, \Gamma)$.

The question arises: under which conditions can $\alpha_V$ be implemented w.r.t. $\pi$ by an $\text{Ad}$-automorphism on the Fock space? That is, we need the existence of a unitary $\Phi(V)$ satisfying

$$\pi \circ \alpha_V = \text{Ad} \Phi(V) \circ \pi.$$ 

This is answered by the well-known implementation criterion (see, for example, [A1]). In general, $\Phi(V)$ is unique mod $T1$ and $\Phi(V)$ is called an implementer of $V$ on the Fock space. The implementation criterion for the special Bogoliubov unitaries $\phi(U)$ says: $\phi(U)$ has an implementer $\Phi(U) := \Phi(\phi(U))$ iff $PU P^\perp$ and $P^\perp UP$ are Hilbert-Schmidt, i.e. members of $\mathcal{L}_2(\mathcal{H}_0)$. Since

$$\Pi \phi(U) \Pi = \text{diag} \{ PUP, \gamma P^\perp UP, \gamma P^\perp \}$$

one obtains

$$0 = \text{ind} \Pi \phi(U) \Pi | \mathcal{H} = \text{ind} PUP | \mathcal{H}_0 + \text{ind} P^\perp UP^\perp | P^\perp \mathcal{H}_0.$$ 

Denote by $\mathcal{U}_{rs} \subset \mathcal{U}(\mathcal{H}_0)$ the subgroup of all implementable unitaries on $\mathcal{H}_0$. In particular, the scalar unitaries $U_\lambda := \lambda 1, \lambda \in T$, are implementable, even without phase ambiguity because $\phi(U_\lambda)$ commutes with $\Pi$. The corresponding unique implementer is denoted by $\Phi(\lambda)$. Thus the ‘identical representation’ $T \ni \lambda \rightarrow \lambda 1$ on $\mathcal{H}_0$ is implemented on the Fock space by $\lambda \rightarrow \Phi(\lambda) \in \mathcal{U}(\mathcal{F}_\Pi(\mathcal{H}, \Gamma))$ whereas $\mathcal{U}_{rs} \ni U \rightarrow \Phi(U)$ is a projective unitary representation.
To each $U \in \mathcal{U}_{res}$ there corresponds uniquely an index $q(U) \in \mathbb{Z}$, defined by

$$\Phi(\lambda)\Phi(U)\Phi(\lambda)^{-1} = \lambda^{q(U)}\Phi(U), \quad \lambda \in T.$$  \hfill (8)

According to a result of Carey, Hurst, O’Brien [CHO1] $q(U)$ is given by

$$q(U) = \mathrm{ind} \, PUP|\mathcal{P}H_0.$$  

This implies that there are unitaries $V_i \in \mathcal{U}_{res}$ satisfying $q(V_1) = 1$. Then $q(V_i^n) = n$, $n \in \mathbb{Z}$, follows. Note that $q(U_1U_2) = q(U_1) + q(U_2)$ for $U_1, U_2 \in \mathcal{U}_{res}$. The set of all $U \in \mathcal{U}_{res}$ with $q(U) = n$ is denoted by $\mathcal{U}_{res}(n)$. Thus $\mathcal{U}_{res}(n) \neq \emptyset$. If $V_i \in \mathcal{U}_{res}(n)$ then

$$\mathcal{U}_{res}(n) = \{UV_i : U \in \mathcal{U}_{res}(0)\} = \{V_iU : U \in \mathcal{U}_{res}(0)\}.$$  

Putting $\alpha_\lambda := \mathrm{Ad} \, \Phi(\lambda)|C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res}))$ we obtain a "maximal" Hilbert system $\{C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res})), \alpha_T\}$ where $\mathcal{A} = C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res}(0)))$. Note that

$$V_iUV_i^{-1}, V_i^{-1}UV_i \in \mathcal{U}_{res}(0)$$  

if $U \in \mathcal{U}_{res}(0), V_i \in \mathcal{U}_{res}(1)$, i.e.

$$\mathrm{Ad} \, V_i|C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res}(0))) \in \mathrm{aut} \, C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res}(0))).$$

