Verdier–Riemann–Roch for Chern Class
and Milnor Class

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ABSTRACT. The original Verdier-Riemann-Roch says that for a local complete intersection
morphism \( f : X \to Y \) the pullback of \( K \)-theory \( f^* : K_0(Y) \to K_0(X) \) and the homology
pullback \( Id(T_f) \cap f^* : H_*(Y) \to H_*(X) \) commute with the Baum-Fulton-MacPherson's
Riemann-Roch transformation \( \tau : K_0 \to H_0 \). In this paper we deal with a Chern class
version of this, with the \( K \)-theory replaced by the constructible function functor \( F \) and \( \tau \)
replaced by the Chern-Schwartz-MacPherson class transformation \( c_* : F \to H_* \).

§1 INTRODUCTION

In [BFM] Baum, Fulton and MacPherson formulated a singular Riemann-Roch, exten-
ting the Grothendieck-Riemann-Roch to possibly singular varieties. That is the
unique natural transformation from the \( K \)-theory to the rational homology theory

\[(BFM) \quad \tau : K_0 \to H_0 \]

satisfying the normalization condition that \( \tau(\mathcal{O}_X) = Id(T_X) \cap [X] \) for a non-singular
variety \( X \) with \( \mathcal{O}_X \) the structure sheaf and \( Id(T_X) \) the total Todd class of the tangent
bundle \( T_X \). This is a covariant aspect of the two theories \( K_0, H_0 \). As to the contravariant
aspect of these two theories, we have the Verdier-Riemann-Roch, which was conjectured
in [BFM] and proved affirmatively by J.-L. Verdier [V, Theorem 18.2 (3), p. 349], i.e., the,
following commutative diagram for a local complete intersection morphism \( f : X \to Y \):

\[
\begin{array}{ccc}
K_0(Y) & \xrightarrow{\tau_Y} & H_*(Y)_0 \\
f^* \downarrow & & \downarrow Id(T_f) \cap f^* \\
K_0(X) & \xrightarrow{\tau_X} & H_*(X)_0
\end{array}
\]

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Roch formula

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where $td(T_f)$ is the total Todd class of the virtual relative tangent bundle $T_f$ of the morphism $f$.

\[
\begin{array}{ccc}
K^0(X) & \xrightarrow{ch} & H_*(X)_\mathbb{Q} \\
\downarrow f_* & & \downarrow f_*(td(T_f)\cup ) \\
K^0(Y) & \xrightarrow{ch} & H_*(Y)_\mathbb{Q}
\end{array}
\]

(SGA 6)

These three Riemann-Roch theorems follow from the Grothendieck transformation (see [FM, II, §1])

\[\tau : \mathbb{K}_{\text{alg}} \to \mathbb{H}_\mathbb{Q}\]

from the bivariant algebraic $K$-theory to the bivariant homology theory with rational coefficients and the bivariant-theoretic “Riemann-Roch formula” ([FM, 1.4])

\[\tau(\mathcal{O}_f) = td(T_f) \cdot U_f\]

of the canonical orientations $\mathcal{O}_f \in \mathbb{K}_{\text{alg}}(X \xrightarrow{f} Y)$ and $U_f \in \mathbb{H}(X \xrightarrow{f} Y)_\mathbb{Q}$ for a local complete intersection morphism $f : X \to Y$ (see [FM, II, 0.2]).

As remarked in [BFM, (0.3)], the motivation of BFM’s Riemann-Roch is the Chern-Schwartz-MacPherson class theory, i.e., the unique natural transformation

\[c_* : F \to H_*\]

from the covariant functor $F$ of constructible functions to the homology functor, satisfying the normalization condition that $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ for a nonsingular variety $X$ with $\mathbb{1}_X$ being the characteristic function on $X$. The natural transformation $c_* : F \to H_*$ is nothing but “Grothendieck-Riemann-Roch” for the Chern class (cf. [G]). It is, therefore, quite natural and reasonable to think of the “contravariant aspect” of the Chern-Schwartz-MacPherson class theory; i.e., “Verdier-Riemann-Roch” for Chern class, since the original Verdier-Riemann-Roch is one for Todd class. Thus, first of all, a naïve or simple-minded question is to ask about the commutativity of the following diagram:

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
\downarrow f^* & & \downarrow (c(T_f) \cap f^*) \\
F(X) & \xrightarrow{c_*} & H_*(X; \mathbb{Z})
\end{array}
\]

(1.1)

Here, $f^* : F(Y) \to F(X)$ is the usual functional pullback of constructible functions and $c(T_f)$ is the total Chern class of the bundle $T_f$. The above diagram (1.1) is commutative if $f$ is smooth ([Y1]), otherwise it is not. Consider the very simple case when $f : X \to pt$
is a local complete intersection morphism from a singular variety \( X \) to a point, which means that the variety \( X \) is a singular local complete intersection in a smooth variety. Then the problem of the commutativity of the diagram (1.1) is equivalent to that of whether the Chern-Schwartz-MacPherson class \( c_*(X) \) of \( X \) and the Fulton-Johnson class or Fulton-Chern class \( c^{FJ}_*(X) \) ([F], [FJ]) are the same or not. They are not the same and the difference of these two classes, which is called Milnor class, has been recently studied well from different motivations (e.g., see [A3], [BLSS1,2], [PP3], [Su1], [Y2], etc.). Note that in the case of a singular plane curve, it is already implicitly observed in [P, §6, Comparaison des classes] that the difference of these two classes is the sum of Milnor numbers of the singularities (cf. [F, Example 4.2.6 (b)]).

From a bivariant-theoretic viewpoint, a more natural question to ask is then whether or not there exists a certain constructible function \( \alpha \in F(X) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
\alpha \cdot f^* & | & (c(T_f) \cap f^*) \\
F(X) & \xrightarrow{c_*} & H_*(X; \mathbb{Z}).
\end{array}
\]

(1.2)

The starting point of the present work is the observations that the constructible function \( F \) itself can be a bivariant theory and furthermore that there are several bivariant theories of constructible functions [Y4, 5]. For example, roughly speaking, a constructible function \( \alpha \in F(X) \) making the diagram (1.2) commutative, with \( c(T_f) \cap f^* \) replaced by a certain homomorphism \( \theta_f : H_*(Y; \mathbb{Z}) \to H_*(X; \mathbb{Z}) \), is also “bivariant” (see Theorem (2.5) below for more details).

In this paper we show that for a blow-up map (which is a non-smooth local complete intersection morphism) there does not exist any constructible function \( \alpha \in F(X) \) such that the diagram (1.2) is commutative, but that it is true for a Zariski locally trivial fiber bundle with the fiber being a local complete intersection in a smooth variety, and thus we speculate that it would be true even for fiber bundles in the usual topology.

