Homogeneous Hyper-Hermitian Metrics
Which Are Conformally HyperKähler

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Supported by Federal Ministry of Science and Transport, Austria
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MARÍA LAURA BARBERIS

Abstract. Let \( g \) be a hyper-Hermitian metric on a simply connected hypercomplex four-manifold \((M, \mathcal{H})\). We show that when the isometry group \( I(M, g) \) contains a subgroup acting simply transitively on \( M \) by hypercomplex isometries then the metric \( g \) is conformal to a hyper-Kähler metric. We describe explicitly the corresponding hyper-Kähler metrics and it follows that, in four dimensions, these are the only hyper-Kähler metrics containing a homogeneous metric in its conformal class.

1. Preliminaries

A hypercomplex structure on a \( 4n \)-dimensional manifold \( M \) is a family \( \mathcal{H} = \{ J_\alpha \}_{\alpha=1,2,3} \) of fibrewise endomorphisms of the tangent bundle \( TM \) of \( M \) satisfying:

\[
\begin{align*}
J_\alpha^2 &= -I, \quad \alpha = 1, 2, 3, \\
J_1 J_2 &= -J_2 J_1 = J_3,
\end{align*}
\]

where \( I \) is the identity on the tangent space \( T_p M \) of \( M \) at \( p \) for all \( p \) in \( M \) and \( N_\alpha \) is the Nijenhuis tensor corresponding to \( J_\alpha \):

\[
N_\alpha(X,Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha([X, J_\alpha Y] + [J_\alpha X, Y])
\]

for all \( X, Y \) vector fields on \( M \). A differentiable map \( f : M \to M \) is said to be hypercomplex if it is holomorphic with respect to \( J_\alpha, \alpha = 1, 2, 3 \). The group of hypercomplex diffeomorphisms on \((M, \mathcal{H})\) will be denoted by \( \text{Aut}(\mathcal{H}) \).

A riemannian metric \( g \) on a hypercomplex manifold \((M, \mathcal{H})\) is called hyper-Hermitian when \( g(J_\alpha X, J_\alpha Y) = g(X, Y) \) for all vector fields \( X, Y \) on \( M, \alpha = 1, 2, 3 \).

Given a manifold \( M \) with a hypercomplex structure \( \mathcal{H} = \{ J_\alpha \}_{\alpha=1,2,3} \) and a hyper-Hermitian metric \( g \) consider the 2-forms \( \omega_\alpha, \alpha = 1, 2, 3 \), defined by

\[
\omega_\alpha(X, Y) = g(X, J_\alpha Y).
\]

The metric \( g \) is said to be hyper-Kähler when \( d\omega_\alpha = 0 \) for \( \alpha = 1, 2, 3 \).

It is well known that a hyper-Hermitian metric \( g \) is conformal to a hyper-Kähler metric \( \tilde{g} \) if and only if there exists an exact 1-form \( \theta \in \Lambda^1 M \) such that

\[
d\omega_\alpha = \theta \wedge \omega_\alpha, \quad \alpha = 1, 2, 3
\]

where, if \( g = e^f \tilde{g} \) for some \( f \in C^\infty(M) \), then \( \theta = df \).

We prove the following result:

1991 Mathematics Subject Classification. Primary 53C15, 53C25, 53C30.
Key words and phrases. hyper-Hermitian metric, hypercomplex manifold, conformally hyper-Kähler metric.

The author was partially supported by CONICET, ESI (Vienna) and FOMEC (Argentina).
Theorem 1.1. Let \((M, \mathcal{H}, g)\) be a simply connected hyper-Hermitian 4-manifold. Assume that there exists a Lie group \(G \subset \text{Isom}(M, g) \cap \text{Aut}(\mathcal{H})\) acting simply transitively on \(M\). Then \(g\) is conformally hyper-Kähler.

We conclude that one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz \([2]\) contains a homogeneous hyper-Hermitian metric in its conformal class. This hyper-Hermitian metric is not symmetric and has negative sectional curvature \([1]\).

