Weighted Bergman Kernels and Quantization

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WEIGHTED BERGMAN KERNELS AND QUANTIZATION

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Abstract. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^N$, $\phi, \psi$ two positive functions on $\Omega$ such that $-\log \psi, -\log \phi$ are plurisubharmonic, $z \in \Omega$ a point at which $-\log \phi$ is smooth and strictly plurisubharmonic, and $M$ a nonnegative integer. We show that as $k \to \infty$, the Bergman kernels with respect to the weights $\phi^k \psi^M$ have an asymptotic expansion

$$K_{\phi^k \psi^M}(x, y) = \frac{k^N}{\pi^N \phi(x, y) \psi(x, y)^M} \sum_{j=0}^{\infty} b_j(x, y) \phi^{-j}, \quad b_0(x, y) = \det \left[ -\frac{\partial^2 \log \phi(x)}{\partial z_j \bar{\sigma}_k} \right]$$

for $x, y$ near $z$, where $\phi(x, y)$ is an almost-analytic extension of $\phi(x) = \phi(x, x)$ and similarly for $\psi$. If in addition $\Omega$ is of finite type, $\phi, \psi$ behave reasonably at the boundary and $-\log \phi, -\log \psi$ are strictly plurisubharmonic on $\Omega$, we obtain also an analogous asymptotic expansion for the Berezin transform and give applications to the Berezin quantization. Finally, for $\Omega$ smoothly bounded and strictly pseudoconvex and $\phi$ a smooth strictly plurisubharmonic defining function for $\Omega$, we also obtain results about the Berezin-Toeplitz quantization on $\Omega$.

Let $\Omega$ be a domain in $\mathbb{C}^N$, $\rho$ a positive continuous function on $\Omega$, and $K_{\rho}$ the reproducing kernel of the weighted Bergman space $A^2(\Omega, \rho)$ of all holomorphic functions on $\Omega$ square-integrable with respect to the measure $\rho(z)^{1/2} d\omega$, $d\omega$ being the Euclidean volume element in $\mathbb{C}^N$; we call $K_{\rho}$ the weighted Bergman kernel corresponding to $\rho$, and for $\rho \equiv 1$ we will speak simply of the Bergman kernel $K_{\rho}$ of $\Omega$. The Berezin transform $B_{\rho}$ is the integral operator defined by

$$B_{\rho} f(y) = \int_{\Omega} f(x) \frac{|K_{\rho}(x, y)|^2}{K_{\rho}(y, y)} / \rho(x) \, dx$$

for all $y$ for which $K_{\rho}(y, y) \neq 0$. In terms of the operator $M_f$ of multiplication by $f$ on the space $L^2(\Omega, \rho \, d\omega)$ this can be rewritten as

$$B_{\rho} f(y) = \frac{\langle M_f K_{\rho}(. , y), K_{\rho}(\cdot , y) \rangle}{\|K_{\rho}(\cdot , y)\|^2},$$

from which it is immediate that the integral (1) converges, for instance, for any bounded measurable function $f$.

The Berezin transform was first introduced by F.A. Berezin [Ber] in the context of quantization of Kähler manifolds. More specifically, let $\phi$ be a positive function on $\Omega$ such that $-\log \phi$ is strictly plurisubharmonic, and set

$$g, \phi = \partial^2 (-\log \phi) / \partial z_j \partial \bar{\sigma}_k$$

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and $\chi = \det(g_{jk})$ (so that $ds^2 = g_{jk} dz_j d\bar{z}_k$ is the Kähler metric with potential $-\log \phi$ and $\chi$ the corresponding volume density). For $\Omega$ a bounded symmetric domain in $\mathbb{C}^N$ and $\phi(z) = 1/K_{\Omega}(z, z)$ (so that $ds^2$ is the Bergman metric), Berezin showed that for all $m \geq 1$ it holds that

$$K_{\phi^m \chi}(x, y) = p(m)\phi(x, y)^{-m} \quad (3)$$

where $\phi(x, y) = 1/K_{\Omega}(x, y)$ is a function on $\Omega \times \Omega$ holomorphic in $x, \bar{y}$ such that $\phi(x, x) = \phi(x)$, and $p$ is a polynomial of degree $N$ which depends only on $\Omega$; and that

$$B_{\phi^m \chi} f(y) = f(y) + \frac{1}{m} \Delta f(y) + O\left(\frac{1}{m^2}\right) \quad (4)$$

as $m \to \infty$, where $\Delta$ is the Laplace-Beltrami operator of the metric $ds^2$ on $\Omega$. Using (4) it suffices that (3) hold only asymptotically as $m \to \infty$ in a certain sense and used this to extend the range of applicability of Berezin’s original procedure to all plane domains with the Poincaré metric [E1], to some complete Reinhardt domains in $\mathbb{C}^2$ with natural rotation-invariant Kähler metrics [E2], and finally to any strictly pseudoconvex domain $\Omega$ with real-analytic boundary and $\phi$ a real-analytic defining function for $\Omega$ such that $-\log \phi$ is strictly plurisubharmonic [E6]. In fact, [E6] even dealt with the more general setting of weights of the form $\rho = \phi^m \psi^M$ with $-\phi, -\psi$ two $C^\infty$ defining functions of a strictly pseudoconvex domain $\Omega$ such that $-\log \phi, -\log \psi$ are plurisubharmonic, $M$ fixed and $m \to \infty$. Then (3), with $\phi^m \psi^M$ in place of $\phi^m \chi$, holds asymptotically for $(x, y)$ near the diagonal, and (4) holds for any $f \in L^\infty(\Omega)$ which is smooth in a neighbourhood of $y$.

The aim of the present paper is to improve these results by relaxing the hypotheses of real-analyticity and of $\phi, \psi$ being defining functions.

For a function $f$ on a domain in $\mathbb{C}^n$, we say that $f$ is almost analytic at $x = a$ if $\partial f/\partial \bar{z}_j$, $j = 1, \ldots, n$ vanish at $a$ together with their partial derivatives of all orders. It is known that any $C^\infty$ function $\phi(x)$ possesses a (non-unique) almost analytic extension $\phi(x, y)$ such that $\phi(x, y)$ is almost-analytic in $x$ and $\bar{y}$ at all points of the diagonal $x = y$, and $\phi(x, x) = \phi(x)$. Further, if $\phi(x)$ is real-valued, then the extension may be chosen so that $\phi(y, x) = \overline{\phi(x, y)}$ (just replace $\phi(x, y)$ by $\frac{1}{2}(\phi(x, y) + \overline{\phi(y, x)})$); in the sequel, we will always assume that an extension with this additional property has been chosen for a real-valued $\phi(x)$. We now have the following results.

**Theorem 1.** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^N$, $\phi, \psi$ two bounded positive continuous functions on $\Omega$ such that $-\log \phi, -\log \psi$ are plurisubharmonic, and let $x_0 \in \Omega$ be a point in a neighbourhood of which $\phi$ and $\psi$ are $C^\infty$ and $-\log \phi$ is strictly plurisubharmonic. Fix an integer $M \geq 0$. Then there is a smaller neighbourhood $U$ of $x_0$ such that the asymptotic expansion

$$K_{\phi^k \psi^M}(x, y) = \frac{k^N}{\pi^N \phi(x, y)^k \psi(x, y)^M} \sum_{j=0}^{\infty} b_j(x, y) k^{-j} \quad (5)$$

holds uniformly for all $x, y \in U$ as $k \to \infty$, in the sense that for each $m > 0$,

$$\sup_{x, y \in U} \left| \phi(x)^{k/2} \phi(y)^{k/2} K_{\phi^k \psi^M}(x, y) - \frac{k^N \phi(x)^{k/2} \phi(y)^{k/2}}{\pi^N \phi(x)^k \psi(x)^M} \sum_{j=0}^{N+m-1} b_j(x, y) k^{-j} \right| = O(k^{-m}) \quad (6)$$

as $k \to \infty$. Here $\phi(x, y), \psi(x, y)$ are fixed almost-analytic extensions of $\phi(x)$ and $\psi(x)$ to $U \times U$, respectively. The coefficients $b_j(x, y) \in C^\infty(U \times U)$ are almost-analytic at $x = y$,
and their jets at a point \((x, x)\) on the diagonal depend only on the jets of \(\phi\) and \(\psi\) at \(x\). In particular,

\[
b_0(x, x) = \det \left[ \frac{1}{\phi(x)} \frac{1}{\psi(x)} \right].
\]  

In the situation of the last theorem, consider the domain

\[
\tilde{\Omega} = \{ (z_1, z_2, z_3) \in \Omega \times C^M \times C : \frac{|z_2|^2}{\phi(z_1)} + \frac{|z_3|^2}{\psi(z_1)} < 1 \},
\]

Recall that for a domain \(D\) in \(C^n\), a boundary point \(z \in \partial D\) is called smooth if in some neighbourhood of \(z\), \(\partial D\) is a \(C^\infty\)-submanifold of \(C^n\); the domain \(D\) is called smoothly bounded if it is bounded and all its boundary points are smooth. A smooth boundary point \(z\) is said to be of finite type \(\leq m\) if there is no complex analytic variety passing through \(z\) which has order of contact with \(\partial D\) at \(z\) bigger than \(m\). (Thus, for instance, a strictly pseudoconvex boundary point is of type 2.) Finally, a smoothly bounded domain is said to be of finite type if all its boundary points are of finite type.

**Theorem 2.** Assume that the hypotheses of Theorem 1 are fulfilled, and that in addition \(\tilde{\Omega}\) is smoothly bounded and of finite type. (This implies, in particular, that \(\phi, \psi \in C^\infty(\Omega)\).) Then for any \(f \in L^\infty(\Omega)\) which is \(C^\infty\) in a neighborhood of \(x_0\), there is an asymptotic expansion

\[
B_{\phi^* \psi^*} f(y) = \sum_{j=0}^{\infty} Q_j f(y) \cdot k^{-j},
\]

uniformly for all \(y\) in a neighbourhood of \(x_0\), where \(Q_j\) are linear differential operators whose coefficients involve only the derivatives of \(\phi, \psi\) at \(y\) and \(Q_0\) is the identity operator.

We remark that in the applications to the Berezin quantization, \(- \log \phi\) is the potential of the Kähler metric, and thus is automatically strictly plurisubharmonic on all of \(\Omega\).

The whole approach can also be adapted to arbitrary Kähler manifolds \(\Omega\) in place of domains in \(C^N\) [Pe], and sections of line bundles in place of functions. The function \(\phi\) then defines the metric structure of the line bundle, and \(\partial \bar{\partial} \log \phi\) is the corresponding curvature form. For compact Kähler manifolds and \(- \log \phi\) strictly plurisubharmonic on all of \(\Omega\) (i.e. the line bundle of strictly negative curvature) the analogue of Theorem 1 has been obtained independently by Zelditch [Ze] for \(x = y\) and by Catlin [Ca] for general \(x, y\); and the analogue of Theorem 2 in this setting was established by Karabegov and Schlichenmaier [KS]. In [BMS] and [Sch] the authors also obtain (still in the context of compact manifolds) somewhat stronger results concerning the Berezin-Toeplitz quantization, and we finish by observing that the same results can also be obtained in our noncompact situation.

Recall that, quite generally, the Toeplitz operator \(T_F^{(\rho)}\) with symbol \(F \in L^\infty(\Omega)\) is the operator on \(A^2(\Omega, \rho)\) given by the recipe

\[
T_F^{(\rho)} f(y) = \int_{\Omega} f(x) F(x) K_\rho(y, x) \rho(x) \, dx,
\]

or, equivalently,

\[
T_F^{(\rho)} f = P_\rho(F f)
\]

where \(P_\rho\) is the orthogonal projection of \(L^2(\Omega, \rho)\) onto \(A^2(\Omega, \rho)\). For simplicity, we state our result on the Berezin-Toeplitz quantization only for the weights which are of most interest to us, viz. \(\rho = \phi^n \chi\) with \(\chi = \det[- \partial \bar{\partial} \log \phi]\).
Theorem 3. Let $\Omega$ be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $-\phi$ a smooth defining function for $\Omega = \{ \phi > 0 \}$ such that $-\log \phi$ is strictly plurisubharmonic. Then:

(i) for any $f \in C^\infty(\overline{\Omega})$, $\| T^m f \|_\infty \to \| f \|_\infty$ as $m \to \infty$;
(ii) there exist bilinear differential operators $C_j$ $(j = 0, 1, 2, \ldots)$ such that for any $f, g \in C^\infty(\overline{\Omega})$ and an integer $M$,

$$\left\| T^m f \right\| - \sum_{j=0}^{M} m^{-j} T^{m-j} \left\| C_j (f, g) \right\| = O(m^{-M-1})$$

as $m \to \infty$. Further, $C_0 (f, g) = fg$ and $C_1 (f, g) - C_1 (g, f) = i \{ f, g \}$, the Poisson bracket of $f$ and $g$ with respect to the metric (2).