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§2 Bivariant Theories of constructible functions

First we recall a general theory of bivariant theory due to Fulton and MacPherson (see [FM] for full details). A bivariant theory \( \mathcal{B} \) on a category \( \mathcal{C} \) with values in an abelian category is an assignment to each morphism

\[ X \xrightarrow{f} Y \]
in the category $\mathcal{C}$ a graded abelian group

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations:

(Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.

(Pushforward operations): For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f$ proper, the pushforward operation

$$f_* : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.

(Pullback operations): For a fiber square (which will be sometimes simply denoted by $X' = X \times_Y Y'$)

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y,
\end{array}
$$

the pullback operation

$$g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{g'} Y')$$

is defined.

And these three operations are required to satisfy the following seven axioms (see [FM, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{C}$. Then a Grothendieck transformation from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category $\mathcal{C}$, which preserves the above three basic operations.
They also introduced the notion of operational bivariant theory associated to a homology theory ([FM, Part I, §8]). Let $T_*$ be a covariant functor (or sometimes called a homology theory) on the category $C$. Then the associated operational bivariant theory $\mathcal{OT}$ of $T_*$ is defined as follows. For a morphism $f : X \to Y$, an element $c \in \mathcal{OT}(X \xrightarrow{f} Y)$ is defined to be a collection of homomorphisms

$$c(g) : T_*(Y') \to T_*(X)$$

for all $g : Y' \to Y$ and the fiber square $X' = X \times_Y Y'$. And these homomorphisms $c(g)$ are required to be compatible with proper pushforward, i.e., for a fiber diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{h'} & X' \\
\downarrow f'' & & \downarrow f' \\
Y'' & \xrightarrow{h} & Y'
\end{array}
\]

the following diagram must commute:

\[
\begin{array}{ccc}
T_*(Y'') & \xrightarrow{(c \circ h')^*} & T_*(X'') \\
\downarrow h_* & & \downarrow h_*' \\
T_*(Y') & \xrightarrow{c(g)} & T_*(X')
\end{array}
\]

If $C$ has a final object $pt$ and $T_*(pt)$ has a distinguished element $1$, then the homomorphism $\text{ev} : \mathcal{OT}(X \to pt) \to T_*(X)$ defined by $\text{ev}(c) := (c(\text{id}_{pt}))(1)$ is called the evaluation homomorphism.

Let $\mathbb{B}$ be a bivariant theory. Then the associated operational bivariant theory $\mathbb{B}^{\text{op}}$ of $\mathbb{B}$ is defined to be the operational bivariant theory constructed from the covariant functor $B_*(X) = \mathbb{B}(X \to pt)$. Then we have the following canonical Grothendieck transformation

$$\text{op} : \mathbb{B} \to \mathbb{B}^{\text{op}}$$

defined by, for each $\alpha \in \mathbb{B}(X \to Y)$,

$$\text{op}(\alpha) := \{(g^*\alpha)\bullet : \mathbb{B}(Y' \to pt) \to \mathbb{B}(X' \to p) \cdot g : Y' \to Y\}$$

where $X' = X \times_Y Y'$ is the fiber square.

Now we discuss some bivariant theories of constructible functions. First, the abelian group $F(X)$ of a given analytic variety $X$ consists of all the constructible functions on $X$. The association $X \mapsto F(X)$ becomes a contravariant functor with the usual pullback and at the same time a covariant functor with the pushforward $f_*$ which takes the topological Euler-Poincaré characteristic of the fibers weighted by constructible functions.

The constructible function functor $F$ itself can be a bivariant theory without any geometric or topological requirement on constructible functions as follows:
Proposition (2.1). \([Y5, \text{Proposition (3.1)}]\) For any morphism \(f : X \to Y\) the group \(s\mathbb{F}(X \to Y)\) is defined by
\[
s\mathbb{F}(X \overset{f}{\to} Y) := F(X),
\]
Then this is a bivariant theory with the following operations of product, pushforward and pullback, i.e., they satisfy the seven axioms of the bivariant theory.

(i): the product operation
\[
\bullet : s\mathbb{F}(X \overset{f}{\to} Y) \otimes s\mathbb{F}(Y \overset{g}{\to} Z) \to s\mathbb{F}(X \overset{gf}{\to} Z)
\]
is defined by:
\[
\alpha \bullet \beta := \alpha \cdot f^* \beta.
\]

(ii): the pushforward operation
\[
f_* : s\mathbb{F}(X \overset{gf}{\to} Z) \to s\mathbb{F}(Y \overset{g}{\to} Z)
\]
is the pushforward
\[
f_* : F(X) \to F(Y).
\]

(iii): For a fiber square
\[
\begin{array}{cc}
X' & \overset{g'}{\to} & X \\
\downarrow f' & & \downarrow f \\
Y' & \overset{g}{\to} & Y,
\end{array}
\]
the pullback operation
\[
g^* : s\mathbb{F}(X \overset{f}{\to} Y) \to s\mathbb{F}(X' \overset{g'}{\to} Y')
\]
is the pullback
\[
g'^* : F(X) \to F(X').
\]

This bivariant group is called the simple bivariant group of constructible functions.

It is clear that Axioms (B-2) and (B-3) hold, and to see that these three operations satisfy the other five axioms, we use the following three properties:

(2.2): for the above fiber square in (iii) the following diagram commutes (e.g., see [Er, Proposition 3.5], [FM, Axiom (A23)])
\[
\begin{array}{cc}
F(Y') & \overset{f'^*}{\to} & F(X') \\
\downarrow g' & & \downarrow g'^* \\
F(Y) & \overset{f^*}{\to} & F(X),
\end{array}
\]
(2.3): for a morphism $f : X \to Y$ and constructible functions $\alpha, \beta \in F(Y)$ we have

$$f^*(\alpha \cdot \beta) = f^* \alpha \cdot f^* \beta,$$

(2.4) (projection formula): for a morphism $f : X \to Y$ and constructible functions $\alpha \in F(Y)$ and $\beta \in F(X)$ we have

$$f_*(f^* \alpha \cdot \beta) = \alpha \cdot f_* \beta.$$

Let $\mathbb{H}$ be the Fulton-MacPherson’s bivariant homology theory, constructed from the cohomology theory, which is defined as follows. For a morphism $f : X \to Y$, choose a morphism $\phi : X \to \mathbb{R}^n$ such that $\Phi := (f, \phi) : X \to Y \times \mathbb{R}^n$ is a closed embedding; simply we say that $\phi$ “factors” $f$. (Such a morphism always exists, because, for example, we can take $\phi$ itself to be a closed embedding, thanks to the Whitney’s embedding theorem.)