As a consequence of Theorem 1.1 and the results in \([1]\) we obtain that the following symmetric riemannian metrics are conformally hyper-Kähler:

- the riemannian product of the canonical metrics on \(\mathbb{R} \times S^3\);
- the riemannian product of the canonical metrics on \(\mathbb{R} \times \mathbb{R} H^3\), where \(\mathbb{R} H^3\) denotes the real hyperbolic space;
- the canonical metric on the real hyperbolic space \(\mathbb{R} H^4\).

Acknowledgements. I would like to thank the organizers of the program Holonomy Groups in Differential Geometry for giving me the opportunity to visit the Erwin Schrödinger Institute, Vienna. I am also grateful to D. Alekseevsky, L. Dotti Miatello, L. Ornea and S. Salamon for useful conversations.

2. Proof of the main theorem

Proof of Theorem 1.1. Since \(G\) acts simply transitively on \(M\) then \(M\) is diffeomorphic to \(\mathbb{R} \times S^3\) and therefore the hypercomplex structure and hyper-Hermitian metric can be transferred to \(G\) and will also be denoted by \(\{J_\alpha\}_{\alpha=1,2,3}\) and \(g\), respectively. Since \(G\) acts by hypercomplex isometries it follows that both \(\{J_\alpha\}_{\alpha=1,2,3}\) and \(g\) are left invariant on \(G\). All such simply connected Lie groups were classified in \([1]\), where it is shown that the Lie algebra \(\mathfrak{g}\) of \(G\) is either abelian or isomorphic to one of the following Lie algebras (we fix an orthonormal basis \(\{e_j\}_{j=1,\ldots,4}\) of \(\mathfrak{g}\)):

1. \([e_2, e_3] = e_4, \ [e_3, e_4] = e_2, \ [e_4, e_2] = e_3, \ e_1 \text{ central};
2. \([e_1, e_2] = e_1, \ [e_2, e_3] = e_2, \ [e_3, e_4] = e_2, \ [e_2, e_4] = -e_1;
3. \([e_1, e_j] = e_j, \ j = 2, 3, 4;
4. \([e_3, e_4] = \frac{1}{2} e_2, \ [e_1, e_2] = e_2, \ [e_1, e_j] = \frac{1}{2} e_j, \ j = 3, 4.

Observe that in case 1 above \(M\) is diffeomorphic to \(\mathbb{R} \times S^3\) while in the remaining cases it is diffeomorphic to \(\mathbb{R}^4\), therefore in all cases any closed form on \(M\) is exact. We now proceed by finding in each case a closed form \(\theta \in \Lambda^1 \mathfrak{g}^*\) satisfying (1.4). Note that we work on the Lie algebra level since \(g\) and \(\omega_{\alpha}\) are all left invariant on \(G\). Let \(\{e_j\}_{j=1,\ldots,4} \subset \Lambda^1 \mathfrak{g}^*\) be the dual basis of \(\{e_j\}_{j=1,\ldots,4}\). From now on we will write \(e^{j_1 \cdots j_r}\) to denote \(e^{j_1} \wedge e^{j_2} \wedge \cdots\). In all the cases below the 2-forms \(\omega_{\alpha}\) are determined from (1.3) in terms of the hypercomplex structures constructed in \([1]\).

Case 1. The 2-forms \(\omega_{\alpha}\) are given as follows:

\[
\omega_1 = -\epsilon^{12} - \epsilon^{34}, \quad \omega_2 = -\epsilon^{13} + \epsilon^{24}, \quad \omega_3 = -\epsilon^{14} - \epsilon^{23}.
\]

To calculate \(d \omega_{\alpha}\) we obtain first \(de^j\) (recall that \(d \sigma(x, y) = -\sigma[x, y]\) for \(\sigma \in \Lambda^1 \mathfrak{g}^*\)):

\[
d e^1 = 0, \quad d e^2 = -\epsilon^{24}, \quad d e^3 = \epsilon^{24}, \quad d e^4 = -\epsilon^{23}.
\]

These equations and the fact that \(d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^r \sigma \wedge d\tau\) for all \(\sigma \in \Lambda^r \mathfrak{g}^*\) give the following formulas:

\[
d \omega_1 = -\epsilon^{134}, \quad d \omega_2 = \epsilon^{134}, \quad d \omega_3 = -\epsilon^{134}.
\]
from which we conclude that \( \phi = e^1 \), which is closed and therefore exact since \( G \) is diffeomorphic to \( \mathbb{R} \times S^3 \). We conclude that this hyper-Hermitian metric, which, as shown in [1], is homothetic to the riemannian product of the canonical metrics on \( \mathbb{R} \times S^3 \), is conformal to a hyper-Kähler metric.