Let $V_i \in \mathcal{U}_{res}(1)$ be fixed. We put

$$\mathcal{F}_{V_i} := C^*_{\mathcal{A}}(\Phi(T), \Phi(V_i)).$$

Then $\{\mathcal{F}_{V_i}, \alpha_T|\mathcal{F}_{V_i}\}$ is a "minimal" Hilbert system, because from (8) we obtain

$$\Phi(V_i)\Phi(\lambda)\Phi(V_i)^{-1} = \lambda^{-1}\Phi(\lambda)$$

and this means that $\mathrm{Ad} \, V_i|C^*_{\mathcal{A}}(\Phi(T)) \in \mathrm{aut} \, C^*_{\mathcal{A}}(\Phi(T))$ which implies that $\mathcal{F}_{V_i} \cap \Phi(T)' = C^*_{\mathcal{A}}(\Phi(T))$. Therefore we obtain a preliminary result which corresponds to Proposition 2:

**PROPOSITION 3.** Let $\mathcal{F}$ be a $C^*$-algebra on the Fock space $\mathcal{F}_{\Pi}(\mathcal{H}, \Gamma)$ with

$$C^*_{\mathcal{A}}(\Phi(T), \Phi(V_i)) \subseteq \mathcal{F} \subseteq C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res})).$$

Then:

(i) $\alpha_\lambda := \mathrm{Ad} \, \Phi(\lambda)|\mathcal{F} \in \mathrm{aut} \, \mathcal{F}, \lambda \in T$, and $\lambda \to \alpha_\lambda$ is continuous w.r.t. pointwise norm convergence.

(ii) $\{\mathcal{F}, \alpha_T\}$ is Hilbert and the spectral subspaces $\mathcal{F}_n$ are given by

$$\mathcal{F}_n = \mathcal{A}_n^\Phi(V_i^n), \quad n \in \mathbb{Z},$$

where $\mathcal{A}$ denotes the fixed point algebra

$$\mathcal{A} := \mathcal{F}_0 = \mathcal{F} \cap \Phi(T)' \subseteq C^*_{\mathcal{A}}(\Phi(\mathcal{U}_{res}(0))).$$
(II) The center $\mathcal{Z}$ of $\mathcal{A}$ satisfies

$$\mathcal{Z} = \mathcal{A}' \cap \mathcal{F}$$

and

$$C^*(\Phi(T)) \subseteq \mathcal{Z}.$$  \hspace{1cm} (9)

**Proof.** Immediate from the preceding discussion. $\Box$

If $\mathcal{F}$ is the "minimal" Hilbert system mentioned above then $\mathcal{A} = C^*(\Phi(T))$ is abelian and we have equality in (9). The problem is to construct proper intermediate Hilbert systems $\{\mathcal{F}, \alpha_T\}$ such that $\mathcal{Z} = C^*(\Phi(T))$. In the following we present an abstract approach to produce such Hilbert systems.

### 4.1.2 The Schwinger term

According to 3.1.1 we have $T1_{\mathcal{H}_0} \subseteq \mathcal{U}_{\mathcal{F}_0}(0)$. Now we consider subgroups $\mathcal{U}_0 \subseteq \mathcal{U}_{\mathcal{F}_0}(0)$ with $T1_{\mathcal{H}_0} \cap \mathcal{U}_0 = \{1_{\mathcal{H}_0}\}$ and which are generated by their 1-parameter subgroups $e^{itA}$ where the generators $A$ satisfy $A = A^* \in \mathcal{L}(\mathcal{H}_0)$ and $\gamma A \gamma = A$. Then $\phi(e^{itA}) = e^{it\phi(A)}$ where $\phi(A)$ is defined by

$$\phi(A)(f, g) := (Af, -Ag), \quad (f, g) \in \mathcal{H}.$$ 

$\phi(A)$ satisfies $\Gamma \phi(A) \Gamma = -\phi(A)$. By $\mathcal{L}$ we denote the real-linear subspace of all these generators. In what follows we refer to Araki [A1] and quote some results, for convenience. The first requirement for $\mathcal{L}$ is $\Pi \phi(A) \Pi \phi(A) \subseteq \mathcal{L}_2(\mathcal{H})$ for $A \in \mathcal{L}$, which means $PA \bar{P} \subseteq \mathcal{L}_2(\mathcal{H}_0)$. Then, according to [A1, Theorem 6.10, p.78] one has the following result:

The group $e^{itA}$ has a special continuity property (P-norm continuity) and the implementation $\Phi(e^{itA})$ is a unitary strongly continuous group

$$\Phi(e^{itA}) = e^{it\Phi(\phi(A))}, \quad \Phi(\phi(A)) \text{ selfadjoint on } \mathcal{F}_c(\mathcal{H}, \Gamma), \quad A \in \mathcal{L},$$

(where we use for $\phi(A)$ the same implementation symbol as for unitaries). The set of all generators $\Phi(\phi(A))$ has a common dense domain and on this domain the commutator $[\Phi(\phi(A_2)), \Phi(\phi(A_1))]$ can be calculated:

$$[\Phi(\phi(A_2)), \Phi(\phi(A_1))] = i\Phi([\phi(A_2), \phi(A_1)]) +$$

$$\frac{1}{2} \text{tr} \{\Pi \phi(A_1) \Pi \phi(A_2) \Pi \phi(A_1) \Pi - \Pi \phi(A_2) \Pi \phi(A_1) \Pi \} =$$

$$i\{\Phi(\phi([A_2, A_1])) + \text{Im } \text{tr}(PA_1 \bar{P} A_2 \bar{P})\}$$

where the second term is known as the Schwinger term. We put

$$s(A_1, A_2) := 2\text{Im } \text{tr}(PA_1 \bar{P} A_2 \bar{P}), \quad A_1, A_2 \in \mathcal{L}.$$ 

$s(\cdot, \cdot)$ is symplectic bilinear on $\mathcal{L}$. Further we put

$$\langle A_1, A_2 \rangle := \text{tr}(PA_1 \bar{P} A_2 \bar{P}), \quad A_1, A_2 \in \mathcal{L}.$$
Then \( \langle \cdot, \cdot \rangle \) is a semi-scalar product on \( \mathcal{L} \) and \( \langle A, A \rangle = 0 \) iff \( P A P^\perp = P^\perp A P = 0 \).

The second requirement is that \( \mathcal{V}_0 \) is abelian, or equivalently, that the generators \( A \in \mathcal{L} \) all commute. Then

\[
[\Phi(\phi(A_2)), \Phi(\phi(A_1))] = i \text{Im} \text{tr}(PA_1 P^\perp A_2 P), \quad A_1, A_2 \in \mathcal{L}.
\]

The third requirement is: \( \langle \cdot, \cdot \rangle \) is a scalar product on \( \mathcal{L} \). This implies that \( s \) is non-degenerate on \( \mathcal{L} \). Let \( \mathcal{W} := CCR(\mathcal{L}, s) \) be the Weyl algebra for the parameters \( \{\mathcal{L}, s\} \). Then, according to [A1, Theorem 7.1, p.121] or [CR1] one has:

The representation \( \rho \) of \( \mathcal{W} \) defined by

\[
\rho(W(A)) := \Phi(e^{i A}) = e^{i \Phi(\phi(A))}, \quad A \in \mathcal{L},
\]
on the Fock space \( \mathcal{F}_\Pi(\mathcal{H}, \Gamma) \) is the Fock representation w.r.t. the generating functional

\[
f(A) := e^{-\frac{i}{2} \Phi(\phi(A))},
\]
where \( W(A) \) denotes the abstract Weyl generators for \( \mathcal{W} \).

Note that \( \rho(W) \) is simple and therefore it has trivial center. Now recall that \( T_{1_{\mathcal{H}_0}} \cdot \mathcal{V}_0 \) is a direct product. This gives

\[
\mathcal{A} := C^*(\Phi(T), \rho(W)) \cong C^*(\Phi(T)) \otimes \rho(W)
\]
and

\[
\mathcal{Z} = \mathcal{Z}(\mathcal{A}) \cong \mathcal{Z}(C^*(\Phi(T))) \otimes \mathcal{Z}(\rho(W)) \cong C^*(\Phi(T)).
\]
The last requirement refers to some \( V_1 \in \mathcal{U}_{\mathcal{H}_0}(1) \) and its connection to \( \mathcal{L} \). It says that

\[
V_1 \mathcal{L} V_1^{-1} = \mathcal{L},
\]
i.e. \( \text{Ad } V_1 \) acts on \( \mathcal{L} \) as an automorphism. This implies

\[
\Phi(V_1) \rho(W) \Phi(V_1)^{-1} = \rho(W),
\]
i.e. \( \text{Ad } \Phi(V_1) | \rho(W) \in \text{aut } \rho(W) \) hence

\[
\text{Ad } \Phi(V_1) | \mathcal{A} \in \text{aut } \mathcal{A}
\]
follows and we obtain