The result of the kind appearing in Theorem 3 was first obtained for $\Omega$ a domain in $\mathbb{C}$ with the Poincaré metric by Klimek and Lesniewski [KL], using uniformization techniques, and for $\Omega$ a bounded symmetric domain with the invariant (Bergman) metric and $\Omega = \mathbb{C}^n$ with the Euclidean metric by Borthwick, Lesniewski and Upmeier [BLU] and Coburn [Co], respectively, in both cases with the aid of the computational machinery available thanks to the specific nature of the domain and metric. In our case (i) is a fairly straightforward consequence of Theorem 2, while (ii) follows, as in [BMS] and [Sch], from the Bott de Monvel-Guillemin calculus of generalized Toeplitz operators [BG]; see also [Gu].

As in [E6], our method of proof of Theorem 1 is based on the analysis of the Bergman kernel $\tilde{K}$ of the Forelli-Rudin domain (8) over $\Omega$; (3) is then obtained from Fefferman’s asymptotic expansion of $\tilde{K}$ near the boundary. This is done in Section 2, after establishing some localization theorems for the Bergman kernel in Section 1. Theorem 2 is proved in Section 3, and its applications to quantization are described in Section 4. The Berezin-Toeplitz quantization is discussed in Section 5. At the end of each section we provide various remarks, comments on related developments, open problems, etc.

Throughout the paper, “psh” is an abbreviation for “plurisubharmonic”.

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1. Preliminaries

Our starting point is the following proposition, reproduced here from [E6] (see also [BFS]), which relates the weighted Bergman kernels $K_{\phi, \psi, M}$ on $\Omega$ to the unweighted Bergman kernel of the domain $\Omega$.

Proposition 4. Let $\Omega$ be an arbitrary domain in $\mathbb{C}^n$ (it need not be bounded), $\phi, \psi$ two positive continuous functions on $\Omega$, $M$ a nonnegative integer, and $\Omega'$ the domain defined by (8). Then the Bergman kernel $K := K_{\tilde{\Omega}}$ of $\Omega$ is given by

$$\tilde{K}(z; t) = \sum_{k,l=0}^{\infty} \frac{(k+l+M+1)!}{k!l! \pi^{M+1}} K_{\psi + M \rho^{k+1}} (z_1, t_1, z_2, t_2) \{z_1, z_2\} \left(z_1, z_2, t_1, t_2, z_1, t_1, z_2, t_2\right).$$

The series converges uniformly on compact subsets of $\Omega'$.

Note that by the familiar criterion for Hartogs domains, $\tilde{\Omega}$ is pseudoconvex if and only if $\Omega$ is pseudoconvex and $-\log \phi, -\log \psi$ are psh.
Proof. Arguing as in [Lig], Proposition 0 shows that

$$\mathring{K}(z; t) = \sum_{\alpha, \beta} K_{\alpha, \beta}(z_1, t_1) z_2^\beta \bar{z}_2^\beta \bar{z}_3^\alpha \bar{z}_3^\alpha$$

where the summation is over all multiindices $\alpha \in \mathbb{N}, \beta \in \mathbb{N}^M$, $\mathbb{N} = \{0, 1, 2, \ldots \}$, and

$$g_{\alpha, \beta}(z_1) = \int \left\{ \frac{|z_2|^2}{\psi(z_1)} + \frac{|z_3|^2}{\phi(z_1)} \leq 1 \right\} \frac{1}{|\alpha|! \beta!} |z_2|^{|\alpha|+1} |z_3|^{|\beta|+M} d\bar{z}_2 d\bar{z}_3.$$

Since $\sum_{|\alpha|=k} x^\alpha \overline{x}^\beta / |\alpha|! = \langle x, y \rangle^k / k!$, the required assertion follows. \qed

The construction similar to (8) was first used by Forelli and Rudin [For], [FR], [Rud]; for other applications, see [Lig], [KLR] and the references therein.

Let us recall the asymptotic formula for the boundary behaviour of the Bergman kernel due to Fefferman [Fef] and Boutet de Monvel–Sjöstrand [BS]. Let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $-\phi$ a $C^\infty$ defining function for $\Omega$, i.e. $\Omega = \{ z : \phi(z) > 0 \}$, $\phi$ is $C^\infty$ in a neighborhood of $\overline{\Omega}$, $\nabla \phi \neq 0$ on $\partial \Omega$, and the Levi matrix $(-\partial^2 \phi / \partial z_j \partial z_k)$ is positive definite on the complex tangent space (the last condition is equivalent to the Monge-Ampère matrix in (13) below having a positive and 1 negative eigenvalue, for any $z \in \partial \Omega$). Then there exist functions $a(x, y), b(x, y), \phi(x, y) \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that

(a) $a(x, y), b(x, y), \phi(x, y)$ are almost-analytic in $x, \bar{y}$ in the sense that $\partial \phi(x, y) / \partial \bar{x}$ and $\partial \phi(x, y) / \partial y$ have a zero of infinite order at $x = y$, and similarly for $a(x, y)$ and $b(x, y)$;
(b) $\phi(x, x) = \phi(x)$;
(c) for $x \in \partial \Omega$,

$$a(x, x) = \frac{n!}{\pi^n} J[\phi](x) > 0 \quad (12)$$

where $J[\phi]$ is the Monge-Ampère determinant

$$J[\phi] = -\det \begin{bmatrix} -\phi & -\partial \phi / \partial \bar{z}_k \\ -\partial \phi / \partial z_j & -\partial^2 \phi / \partial z_j \partial \bar{z}_k \end{bmatrix} \quad (13)$$

whose positivity follows from the strong pseudoconvexity of $\partial \Omega$;
(d) the Bergman kernel of $\Omega$ is given by the formula

$$K(x, y) = \frac{a(x, y)}{\phi(x, y)^{n+1}} + b(x, y) \log \phi(x, y) \quad (14)$$

for $(x, y) \in \Omega, = \{ |x - y| < \varepsilon, \text{dist}(x, \partial \Omega) < \varepsilon \}$, where $\varepsilon > 0$ is sufficiently small;
(e) outside any $\Omega, \varepsilon$, the Bergman kernel is $C^\infty$ up to the boundary of $\overline{\Omega} \times \overline{\Omega}$;
(f) if the boundary $\partial \Omega$ is even real-analytic, then the functions $a(x, y), b(x, y)$ and $\phi(x, y)$ can in fact be chosen to be holomorphic in $x, \bar{y}$ in a neighborhood of the boundary diagonal $\{ (x, x) : x \in \partial \Omega \}$ in $\mathbb{C}^n$, and outside any $\Omega, \varepsilon$ the Bergman kernel is holomorphic in $x, \bar{y}$ in a neighborhood of $\overline{\Omega} \times \overline{\Omega}$.

The original proofs in [Fef] and [BS] deal only with (a)-(e); part (f) is due to Kashiwara [Kas] and Bell [Bell].
Observe that if \( \phi'(x, y) \) is another function satisfying (a) and (b), then \( h = (\psi'/\psi) - 1 \) vanishes at \( x = y \) to an infinite order; thus (14) remains in force with \( \phi' \) and \( a' = (1 + h)^{t+1}a + \phi'^{t+1}b\log(1 + h) \) in the place of \( \phi \) and \( a \). It follows that even for any function \( \phi(x, y) \) satisfying (a) and (b) there exist \( a(x, y), b(x, y) \) such that the conclusions (a)-(d) hold. This allows us to work with a convenient \( \phi(x, y) \) in concrete situations later on; for instance, if \( \phi(x) \) is of the form \( |x|^2 + (a \text{ function of } x_2, \ldots, x_n) \), we can take \( \phi(x, y) = x_1 \bar{y}_1 + (a \text{ function of } x_2, \ldots, x_n, y \ldots) \).

We will find convenient the following two (probably well-known) "localization lemmas", which can be used to obtain a local variant of Fefferman's theorem (see [E6]).

**Lemma 5.** Let \( \Omega_1 \subset \Omega \) be two bounded pseudoconvex domains and \( U \) a neighborhood of a point \( x_0 \in \partial \Omega \) such that \( U \cap \partial \Omega_1 = U \cap \partial \Omega \) and the piece of common boundary \( U \cap \partial \Omega \) is smooth and strictly pseudoconvex. Then the difference \( K_{\Omega_1}(x, y) - K_{\Omega}(x, y) \) is \( C^\infty \) on \( (U \cap \overline{\Omega_1}) \times (U \cap \overline{\Omega_1}) \).

**Lemma 6.** Let \( \Omega \) be a pseudoconvex domain (possibly unbounded) and \( x_0 \in \partial \Omega \) a strictly pseudoconvex point of its boundary. Then there exists a bounded strictly pseudoconvex domain \( \Omega_1 \subset \Omega \) such that \( \partial \Omega \) and \( \partial \Omega_1 \) coincide in a neighborhood of \( x_0 \). Further, if \( x_0 \) is a smooth boundary point, then \( \Omega_1 \) can be chosen to be smoothly bounded.

**Proof of Lemma 5.** For \( \Omega, \Omega_1 \) smoothly bounded and strictly pseudoconvex, this is the content of Lemma 1 on p. 6 in [Fef]. The local version given here follows in the same way by J.J. Kohn's local regularity theorems for the \( \overline{\partial} \)-operator and subelliptic estimates at \( x_0 \) [Ko, Theorems 1,13 and 1,16] by the argument as on p. 469 in [Be2], cf. in particular the formula (2,1) there. \( \square \)

**Proof of Lemma 6.** Let \( u \) be a defining function for \( \Omega = \{u < 0\} \) strictly-psh in a neighborhood \( B(x_0, \delta) \) of \( x_0 \) (see e.g. [Kn], Proposition 3.2.1). Choose a \( C^\infty \) function \( \theta : [0, 1) \to \mathbb{R}^+ \) such that \( \theta \equiv 0 \) on \([0, \frac{1}{\delta}])\), \( \theta' > 0 \) on \([\frac{1}{\delta}, 1)\) and \( \theta(1) = +\infty \). Set \( \Omega_1 = \{x : u(x) + \theta(|x - x_0|^2/\delta^2) < 0\} \). Then \( \Omega_1 \subset \Omega \cap B(x_0, \delta) \), \( \partial \Omega_1 \) coincides with \( \partial \Omega \) in \( B(x_0, \delta/2) \), and as \( \theta'' \geq 0 \), \( \theta(|x - x_0|^2/\delta^2) \) is psh, so \( \Omega_1 \) is strictly pseudoconvex. Finally, if \( u \) is \( C^{\infty} \) in \( B(x_0, \delta) \), then \( \Omega_1 \) is smoothly bounded. \( \square \)

It turns out that the boundedness hypothesis on \( \Omega \) in Lemma 5 is, in fact, unnecessary: see [E7], Section 4, where also the full details of the proof can be found. The conclusion of the lemma fails, however, if \( U \cap \partial \Omega = U \cap \partial \Omega_1 \) is only assumed to be weakly pseudoconvex: for instance, take \( \Omega = \{\max(|z_1|, |z_2|) < 1\} \subset \mathbb{C}^2 \), \( \Omega_1 = \{\max(|z_1|, |2z_2|) < 1\} \), and \( x_0 = (1, 0) \). Similarly, the hypothesis that \( \Omega \) be pseudoconvex cannot be dispensed with: an example is \( \Omega_1 = \{z \in \mathbb{C}^2 : |z| < 2\} \), \( \Omega = \Omega_1 \cup \{|z_1| < 3, 1 < |z_2| < 3\} \), \( x_0 = (2, 0) \). On the other hand, the hypothesis that \( \Omega_1 \) be pseudoconvex is not needed in the proof and can be omitted (but we won't have any use for this refinement in the sequel).

A similar construction as in Lemma 6 was used by Bell [Be2] (cf. also the references therein).

**Remark.** There is also a "local version" of part (f) of Fefferman's theorem: namely, if \( \Omega \) is bounded pseudoconvex and \( z \in \partial \Omega \) is a strictly pseudoconvex and real-analytic boundary point (i.e. \( \partial \Omega \) is a \( C^\omega \)-submanifold of \( \mathbb{C}^n \) in some neighborhood of \( z \) ), then there exists a neighborhood \( U \) of \( z \) and functions \( a(x, y), b(x, y) \) and \( \phi(x, y) \) on \( U \times U \), holomorphic in \( x, y \), such that \( -\phi(x, y) \) is a local defining function for \( \Omega \) on \( U \) and (12) and (14) hold. See [Kan], §9, in particular the Theorem on p. 94. (The author is obliged to M. Kashiwara and Gen Komatsu for this information.)