Case 2. In this case we have the following equations for \( \omega_\alpha \):

\[
\omega_1 = e^{14} - e^{23}, \quad \omega_2 = -e^{12} + e^{34}, \quad \omega_3 = -e^{13} - e^{24},
\]

and we calculate

\[
(2.2) \quad d\epsilon_1 = -e^{13} + e^{24}, \quad d\epsilon_2 = -e^{23} + e^{14}, \quad d\epsilon_3 = 0, \quad d\epsilon_4 = 0,
\]

\[
(2.3) \quad d\omega_1 = -2e^{134}, \quad d\omega_2 = -2e^{123}, \quad d\omega_3 = 2e^{234}
\]

so that (1.4) is satisfied for \( \phi = 2e^3 \), which again is closed, so this hyper-Hermitian metric is also conformal to a hyper-Kähler metric. In this case the hyper-Hermitian metric is homothetic to the riemannian product of the canonical metrics on \( \mathbb{R} \times \mathbb{R} H^3 \), where \( \mathbb{R} H^3 \) denotes the real hyperbolic space.

Case 3. In this case the 2-forms \( \omega_\alpha \) are given as follows:

\[
\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}
\]

and a calculation of exterior derivatives gives:

\[
(2.4) \quad d\epsilon_1 = 0, \quad d\epsilon_j = -e^{1j}, \quad j = 2, 3, 4
\]

\[
(2.5) \quad d\omega_1 = 2e^{134}, \quad d\omega_2 = -2e^{124}, \quad d\omega_3 = -2e^{123}
\]

so that (1.4) is satisfied for \( \phi = -2e^1 \). This hyper-Hermitian metric is homothetic to the canonical metric on the real hyperbolic space \( \mathbb{R} H^4 \).

Case 4. In this case we have the following equations for \( \omega_\alpha \):

\[
\omega_1 = -e^{13} + e^{24}, \quad \omega_2 = -e^{12} - e^{34}, \quad \omega_3 = e^{14} - e^{23}
\]

and we calculate

\[
(2.6) \quad d\epsilon_1 = 0, \quad d\epsilon_2 = -e^{12} - \frac{1}{2}e^{34}, \quad d\epsilon_j = -\frac{1}{2}e^{1j}, \quad j = 3, 4
\]

\[
(2.7) \quad d\omega_1 = -\frac{3}{2}e^{134}, \quad d\omega_2 = \frac{3}{2}e^{124}, \quad d\omega_3 = \frac{3}{2}e^{123}
\]

so that (1.4) is satisfied for \( \phi = -\frac{3}{2}e^1 \). This hyper-Hermitian metric is not symmetric and has negative sectional curvature (cf. [1]).

Remark 2.1. All the hyper-Hermitian manifolds \( (M, \mathcal{H}, g) \) considered above admit a connection \( \nabla \) such that:

\[
\nabla g = 0, \quad \nabla J_\alpha = 0, \quad \alpha = 1, 2, 3
\]

and the \((3, 0)\) tensor \( \epsilon(X, Y, Z) = g(X, T(Y, Z)) \) is totally skew-symmetric, where \( T \) is the torsion of \( \nabla \). Such a connection is called an HKT connection (cf. [3]). In case \( M \) is diffeomorphic to \( \mathbb{R} \times S^3 \) it can be shown that, moreover, the corresponding 3-form \( \epsilon \) is closed.
3. Coordinate description of the Hyper-Kähler metrics

In this section we will use global coordinates on each of the Lie groups considered in the previous section to describe the corresponding hyper-Kähler metrics. This will allow us to identify the hyper-Kähler metric in §2, Case 4, with one constructed by the Gibbons-Hawking ansatz [2].