Then the $i$-th bivariant homology group $\mathbb{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^i(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_{\phi}),$$

where $X_{\phi}$ is defined to be the image of the morphism $\Phi = (f, \phi)$. It turns out that the definition is independent of the choice of $\phi$ factoring $f$, i.e., if $\psi : X \to \mathbb{R}^m$ is another factorization of $f$, then we have

$$H^i(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_{\phi}) \cong H^i(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \setminus X_{\psi}).$$

Note that $\mathbb{H}^i(X \xrightarrow{id} Y) \cong H^i(Y)$ and $\mathbb{H}^i(X \to pt) \cong H_i(X)$ by the Alexander isomorphism. The bivariant group $\mathbb{H}^i(X \xrightarrow{f} Y)$ has the well-defined bivariant operations of product, pushforward and pullback. See [FM, §3.1] for full details of $\mathbb{H}$.

A key feature of the simple bivariant group $sF$ is the following result, which gives a counterexample to [FM, Part I, §8.2, pp.90-91]:

**Theorem (2.5).** ([Y5, Remark (3.3)]) Let $\mathbb{H}^p$ be the operational bivariant homology theory associated to the homology theory $H_*$. There does not exist a Grothendieck transformation

$$\gamma : sF \to \mathbb{H}^p$$

such that for each morphism $X \to pt$ the associated homomorphism followed by the evaluation homomorphism

$$ev \circ \gamma : F(X) = sF(X \to pt) \to \mathbb{H}^p(X \to pt) \to H_*(X)$$

is the Chern-Schwartz-MacPherson class homomorphism $c_*$.  

And similarly we can show the following:
**Theorem (2.6).** ([Y5, Theorem (3.2)]) There does not exist a Grothendieck transformation

\[ \gamma^s : sF \to \mathbb{H} \]

such that \( \gamma^s(\mathbb{1}_\pi) = c(TX) \cap [X] \) for \( X \) smooth, where \( \pi : X \to pt \) and \( \mathbb{1}_\pi = \mathbb{1}_X \).

The proof of these theorems tells us that the simple bivariant group, i.e., the constructible function group is certainly too large for the possible existence of a Grothendieck transformation from this bivariant theory of constructible functions to the bivariant homology theory. And there are more finer bivariant theories of constructible functions (see Theorem (2.7) below and also see [Y5]), but the most interesting bivariant theory of constructible functions, which is conjecturally best-fit, for the possible existence of a Grothendieck transformation, as supported by the Brasselet’s theorem [B1], is Fulton-MacPherson’s bivariant theory of constructible functions, i.e., \( \mathbb{F}(X \overset{f}{\to} Y) \) consists of all the constructible functions on \( X \) which satisfy the local Euler condition with respect to \( f \) (see [B], [FM], [Sa], [Z]). Here a constructible function \( \alpha \in F(X) \) is said to satisfy the local Euler condition with respect to \( f \) if for any point \( x \in X \) and for any local embedding \( (X, x) \to (C^N, 0) \) the following equality holds

\[ \alpha(x) = \chi(B_\epsilon \cap f^{-1}(z); \alpha), \]

where \( B_\epsilon \) is a sufficiently small open ball of the origin 0 with radius \( \epsilon \) and \( z \) is any point close to \( f(x) \) (cf. [B], [Sa]). The three operations on \( \mathbb{F} \) are the same as above in \( s\mathbb{F} \) and it is known that these three operations are well-defined for \( \mathbb{F} \) (e.g., see [BY], [Sa], [Z]). Note that \( \mathbb{F}(X \overset{id}{\to} X) \) consists of all locally constant functions and \( \mathbb{F}(X \to pt) = F(X) \).

Suppose that there is a Grothendieck transformation

\[ \gamma : \mathbb{F} \to \mathbb{H} \]

satisfying the normalization condition that \( \gamma(\mathbb{1}_\pi) = c(TX) \cap [X] \) for \( X \) smooth, where \( \pi : X \to pt \) and \( \mathbb{1}_\pi = \mathbb{1}_X \). In [B1], J.-P. Brasselet constructed such a Grothendieck transformation \( \gamma^{B_\epsilon} : \mathbb{F} \to \mathbb{H} \) in the category whose objects are complex analytic varieties and whose morphisms are cellular. Then, since Grothendieck transformations preserve product, pushforward and pullback, and since the Chern-Schwartz-MacPherson class transformation is unique, it follows that for any bivariant constructible function \( \alpha \in \mathbb{F}(X \to Y) \) we get the following commutative diagram (Vandier-Riemann-Roch for Chern classes associated to the bivariant constructible function \( \alpha \)):

\[
\begin{array}{ccc}
F(Y) & \overset{c_*}{\longrightarrow} & H_*(Y; \mathbb{Z}) \\
\alpha \cdot & & \gamma(\alpha) \cdot \\
F(X) & \overset{c_*}{\longrightarrow} & H_*(X; \mathbb{Z}).
\end{array}
\]

Here we emphasize that \( \alpha \bullet \) means \( \alpha \cdot f^* \) for \( f : X \to Y \).
Note that no matter which bivariant theory of constructible functions we consider, as long as we assume the existence of a Grothendieck transformation from that to the homology theory satisfying the above normalization condition, we get the above commutative diagram.

Such a commutative diagram as above, without requiring $\alpha \in F(X \to Y)$ and with $\gamma(\alpha) \circ \theta$ replaced simply by a certain homomorphism, already requires some strong condition on the constructible function $\alpha \in F(X)$, as observed below.

Observation (2.7). Suppose that for a morphism $f : X \to Y$ we have the following commutative diagram:

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
\downarrow \alpha \cdot f^* & & \downarrow \theta_f \\
F(X) & \xrightarrow{c_*} & H_*(X; \mathbb{Z}).
\end{array}
\]

(2.7.1)

with a constructible function $\alpha \in F(X)$ (not necessarily $\alpha \in F(X \to Y)$) and a homomorphism $\theta_f : H_*(Y; \mathbb{Z}) \to H_*(X; \mathbb{Z})$. Then we can see that this commutative diagram always implies the following things:

(i) If $f : X \to Y$ is not surjective, then $\alpha \equiv 0$.

(ii) The Chern-Schwartz-MacPherson classes $c_*(f^{-1}(y); \alpha) := c_*(\alpha|_{f^{-1}(y)})$ of the fiber weighted by the constructible function $\alpha$ are locally constant, considered as the homology classes in the total variety $X$. In particular, the pushforward $f_* \alpha \in F(Y)$ is locally constant, or equivalently, the Euler-Poincaré characteristics $\chi(f^{-1}(y); \alpha) := \chi(\alpha|_{f^{-1}(y)})$ of the fiber weighted by the constructible function $\alpha$ are locally constant.

(iii) If $f : \tilde{X} \to X$ is a blow-up of $X$ along a subvariety $V \subset X$ and let $E$ be the exceptional divisor, then $\alpha$ is constant on $\tilde{X} \setminus E = f^{-1}(X \setminus V)$.