2. **Weighted Bergman kernels**

We now use Fefferman's asymptotic expansion together with Proposition 4 to determine
the asymptotics of $K_{\phi^m \phi^M}(z, z)$ as $z$ and $M$ are fixed and $m \to \infty$. Let us start with two technical lemmas.

Let $\phi(x, y)$ be a function in $C^\infty(\Omega \times \Omega)$ almost-analytic in $x, y$ on the diagonal and such that $\phi(x, y) = \phi(y, x)$, $\phi(x, x) =: \phi(x) > 0$, and $-\log \phi(x)$ is strictly psh at some point $x_0$. The last condition implies that there exists $c > 0$ and a small ball $U$ centered at $x_0$ such that

$$\frac{\phi(x)\phi(y)}{[\phi(x, y)]^2} \leq 1 - c|x - y|^2 \quad \forall x, y \in U.$$

Denote

$$D = \{(x, y, \tau) \in U \times U \times \mathbb{C} : |\tau|^2 < \frac{\phi(x)\phi(y)}{[\phi(x, y)]^2}\}.$$

Then

$$|x - y|^2 < \frac{2}{c} (1 - \tau) \quad \forall (x, y, \tau) \in D.$$

Let $\mathcal{G}^m$ and $\mathcal{G}$ be the set of all functions in $C^m(\overline{D})$ and $C^\infty(\overline{D})$, respectively, which are almost-analytic at all points $(x, x, 1), x \in U$.

Denote

$$u_k(\tau) = \begin{cases} (1 - \tau)^k & (k < 0), \\ (1 - \tau)^k \log \frac{1}{1 - \tau} & (k \geq 0). \end{cases}$$

**Lemma 7.** Let $f(x, y, \tau)$ be a function in $\mathcal{G}$ which is in fact holomorphic in $\tau$ on $D$. Assume that

$$f(x, y, \tau) = \sum_{j=-n-1}^n \sum_{j=0}^m a_j(x, y)u_j(\tau) + g(x, y, \tau)$$

where $g \in \mathcal{G}^m$. Then the Taylor coefficients $f_k(x, y)$ of $f$ with respect to $\tau$ satisfy

$$\left|f_k(x, y) - \sum_{j=0}^n \binom{j+k}{k} a_{j-1}(x, y) - \sum_{j=0}^m \frac{(-1)}{j+1} \binom{k}{j+1} a_j(x, y)\right| \leq \frac{||\partial^m_{\tau^i} g||_\infty}{k(k-1)\ldots(k-m)} \left(\frac{||\phi(x, y)||}{\sqrt{\phi(x, y)\phi(y)}}\right)^k$$

as $k \to \infty$.

(Here and elsewhere $\partial^m_{\tau^i}$, etc., is a shorthand for $\partial^m/\partial \tau^m$, etc.)

**Proof.** This is immediate from the formulas

$$(1 - \tau)^{-j-1} = \sum_{k=0}^{\infty} \binom{k+j}{k} \tau^k \quad (j \geq 0),$$

$$(1 - \tau)^j \log \frac{1}{1 - \tau} = \tau^{j+1} - \tau^2 (\tau - 1)^j + \sum_{k=j+1}^{\infty} \frac{(-1)^j}{k(k-1)\ldots(k-j)} \tau^k \quad (j \geq 0),$$

and the Cauchy estimates for the Taylor coefficients of $g (r = \frac{\sqrt{\phi(x, y)\phi(y)}}{[\phi(x, y)]})$:

$$(k+1)\ldots(k+m)g_{k+m}(x, y)r^{k+m} = \left|\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^m g}{\partial \tau^m}(x, y, re^{i\theta}) e^{-ik\theta} \ d\theta\right| \leq \left\|\frac{\partial^m g}{\partial \tau^m}\right\|_\infty.$$
Lemma 8. Each $G \in \mathcal{G}$ admits a decomposition

$$G(x, y, \tau) = G_0(x, y) + (\tau - 1)g_1(x, y, \tau)$$

(18)

with $G_0 \in C^\infty(U \times U)$ almost-analytic on the diagonal, and $g_1 \in \mathcal{G}$.

Proof. Extend $G$ to a $C^\infty$ function on $\overline{U} \times \overline{U} \times \mathbb{C}$ (still denoted by $G$) and set

$$G_0(x, y) := G(x, y, 1), \quad g_1(x, y, \tau) := \frac{G(x, y, \tau) - G(x, y, 1)}{\tau - 1}, \quad (x, y, \tau) \in \mathcal{D}.$$ 

Clearly the relation (18) is satisfied, and $G_0(x, y)$ is almost-analytic on the diagonal. Thus we only need to show that for any multiindices $a, b, c, d$ and nonnegative integers $j, k$, the function

$$\partial^a_x \partial^b_y \partial^j_y \partial^l \partial^k g_1(x, y, \tau)$$

extends continuously to the points where $\tau = 1$, and vanishes there unless $|b| = |c| = k = 0$.

For brevity, let us introduce the shorthand

$$H_{a, b, c, d, j + 1, k}(x, y, \tau).$$

We have

$$G(x, y, \tau) - G(x, y, 1) = \int_0^1 [(\tau - 1) \partial_\tau G(x, y, \tau_i) + (\tau - 1) \partial_\tau G(x, y, \tau_i)] d\tau.$$

where $\tau_i = 1(\tau - 1)t$. Hence

$$g_1(x, y, \tau) = \int_0^1 \left[ \partial_\tau G(x, y, \tau_i) + \frac{\tau - 1}{\tau - 1} \partial_\tau G(x, y, \tau_i) \right] d\tau.$$

Applying $\partial^a_x \partial^b_y \partial^j_y \partial^l \partial^k$ to both sides, the first term in the integrand becomes just

$$t^{j + k} H_{a, b, c, d, j + 1, k}(x, y, \tau_i),$$

while the second one yields an expression of the form

$$\sum_{i=0}^j \sum_{l=0}^k t^{i + l} c_{i, l} \frac{(\tau - 1)^{\alpha_i}}{(\tau - 1)^{\beta_i}} H_{a, b, c, d, i, l + 1}(x, y, \tau_i),$$

(19)

with $c_{i, l} \in \mathbb{C}$, $\alpha_i \in \{0, 1\}$, and $\beta_i \in \{1, \ldots, j + 1\}$. Consider the function $\gamma(x, y, \tau) := H_{a, b, c, d, j + 1, k}(x, y, \tau)$. By the Taylor formula, for any integer $N \geq 0$ and $(x, y, \tau) \in U \times U \times \mathbb{C}$,

$$\gamma(x, y, \tau) = \sum_{m=0}^N \frac{1}{m!} \nabla^m \gamma(x, y, \tau) \left( \frac{x - y}{2}, \frac{x - y - x}{2}, \tau - 1 \right)$$

$$+ \frac{1}{(N + 1)!} \nabla^{N + 1} \gamma_{X_\theta} \left( \frac{x - y}{2}, \frac{x - y - x}{2}, \tau - 1 \right),$$

where $X_\theta := (\frac{x - y}{2}, \frac{x - y}{2}, 1) + \theta (\frac{x + y}{2}, \frac{x - y}{2}, \tau - 1)$, for some $\theta \in [0, 1]$. Here $\nabla^m \gamma_X Y$ stands for the $m$-th total differential of $\gamma$ at the point $X$ evaluated on the $m$-vector $(Y, \ldots, Y)$. But in view of the almost-analyticity of $G$, we have $\nabla^m \gamma(x, y, \tau) \equiv 0$ for all $m$ and $z$; hence

$$|\gamma(x, y, \tau)| \leq \frac{1}{(N + 1)!} \sup_{X \in U \times U \times [\tau, 1]} \|\nabla^{N + 1} \gamma_{X}\| \cdot \left( \frac{|x - y|^2}{2} + |1 - \tau|^2 \right)^{\frac{N + 1}{2}}.$$
where \([\tau, 1]\) denotes the line segment in \(\mathbb{C}\) with endpoints \(\tau\) and 1. Replacing \(\tau\) by \(\tau_i\), noting that \(|1 - \tau_i| = t|1 - \tau|\), and letting \(i\) and \(l\) vary, it follows that the expression (19) is bounded in modulus by

\[
C_{abcdejkN} \frac{|x - y|^{N+1} + |1 - \tau|^{N+1}}{|1 - \tau|^{j+1}},
\]

uniformly for \(x, y \in \mathcal{T}\), \(t \in [0, 1]\), and \(\tau\) in compact subsets of \(\mathbb{C}\). However, for \((x, y, \tau) \in \mathcal{D}\) we have the estimate (16), hence the last quantity is in that case estimated by

\[
C_{abcdejkN} |1 - \tau|^{-N-\frac{j+1}{2}}.
\]

Consequently,

\[
\left| \partial_z^{d} \partial_y^{d'} \partial_y^{d''} \partial_y^{d'''} g_1(x, y, \tau) - \int_0^1 H_{a,b,c,d,j+1,k}(x, y, \tau_t) \, dt \right| \leq C_{abcdejkN} \frac{|1 - \tau|^{N-\frac{j+1}{2}}}{\sqrt{t}}
\]

\(\forall (x, y, \tau) \in \mathcal{D}\). As \(N\) is arbitrary, it follows that the limit

\[
\lim_{D \ni (x,y,\tau) \to (z,\zeta,1)} \partial_z^{d} \partial_y^{d'} \partial_y^{d''} \partial_y^{d'''} g_1(x, y, \tau)
\]

exists for each \(z \in U\), and equals

\[
H_{a,b,c,d,j+1,k}(z, \zeta, 1),
\]

which vanishes unless \(|b| = |c| = k = 0\). This completes the proof of the lemma. \(\square\)

**Proof of Theorem 1.** The hypotheses ensure that \(\tilde{\Omega}\) is a bounded pseudoconvex domain in \(\mathbb{C}^{N+M+1}\), the points \(z \in \partial \Omega\) with \(z_1 = x_0\) and \(z_2 = 0\) are smooth strictly pseudoconvex boundary points, and \(u(z) = |z_1|^2 + g(z_1)|z_2|^2 - \phi(z_1)\), with \(g = \phi/\psi\), is a smooth local defining function near any such point. By continuity, there is \(\delta > 0\) such that all points of \(\partial \tilde{\Omega}\) with \(|z_1 - x_0|^2 + |z_2|^2 < \delta\) are strictly pseudoconvex. We may assume that \(\delta < \text{dist}(x_0, \partial \Omega)\). Choose a \(C^\infty\) function \(\theta\) on \([0, 1]\) such that \(\theta'' \geq 0\), \(\theta \equiv 0\) on \([0, 2/3]\) and \(\theta(t) = -\log(1 - t)\) on \((3/4, 1)\), and define \(u'(z) = |z_1|^2 + g(z_1)|z_2|^2 - \exp[-\theta(|z_1 - x_0|^2 + |z_2|^2)/\delta]\phi(z_1)\) and \(\Omega' = \{u' < 0\}\). A similar argument as in the proof of Lemma 6 shows that \(\Omega'\) is a strictly pseudoconvex domain \(\subset \Omega\) and that \(\Omega\) and \(\tilde{\Omega}\) coincide in a neighborhood of \(\Pi = \{z : |z_1 - x_0|^2 + |z_2|^2 < \delta/2\}\). Thus the conclusions (a)–(e) of Fefferman’s theorem are applicable to \(\Omega'\), and also by Lemma 5 \(K_{\tilde{\Omega'}} - K\) is \(C^\infty\) on \(\mathbb{P} \cap \tilde{\Omega} \times \mathbb{P} \cap \tilde{\Omega}\). It follows that \(\tilde{K}\) is \(C^\infty\) on \(\mathbb{P} \cap \tilde{\Omega} \times \mathbb{P} \cap \tilde{\Omega}\) minus the boundary diagonal \(S = \{(z, z) : z \in \mathbb{P} \cap \partial \tilde{\Omega}\}\), while near \(S\) it is of the form

\[
\tilde{K}(z, t) = \frac{a(z, t)}{(-u(z, t))^{N+M+2}} + b(z, t) \log[-u(z, t)],
\]

(20)

with some almost-analytic \(C^\infty\) functions \(a\) and \(b\) satisfying \(u(z, z) = u(z)\) and

\[
a(z, z) = \frac{(N + M + 1)!}{\pi^{N+M+1}} J[-u](z), \quad z \in \mathbb{P} \cap \tilde{\Omega},
\]