PROPOSITION 4. Let \( \mathcal{F} := C^*(\mathcal{A}, V_1) \in \mathcal{L}(\mathcal{F}_\Pi(\mathcal{H}, \Gamma)) \) and \( \alpha_\lambda := \text{Ad } \Phi(\lambda) | \mathcal{F}, \lambda \in T \). Then \( \alpha_T \subseteq \text{aut } \mathcal{F} \) and \( \{\mathcal{F}, \alpha_T\} \) is a Hilbert system where the fixed point algebra coincides with \( \mathcal{A} \). The center \( \mathcal{Z} \) of \( \mathcal{A} \) satisfies

\[
\mathcal{Z} = \mathcal{A}' \cap \mathcal{F}
\]
and

\[
\mathcal{Z} = C^*(\Phi(T)).
\]

Proof. This follows immediately from the discussion preceding the statement of the proposition, in particular (10). \( \square \)
4.2 An example using the implementation of the loop group

In this section we present an example where all requirements of the foregoing procedure are satisfied. Put

\[ \mathcal{H}_0 := L^2(S^1), \]
\[ \gamma: (\gamma(x))(\zeta) := x(\overline{\zeta}), \quad \text{complex conjugation}, \]
\[ P := P_{\geq 0}, \]

where \( P_{\geq 0} \) denotes the projection onto the “nonnegative terms” of the regular representation of \( T \) on \( L^2(S^1) \), i.e., \( P_{\geq 0} = \sum_{n \geq 0} P_n \), where the \( P_n \) are the 1-dimensional projections of Section 2. Let \( \mathcal{U}_{r,s} \) be as before. In this example first we choose a subgroup \( \mathcal{V}_0 \subset \mathcal{U}_{r,s} \) which is defined as follows: Let \( L(T) := C^\infty(S^1 \rightarrow T) \) be the loop group of \( T \). The functions \( f \in L(T) \) are interpreted as multiplication operators

\[ f \rightarrow U_f : (U_f(x))(\zeta) := f(\zeta)x(\zeta), \quad x \in L^2(S^1). \]

It is well known that \( U_{L(T)} \subset \mathcal{U}_{r,s} \) (see, for example, [PS]). Further we have

\[ \text{ind } PU_{U_f}P|\mathcal{H}_0 = w(f) \]

where \( w(f) \) is the winding number of \( f \) (see [CH]), i.e., we have \( q(U_f) = w(f) \).

Now we put

\[ \mathcal{V}_0 := \{ U_f \in U_{L(T)} : w(f) = 0 \}. \]

Obviously \( w(f) = 0 \) implies \( f(\zeta) = e^{i\alpha} \), where \( \zeta = e^{i\alpha}, 0 \leq \alpha \leq 2\pi, h(0) = h(2\pi), 0 \leq h(0) < 2\pi \). The real-valued function \( \alpha \rightarrow h(\alpha) \) is smooth. The subgroup \( \mathcal{V}_0 \) of the general procedure is then defined by the equation

\[ \mathcal{V}_0 = T1_{\mathcal{H}_0} \cdot \mathcal{V}_0 \cong T1_{\mathcal{H}_0} \times \mathcal{V}_0 \]

so that for the functions \( U_f \in \mathcal{V}_0 \) the zero-Fourier coefficient of \( h \) vanishes, i.e., \( h_0 = 0 \) or \( \int_0^{2\pi} h(\alpha)\,d\alpha = 0 \), where \( h(\alpha) = \sum_{n \in \mathbb{Z}} h_n e^{in\alpha} \cdot h(\cdot) \) acts as a multiplication operator \( A \) on \( L^2(S^1) \). Then \( \gamma A = A \) and \( \mathcal{L} \) is defined as the set of all \( A \) belonging to \( f \) with \( U_f \in \mathcal{V}_0 \).