Case 1. $G = \mathbb{R}^* = GL(1, \mathbb{R}) = \left\{ \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} : (x, y, z, t) \in \mathbb{R}^4 - \{0\} \right\}$.

We obtain a basis of left invariant 1-forms on $G$ as follows. Set $r^2 = x^2 + y^2 + z^2 + t^2$, $r > 0$, and $\Omega = g^{-1} dg$ for $g \in G$, that is,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} \quad \text{then} \quad \Omega = \begin{pmatrix} \sigma_1 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\ \sigma_2 & \sigma_1 & -\sigma_4 & \sigma_3 \\ \sigma_3 & \sigma_4 & \sigma_1 & -\sigma_2 \\ \sigma_4 & -\sigma_3 & \sigma_2 & \sigma_1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}.$$

Then $\sigma_j, 1 \leq j \leq 4$, is a basis of left invariant 1-forms on $G$ and it follows from $d\Omega = \Omega \wedge \Omega = 0$ that

$$d\sigma_1 = 0, \quad d\sigma_2 = -2\sigma_3 \wedge \sigma_4, \quad d\sigma_3 = 2\sigma_2 \wedge \sigma_4, \quad d\sigma_4 = -2\sigma_2 \wedge \sigma_3.$$

Setting

$$\epsilon_1 = 2\sigma_1, \quad \epsilon_2 = 2\sigma_2, \quad \epsilon_3 = 2\sigma_3, \quad \epsilon_4 = 2\sigma_4,$$

so that $\{\epsilon_j\}_{1 \leq j \leq 4}$ satisfy (2.1), the left-invariant hyper-Hermitian metric is

$$g = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2 + (\epsilon_4)^2 = \frac{4}{r^2} (dx^2 + dy^2 + dz^2 + dt^2)$$

and since the Lee form is $\theta = \epsilon_1 = d(2 \log r)$ the corresponding hyper-Kähler metric is $\check{g} = \epsilon_1 \log r g$, that is,

$$\check{g} = \frac{4}{r^2} \left( \frac{(dr)^2}{r^2} + (\sigma_2)^2 + (\sigma_3)^2 + (\sigma_4)^2 \right) = \frac{4}{r^2} (dx^2 + dy^2 + dz^2 + dt^2)$$

Case 2. Define a product on $\mathbb{R}^4$ as follows:

$$(x, y, z, t)(x', y', z', t') = (x + \epsilon^2 (x' \cos t - y' \sin t), y + \epsilon^2 (x' \sin t + y' \cos t), z + z', t + t').$$

This defines a Lie group structure on $\mathbb{R}^4$ that makes it isomorphic to the Lie group considered in §2, Case 2. The following 1-forms are left-invariant with respect to the above product:

$$\begin{align*}
\epsilon_1 &= -\epsilon^2 \cos t dx + \epsilon^2 \sin t dy, \\
\epsilon_2 &= -\epsilon^2 \sin t dx + \epsilon^2 \cos t dy, \\
\epsilon_3 &= -dz, \\
\epsilon_4 &= -dt.
\end{align*}$$
These forms satisfy relations (2.2). The hyper-Hermitian metric is therefore given as follows:
\[ g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2z}(dx^2 + dy^2) + dz^2 + dt^2 \]
and the Lee form is \( \theta = 2e^3 = -2dz \), so that the hyper-Kähler metric becomes
\[ \tilde{g} = e^{2z} g = (dx^2 + dy^2) + e^{2z}(dz^2 + dt^2). \]
Observe that the change of coordinates \( s = e^z \) gives the following simple form for \( \tilde{g} \) on \( \mathbb{R}^+ \times \mathbb{R}^3 \):
\[ \tilde{g} = dx^2 + dy^2 + ds^2 + s^2 dt^2. \]