(Here and in (20) we have used the fact that \(u' = u\) on \(\mathbb{P} \cap \tilde{\Omega}\).) Let us now specialize to \(z, t \in \mathbb{P} \cap \tilde{\Omega}\) with \(z_2 = t_2 = 0\), i.e. to points of the form \((x, 0, z_3)\) with \((x, z_3) \in D := \{|x - x_0|^2 < \delta/2, |z_3|^2 < \phi(x)\} \subset \mathbb{C}^{N+1}\). It follows that the function

\[
g(x, z_3; y, t_3) = \tilde{K}(x, 0, z_3; y, 0, t_3)
\]
is $C^\infty$ on $\overline{D \times D}$ minus the boundary diagonal $\Delta = \{(x, z_3; y, t_3) : x = y, z_3 = t_3, |z_3| = \sqrt{\phi(x)}\}$, while near $\Delta$ it is of the form
\[
g(x, z_3; y, t_3) = \frac{a'(x, z_3; y, t_3)}{[\phi(x, y) - \overline{z_3}t_3]^{N+M+2}} + b'(x, z_3; y, t_3) \log[\phi(x, y) - \overline{z_3}t_3]
\]
where $a', b'$ — the restrictions of $a, b$ to $z_2 = t_2 = 0$ — are holomorphic in $x, \overline{y}, z_3, t_3$ on $\Delta$. Switching to the variable
\[
\tau = \frac{\overline{z_3}t_3}{\phi(x, y)}
\]
we thus see that
\[
g(x, z_3; y, t_3) = F(x, y, \tau)
\]
for a function $F$ on the domain
\[
D_1 = \{(x, y, \tau) : x, y \in U, |\tau|^2 < \frac{\phi(x)\phi(y)}{|\phi(x, y)|^2}\},
\]
where $U = \{x : |x - x_0|^2 < \delta^2/2\}$, such that $F$ is $C^\infty$ on $\overline{D_1}$ minus the set $\Delta_1 = \{(x, y, \tau) : x = y, \tau = 1\}$, while near $\Delta_1$ it is of the form
\[
F(x, y, \tau) = \frac{G(x, y, \tau)}{(1 - \tau)^{N+M+2}} + H(x, y, \tau) \log(1 - \tau), \tag{21}
\]
where $H(x, y, \tau) = b''(x, y, \tau \phi(x, y))$ and
\[
G(x, y, \tau) = \frac{a''(x, y, \tau \phi(x, y))}{\phi(x, y)^{N+M+2}} + (1 - \tau)^{N+M+2} b''(x, y, \tau \phi(x, y)) \log[\phi(x, y)^{N+M+2}]
\]
are functions in $C^\infty(\overline{D_1})$ almost-analytic in $x, \overline{y}, \tau$ on $\Delta_1$. Shrinking $U$ if necessary so that (15) is satisfied, we are in a position to apply Lemma 8; thus there exist functions $G_0, G_1, G_2, \ldots$ in $C^\infty(U \times U)$, almost-analytic on the diagonal, such that for each integer $m \geq 0$,
\[
G(x, y, \tau) = \sum_{j=0}^{m} G_j(x, y) \cdot (1 - \tau)^j + (1 - \tau)^{m+1} g_m(x, y, \tau)
\]
with $g_m \in G$, and similarly for $H$. From (21) we therefore obtain, for any $m \geq 0$,
\[
F(x, y, \tau) = \sum_{j=0}^{N+M+1} j_j(x, y) \cdot (1 - \tau)^{-j-1} + \sum_{j=0}^{m} H_j(x, y)(1 - \tau)^j \log(1 - \tau) + R_m(x, y, \tau)
\]
where the function $R_m \in G^m$ is holomorphic in $\tau$ on $D$. Thus Lemma 7 can be applied to $F(x, y, \tau)$, with $n = N + M + 1, m$ any integer $\geq 0$, and
\[
a_{l-N-M-1}(x, y) = G_l(x, y), \quad l = 0, \ldots, N + M + 1,
\]
\[
a_{l}(x, y) = H_l(x, y), \quad l \geq 0.
\]
As the Taylor coefficients of $F(x, y, \cdot)$ are, in view of Proposition 4, given by
\[
f_k(x, y) = \frac{(k + M + 1)!}{\pi^{M+1} k!} K_{\psi_M} (x, y) \phi(x, y)^k,
\]
we thus conclude that
\[
\left| \frac{(k + M + 1)!}{\pi^{M+1} k!} K_{\phi^{k+1} \psi^M}(x, y) \phi(x, y)^k - \sum_{j=0}^{N+M+1} \binom{k+j}{j} G_{M+N+1-j}(x, y) \right|
\]
\[
- \sum_{j=0}^{m} \frac{(-1)^j}{(j+1)(k+j+1)} H_j(x, y) \leq C_m \frac{k^N}{\sqrt{\phi(x) \phi(y)}} \frac{\phi(x, y)}{\phi(x) \phi(y)}
\]
as \(k \to \infty\), for any \(m \geq 0\), uniformly for \(x, y \in U\). As \(\binom{k+j}{j} \sim (k+1)^j\) and \(\binom{k}{j+1} \sim (k+1)^{j+1}\) for each fixed \(j\), we thus obtain (6) and (5). The claim concerning the local character of the coefficients \(b_j(x, y)\) follows from the fact that the jet of the function \(b\) and the \((N + d + M)\)-jet of the function \(a\) in (20) at a point \((z, z)\), \(z \in \partial \Omega\), are known to depend only on the jet of the boundary \(\partial \Omega\) at the point \(z\), hence for \(z = (x, 0, \sqrt{\phi(x)})\) only on the jets of \(\phi\) and \(\psi\) at \(x\) (see e.g. [BFG], p. 312); and, consequently, so do, in turn, \(a'\) and \(b', a''\) and \(b''\), \(G\) and \(H\), \(G_j\) and \(H_j\), and \(b_j\). In particular, for \(x = y\) the leading order term is
\[
k^N b_N(x, x) = \frac{k^N \pi^{M+1} G_0(x, x)}{\phi(x) \phi(x)} \cdot \frac{1}{(N + M + 1)!} \frac{d^M}{d \phi(x)^M} \frac{\phi(x)}{\phi(x)}
\]
\[
= \frac{k^N \pi^{M+1} a'(x, \sqrt{\phi(x)}; x, \sqrt{\phi(x)})}{\phi(x)^N M + k + 2} = \frac{k^N \pi^{M+1} a(x, 0, \sqrt{\phi(x)}; x, 0, \sqrt{\phi(x)})}{\phi(x)^N M + k + 2}
\]
\[
= \frac{k^N J[-u](x, 0, \sqrt{\phi(x)})}{\sqrt{\phi(x)}}
\]

Standard matrix manipulations show that
\[
J[-u](z) = \frac{\phi^{N+1}}{\psi} \cdot \det \left[ \left(1 - \frac{|z|^2}{\psi} \right) \cdot \partial \phi \log \frac{1}{\psi} + \frac{|z|^2}{\psi} \cdot \partial \theta \log \frac{1}{\psi} \right],
\]
so
\[
\frac{b_N(x, x)}{\phi(x) \psi} = \frac{\det[-\partial \phi \log \phi(x)]}{\phi(x) \psi^{k+1}},
\]
whence (7) follows. This completes the proof of Theorem 1. □

**Corollary 9.** Let \(\Omega\) be a domain in \(\mathbb{C}^N\) and \(\phi\) a positive function on \(\Omega\). Assume that
- \(\Omega\) is bounded and pseudoconvex,
- \(\phi\) and \(J[\phi]\) are bounded, and \(-\log \phi\) is psh on \(\Omega\),
- there exists an integer \(M \geq 0\) such that \(-\log \phi - \frac{1}{M} \log J[\phi]\) is psh on \(\Omega\), and
- \(-\log \phi\) is smooth and strictly psh at a point \(x_0 \in \Omega\).

Let \(\chi = \det[-\partial \phi \log \phi]\). Then as \(k \to \infty\), there is an asymptotic expansion
\[
K_{\phi^{k+1}}(x, y) = \frac{k^N}{\phi(x, y)^k} \sum_{j=0}^{m} \beta_j(x, y) k^{-j} \leq C_m U \frac{k^N - m - 1}{\phi(x)^{k/2} \phi(y)^{k/2}}
\]
for \(x, y\) in a small neighbourhood \(U\) of \(x_0\) and any \(m \geq 0\), where the coefficients \(\beta_j(x, y) \in C^\infty (U \times U)\) are almost analytic on the diagonal, their jets at \((x_0, x_0)\) depend only on the jet of \(\phi(x)\) at \(x_0\), and \(\beta_0 = 1\).

**Proof.** Apply the previous theorem with \(\psi = \phi \cdot J[\phi]^{1/M}\), observe that
\[
\chi = \frac{J[\phi]}{\phi^{N+1}},
\]
and
and replace \( k + M + N + 1 \) by \( k \). □

Remarks. (1.) The boundedness assumptions on \( \Omega, \phi, \) and \( \psi \) or \( J[\phi] \) in Theorem 1 and Corollary 9 (which are equivalent to the boundedness of the domain \( \Omega \)) are in fact unnecessary, cf. the remark after Lemma 5.

(2.) In the applications to the Berezin quantization, one takes for \(-\log \phi\) the Kähler potential of the metric (2); hence \(-\log \phi\) is automatically smooth and strictly plurisubharmonic at all points of \( \Omega \). The third condition in Corollary 9 can in this context be rephrased in terms of the Ricci tensor: by (22), it is equivalent to the plurisubharmonicity of \(- (M + N + 1) \log \phi - \log \chi\); but

\[
\frac{\partial^2 \log \chi}{\partial z_j \partial \overline{z}_k} = \text{Ric}_{\overline{z}z}
\]

is the Ricci tensor of the metric \( g_{\overline{z}z} \). Thus the condition says that

\[
\text{Ric}_{\overline{z}z} \leq (M + N + 1) g_{\overline{z}z}
\]

at all points of \( \Omega \), in the sense that the difference of the right-hand and the left-hand side is positive definite.

(3.) Theorem 1 fails if \(-\log \phi\) is only assumed to be psh but not strictly psh at \( x_0 \). Indeed, examples given at the end of Section 2 in [E6] suggest that in that case one gets in (5) an asymptotic expansion not in the negative powers of \( k \), but instead in negative powers of \( k^2/m \), where \( m \) is the type (assumed to be finite) of the boundary point \((x_0, 0, \sqrt{\phi(x_0)}) \in \partial \Omega\); and similarly it seems that

\[
\beta_0(x, x) = \lim_{k \to \infty} \left( \frac{\pi}{k} \right)^N \phi(x)^k K_{\phi, \chi}(x, x) = \frac{2}{m},
\]

with the right-hand side interpreted as zero for points of infinite type.

(4.) Theorem 1 fails even more drastically if \(-\log \phi\) is not even psh at \( x_0 \). Indeed, it is an immediate consequence of (6) that

\[
\lim_{k \to \infty} K_{\phi, \psi^M}(x_0, x_0)^{1/k} = 1/\phi(x_0).
\]  
\[ (23) \]

On the other hand, for arbitrary positive lower-semicontinuous functions \( \phi, \psi \) on \( \Omega \) (not necessarily such that \(-\log \phi\) or \(-\log \psi\) are psh), it follows from Proposition 4, the formula for the radius of convergence, and the fact that the domain of convergence of a power series is always a log-convex complete Reinhardt domain, that

\[
\limsup_{k \to \infty} K_{\phi, \psi^M}(x, x)^{1/k} = \frac{1}{\phi^*(x)}
\]

where \( \log(1/\phi^*) \) is the greatest psh minorant of \( \log(1/\phi) \); in particular, (23) cannot hold if \(-\log \phi\) is not psh. See [E3] and the references therein for related matters.

(5.) If \( \phi, \psi \) are assumed to be not only \( C^\infty \) but real-analytic (\( C^\omega \)) near \( x_0 \), the assertion (6) of Theorem 1 can be substantially sharpened, namely to

\[
\sup_{x, y \in U} \left| \phi(x, y)^k K_{\phi^+ \psi^M}(x, y) - \frac{k^N}{\pi^N \psi(x, y)^M} \sum_{j=0}^{N+m-1} b_j(x, y) k^{-j} \right| = O(k^{-m}).
\]  
\[ (24) \]
Also, the coefficients $b_j(x, y)$ are not merely almost-analytic, but holomorphic in $x, \overline{y}$ on $U \times U$. For $x \neq y$, the estimate (24) is better than (6) by an exponential factor, cf. (15). For $\Omega$ strictly pseudoconvex with $C^\omega$ boundary and $\phi, \psi$ two $C^\omega$ defining functions for $\Omega$, (24) was proved in [E6], Theorem 11; the local version stated above follows exactly in the same way from the corresponding local variant of the part (f) of Fefferman's theorem mentioned in the end of Section 1.