The first requirement of the last subsection is \( PAP_1 \perp P_1 AP \in \mathcal{L}(\mathcal{H}_0) \). This is true because with \( A = \sum_{n} A_n \zeta^n \) one obtains

\[ \sum_{n \in \mathbb{Z}} ||P_1 AP_1 \zeta_n||^2 = \sum_{m=1}^{\infty} m |A_m|^2 < \infty. \]

The second requirement of the last subsection, that “\( \mathcal{V}_0 \) is abelian” is obvious. The third requirement says: \( \langle \cdot, \cdot \rangle \) is a scalar product. This is true because of

\[ \text{tr} (PAP_1 \perp AP) = \sum_{m=1}^{\infty} m |A_m|^2, \]
so that $A_m = 0$ for $m \geq 1$ implies $A_{-m} = A_{m} = 0$, $m \geq 1$. Moreover $A_0 = 0$. The fourth requirement can be fulfilled choosing $V_1 := U_{\beta}$, where, for example, $f_1(\zeta) := \zeta$. Then $V_1L_1^{-1} = \mathcal{L}$ is obvious. Therefore the general approach works and we arrive at the following

**PROPOSITION 5.** Let $\mathcal{F} := C^*(\Phi(U_{L(T)}))$ be the $C^*$-algebra generated by the implementers of the loop group and put $\alpha_\theta := \text{Ad} \Phi(\lambda)|\mathcal{F}$, $\lambda \in T$. Then $\{\mathcal{F}, \alpha_T\}$ is Hilbert, $\mathcal{A} \cong C^*(\Phi(T)) \otimes \rho(\text{CCR}(\mathcal{L}, s))$, $\mathcal{A} \cap \mathcal{F} = \mathcal{Z}$, $\mathcal{Z} = C^*(\Phi(T))$, and $C(\text{spec } \mathcal{Z} \to T) \cong \text{stab } \mathcal{A}$.

### 4.3 The case spec $\mathcal{Z} = S^1$

We refer to the Weyl-algebra $\mathcal{B} := \text{CCR}(\mathcal{L}, s)$ of the foregoing section, where $\nu$ is the special automorphism, given by $\text{Ad} \Phi(V_1)$ in the representation $\rho$. Recall that $(\nu^n, \iota) = \{0\}$ for all $n \in \mathbb{Z}$, $\mathcal{B}$ is equipped with a natural net structure w.r.t. $S^1$ : if $\Delta \subset S^1$ is an open interval with $\text{clo } \subset S^1$ then the assigned $C^*$-algebra

$$\mathcal{B}(\Delta) := C^*(W(A), \text{supp } A \subset \Delta)$$

has the properties

(I) $\Delta_1 \subset \Delta_2$ implies $\mathcal{B}(\Delta_1) \subset \mathcal{B}(\Delta_2)$,

(II) if $\text{clo } \Delta_1 \cap \text{clo } \Delta_2 = \emptyset$ then $\mathcal{B}(\Delta_1)$ and $\mathcal{B}(\Delta_2)$ commute elementwise.

(III) $\mathcal{B} = \text{clo } \bigcup_{\Delta \subset S^1} \mathcal{B}(\Delta)$.

Concerning property (II) recall that the symplectic form $s$ can be rewritten as

$$s(A, B) = \frac{1}{2\pi} \int_0^{2\pi} A(\alpha)B'(\alpha) d\alpha.$$

Further let $\delta$ be an automorphism of $C(S^1)$. By $\sigma$ we denote the corresponding homeomorphism of $S^1 : (\delta f)(\mu) = f(\sigma(\mu))$, $\mu \in S^1$, $f \in C(S^1)$. Putting