Finally we note the following theorem:

Theorem (2.8). ([Y5, Theorem (3.6)]) For a morphism $f : X \to Y$, we define $\nu F(X \to Y)$ to be the set of all constructible functions $\alpha \in F(X)$ satisfying the following condition: for any morphism $g : Y' \to Y$ and fiber square

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

we have the following commutative diagram

\[
\begin{array}{ccc}
F(Y') & \xrightarrow{c_*} & H_*(Y'; \mathbb{Z}) \\
\downarrow g^* \alpha \cdot f'^* & & \downarrow \theta_f' \\
F(X') & \xrightarrow{c_*} & H_*(X'; \mathbb{Z}).
\end{array}
\]
with a certain homomorphism \( \theta_p : H_*(Y'; \mathbb{Z}) \to H_*(X'; \mathbb{Z}) \). Then we have

(i) \( vF \) becomes a bivariant theory with the same operations of product, pushforward and pullback as in \( sF \),

(ii) \( vF(X \xrightarrow{\text{id}} X) \) consists of all locally constant functions, and

(iii) \( vF(X \to pt) = F(X) \).

Remark (2.9). (i) In general \( vF(X \to Y) \neq F(X \to Y) \). Indeed, consider a blow-up \( \pi : \tilde{X} \to X \) of \( X \) (\( \dim X > 1 \)) along a nonsingular subvariety \( V \subset X \) whose codimension is \( > 1 \) and let \( E \) be the exceptional divisor, which is a fiber bundle over \( V \) with the fiber being the projective space of dimension \( = \text{codimension of } V - 1 \). Let us take one point \( v_0 \in V \) and choose any two different points \( x, y \) from the fiber \( \pi^{-1}(v_0) = \mathbb{P}^{\text{codim } V - 1} \). And we set \( \alpha = 1_x - 1_y \in F(\tilde{X}) \). Then it is clear that we have the following commutative diagram with \( \theta_\pi \) being the zero homomorphism

\[
\begin{array}{ccc}
F(X) & \xrightarrow{c_*} & H_*(X; \mathbb{Z}) \\
\downarrow{\alpha \cdot \pi^*} & & \downarrow{0} \\
F(\tilde{X}) & \xrightarrow{c_*} & H_*(\tilde{X}; \mathbb{Z}).
\end{array}
\]

In fact, we can see that for any morphism \( g : Y' \to Y \) and the fiber square

\[
\begin{array}{ccc}
\tilde{X}' & \xrightarrow{g'} & \tilde{X} \\
\downarrow{\pi} & & \downarrow{\pi} \\
X' & \xrightarrow{g} & X,
\end{array}
\]

we have the following commutative diagram

\[
\begin{array}{ccc}
F(X') & \xrightarrow{c_*} & H_*(X'; \mathbb{Z}) \\
\downarrow{g'^* \alpha \cdot \pi'^*} & & \downarrow{0} \\
F(\tilde{X}') & \xrightarrow{c_*} & H_*(\tilde{X}'; \mathbb{Z}).
\end{array}
\]

Hence \( \alpha \in vF(\tilde{X} \xrightarrow{\pi} X) \). On the other hand we have \( \alpha \notin F(\tilde{X} \xrightarrow{\pi} X) \).

(ii) One might be tempted to think or guess that even if in the definition of \( vF \) we just require the constructible function \( \alpha \in F(X) \) to satisfy the commutativity of the diagram (2.7.1) instead of considering all the fiber squares as above we would get a bivariant theory, but it is not the case. It is because the pullback is not well-defined.

\section*{§3 Results}

Recall that a local complete intersection morphism \( f : X \to Y \) is the composite \( f = p \circ r \) of a regular embedding \( r : X \to P \) (i.e., \( X \) is a local complete intersection in
and a smooth morphism $p : P \to Y$. Local complete intersection morphisms which we deal with are blowups (see [E, 6.7] and [FM, 9.2.2]). Let $Z$ be a regularly embedded closed subscheme of a scheme $Y$, and let $\tilde{Y}$ be the blow-up of $Y$ along $Z$, then the projection $f : \tilde{Y} \to Y$ is a local complete intersection morphism of relative codimension 0.

**Theorem (3.1).** Let $f : \overline{\mathbb{P}}^n \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at a single point $P$. Then there is no constructible function $\alpha \in F(\overline{\mathbb{P}}^n)$ such that the diagram (1.2) is commutative.

**Proof.** Let us suppose that there exist some constructible function $\alpha \in F(\overline{\mathbb{P}}^n)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F(\mathbb{P}^n) & \xrightarrow{\alpha \cdot f^*} & H_*(\mathbb{P}^n; \mathbb{Z}) \\
\downarrow c_* & & \downarrow c(T_f) \cap f^* \\
F(\overline{\mathbb{P}}^n) & \xrightarrow{c_*} & H_*(\overline{\mathbb{P}}^n; \mathbb{Z}).
\end{array}
$$

Then it follows from the commutativity of the above diagram that $\alpha = 1$ on $\overline{\mathbb{P}}^n \setminus E$, where $E(\cong \mathbb{P}^{n-1})$ denotes the exceptional divisor. Indeed, let $Q$ be any point different from $P$ and set $Q' = f^{-1}(Q)$. Then we have

$$
\alpha(Q') = c_*(\alpha(Q') \cdot 1_Q) = c_*(\alpha \cdot f^*(1_Q)) = c(T_f) \cap f^*(c_*(1_Q)) = c(T_f) \cap f^*([Q]) = c(T_f) \cap [Q'] = [Q'] = 1.
$$

Hence $\alpha$ can be expressed as

$$
\alpha = 1_{\overline{\mathbb{P}}^n \setminus E} + \beta_E
$$

with some constructible function $\beta_E$ supported on $E$. Then we have

$$
c_* (\alpha \cdot f^* (1_P)) = c_*(\alpha \cdot 1_E) = c_*(\beta_E).
$$

On the other hand, by the same argument as above, we get that

$$
c_* (\alpha \cdot f^* (1_P)) = c(T_f) \cap f^* (c_*(1_P)) = c(T_f) \cap f^*([P]) = 1.
$$