6. A consequence of (24) is that $K_{\phi^k, \psi^M}(x, y)$ is zero-free on $U \times U$ as soon as $k$ is sufficiently large, and

$$K_{\phi^k, \psi^M}(x, y) \to \phi(x, y)$$

on $U \times U$ as $k \to \infty$ when the holomorphic branches of the roots are chosen appropriately. Note that for $\phi, \psi$ not $C^\omega$ but merely $C^\infty$, (6) is too weak to yield this (unless $x = y$); on the other hand, it can be shown that the sequence of the absolute values

$$|K_{\phi^k, \psi^M}(x, y)|^{1/k}$$

is always locally uniformly bounded, on all of $\Omega \times \Omega$ (1), for any positive lower semicontinuous functions $\phi, \psi$ (see [E3]). Almost nothing seems to be known about the limiting behaviour of this sequence, however. For plane domains $\Omega$ with $-\log \phi$ the Kähler potential of the Poincaré metric and $M = 0$, this problem was studied by the present author in [E4]; it turns out that in that case the limit (25) exists for all $(x, y)$ not in the cut locus of $\Omega$ (i.e. if there is a unique shortest geodesic connecting $x$ to $y$), while for $(x, y)$ in the cut locus the sequence (26) can display an oscillatory behaviour as $k \to \infty$. Understanding the limiting behaviour of (26) in the general case seems quite intriguing.

7. For $\Omega$ a compact manifold, Catlin [Ca] has obtained the estimates (6) even for the derivatives of $K_{\phi^k, \psi^M}$ with respect to the $x$ and $y$ variables. Our methods can also be used to give a similar result in the present context as well, as it is easily seen that the formula (11) in Proposition 4 can be differentiated termwise with respect to $x, y$ any numbers of times. We omit the details.

8. The asymptotics as $k \to \infty$ of the integral

$$I_k(f) = \int_{\Omega} f(x) K_{\phi^k}(x, x) \phi(x)^k dx$$

are of interest in the study of Pauli operators (quantum Hamiltonians) in magnetic fields; see [Erd], [Rai].

3. The Berezin Transform

We begin by recalling the following generalization of the part (e) of Fefferman's theorem (that part is actually due to Kerzman [Ke]).

**Proposition 10.** Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $x, y \in \partial D$ two smooth boundary points of finite type. Then there exist neighbourhoods $U, V$ of $x$ and $y$, respectively, in $\mathbb{C}^n$ such that the (unweighted) Bergman kernel $K(x, y)$ of $D$ extends to a $C^\infty$ function on $(\overline{D} \cap U) \times (\overline{D} \cap V)$.

The proof of Proposition 10 can be found in [Be2] and [Bo]; again, the boundedness hypothesis can be dropped [E7].

**Proof of Theorem 2.** By Theorem 1, there exists a neighborhood $U$ of $x_0$ such that the asymptotic expansion (6) holds for $x, y$ in a neighborhood of $U$. This implies, first of all, that $K_{\phi^k, \psi^M}(y, y) \neq 0$ as soon as $k$ is large enough; thus the Berezin transform

$$B_{\phi^k, \psi^M} f(y) = \int_{\overline{D}} f(x) \frac{|K_{\phi^k, \psi^M}(x, y)|^2}{K_{\phi^k, \psi^M}(y, y)} \phi(x)^k \psi(x)^M dx$$

(27)
is always defined for all $y \in \overline{U}$ as soon as $k$ is sufficiently large. Split the integration in (27) into integration over $U$ and over $\Omega \setminus U$. Consider the function

$$f(x, y, s) = \bar{K}(x, 0, s; y, 0, \sqrt{\phi(y)}).$$

Let $V$ be another neighbourhood of $x_0$ whose closure is contained in $U$. By Proposition 10 and the finite type hypothesis, $f$ is a $C^\infty$ function on the set compact $W = \{(x, y, s) : x \in \bar{\Omega} \setminus U, y \in \overline{V}, |s|^2 \leq \phi(x)\}$. Thus for any integer $j \geq 0$,

$$\sup_{(x, y, s) \in W} \left| \frac{\partial_j f}{\partial s^j} \right| = c_j < +\infty.$$

On the other hand, By Proposition 4,

$$f(x, y, s) = \sum_{k=0}^{\infty} \frac{(k + M + 1)!}{k! \pi^{M+1}} K_{\phi^{k+1} \psi M}(x, y) \phi(y)^{k/2} s^k. \quad (28)$$

Applying Cauchy estimates to the function $s \mapsto \partial_j f(x, y, s)/\partial s^j$, holomorphic in the disc $|s| < \sqrt{\phi(x)}$, we thus obtain

$$|K_{\phi^{k+1} \psi M}(x, y) \phi(x)^{k/2} \phi(y)^{k/2}| \leq \frac{(k - j)! \pi^{M+1}}{(k + M + 1)!} c_j,$$

for all $x \in \Omega \setminus U, y \in \overline{V}$ and $k \geq j$; that is,

$$|K_{\phi \psi M}(x, y)|^2 \phi(x)^k \phi(y)^k \leq c_j^k k^{-2(M+j+1)} \quad \forall x \in \Omega \setminus U, y \in \overline{V}, k \geq j + 1,$$

and, upon invoking (6) with $x = y$,

$$\frac{|K_{\phi \psi M}(x, y)|^2}{K_{\phi \psi M}(y, y)} \phi(x)^k \psi(x)^M \leq \frac{c_j^k \psi(x)^M}{k^{N+2j+2} M+2}$$

for all $x \in \Omega \setminus U, y \in \overline{V}$ and $k \geq j + 1$. It follows that the integral over $\Omega \setminus U$ is $O(k^{-j})$, uniformly as $y \in \overline{V}$, for any $j$.

It remains to deal with the integral over $U$. In that case, by (6) we have the asymptotic formula

$$K_{\phi^{k+1} \psi M}(x, y) \cdot \phi(x)^{k/2} \phi(y)^{k/2} = \frac{k^N}{\pi^N} \phi(x)^k \phi(y)^k \psi(x)^M \psi(y)^M \sum_{j=0}^{m+N} b_j(x, y) k^{-j} + k^{-m-1} C_m(x, y, k)$$

for $m$ any integer $\geq 0$, with $\sup_{x, y \in U, k \geq 1} |C_m(x, y, k)| < \infty$. Combining this with the similar estimates with $x$ and $y$ interchanged and with $x = y$, respectively, we arrive at the asymptotic expansion

$$\frac{|K_{\phi \psi M}(x, y)|^2}{K_{\phi \psi M}(y, y)} \phi(x)^k \psi(x)^M = \frac{k^N}{\pi^N} \frac{\phi(x)^k \phi(y)^k \psi(x)^M \psi(y)^M}{|\phi(x, y)|^2 k \psi(x, y) |\phi(x, y)|^{2M}} \sum_{j=0}^{m+N} \gamma_j(x, y) k^{-j} + k^{N-m-1} C'_m(x, y, k)$$
with \( \sup_{x,y \in U, k \geq 1} |C^j_m(x,y,k)| < \infty \), and with \( \gamma_0 = 1 \). We thus see that as \( k \to \infty \) the integral over \( U \) has the same asymptotic expansion as

\[
\left( \frac{k}{\pi} \right)^N \sum_{j=0}^\infty \frac{k^{-j}}{j!} \int_U f(x) \beta_j(x,y) \frac{\psi(x) M \psi(y)^M}{\psi(x)^M} \frac{b_0(x,y)^j}{b_0(y)} \frac{[\phi(x) \phi(y)]^k}{[\phi(x),\phi(y)]^2} \, dx.
\]

Finally, recall the familiar formula for the asymptotics of Laplace integrals: if \( D \) is a bounded region in \( \mathbb{R}^n \), \( F \) a complex-valued and \( S \) a real-valued functions in \( C^\infty(D) \), and \( S \) peaks at a single point \( x_0 \in D \), then as \( \lambda \to +\infty \)

\[
\int_D F(x) e^{\lambda S(x)} \, dx = \frac{e^{\lambda S(x_0)}}{\sqrt{\text{Hess } S(x_0)}} \sum_{j=0}^\infty a_j \lambda^{-j},
\]

where the coefficients \( a_j \) depend only on the derivatives of \( F \) and \( S \) at \( x_0 \), \( a_0 = F(x_0) \), and

\[
\text{Hess } S(x_0) = \det \left[ \frac{\partial^2 S}{\partial x_j \partial x_k}(x_0) \right].
\]

Moreover, if \( F \) and \( S \) (and \( x_0 \)) depend in addition smoothly on some additional parameter \( y \in D^l \subset \mathbb{R}^n \), then the asymptotic expansion (30) holds uniformly as \( y \) ranges over a compact subset of \( D^l \). (See [Fed], Theorems 11.2.11 and 11.2.4, or [BH], Section 8.3.) As we have already observed in (15), owing to the strict plurisubharmonicity of \( -\log \phi \) the function \( x \mapsto \frac{\phi(x) \phi(y)}{\phi(x,y)^2} \) has a strict local maximum at \( x = y \). Diminishing \( U \) if necessary, we may thus assume that the function

\[
S(x) = \log \frac{\phi(x) \phi(y)}{[\phi(x,y)]^2}
\]

peaks only at \( x = y \) on \( \overline{U} \times \overline{U} \); shrinking \( U \) further if needed we may likewise assume that \( f \) is \( C^\infty \) on \( \overline{U} \). Consequently, the formula (30) can be applied to the integrals in (29); and since a short computation reveals that

\[
\text{Hess } S(y) = 4^N (\det[-\partial \partial \log \phi])^2 = 4^N b_0(y)^2,
\]

the assertion of the theorem follows.

**Remarks.** (1.) Though the coefficients \( a_j \) in (30) and the operators \( Q_j \) in (9) can in principle be evaluated explicitly, the required computations are extremely cumbersome (cf. the discussion after Theorem 12 in [E6]). In the case which occurs in the Berezin quantization, namely for \( B_{\phi^k \chi} \) with \( \chi = \det[-\partial \partial \log \phi] \), the following general description is therefore of interest. It also shows that the operators \( Q_j \) are very natural objects from the point of view of Kähler geometry.

**Theorem 11.** The operators \( Q_m \) are finite sums of differential operators of the form

\[
f \mapsto \sum_{i_1, \ldots, i_k, j_1, \ldots, j_l} C^{i_1 \ldots i_k, j_1 \ldots j_l} f_{j_1 \ldots j_l i_1 \ldots i_k}
\]

with \( k, l \leq m \), where the slash stands for the covariant differentiation and \( C^{i_1 \ldots i_k j_1 \ldots j_l} \) are tensor fields on \( \Omega \), symmetric in \( i_1, \ldots, i_k \) and in \( j_1, \ldots, j_l \), that are contractions of tensor products of the contravariant metric tensor \( g^{jk} \), the curvature tensor \( R_{j^k i^m} \), and the latter’s covariant derivatives. In particular, \( Q_0 = I \) (the identity operator),

\[
Q_1 = \Delta \equiv \sum_{j,k} g^{jk} \frac{\partial^2}{\partial z_j \partial \bar{z}_k},
\]
the Laplace-Beltrami operator corresponding to the metric (2), and

$$Q_j = \frac{1}{2} \Delta^2 + \frac{1}{2} \sum_{j,k} \text{Ric}_{jk} \frac{\partial^2}{\partial x_j \partial x_k},$$

with $\text{Ric}_{jk}$ the contravariant Ricci tensor, i.e. $\text{Ric}_{jk} = \sum_{l,m} g^{il} \frac{\partial^2}{\partial x^i \partial x^m} \log \chi$, where $\chi = \det[g_{jk}]$ and $g^{-i}$ is the inverse matrix to $g_{jk}$.

For the proof, see [E5], Theorem 4. (The inequalities $k, l \leq m$ are not mentioned there explicitly, but they follow easily from the formula (2.22) there.) In [E6] it is also shown how to evaluate the coefficients $b_j(x,y)$ in (5), for instance,

$$b_1(x,y) = \Delta \log \frac{\sqrt{\chi(x,y)}}{\psi(x,y)^M}.$$

(2.) The condition of finite type, and even of smooth boundedness, of $\tilde{\Omega}$ in Theorem 2 is rather unsatisfactory, since none of the standard metrics associated with a domain $\Omega$ — the Bergman and the Cheng-Yau metric, for instance — satisfy it in general. Indeed, the Bergman metric corresponds to the choice $\phi(z) = 1/K_\Omega(z,z)$, with $K_\Omega(z,z)$ the unweighted Bergman kernel of $\Omega$, which is almost never $C^\infty$ up to the boundary, even for $\Omega$ strictly pseudoconvex, owing to the presence of the logarithmic term in (14). Similarly, for the Cheng-Yau metric on a strictly pseudoconvex domain (the unique Kähler-Einstein metric for which $\text{Ric}_{jk} = -g_{jk}$) the potential has a similar logarithmic singularity at the boundary, by a result of Lee and Melrose [LM]. Thus in both cases, $\tilde{\Omega}$ fails to be $C^\infty$ on the “equator” $\{(x,0) : x \in \partial \Omega\} \subset \partial \Omega$.