**Case 3.** We endow \( \mathbb{R}^4 \) with the following product:
\[ (x, y, z, t)(x', y', z', t') = (x + e^1x', y + e^2y', z + e^3z', t + t') \]
thereby obtaining the Lie group structure considered in \( \S 2 \), Case 3, with corresponding left-invariant 1-forms:
\[ e^1 = dt, \quad e^2 = e^{-1}dx, \quad e^3 = e^{-1}dy, \quad e^4 = e^{-1}dz. \]
The hyper-Hermitian metric is therefore
\[ g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2t}(dx^2 + dy^2 + dz^2) + dt^2 \]
with corresponding Lee form \( \theta = -2e^1 = -2dt \), yielding the following hyper-Kähler metric:
\[ \tilde{g} = e^{2t} g = dx^2 + dy^2 + dz^2 + e^{2t} dt^2. \]
Setting \( s = e^t \), \( \tilde{g} \) is the euclidean metric \( ds^2 + dx^2 + dy^2 + dz^2 \) on \( \mathbb{R}^+ \times \mathbb{R}^3 \).

**Case 4.** Consider the following product on \( \mathbb{R}^4 \):
\[ (x, y, z, t)(x', y', z', t') = (x + e^{\frac{1}{4}}x', y + e^{\frac{3}{4}}y', z + e^{\frac{3}{4}}z' + \frac{e^{\frac{3}{4}}}{4}(xy' - yx'), t + t') \]
which yields the Lie group structure considered in \( \S 2 \), Case 4. It is easily checked that the following left-invariant 1-forms satisfy (2.6):
\[ e^1 = dt, \quad e^2 = e^{-\frac{1}{2}}(dz - \frac{1}{4}xy + \frac{1}{4}ydx), \quad e^3 = e^{-\frac{1}{4}}dx, \quad e^4 = e^{-\frac{1}{4}}dy. \]
The hyper-Hermitian metric is now obtained as in the above cases:
\[ g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = dt^2 + e^{-\frac{1}{2}}(dx^2 + dy^2) + e^{-\frac{1}{2}}(dz - \frac{1}{4}(xy - ydx))^2 \]
and the Lee form is \( \theta = -\frac{3}{2}dt \), from which we obtain the hyper-Kähler metric as usual:
\[ \tilde{g} = e^{\frac{3}{2}t} dt^2 + e^{-\frac{3}{4}}(dx^2 + dy^2) + e^{\frac{3}{4}}(dz - \frac{1}{4}(xy - ydx))^2. \]
Setting \( s = e^{\frac{3}{4}} \), \( \tilde{g} \) becomes
\[ \tilde{g} = s(ds^2 + dx^2 + dy^2) + \frac{1}{s}(dz - \frac{1}{4}(xy - ydx))^2 \]
on \( \mathbb{R}^+ \times \mathbb{R}^3 \), which allows us to identify \( \tilde{g} \) with one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz [2]. The identification is easily obtained from [4], Proposition 1.
We can now rephrase Theorem 1.1 as follows, where $[h]$ denotes the conformal class of $h$:

**Corollary 3.1.** Let $h$ be a hyper-Kähler metric on a simply connected hypercomplex 4-manifold $(M,\mathcal{H})$ such that there exist $g \in [h]$ and a Lie group $G \subset I(M, g) \cap \text{Aut}(\mathcal{H})$ acting simply transitively on $M$. Then $(M, h)$ is homothetic to either $\mathbb{R}^4$ with the euclidean metric or one of the following riemannian manifolds:

1. $M = \mathbb{R}^4 - \{0\}$, $h = r^{-4}(dx^2 + dy^2 + dz^2 + dt^2)$,
2. $M = \mathbb{R}^+ \times \mathbb{R}^3$, $h = ds^2 + dx^2 + dy^2 + dz^2$,
3. $M = \mathbb{R}^+ \times \mathbb{R}^3$, $h = ds^2 + dx^2 + dy^2 + dz^2$,
4. $M = \mathbb{R}^+ \times \mathbb{R}^3$, $h = s(ds^2 + dx^2 + dy^2) + s^{-1}(dz - \frac{1}{4}(xdy - ydx))^2$.

**References**


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