Thus it follows that $c_*(\beta_E) = 1$. Now, since any constructible function can be expressed as a linear combination of some characteristic functions of subvarieties, it follows from the irreducibility of $E$ that

$$
\beta_E = b \cdot 1_E + \delta
$$

where $b$ is some integer and $\delta$ is another constructible function supported on subvarieties of lower dimensions, i.e., of dimension $< n - 1$. If $b \neq 0$, then

$$
c_*(\beta_E) = b|E| + \cdots,
$$

since $[E]$ cannot vanish, because $H_{2n-2}(\overline{\mathbb{P}}^n) \cong H_{2n-2}(\mathbb{P}^n \setminus P) \oplus H_{2n-2}(E)$ by the Mayer-Vietoris sequence. Hence $c_*(\beta_E) \neq 1$. Therefore $b = 0$, and since $c_*(\beta_E) = 1$ it follows
that $\delta \neq 0$ and that $c_*(\delta) = 1$. Now consider any line $L$ going through the blown-up point $P$, and let $\tilde{L}$ be the proper transform of $L$ and let $[[L]]$ be the corresponding point in the exceptional divisor $E$. Then we have

$$\alpha \cdot f^*(H_L) = (H_{P^\infty\setminus E} + \delta) \cdot (H_{E} + H_{\tilde{L}} - 1_{[[L]]})$$

$$= \delta + 1_{\tilde{L}} - 1_{[[L]]}$$

Therefore we have

$$c_*(\alpha \cdot f^*(H_L)) = c_*(\delta) + c_*(\tilde{L}) - c_*([[L]])$$

$$= c_*(\delta) + c_*(\tilde{L}) - 1$$

$$= c_*(\tilde{L}) \quad \text{(since } c_*(\delta) = 1)$$

$$= [\tilde{L}] + 2 \quad \text{(since } \tilde{L} \cong \mathbb{P}^1).$$

On the other hand it follows from [F, Corollary 6.7.1] and the commutativity of the pullbacks with the cycle map [F, Example 19.2.1] that we have

$$f^*(L) = [\tilde{L}] + [L']$$

with $L'$ is a projective line in $E \cong \mathbb{P}^{n-1}$. Therefore we get that

$$c(T_f) \cap f^*(c_*(L)) = c(T_f) \cap f^*([L] + 2) \quad \text{(since } L \cong \mathbb{P}^1)$$

$$= c(T_f) \cap f^*([L]) + 2 \quad \text{(since } c(T_f) = 1 + \cdots)$$

$$= c(T_f) \cap ([\tilde{L}] + [L']) + 2$$

$$= [\tilde{L}] + [L'] + \text{some integer.}$$

Here we note that $[L']$ does not vanish since $H_2(\mathbb{P}^n) \cong H_2(\mathbb{P}^n \setminus P) \oplus H_2(E)$ by the Mayer-Vietoris sequence. Then the above commutative diagram implies that

$$c_*(\alpha \cdot f^*(H_L)) = c(T_f) \cap f^*(c_*(L)),$$

namely we have to have that

$$[\tilde{L}] + 2 = [\tilde{L}] + [L'] + \text{some integer.}$$

Thus, in particular $[L']$ has to vanish, which is a contradiction. □

More generally, we can construct such an example from any variety $V$ of dimension $n > 2$. 
Corollary (3.2). Let $\tilde{V}$ be the blow-up of $V$ at any smooth point of $V$ and $f : \tilde{V} \to \tilde{V}$ the blow-up of $\tilde{V}$ at any point of the exceptional divisor ($\cong \mathbb{P}^{n-1}$) of $\tilde{V}$. Then there is no constructible function $\alpha \in F(\tilde{V})$ such that the diagram (1.2) is commutative.

This can be seen using the above case of the blow-up $f : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$.

As we can see from the above argument, as to the cohomology class $c(T_f)$, we use only the fact that the 0-dimensional part of the cohomology class $c(T_f)$ is equal to 1, thus as a corollary we get

Corollary (3.3). Let $f : \tilde{X} \to X$ be a blow-up map as above. Then there are no constructible function $\alpha \in F(\tilde{X})$ and no total cohomology class $\Phi(\tilde{X})$ of $\tilde{X}$ whose 0-dimensional part is 1, i.e., no total cohomology class $\Phi(\tilde{X}) \in 1 + \bigoplus_{i \geq 0} H^{2i}(\tilde{X}; \mathbb{Z})$, such that the following diagram is commutative:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\alpha} & H_*(X; \mathbb{Z}) \\
\downarrow \alpha \cdot f^* & & \downarrow \Phi(\tilde{X}) \cap f^* \\
F(\tilde{X}) & \xrightarrow{c_*} & H_*(\tilde{X}; \mathbb{Z}).
\end{array}
$$

Remark (3.4). For a blow-up map $\pi : \tilde{X} \to X$ along any subvariety of $X$ there is no bivariant constructible function $\alpha \in F(\tilde{X} \to X)$ whose value is generically equal to 1, because of the local Euler condition imposed on the constructible function $\alpha$ (e.g., see [Sa, (1.3) Remarque]). As to the bivariant-theoretic “Riemann-Roch formula”, it is therefore obvious that for any blow-up map $\pi : \tilde{X} \to X$ there is no canonical orientation $\alpha_f \in F(X \to Y)$ and no total cohomology class $\Phi(\tilde{X})$ of $\tilde{X}$ whose 0-dimensional part is 1 such that

$$
\gamma(\alpha) = \Phi(\tilde{X}) \bullet U_f
$$

for any Grothendieck transformation $\gamma : F \to \mathbb{H}$ satisfying the normalization condition that $\gamma(\mathbb{1}_\pi) = c(TX) \cap [x]$ for $X$ smooth with $\pi : X \to pt$ and $\mathbb{1}_\pi := \mathbb{1}_X$.

Next, we consider cases of non-smooth local complete intersection morphisms for which the diagram (1.2) is commutative.

Let $X$ be a possibly singular variety embeddable into a nonsingular variety $M$. Then the Fulton-Chern class or the canonical class $c^F(X)$ of $X$ ([F, Example 4.2.6]) is defined by

$$
c^F(X) := c(TM|_X) \cap s(X, M),
$$

where $s(X, M)$ is the relative Segre class of $X$ in $M$ ([F, §4.2]). If $X$ is a local complete intersection in a smooth variety $M$, then the Fulton-Johnson class $c^{FJ}(X)$ of $X$ ([FJ]) and the Fulton-Chern class of $X$ ([F, Example 4.2.6]) are the same and they are expressed by

$$
c^{FJ}(X) = c^F(X) = c(TM|_X) c(NX M)^{-1} \cap [X] = c(TX) \cap [X],
$$
where $T_X = TM|_X - N_X M$ and $N_X M$ is the normal bundle of $X$ in $M$. Since the Chern-Schwartz-MacPherson class homomorphism $c_* : F(W) \to A_*(W)$ is always surjective for any variety $W$, any polynomial or power series of Chern classes of vector bundles acts on the Chow homology group and the action commutes with the cycle map (see [F, Remark 3.2.2, Proposition 19.1.2]), there exists a constructible function $\xi^{FJ} \in F(X)$ (which shall be called a *Fulton-Johnson constructible function*) such that

$$c^{FJ}(X) = c_*(\xi^{FJ}).$$

Note that there are of course infinitely many such constructible functions. To obtain the constructible function $\xi^{FJ}$ in a canonical form, we note that the difference between the two classes $c^{FJ}(X)$ and $c_*(X)$ is supported on the singular locus (e.g., see [Su1] for a rigorous proof of this). Hence a reasonable canonical constructible function $\xi^{FJ}$ is of the form

$$\xi^{FJ} = 1_X + \varepsilon_{X_{\text{Sing}}}.$$ 

with a constructible function $\varepsilon_{X_{\text{Sing}}}$ supported on the singular locus $X_{\text{Sing}}$ of $X$, and up to sign, the Chern-Schwartz-MacPherson class of the extra constructible function $\varepsilon_{X_{\text{Sing}}}$, considered as the homology class of the ambient variety $X$, is the so-called *Milnor class* of $X$ and usually denoted by $\mathcal{M}(X)$. More precisely,

$$\mathcal{M}(X) = (-1)^{\dim X} c_*(\varepsilon_{X_{\text{Sing}}}).$$

See [A3], [BLSS1, 2], [PP3], [Su1], etc., for some fundamental results and formulas of the Milnor class. With this definition we have