(3.) What prevents us from getting Theorem 2 without the smooth boundedness and finite type hypotheses is only Proposition 10, which was needed in the proof to estimate the “tail” of the integral (27) defining the Berezin transform. However, examples suggest that even though Proposition 10 obviously fails in the absence of the above-mentioned hypotheses, the required estimate for the “tail” need not. Hence, Theorem 2 could be extended to all pseudoconvex domains $\Omega$ and positive functions $\phi, \psi \in C^\infty(\Omega)$ such that $-\log \phi, -\log \psi$ are psh if we had a direct proof of the following assertion:

**Problem.** If $\Omega, \phi, \psi$ are as above and $M$ is a nonnegative integer, then for any $x \in \Omega$ and any neighbourhood $U$ of $x$ there exists $\delta > 0$ and an integer $m_0$ such that

$$\sup_{y \in \Omega \setminus U} \left[ \frac{\left| K_{\phi^m \psi^M}(x,y) \right|^2}{K_{\phi^m \psi^M}(x,x)K_{\phi^m \psi^M}(y,y)} \right]^{1/m} \leq 1 - \delta \quad \forall m \geq m_0.$$

(In view of (5), one may replace $1/K_{\phi^m \psi^M}(x,x)$ by $\phi^m(x)$, and similarly for $y$.)

Obviously, this problem is related to the problem of understanding the asymptotic behaviour of $\left| K_{\phi^m \psi^M}(x,y) \right|^{1/m}$ as $m \to \infty$ away from the diagonal $x = y$, mentioned in Remark (6.) in Section 2 above.

4. The Berezin star-product

As this seems not to be done in full anywhere in the literature, we now describe briefly how to construct a star product from Theorem 2. For the sake of brevity, denote $\hbar = 1/m$ and $A_{\hbar} = A^2(\Omega, \phi^m \psi^M)$, $K_{\hbar} = K_{\phi^m \psi^M}$, $B_{\hbar} = B_{\phi^m \psi^M}$ ($m = 1, 2, \ldots$). For $T \in B_{\hbar}$, the
Banach algebra of all bounded linear operators on $A_h$, define the function $T(x, y)$ — the covariant symbol of $T$ — on $\Omega \times \Omega$ by

$$ T(x, y) = \frac{\langle TK_h(\cdot, y), K_h(\cdot, x) \rangle_{A_h}}{K_h(x, y)}, \quad x, y \in \Omega. \quad (32) $$

Then $T$ is representable as the integral operator on $\Omega$ with kernel $T(x, y)K_h(x, y)$ with respect to the measure $\phi(x)^m \psi(y)^M dx$, and for $T_1, T_2 \in B_h$,

$$ (T_1 T_2)(x, y) = \int_\Omega T_1(x, z)T_2(z, y) \frac{K_h(x, z)K_h(z, y)}{K_h(x, y)} \phi(z)^m \psi(z)^M dz. \quad (33) $$

Set $\hat{T}(x) := T(x, x)$ and let $A_h$ be the vector space of all functions of the form $\hat{T}(x)$ with $T \in B_h$. Since the correspondence $T \mapsto \hat{T}$ is one-to-one, one can transfer the algebraic operations and operator involution from $B_h$ into $A_h$, which endows $A_h$ with the structure of an involutive complex algebra, with complex conjugation as the involution, and with an (associative, but not commutative) product which we denote by $*$. See [Ber] or [E2, p. 415] for details.

Let $\mathcal{A}$ be the algebraic direct sum of all $A_h$, $h = 1, 1/2, \ldots$, and let $\tilde{\mathcal{A}} \subset \mathcal{A}$ be the subset of all elements of the form $f(h; x) = \hat{T}_h(x)$, where $\hat{T}_h \in B_h$ for each $h$, for which there exist functions $f_j(x, y)$ on $\Omega \times \Omega$, for $j = 1, 2, \ldots$, holomorphic in $x$ and $y$, such that for each $N \in \mathbb{N}$

$$ T_h(x, y) = f_0(x, y) + hf_1(x, y) + \cdots + h^Nf_N(x, y) + h^{N+1}F_N(h; x, y) \quad (34) $$

where

$$ \sup_x |f_j(x, y)| < \infty, \quad \sup_x |f_j(y, x)| < \infty, \quad \sup_{x, h} |F_N(h; x, y)| < \infty, \quad (35) $$

$$ \forall y \in \Omega, \ j = 0, 1, \ldots, N, \ h = 1, 1/2, 1/3, \ldots. $$

Clearly $\tilde{\mathcal{A}}$ is closed under addition, scalar multiplication and complex conjugation. We denote the product in $\mathcal{A}$ by $*$.

**Theorem 12.** Let $f, g \in \tilde{\mathcal{A}}$. Then for each $z \in \Omega$, we have an asymptotic expansion as $h \to 0$

$$ (f * g)(h; z) = \sum_{i, j, k \geq 0} C_k(Rf_i, Rg_j) h^{i+j+k} \quad (36) $$

where $R : C^\infty(\Omega \times \Omega) \to C^\infty(\Omega)$ is the operator of restriction to the diagonal $x = y$, and $C_k$ are bilinear differential operators of the form

$$ C_k(u, v) = \sum_{\alpha, \beta \geq 0} \sum_{i, j, k \geq 0} T^\alpha_{i, j, k} u^\alpha v^\beta, \quad u, v \in C^\infty(\Omega), \quad (37) $$

where $\alpha, \beta$ are multindices, the slash stands for covariant differentiation with respect to the metric (2), and $T^{\alpha\beta}_{i, j, k}$ are, for each fixed $i, j, k \geq 0$, $i + j \leq k$, tensor fields on $\Omega$, symmetric in the entries of $\alpha$ and of $\beta$, of the same form as in Theorem 11. In particular,

$$ C_0(u, v) = uv, \quad C_1(u, v) = \sum_{i, j} g^{i\beta} u^\beta v^j, \quad C_2(u, v) = \frac{1}{2} \sum_{i, j, k, l, m} g^{i\beta} g^{l\alpha} u^\alpha v^j w^0. \quad (38) $$
where as before $\tilde{g}^k$ is the inverse matrix to $g^k_\phi = -\partial \phi \log \phi$.

Proof. Denote $u(h; x) = f(h; z) g(h; x, z)$, where $f(h; z, x) = T_h(z, x)$ and similarly for $g$. Owing to (33) we then have

$$(f * g)(h; z) = \int_\Omega u(h; x) \frac{|K_h(x, z)|^2}{K_h(z, z)} \phi(x)^{1/h} \psi(x)^M dx = (B_h u(h; \cdot))(z).$$

In view of (34), for each $N \in \mathbb{N}$,

$$u(h; x) = u_0(x) + h u_1(x) + \cdots + h^N u_N(x) + h^{N+1} U_N(h; x),$$

where

$$u_k(x) = \sum_{i+j=k} f_i(z, x) g_j(x, z), \quad k = 0, 1, \ldots, N,$$

$$U_N(h; x) = \sum_{i+j > N \atop i, j \leq N} h^{i+j-N-1} f_i(z, x) g_j(x, z) + f_0(z, x) G_N(h; x, z) + F_N(h; x) g_0(x, z).$$

The hypotheses on $\tilde{A}$ ensure that $u_1, \ldots, u_N$ and $U_N(h; \cdot)$ are bounded on $\Omega$, the last uniformly in $h$. By Theorem 2, we therefore get

$$(B_h u(h; \cdot))(z) = \sum_{k=0}^N h^k B_h u_k(z) + h^{N+1} O(||U_N(h; \cdot)||_\infty)$$

$$= \sum_{k=0}^N h^k \sum_{l=0}^\infty h^l Q_l u_k(z) + h^{N+1} O(1)$$

$$= \sum_{l+k \geq 0 \atop l+k \leq N} h^{l+k} Q_l u_k(z) + O(h^{N+1})$$

$$= \sum_{l+i+j \geq 0 \atop l+i+j \leq N} h^{l+i+j} Q_l [f_i(z, x) g_j(x, z)]_{z \to z} + O(h^{N+1}),$$

where in the last expression the operators $Q_l$ act on the $x$ variable. As $N$ is arbitrary, we thus obtain an asymptotic expansion

$$(f * g)(h; z) = \sum_{i,j,l \geq 0} h^{i+j+l} \Gamma_l (f_i, g_j)(z),$$

with

$$\Gamma_l (f_i, g_j)(z) := Q_l [f_i(z, \cdot) g_j(\cdot, z)]_{z \to z}.$$  

We have to show that this coincides, in fact, with $C_k(R f_i, R g_j)(z)$ for certain differential operators $C_k$ of the form (37).

Consider thus, quite generally, two functions $f, g$ on $\Omega \times \Omega$ holomorphic in the first variable and anti-holomorphic in the second, fix a point $z \in \Omega$, and let us show that $Q_k[f(z, \cdot) g(\cdot, z)]$, evaluated at $z$, comes as a sum

$$\sum_{i,j=0}^k \sum_{|\alpha|=i \atop |\beta|=j} S_{(i,j)}^{(\alpha)} (R f)_{\alpha} (R g)_{\beta} \quad \text{evaluated at } z,$$
where $S_{ij}^{(j)}$, $i, j = 0, \ldots, k$, are tensor fields on $\Omega$ of the form described in the theorem. For brevity, denote $F = f(z, \cdot)$ and $G = g(\cdot, z)$; thus $G$ is holomorphic and $F$ is antiholomorphic on $\Omega$. In view of Theorem 11, it is sufficient to show that each covariant derivative of the product $FG$, evaluated at $z$, is a sum of expressions of the form

$$S_{ij}^{(j)} F_{ij} G_{ij} \big|_z,$$

(39)

with $S_{ij}^{(j)}$ being components of tensor fields involving only the metric tensor, the curvature tensor, and the latter’s covariant derivatives; the assertion will then follow since, as $G$ is holomorphic,

$$G_{ij}^{(j)}(z) = (\delta g)_{ij}^{(j)}(z),$$

and similarly for $F_{ij}$. Now by the product rule, $(FG)_{ij}$ comes as a sum of products $F_{ij} G_{ij}$ of covariant derivatives of $F$ and $G$. Using the Ricci formula, which exhibits the commutator $(T_{\alpha \beta \gamma \delta})_{ij} = (T_{\alpha \beta \gamma \delta})_{ij} - (T_{\alpha \beta \gamma \delta})_{ij}$ of two successive covariant differentiations of a tensor $T_{\alpha \beta \gamma \delta}$ as a sum of contractions of $T_{\alpha \beta \gamma \delta}$ against the curvature tensor (see e.g. [E5], formula (2.17)), we further reduce to showing

$$F_{ij} G_{ij} \big|_z,$

is of the form (39), for any multiindices $\alpha, \beta, \gamma$ and $\delta$. However, in view of the holomorphy of $F$ and $G$, we have $F_{ij} = G_{ij} = 0$ for any $l$, hence the last expression vanishes unless $|\gamma| = |\delta| = 0$. Thus the required assertion follows.

Finally, recalling that $Q_0 = 1$ and $Q_1 = \Delta$, we have $\Gamma_0(f, g)(z) = F(z)G(z)$ and $\Gamma_1(f, g)(z) = \sum_i g_i F_{ij} G_{kj} |_z$, hence in terms of $u := Rf$ and $v := Rg$ we get $C_0(u, v) = uv$ and $C_1(u, v) = \sum_i g_i F_{ij} G_{jk} |_z$. The formula for $C_2$ follows similarly from the formula for $Q_2$ in Theorem 11. This completes the proof. $\square$

Consider now the ring (over the field of complex numbers) $C^\infty(\Omega)[[h]]$ of all formal power series

$$f(h; x) = \sum_{j=0}^{\infty} f_j(x)h^j, \quad f_j \in C^\infty(\Omega) \forall j,$$

with the usual algebraic operations, and endow it with the $C[[h]]$-linear product $\ast$ given by

$$(\sum_{i \geq 0} f_i h^i) \ast (\sum_{j \geq 0} g_j h^j) := \sum_{i, j \geq 0} C_k(f_i, g_j) h^{i+j+k}$$

(40)

with the operators $C_k$ from (36).