$$\mathcal{A} := C(S^1) \otimes \mathcal{B}$$

and

$$\kappa := \delta \otimes \nu$$

then $\kappa \in \text{aut } \mathcal{A}$, $(\kappa^n, \iota) = \{0\}$ for all $n \in \mathbb{Z}$ follows. According to general extension results (see [BL2] for example) there is a Hilbert system $\{\mathcal{F}, \alpha_T\}$ w.r.t. the action group $T$ where $\mathcal{A} \subset \mathcal{F}$, $\mathcal{A}$ the fixed point algebra of $\alpha_T$. This means there is a unitary $V \in \mathcal{F}$ with $\alpha_\lambda V = \lambda V$ for all $\lambda \in T$. Obviously, $\mathcal{Z} := \mathcal{Z}(\mathcal{A}) \cong C(S^1)$. Recall that

$$\mathcal{F} = \text{clo } \bigcup_{\mu \in \mathbb{Z}} \{ \sum_{n \in \mathbb{Z}} A_n V^n, A_n \in \mathcal{A}, \text{ finite sum} \}. \quad (11)$$

The net structure of $\mathcal{B}$ can be extended to $\mathcal{A}$ by

$$\mathcal{A}(\Delta) := \mathcal{Z} \otimes \mathcal{B}(\Delta), \quad \Delta \subset S^1.$$
This means that the elements of $A(\Delta)$ are continuous functions

$$S^1 \ni \mu \to B(\mu), B(\mu) \in B(\Delta) \text{ for all } \mu \in S^1.$$  

Furthermore, using (11) the net structure can be extended to $\mathcal{F}$.

The property $\text{stab} \mathcal{A} \cong C(S^1 \to T)$ means that the action of an element $f$ of the group $C(S^1 \to T)$ on the algebra $\mathcal{F}$ is well-defined if one adopts the point of view to consider $f$ as an element of $\text{stab} \mathcal{A}$ which is justified by the mentioned isomorphism. Recall that the action of $\beta \in \text{stab} \mathcal{A}$ is given by

$$\beta(V) = VZ$$

where $Z \in \mathcal{U}(\mathcal{Z})$ is the central unitary, corresponding to the function $f \in C(S^1)$. Then one obtains

$$\beta(V^n) = (VZ)^n = \kappa(Z)\kappa^2(Z)\ldots\kappa^n(Z)V,$$

or, written as a function on $S^1$,

$$S^1 \ni \mu \to f(\sigma(\mu)) f(\sigma^2(\mu)) \ldots f(\sigma^n(\mu)) V^n.$$  

This leads to the following formula for $\beta(\sum_n A_n V^n)$:

$$S^1 \ni \mu \to \sum_n f(\sigma(\mu)) f(\sigma^2(\mu)) \ldots f(\sigma^n(\mu)) B_n(\mu) V^n.$$  

This formula shows that the action of $f$ is compatible with the net structure of the model.

REMARK. The example in this subsection arises in the study of two-dimensional electrodynamics (the Schwinger model) although in a slightly more complicated form. In [CW] the field algebra of the Schwinger model is constructed. In this model there is group of so-called ‘local gauge transformations’ which contains $T$, the global gauge group. The subalgebra of the field algebra, invariant under local gauge transformations, is of the form $A_0 = Z_0 \otimes B_0$ where $Z_0$ is the centre of $A_0$. In [CW] it is further observed that one may pass to a quotient algebra of $A_0$ to obtain the algebra of observables. This latter algebra is of the form $C(S^1) \otimes \mathcal{C}$ where $\mathcal{C}$ is the algebra of the CCR for a free massive boson field and hence is a simple algebra. The discussion above requires no more than simplicity of $\mathcal{B}$ and hence applies to $C(S^1) \otimes \mathcal{C}$. The presence of the algebra $C(S^1)$ in this tensor product is a remnant of the local gauge symmetry in the model in the sense that the generator of this $C(S^1)$ acts in the full Hilbert space of the model as a gauge transformation. The interpretation in this case is to the existence of a vacuum degeneracy in the Schwinger model (the so-called theta-vacua). Specifically, the representation of $C(S^1) \otimes \mathcal{C}$ which arises in the Schwinger model from the construction in [CW] can be decomposed as a direct integral over the spectrum of the centre, $S^1$, into irreducibles and the GNS cyclic vector in each of these irreducibles has the interpretation of a vacuum state. This interpretation is also available in the abstract setting of Hilbert systems above where the representation of the algebra $A$ on the Hilbert space for the field algebra is also decomposable (due to the non-trivial centre).
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6 References

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