**Proposition (3.5).** Let $X$ be a singular local complete intersection in a smooth variety $M$ and let $X \subseteq X \times Y \xrightarrow{p} Y$ be the projections to each factor.

(i) The following diagram commutes:

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
\downarrow{(q^*\xi^{FJ}) \cdot p^*} & & \downarrow{c(T_p \cap p^*)} \\
F(X \times Y) & \xrightarrow{c_*} & H_*(X \times Y; \mathbb{Z})
\end{array}
$$

(ii) If we let $p^* := \gamma(\mathbb{1}_{X \times Y}) \bullet : H_*(Y; \mathbb{Z}) \to H_*(X \times Y; \mathbb{Z})$ for any Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ satisfying the normalization condition, then for any homology class $b \in H_*(Y; \mathbb{Z})$ we have

$$p^*(b) = c(T_p) \cap p^*(b) - (-1)^{\dim X} \mathcal{M}(X) \times b \quad \text{or}$$

$$p^*(b) = c(T_p) \cap p^*(b) - \gamma(\varepsilon_{X_{\text{Sing}}} \times \mathbb{1}_Y) \bullet b.$$
Proof. (i) The relative tangent bundle $T_p$ is $q^*T_X$ and the homology pullback $p^* : H_*(Y) \to H_*(X \times Y)$ is given by $p^*(b) = [X] \times b$, the homology cross product. Hence for any homology class $b \in H_*(Y)$ we have

\begin{equation}
(3.5.1) \quad c(T_p) \cap p^*(b) = c(q^*T_X) \cap ([X] \times b) \\
= q^*c(T_X) \cap ([X] \times b) \\
= (c(T_X) \times 1) \cap ([X] \times b) \\
= (c(T_X) \cap [X]) \times b \quad \text{(see Note below)} \\
= c_{FJ}(X) \times b.
\end{equation}

**Note:** In general, for cohomology classes $\alpha_1 \in H^d(X), \alpha_2 \in H^e(Y)$ and homology classes $a_1 \in H_m(X), a_2 \in H_n(Y)$, we have $(\alpha_1 \times \alpha_2) \cap (a_1 \times a_2) = (-1)^{d(n-e)}(\alpha_1 \cap a_1) \times (\alpha_2 \cap a_2)$. In our case $d(n - e)$ is even.

Therefore, for any constructible function $\beta \in F(Y)$ we have

\[ c(T_p) \cap p^*(c_*(\beta)) = c_*(\xi_{FJ}^* \times c_*(\beta)). \]

The cross product $\omega \times \zeta$ of two constructible functions $\omega \in F(W)$ and $\zeta \in F(Z)$ is defined to be $(\omega \times \zeta)(w, z) := \omega(w)\zeta(z)$. Then, the Kwieciński’s cross product formula [K1, Théorème 1] (cf. [KY, Theorem 4]) says that $c_*(\omega \times \zeta) = c_*(\omega) \times c_*(\zeta)$. Therefore we have

\[ c(T_p) \cap p^*(c_*(\beta)) = c_*(\xi_{FJ}^* \times \beta) = c_*(q^* \xi_{FJ}^* \cdot p^* \beta), \]

which is nothing but the commutativity of the above diagram.

(ii) First of all we observe that for any constructible function $\alpha \in F(X) = F(X \to p)$ and $b \in H_*(Y; Z)$ we have

\begin{equation}
(3.5.2) \quad c_*(\alpha) \times b = \gamma(q^* \alpha) \bullet b \in H_*(X \times Y; Z). 
\end{equation}

Here we should note that the right-hand-side actually means

\[ A_X \times Y(\gamma(q^* \alpha) \bullet A_Y^{-1}(b)), \]

where $A_Z : H^*(N, N \setminus Z) \xrightarrow{\gamma} H_*(Z)$ is the Alexander duality isomorphism for a variety $Z$ embedded in an (in fact, any) smooth variety $N$. Also note that the Alexander duality isomorphisms commute with homology and cohomology cross products and furthermore for a morphism $X \to pt, c_* = A_X \circ \gamma$, due to the normalization condition imposed on $\gamma$.

Before showing (3.5.2), let us recall the bivariant cross products ([FM, 2.4, p.24]): For morphisms $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$, the bivariant theoretic cross product $\times$

\[ \times : \mathbb{B}(X_1 \xrightarrow{f} X_2) \otimes \mathbb{B}(Y_1 \xrightarrow{g} Y_2) \to \mathbb{B}(X_1 \times Y_1 \xrightarrow{f \times g} X_2 \times Y_2) \]

is defined by

\[ \alpha \times \beta := h^* \alpha \bullet s^* \beta, \]
where \( h : X_2 \times Y_1 \to X_2 \) and \( s : X_2 \times Y_2 \to Y_2 \) are the projections.

In our case consider the following fiber square:

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{p} & Y \\
q \downarrow & & \downarrow \pi_2 \\
X & \xrightarrow{\pi_1} & pt.
\end{array}
\]

Then (3.5.2) can be seen as follows:

\[
c_*(\alpha) \times b = A_{X \times Y}(A_X^{-1}(c_*(\alpha)) \times A_Y^{-1}(b)) \\
= A_{X \times Y}(\gamma(\alpha) \times A_Y^{-1}(b)) \quad \text{(since } c_* = A_X \circ \gamma) \\
= A_{X \times Y}(\pi_2^* \gamma(\alpha) \bullet A_Y^{-1}(b)) \quad \text{(by the bivariant cross product)} \\
= A_{X \times Y}(\gamma(\pi_2^* \alpha) \bullet A_Y^{-1}(b)) \quad \text{(since } \gamma \text{ commutes with pullbacks)} \\
= A_{X \times Y}(\gamma(q^* \alpha) \bullet A_Y^{-1}(b)) \\
= \gamma(q^* \alpha) \bullet b
\]

Therefore (3.5.1) can be expressed as follows:

\[
(3.5.3) \quad c(T_p) \cap p^*(b) = \gamma(q^* \xi^F) \bullet b, \\
= \gamma(q^* \mathbb{1}_X) \bullet b + \gamma(q^* \varepsilon_{X_{\text{Sing}}}) \bullet b, \\
= \gamma(\mathbb{1}_X \times Y) \bullet b + \gamma(\varepsilon_{X_{\text{Sing}} \times Y}) \bullet b.
\]

Thus we get (ii). \( \square \)

Note that Proposition (3.5)(ii) is a solution to [Y1, Problem (3.4)] or [BY, Problem (4.5)] in the case of trivial fiber bundles.