**Theorem 13.** The product $\ast$ is a star-product, i.e. it is associative and satisfies

$$(f \ast g)_0 = f_0 g_0 \quad \text{(the pointwise product in } C^\infty(\Omega)),

\left(\frac{f \ast g - g \ast f}{h}\right)_0 = i\{f, g\} \quad \text{(the Poisson bracket with respect to the metric (2)).}$$

(41)

**Proof.** For $f(h; x) \in \mathcal{A}$, this is immediate from the preceding theorem; as the term at $h^m$ in (40) — namely, $\sum_{i+j+k=m} C_k(f_i, g_j)$ — involves only $\mathcal{T}^g f_i$ and $\partial^\beta g_j$ with $i, j, |\alpha|, |\beta| \leq m$, it therefore suffices to check that for each $f \in C^\infty(\Omega)[[h]]$, $z \in \Omega$ and $N \in \mathbb{N}$, there exists $f' \in \mathcal{A}$ such that

$$\mathcal{T}^f f'_j(z) = \mathcal{T}^f f_j(z) \quad \forall j, |\alpha| \leq N.$$
Indeed, one then has \( C_k(f, g)(z) = C_k(f', g')(z) \) \( \forall k = 0, 1, \ldots, N \), so \( f * g)(h; z) = (f' * g')(h; z) \) mod \( h^{N+1} \), and the validity of (41) for \( f, g \) follows from its validity for \( f', g' \); similarly for the associativity.

We will search for such an \( f' \in \tilde{A} \) in the form

\[
f'(h; z) = \sum_{j=0}^{N} p_j(x) h^j
\]

with \( p_j \) bounded holomorphic functions on \( \Omega \). Then \( f'(x, y) = \overline{p_j(y)} \), so (34) and (35) hold, and (42) is equivalent to \( \partial^\alpha \overline{p_j(z)} = \overline{\partial^\alpha f_j(z)} \) for all \( |\alpha|, j \leq N \). Clearly, such functions \( p_j \) exist (for instance, take the polynomials \( p_j(x) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial^\alpha f_j(z) x^\alpha \)). It remains to show that \( f'(h; x) = \widetilde{T}_h(x) \) for some family of operators \( T_h \in \mathcal{B}_h \) \( (h = 1, 1/2, \ldots) \). However, observe that for \( \widetilde{T} \) the operator on \( A_h \) of multiplication by a bounded analytic function \( p \), one has from (38)

\[
\widetilde{T}(x) = \frac{\langle p K_h(x, y), K_h(x, y) \rangle}{K_h(x, x)} = \frac{p(x) K_h(x, x)}{K_h(x, x)} = p(x),
\]

by the reproducing property of \( K_h \). It follows that \( \widetilde{T}^* = \overline{\widetilde{T}} \). Further, \( T \) is bounded if \( p \) is. Thus the operators

\[
T_h = \text{(multiplication by } \sum_{j=0}^{N} h^j p_j \text{ on } A_h)^*
\]

do the job we need. \( \square \)

Remark. It follows from (37) that the Berezin star product is an example of a “deformation quantization with separation of variables” in the sense of Karabegov [Kar], i.e. \( f * g = fg \) (pointwise product) if \( f \) or \( \overline{f} \) is analytic. It is also easy to show that \( \overline{f} * \overline{g} = g * f \), i.e. \( * \) preserves complex conjugation.

5. Berezin-Toeplitz quantization

In this section we restrict our attention to the case of \( \Omega \) smoothly bounded and strictly pseudoconvex, \( -\phi \) a smooth defining function for \( \Omega \) (i.e. \( \phi \) is \( C^\infty \) in a neighbourhood of \( \overline{\Omega} \), \( \phi > 0 \) on \( \Omega \), \( \phi < 0 \) on the complement of \( \overline{\Omega} \), and \( \phi = 0 \), \( \|\nabla \phi\| \neq 0 \) on \( \partial \Omega \) such that \( -\log \phi \) is strictly psh, and to the weights \( \phi^k \chi \), \( \chi = \det[-\partial \overline{\partial} \log \phi] \). For the proof of part (ii) of Theorem 3, we need to use the theory of Boutet de Monvel-Guillemin Toeplitz operators [BG], in the same way as in [BMS] and [Sch] for compact manifolds; we include a sketch of the proof below for convenience. The point we wish to make here is that the part (i) of Theorem 3 is an easy consequence (not using the Boutet de Monvel-Guillemin theory in any way) of Theorem 2 and the following elementary observation.

Lemma 14. Let \( \Omega \) be a strictly pseudoconvex domain in \( \mathbb{C}^N \) and \( -\phi \) a smooth defining function for \( \Omega \) such that \( -\log \phi \) is strictly psh. Then for any positive continuous function \( g \) on \( \overline{\Omega} \) there exists an integer \( d > 0 \) such that \( -\log \phi - \frac{1}{d} \log g \) is strictly psh on \( \Omega \).

Proof. Observe that

\[
\begin{bmatrix}
-\phi & -\phi_T \\
-\phi_j & -\phi_j T - (\phi/d)(\log g)_{jT}
\end{bmatrix}
= \phi \begin{bmatrix} 1 & 0 \\ \phi_j/\phi & 1 \end{bmatrix}
\begin{bmatrix} -1 & 0 \\ 0 & (\log \phi - (1/d) \log g)_{jT} \end{bmatrix}
\begin{bmatrix} 1 & \phi_T/\phi \\ 0 & 1 \end{bmatrix}
\]
where for brevity we have used the subscripts $j, k$ to denote differentiations by $z_j, \overline{z}_k$. Thus $-\log \phi - (1/d) \log g$ is strictly psh at $z \in \Omega$ if and only if the square matrix
\[
\begin{pmatrix}
-\phi & -\phi \nu \\
-\phi_j & -\phi_j \nu
\end{pmatrix} - \frac{1}{d} \begin{pmatrix}
0 & 0 \\
0 & (\log g)_\nu \phi
\end{pmatrix} \equiv A(z) - \frac{1}{d} B(z)
\]
has 1 negative and $N$ positive eigenvalues. However, in view of the hypotheses on $\phi$ and $g$, both matrix-valued functions $A(z)$ and $B(z)$ are continuous on $\overline{\Omega}$ and $A(z)$ has the required signature $(N, 0, 1)$ for each $z \in \overline{\Omega}$; thus $\sup_{z \in \overline{\Omega}} ||B(z)|| = b < \infty$ and $\mathcal{K} = \{ A(z), z \in \overline{\Omega} \}$ is a compact subset of the open set $\mathcal{N}$ of all Hermitian matrices of signature $(N, 0, 1)$. Taking $d$ so large that $b/d < \text{dist}(\mathcal{K}, \partial \mathcal{N})$, the assertion follows. \(\square\)

**Proof of part (i) of Theorem 3.** Let $J[\phi]$ the Monge-Ampère determinant (13). As $\chi = J[\phi]/\phi^{N+1}$, the strict plurisubharmonicity of $-\log \phi$ implies that $J[\phi] > 0$ on $\Omega$; on the other hand, the fact that $\phi$ is a smooth defining function of the strictly pseudoconvex domain $\Omega$ implies that $J[\phi] \in C^\infty(\overline{\Omega})$ and $J[\phi] > 0$ on $\partial \Omega$. Thus the preceding lemma applies to $g = J[\phi]$. Let $d$ be the constant provided by the lemma, and apply Theorem 2 to $\Omega$, $\phi$, $M = d$, and $\psi = J[\phi]^{1/d}$ $\phi$ (thus $\phi = \phi \psi^{M = o \phi^{N+1}}$). This gives
\[
B_{\phi = \chi} f(x) = f(x) + O(m^{-1}) \quad \text{as } m \to \infty, \tag{43}
\]
for any $f \in L^\infty(\Omega) \cap C^\infty(\Omega)$, uniformly for $x$ in compact subsets. On the other hand, let
\[
k_x^{(m)} = \frac{K_{\phi = \chi}(x, x)}{K_{\phi = \chi}(x, x)^{1/2}}
\]
be the unit vector in the direction of $K_{\phi = \chi}(x, x)$ (the coherent state). Then by the definition of the Toeplitz operator and of the Berezin transform,
\[
B_{\phi = \chi} f(x) = \langle T_f^{(\phi = \chi)}, k_x^{(m)} \rangle_{L^2(\Omega, \phi = \chi)}. \tag{44}
\]
It follows that
\[
|B_{\phi = \chi} f(x)| \leq \|T_f^{(\phi = \chi)}\| \|k_x^{(m)}\|^2 = \|T_f^{(\phi = \chi)}\|, \tag{45}
\]
Combining this with (43), we see that $\liminf_{m \to \infty} \|T_f^{(\phi = \chi)}\| \geq \|f\|_\infty$. On the other hand, for any weight $\rho$,
\[
\|T_f^{(\rho)} h\| = \|P_h(fh)\| \leq \|fh\| \leq \|f\|_\infty \|h\|,
\]
whence $\|T_f^{(\rho)}\| \leq \|f\|_\infty$. Thus the desired assertion follows. \(\square\)

**Proof of part (ii) of Theorem 3.** Let us now consider the domain (8) with $M = 0$:
\[
\overline{\Omega} = \{(z, w) \in \Omega \times \mathbb{C} : |w|^2 < \phi(z)\}. \tag{46}
\]
The hypotheses then mean precisely that $\overline{\Omega}$ is a smoothly bounded, strictly pseudoconvex domain in $\mathbb{C}^n$, $n = N + 1$, and $r(z, w) := |w|^2 - \phi(z)$ is a smooth defining function for $\Omega$. Its boundary $X = \partial \overline{\Omega}$ is a compact manifold, and we denote by $\alpha$ the restriction to $X$ of the 1-form $\Im \partial \overline{\partial} r = (\partial r - \overline{\partial} r)/2i$. Then $\alpha$ is a contact form, i.e. $\alpha \wedge (\alpha^*)^{n-1}$ determines a nonvanishing volume form on $X$. Following [BG], let $L^2$ be the Lebesgue space on $X$ with respect to this measure, and $H^2$ the closure in $L^2$ of the subspace of all functions in $C^\infty(X)$ which extend holomorphically into $\overline{\Omega}$. For each $(z, w) \in \overline{\Omega}$, the evaluation functional $f \mapsto f(z, w)$ turns out to be continuous on $H^2$, hence is given by the scalar product with a certain element $k_{(z, w)}^{(1)} \in H^2$. The function
\[
\overline{K}_{\text{Segal}}(z_1, w_1; z_2, w_2) := \langle k_{(z_2, w_2)}, k_{(z_1, w_1)} \rangle_{H^2}
\]
on $\bar{\Omega} \times \Omega$ is called the Szegö kernel; it extends smoothly to all of $\bar{\Omega} \times \Omega$ minus the boundary diagonal, and the integral operator determined by $K_{\text{Szegö}}$ on $X$ is the Szegö projection $\pi$, i.e. the orthogonal projection in $L^2$ onto $H^2$. For $F \in L^\infty (X)$, the (global) Toeplitz operator $T_F$ is the operator on $H^2$ defined by

$$T_F f = \pi (F f).$$

Let $H_{(m)} \subset H^2 \ (m = 0, 1, 2, \ldots)$ be the subspace of all functions of the form

$$f(z, w) = f(z) w^m \quad ((z, w) \in X).$$

A routine computation shows that (see e.g. [Ran], p. 291)

$$\alpha \wedge (d\alpha)^{n-1} = (n-1)! \frac{\int [r]}{\|d\alpha\|} dS,$$

$dS$ being the surface element (i.e. $(2n-1)$-dimensional Hausdorff measure) on $X$. Introducing the coordinates

$$(z, w) = (z, e^{i\theta} \sqrt{\phi(z)}) \quad (z \in \Omega, \theta \in [0, 2\pi])$$
on $X$, we have $dS = \sqrt{\phi + \|\phi\|^2} \ dz \ d\theta$; and as $\sqrt{\phi + \|\phi\|^2} = \|d\phi\|$ and $J[r] = J[\phi] = \phi^{N+1} \chi$, we thus have

$$\int_X |f(z)w^m|^2 \alpha \wedge (d\alpha)^{n-1} = (n-1)! \int_\Omega \int_0^{2\pi} |f(z)|^2 \phi(z)^m J[r] \ dz \ d\theta$$

$$= 2\pi (n-1)! \int_\Omega \int |f|^2 \phi^{m+N+1} \chi \ dz.$$

It follows that $H_{(m)}$ is isometrically (up to the immaterial factor $(n-1)!2\pi$) isomorphic to the Bergman space $A^2 (\Omega, \phi^{m+N+1} \chi)$. As the subspaces $H_{(m)}$ are pairwise orthogonal and span all of $H^2$, we therefore arrive at the following analogue of Proposition 4 (cf. [Lig], [BFS], [KLR]):

$$K_{\text{Szegö}} (z_1, w_1; z_2, w_2) = \frac{1}{2\pi N!} \sum_{m=0}^\infty K_{\phi^{m+N+1} \chi} (z_1, z_2) \cdot (w_1 \bar{w_2})^m. \quad (46)$$