Now, motivated or hinted by the well-known fact (e.g., see the recent articles [Cr, \S 1.3] and [Gö, Remark 4.1]) that if \( f : Z \to Y \) is a Zariski locally trivial fiber bundle with fiber \( F \), then

\[
[Z] = [Y][F]
\]

in the Grothendieck ring of complex algebraic (or analytic) varieties, just like in the case of \( Z = Y \times F \), by a stratification of the base variety \( Y \) and using the above proposition we can show the following

**Theorem (3.6).** Let \( X \) be a local complete intersection in a smooth variety \( M \) and \( p : X \times Y \to Y \) be a Zariski locally trivial fiber bundle over a possibly singular variety \( Y \) with fiber \( X \). Furthermore we assume that \( X \times Y \to Y \) be a subbundle of a Zariski locally trivial fiber bundle \( M \times Y \to Y \) with fiber \( M \). Let \( \xi^F = \mathbb{1}_X + \varepsilon_{X_{\text{Sing}}} \) be as above, and set \( (\xi^F \times \mathbb{1}_Y)(x, y) := \xi^F(x) \), a constructible function on the total fiber variety \( X \times Y \) (which is in fact a bivariant constructible function, i.e., satisfies the local Euler condition
with respect to the projection $p$.) Let $p^* = \gamma(\mathbb{L}_{\pi Y}) \cdot H_*(Y; \mathbb{Z}) \to H_*(X \tilde{\times} Y; \mathbb{Z})$ for any Grothendieck transformation $\gamma: \mathbb{F} \to \mathbb{H}$ satisfying the normalization condition. Then we have the following commutative diagram

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
\downarrow \rho_*(\xi_{\pi Y} \cdot p^*) & & \downarrow \rho_*(\xi_{\pi Y} \cdot p^*) \\
F(X \tilde{\times} Y) & \xrightarrow{c_*} & H_*(X \tilde{\times} Y; \mathbb{Z}),
\end{array}
$$

Proof. Since $p: X \tilde{\times} Y \to Y$ is a Zariski locally trivial fiber bundle, there exists a stratification $\{Y_i\}$ of $Y$ by constructible sets $Y_i$'s such that the restriction of $p$ to each strata $Y_i$

$$
p_i := p|_{p^{-1}(Y_i)}: X \times Y_i \to Y_i
$$

is a trivial bundle. Note that $F(Y) = \bigoplus F(Y_i)$, i.e., any constructible function of $Y$ can be expressed as a direct sum of constructible functions of the constructible sets $Y_i$. Let $\eta_i: Y_i \to Y$ and $\rho_i: X \times Y_i \to X \tilde{\times} Y$ be the inclusions. Let $c_i \in F(Y_i)$. Then it follows from Proposition (3.5) that the following diagram is commutative

$$
\begin{array}{ccc}
F(Y_i) & \xrightarrow{c_*} & H_*(Y_i; \mathbb{Z}) \\
\downarrow \rho_*(\xi_{\pi Y} \cdot p_i^*) & & \downarrow \rho_*(\xi_{\pi Y} \cdot p_i^*) \\
F(X \times Y_i) & \xrightarrow{c_*} & H_*(X \times Y_i; \mathbb{Z}),
\end{array}
$$

i.e., for any constructible function $\beta_i \in F(Y_i)$

$$
c(T_{\rho_i}) \cap p_i^* (c_*(\beta_i)) = c_*(\xi_{\pi Y} \cdot \eta_i^* p_i^*(\beta_i)).
$$

Now, let $\beta \in F(Y)$. Then we can express $\beta$ as follows:

$$\beta = \sum_i \eta_i^* \beta_i, \quad \beta_i \in F(Y_i).$$

Consider the following fiber square

$$
\begin{array}{ccc}
X \times Y_i & \xrightarrow{\rho_i} & X \tilde{\times} Y \\
\downarrow p_i & & \downarrow p \\
Y_i & \xrightarrow{\eta_i} & Y.
\end{array}
$$
Since $p$ is flat, it follows from [F, Proposition 1.7] that the following diagram is commutative

$$
\begin{array}{ccc}
H_*(X \times Y_l) & \xrightarrow{\rho_i*} & H_*(X \times \bar{Y}) \\
\uparrow p^! & & \uparrow p^*
\end{array}
$$

$$
\begin{array}{cc}
H_*(Y_l) & \xrightarrow{\eta_i*} & H_*(Y).
\end{array}
$$

Here it should be noted that the “constructible function” version of this commutative diagram always holds for any fiber square, namely the formula (2.2) given in §2; thus we do not require the flatness of $p$. Therefore we get

$$
\sum_i \rho_i* (c(T_{p_i}) \cap p_i^* (c_*(\beta_i)))
$$

$$
\begin{aligned}
&= \sum_i \rho_i* (c(\rho_i^* T_p) \cap p_i^* (c_*(\beta_i))) \\
&= \sum_i c(T_p) \cap \rho_i* p_i^* (c_*(\beta_i)) \\
&= \sum_i c(T_p) \cap p^* \eta_i* (c_*(\beta_i)) \\
&= \sum_i c(T_p) \cap p^* (c_*(\eta_i* \beta_i)) \\
&= c(T_p) \cap p^* \left( c_*(\sum_i \eta_i* \beta_i) \right) \\
&= c(T_p) \cap p^* (c_*(\beta)).
\end{aligned}
$$

On the other hand, we get that

$$
\sum_i \rho_i* c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p_i^* \beta_i \right)
$$

$$
\begin{aligned}
&= \sum_i c_* (\rho_i* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p_i^* \beta_i \right)) \\
&= \sum_i c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot \rho_i* p_i^* \beta_i \right) \\
&= \sum_i c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p^* \eta_i* \beta_i \right) \\
&= c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p^* \left( \sum_i \eta_i* \beta_i \right) \right) \\
&= c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p^*(\beta) \right).
\end{aligned}
$$

Hence, for any constructible $\beta \in F(Y)$, we have

$$
c(T_p) \cap p^* (c_*(\beta)) = c_* \left( (\xi^{F,J} \times 1_{Y_l}) \cdot p^*(\beta) \right),
$$
thus the theorem holds. □

It is clear from the above proof that for any homology class \( b \in H_\ast(Y) \) determined by an algebraic or analytic cycle we have

\[
p^\ast(b) = c(T_p) \cap p^\ast(b) - \gamma(\varepsilon_{X_{\Sing}} \times \mathbb{L}_Y) \cdot b,
\]

which is just another way of putting Theorem (3.6). If \( H_\ast(Y) = \bigoplus H_\ast(Y_i) \), then by the same argument above we could conclude that

\[
(3.7)
\]

\[
p^\ast = c(T_p) \cap p^\ast - \gamma(\varepsilon_{X_{\Sing}} \times \mathbb{L}_Y) \cdot b.
\]

However, it is obvious that in general \( H_\ast(Y) \neq \bigoplus H_\ast(Y_i) \), hence we still need to prove the above equality (3.7), if it holds. We speculate that (3.7) would be correct.