For $f \in L^\infty (\Omega)$, let $\hat{f} \in L^\infty (X)$ be defined by

$$\hat{f}(z, w) = f(z) \quad (47)$$

and consider the Toeplitz operator $T_{\hat{f}}$. Then all the subspaces $H_{(m)}$ are invariant under $T_{\hat{f}}$, and the restriction of $T_{\hat{f}}$ to $H_{(m)}$ is (up to the unitary equivalence above) precisely the Toeplitz operator $T_{\phi^{m+N+1} \chi}$. That is,

$$T_{\hat{f}} \simeq \bigoplus_{m=N+1}^\infty T_{\phi^{m} \chi}. \quad (48)$$

Finally, a generalized Toeplitz operator on $H^2$ is the operator $T_P$ given by

$$T_P = \pi P \pi.$$
where $P$ is a pseudodifferential operator on the compact manifold $X$. The order $\text{ord}(T_P)$ and the symbol $\sigma(T_P)$ of $T_P$ are defined as the order of $P$ and the restriction of the symbol $\sigma(P)$ of $P$ to the submanifold

$$\Sigma := \{(x, \xi) : \xi = t\alpha_x, t > 0\}$$

of the cotangent bundle of $X$, respectively. It was then shown in [BG] that these two definitions are unambiguous, and

(P1) the generalized Toeplitz operators form an algebra under composition,

(P2) $\text{ord}(T_1T_2) = \text{ord}(T_1) + \text{ord}(T_2)$, $\sigma(T_1T_2) = \sigma(T_1)\sigma(T_2)$,

(P3) $\sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}$ (the Poisson bracket),

(P4) if $\text{ord}(T) = 0$, then $T$ is a bounded operator on $H^2$, and

(P5) if $\text{ord}(T_1) = \text{ord}(T_2) = k$ and $\sigma(T_1) = \sigma(T_2)$, then $\text{ord}(T_1 - T_2) \leq k - 1$.

Let $\mathcal{T}$ be the subalgebra of all generalized Toeplitz operators which commute with the circle action

$$U_\theta : f(z, w) \mapsto f(z, e^{i\theta}w) \quad ((z, w) \in X, \theta \in \mathbb{R})$$

on $H^2$; clearly, the operators $T_j$ with $\hat{f}$ as in (47) belong to $\mathcal{T}$. Let $D$ be the infinitesimal generator of the semigroup $U_\theta$; one has

$$Dh = imh \quad \forall h \in H_{(m)}, \quad (49)$$

and $D = T_{\partial_j \partial_\theta}$ is a generalized Toeplitz operator of order 1. Using (P1)-(P5) it was then shown in [BMS] (see also [Sch] and [Gu]) that if $T \in \mathcal{T}$ is of order 0, then

$$T = T_j + D^{-1}R$$

for some (uniquely determined) $f \in L^\infty(\Omega)$ and $R \in \mathcal{T}$ of order 0. Repeated application of this formula reveals that, for each $k \geq 0$,

$$T = \sum_{j=0}^{k} D^{-j}T_j + D^{-k-1}R_k,$$

with $f_j \in L^\infty(\Omega)$ and $R_k \in \mathcal{T}$ of order 0. Invoking the boundedness of zeroth order operators, it follows that

$$D^{k+1}\left(T - \sum_{j=0}^{k} D^{-j}T_j\right)$$

is a bounded operator on $H^2$, that is, by (49) and (48),

$$\left\|T|_{H_{(m)}} - \sum_{j=0}^{k} m^{-j}T_j^{(0,m)}\right\| = O(m^{-k-1}).$$

Taking for $T$ the product $T_jT_j$ with $f, g \in L^\infty(\Omega) \cap C^\infty(\Omega)$, we obtain (10). Finally, the assertions concerning $C_0$ and $C_1$ follow from the properties (P2) and (P3) of the symbol; see the references mentioned above for the details. \(\square\)

Remarks. (1.) The same comment as in Remark (2.) at the end of Section 3 applies here as well: namely, the machinery of [BG] makes heavy use of the compactness of the manifold $X = \partial \Omega$, so the last proof does not easily generalize to unbounded domains $\Omega$. The problem is that, first, zeroth order pseudodifferential operators no longer need to be
bounded, and second, that the various smoothing operators which arise as error terms need not be bounded either. The former can be coped with by means of the Calderon-Vaillancourt theorem (by replacing $L^\infty(\Omega) \cap C^\infty(\Omega)$ by the space of all functions on $\Omega$ all of whose derivatives up to a certain order are continuous and uniformly bounded); the latter has so far been dealt with successfully only in the case of $\Omega = \mathbb{C}^n$ with the Euclidean metric (hence with the potential $-\log \phi(z) = |z|^2$), giving rise to the Segal-Bargmann spaces of entire functions square-integrable with respect to the Gaussian measures $e^{-m|z|^2} dz$, see [Bw].

Similarly, the requirement that $X = \partial\Omega$ be smooth prevents us from dealing with the case of $\phi$ not being a defining function (or a power of one), thus excluding the most interesting cases such as the Bergman and the Cheng-Yau (Kähler-Einstein) metric. Apparently, this difficulty will probably be even harder to overcome than the previous one.

We remark that, on the other hand, to some extent the above approach can be generalized to some non-Kähler compact manifolds as well; see [BU].

(2.) On a purely formal level, the Berezin-Toeplitz quantization has been carried out on any Kähler manifold by Reshetikhin and Takhtajan [RT]; however, it is not clear whether these arguments (involving a formal application of the stationary-phase method) can be made rigorous. Cf. also the similar formal expansion by Cornalba and Taylor [CT]. For bounded symmetric domains in $\mathbb{C}^n$ (Hermitian symmetric spaces) with the invariant metric, a similar line of attack — namely, a study of the asymptotics of the weighted Bergman projections $P^m : L^2(\Omega, \phi^m) \to A^2(\Omega, \phi^m)$ as $m$ tends to infinity — has been initiated by Arazy and Ørsted [AO].

(3.) The proofs in Sections 2 and 3 feature an interplay between two subjects: the $\overline{\partial}$-techniques due to Kohn, Catlin, and others, on the one hand, and the theory of Fourier integral operators on the other hand. The latter is more powerful for the applications we have — in particular, the Boutet de Monvel-Guillemin theory of generalized Toeplitz operators in this section relies on it completely, and it is also hidden in Boutet de Monvel and Sjöstrand's proof of the Fefferman theorem which was our departing point for most of the developments in Section 2. On the other hand, the former lend themselves much more easily to localization — they work in a neighbourhood of a smooth strictly pseudoconvex point even if the boundary is bad (nonsmooth, weakly pseudoconvex, even unbounded) away from it. From this point of view, it would be very desirable to have at least the following "noncompact version" of the Boutet de Monvel-Guillemin theory.

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^N$ and $\phi$ a function on $\overline{\Omega}$ which is positive and $C^\infty$ on $\Omega$, vanishes at $\partial \Omega$ (but it is not assumed to be $C^\infty$ up to the boundary), and such that $-\log \phi$ is strictly psh on $\Omega$. Consider the domain $\Omega$ as in (45), and let $X = \{(z, w) \in \Omega \times \mathbb{C} : |w|^2 = \phi(z)\}$ be the part of $\partial \Omega$ which lies over $\Omega$. Then $X$ is a noncompact strictly pseudoconvex CR-manifold; the form $\alpha \wedge (d\alpha)^N$ is, as before, a nonvanishing surface element on $X$, and we define $H^2$ as the closure in $L^2$ of the subspace of functions that extend holomorphically to $\Omega \times \mathbb{C}$. The corresponding Szegö kernel will then still satisfy (46), and one can again introduce the global Toeplitz operators $T_F, F \in L^\infty(X)$, and $T^j, f \in L^\infty(\Omega)$, and the generalized Toeplitz operators $T_p$, with $P$ a pseudodifferential operator on $X$ which we now assume in addition to be bounded on $L^2$. Let again $T$ be the set of all generalized Toeplitz operators commuting with the circle action $U_\theta$. By the Calderon-Vaillancourt theorem, $T^j \in T$ if $f \in BC^\infty(\Omega)$, the subspace in $C^\infty(\Omega)$ of functions all partial derivatives of which belong to $L^\infty(\Omega)$.

**Problem.** Do the properties (P1)-(P5) prevail in this setting?

As has already been mentioned above, this was settled in the affirmative in the case of $\Omega = \mathbb{C}^N$ and $\phi(z) = |z|^2$ in [Bw].
(4.) We conclude by exhibiting how the Berezin-Toeplitz star product implicit in Theorem 3, namely,
\[ f \ast g := \sum_{j=0}^{\infty} h^j C_j(f, g), \quad f, g \in C^\infty(\Omega)[[h]], \quad (50) \]
with \( C_j \) given by (10) and extended \( \mathbb{C}[[\hbar]] \)-linearly from \( C^\infty(\Omega) \) to \( C^\infty(\Omega)[[\hbar]] \), is related to the Berezin star product constructed in Section 4. Let us for the moment write \( *_{BT} \) for the former and \( *_B \) for the latter, and as in Section 4 set again \( \hbar = 1/m \) and \( T_j^{(h)} = T_j(\delta^{m, x}) \), \( k_x^{(h)} = k_x^{(m)} \), etc. For \( f, g \in C^\infty(\Omega) \), we thus have
\[ (B_h f *_B B_h g)(x) = \sum_{k=0}^{\infty} C^B_k (B_h f, B_h g)(x) h^k, \]
where \( C^B_k \) are the operators \( C_k \) from (40), extended to \( C^\infty(\Omega)[[\hbar]] \) by \( \mathbb{C}[[\hbar]] \)-linearity, and we regard
\[ B_h f = \sum_{j=0}^{\infty} Q_j f h^j \quad (51) \]
as an element of \( C^\infty(\Omega)[[\hbar]] \). On the other hand, by the definition of \( *_B \) (cf. (33), (44) and (36)) we have
\[ (B_h f *_B B_h g)(x) = (T_j^{(h)} T_j^{(h)} k_x^{(h)}, k_x^{(h)})_{A_x}. \]
But in view of (44) and (50), the last expression is equal to
\[ B_h \sum_{k=0}^{\infty} C^B_k(f, g) h^k \]
where \( C^B_k \) are the operators \( C_k \) from (50). Thus the two star products are related by
\[ \sum_{k=0}^{\infty} B_h C^B_k(f, g) h^k = \sum_{k=0}^{\infty} C^B_k(B_h f, B_h g) h^k. \quad (52) \]
As \( Q_0 = I \), it follows from (51) that \( B_h \), regarded as a formal power series in \( h \) with differential operators as coefficients, has an inverse \( B_h^{-1} \); one can thus rewrite (52) as
\[ \sum_{k=0}^{\infty} C^B_k(f, g) h^k = B_h^{-1} \sum_{k=0}^{\infty} C^B_k(B_h f, B_h g) h^k, \]
or, regarding \( B_h \) as a linear operator on \( C^\infty(\Omega)[[\hbar]] \) (i.e. as a formal differential operator),
\[ B_h(f *_{BT} g) = B_h f *_{B} B_h g. \quad (53) \]
Thus the two star products differ only by a “change of ordering” \( f \leftrightarrow B_h f \). In the terminology of [Kar], (53) says that the star products \( *_B \) and \( *_{BT} \) are duals of one another.

(5.) Using the formulas for \( C^B_1 \) and \( C^B_2 \) in Theorem 12, it is possible to derive from (53) analogous formulas for the operators \( C^B_1 \) and \( C^B_2 \):
\[ C^B_1(u, v) = -\sum_{j, k} g^{jk} a_{jk} v_j, \]
\[ C^B_2(u, v) = \frac{1}{2} \sum_{j, k, l, m} g^{jk} g^{lm} u_{jkm} v_j v_l - \sum_{j, k} \text{Ric} g^{jk} a_{jk} v_j. \]

(6.) Again, the Berezin-Toeplitz star product is an example of a deformation quantization with the separation of variables, but this time with the role of holomorphic and anti-holomorphic variables swapped (i.e. \( f *_{BT} g = fg \) if \( g \) is holomorphic or \( f \) is anti-holomorphic); see [KS] for the identification of \( *_{BT} \) within this scheme.
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References


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