Now it is reasonable to make the following

**Conjecture (3.8).** Even if the Zariski topology is replaced by the usual topology in Theorem (3.6), (i) the above diagram is commutative and (ii) (3.7) also hold.

Note that by the induction on dimensions of subvarieties of \( Y \) we can see that the statement (i) of the above conjecture holds if and only if for the above bundle

\[
c_\ast(\xi^F \times \mathbb{L}_Y) = c(T_p) \cap p^\ast(c_\ast(Y)).
\]

Furthermore we note that

\[
c_\ast(\xi^F \times \mathbb{L}_Y) = \gamma(\xi^F \times \mathbb{L}_Y) \cdot c_\ast(Y).
\]

**Remark (3.9).** Since \( c(T_p) \cap p^\ast(b) = c(T_p) \cdot U_p \cdot b \), we speculate that

\[
\gamma(\xi^F \times \mathbb{L}_Y) = c(T_p) \cdot U_p.
\]

Otherwise this simple example would show the non-uniqueness of the Grothendieck transformation \( \gamma : F \to \mathbb{L} \) satisfying the normalization condition.

**Remark (3.10).** In this remark we discuss the extra constructible function \( \varepsilon_{X_{\Sing}} \) a bit more.

Let \( E \) be a vector bundle of rank \( k \) over a nonsingular variety \( M \) of dimension \( n + k \) and \( s : M \to E \) a regular section, and let \( X := s^{-1}(0) \) be the zero of the section, which is a local complete intersection of dimension \( n \). Let \( i : X \to M \) be the inclusion. Then the Fulton-Johnson class \( c^F(X) \) is nothing but \( \frac{i^\ast c(TM)}{i^\ast c(E)} \cap [X] \) and the Milnor class is by definition

\[
\mathcal{M}(X) := (-1)^{\dim X} \left( \frac{i^\ast c(TM)}{i^\ast c(E)} \cap [X] - c_\ast(X) \right) = (-1)^{\dim X} c_\ast(\varepsilon_{X_{\Sing}}).
\]
When $X$ has isolated singularities $x_1, x_2, \ldots, x_r$, it follows from [Su1] that we can set

$$\varepsilon_{\text{Sing}} = \sum_{i+1}^{r} \mu_{x_i} \pi_{x_i},$$

where $\mu_{x_i}$ denoting the Milnor number of $X$ at the isolated singularity $x_i$.

In [A3, Theorem I.5] P. Aluffi expressed the Milnor class in terms of his $\mu$-class [A2] in the case of hypersurfaces; very roughly speaking, he described the Milnor class as follows ([A3, Theorem I.4]):

$$\mathcal{M}(X) = \frac{1}{c(L)} \cap (\text{some homology classes supported on the singular locus of } X),$$

where $E = L$ is a line bundle since we are now dealing with the hypersurface case. More precisely

$$\mathcal{M}(X) = (-1)^{\dim X} \frac{1}{c(L)} \cap (s(Y, M)^{\vee} \otimes L),$$

where $Y$ is the singular locus of $X$ and $s(Y, M)$ is the relative Segre class [F, §4.2] and see [A1], [A2] and [A3] for the other notation.

In [PP3] A. Parusiński and P. Pragacz described the Milnor class in the case of hypersurfaces, using some data coming from Whitney stratifications of the hypersurface $X$. They described it as follows:

$$\mathcal{M}(X) = \frac{1}{c(E)} \cap \sum_{S \in \mathcal{S}_X \subseteq \text{Sing}(X)} \alpha_S 
\quad \mathcal{C} \quad \mathcal{S}_X \subseteq \text{Sing}(X).$$

Here $\mathcal{S}_X$ is a (in fact, any) Whitney stratification of $X$ and $\alpha_S$ is a certain integer attached to each stratum $S$ obtained by using the Milnor numbers of the strata. (The degree 0-part of this formula, i.e., a formula for the Euler-Poincaré characteristic $\chi(X)$ of $X$ was obtained in [PP2, Theorem 4].)

In [BLSS2] (see [BLSS1] for its summary) J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa have expressed the Milnor class by the so-called localized Milnor classes of the connected components of the singular locus. In the special case when the connected components are all nonsingular, they describe it very explicitly involving some kind of cohomology classes “$a$” [BLSS2, Lemma 7.5, Theorem 7.6 and Corollary 7.7]; which is, roughly speaking, as follows:

$$\mathcal{M}(X) = \sum_{S} (\text{some cohomology class}) \frac{1}{c(E)} \cap \mathcal{C} \quad \mathcal{S}_X \subseteq \text{Sing}(X),$$

where $S$'s are the connected components of the singular locus and assumed to be non-singular.
So one could expect that a general formula would be of the following form

\[ \mathcal{M}(X) = \sum_{\substack{S \in S_X \\ \text{S } \subseteq \text{Sing}(X)}} \frac{\Theta_S}{c(E)} \cap (i_{\pi_X})_* c_*(S), \]

where \( i_{A,B} : A \to B \) is the inclusion and \( S_X \) is a Whitney stratification of \( X \) and \( \Theta_S \) is a certain cohomology class involving not only the bundle \( E \) but also the subvariety \( S \), and the problem is of course to determine the cohomology class \( \Theta_S \). (Note that in general \( \Theta_S \) cannot be expressed as a polynomial in the individual Chern classes of \( E \) [OY] and surely involves some classes of the variety \( S \) itself as seen in [BLSS2].) Thus, we could express the extra constructible function \( \varepsilon_{X_{\text{Sing}}} \), as a “cohomology valued” constructible function, as follows:

\[ \sum_{\substack{S \in S_X \\ \text{S } \subseteq \text{Sing}(X)}} \frac{\Theta_S}{c(E)} \cdot \mathbb{1}_S. \]

Certainly the action of \( c_* \) on a \( H^*(X) \)-valued constructible function is defined to be

\[ c_*(\sum_W \Theta_W \cdot \mathbb{1}_W) := \sum_W \Theta_W \cap c_*(W), \]

with \( \Theta_W \) denoting a cohomology class of \( X \).

References


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