Theory of Finite Pseudoalgebras

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References
1. Introduction

Since the seminal papers of Belavin, Polyakov and Zamolodchikov [BPZ] and of Borcherds [Bo1] there has been a great deal of work towards understanding the algebraic structures underlying the notion of operator product expansion (OPE) of chiral fields in conformal field theory.

In physics literature the OPE of local chiral fields $\varphi$ and $\psi$ is written in the form [BPZ]:

$$
\varphi(z)\psi(w) = \sum_{j \in \mathbb{Z}} \frac{\varphi(w)_{(j)}\psi(w)}{(z-w)^{j+1}},
$$

(1.1)

where $\varphi(w)_{(j)}\psi(w)$ are some new fields, which may be viewed as bilinear products of fields $\varphi$ and $\psi$ for all $j \in \mathbb{Z}$ (see e.g. [K2] for a rigorous interpretation of (1.1)).

If now $V$ is a space of pairwise local chiral fields which contains 1, is invariant with respect to the derivative $\partial = \partial_w$, and is closed under all $j$th products, $j \in \mathbb{Z}$, we obtain an algebraic structure which physicists (respectively mathematicians) call a chiral (respectively vertex) algebra. In more abstract terms, $V$ is a module over $\mathbb{C}[\partial]$ with a marked element 1 and infinitely many bilinear over $\mathbb{C}$ products $\varphi_{(j)}\psi$, $j \in \mathbb{Z}$, satisfying a certain system of identities, first written down by Borcherds [Bo1]. (An equivalent system of axioms, which is much easier to verify, may be found in [K2].)

One of the important features of the OPE (1.1) is that its singular part encodes the commutation relations of fields, namely one has (see e.g. [K2]):

$$
[\varphi(z), \psi(w)] = \sum_{j \geq 0} \left( \varphi(w)_{(j)}\psi(w) \right) \partial_w^j \delta(z-w)/j!,
$$

(1.2)

where $\delta(z-w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$ is the delta-function. This leads to the notion of a Lie conformal algebra, which is a $\mathbb{C}[\partial]$-module with $\mathbb{C}$-bilinear products $\varphi_{(j)}\psi$ for all non-negative integers $j$, subject to certain identities [K2]. In order to write down these identities in a compact form, it is convenient to consider the formal Fourier transform of (1.2), called the $\lambda$-bracket (where $\lambda$ is an indeterminate):

$$
[\varphi, \psi] = \sum_{j \geq 0} \lambda^j/j!(\varphi_{(j)}\psi).
$$

Then a Lie conformal algebra $L$ is defined as a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map

$$
L \otimes L \to \mathbb{C}[\lambda] \otimes L, \quad a \otimes b \mapsto [a, b]
$$

satisfying the following axioms [DK] ($a, b, c \in L$):

- (sesquilinearity) $[\partial a, b] = -\lambda[a, b], \quad [a, \partial b] = (\partial + \lambda)[a, b],$
- (skew-commutativity) $[b, a] = -[a, \partial b],$
- (Jacobi identity) $[a, [b, c]] = [[a, b], c] + [b, [a, c]].$

In the past few years a structure theory [DK], representation theory [CK, CKW] and cohomology theory [BKV] of finite (i.e., finitely generated as $\mathbb{C}[\partial]$-modules) Lie conformal algebras have been worked out. For example, one of the main results of [DK] states that any finite simple Lie conformal algebra is isomorphic either to the Virasoro conformal algebra:

$$
\text{Vir} = \mathbb{C}[\partial]\ell, \quad [\ell, \ell] = (\partial + 2\lambda)\ell
$$
or to the current conformal algebra associated to a simple finite-dimensional Lie algebra \( g \):

\[
\text{Cur } g = \mathbb{C}[\partial] \otimes g, \quad [a_\lambda b] = [a, b], \quad a, b \in g.
\]

The objective of the present paper is to develop a theory of “multi-dimensional” conformal algebras, i.e., a theory where the algebra of polynomials \( \mathbb{C}[\partial] \) is replaced by a “multi-dimensional” associative algebra \( H \). In order to explain the definition, let us return to the singular part (1.2) of the OPE. Choosing a set of generators \( a^i \) of the \( \mathbb{C}[\partial] \)-module \( L \), we can write:

\[
[a^i, a^j] = \sum_k Q^{ij}_k (\lambda, \partial) a^k,
\]

where \( Q^{ij}_k \) are some polynomials in \( \lambda \) and \( \partial \). The corresponding singular part of the OPE is:

\[
[a^i(z), a^j(w)] = \sum_k Q^{ij}_k (\partial_z, \partial_w)(a^k(t)\delta(z-w))|_{t=w}.
\]

Letting \( P^{ij}_k (x, y) = Q^{ij}_k (-x, x + y) \), we can rewrite this in a more symmetric form:

\[
(a^i(z), a^j(w)) = \sum_k P^{ij}_k (\partial_z, \partial_w)(a^k(w)\delta(z-w)).
\]

We thus obtain an \( H = \mathbb{C}[\partial] \)-bilinear map (i.e., a map of \( H \otimes H \)-modules):

\[
L \otimes L \to (H \otimes H) \otimes_H L, \quad a \otimes b \mapsto [a \ast b]
\]

(1.3)

(1.3)

(where \( H \) acts on \( H \otimes H \) via the comultiplication map \( \Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial \), defined by

\[
[a^i \ast a^j] = \sum_k P^{ij}_k (\partial \otimes 1 + 1 \otimes \partial) \otimes_H a^k.
\]

Hence the notion of \( \lambda \)-bracket \([a, b]_\lambda\) is equivalent to the notion of the \( \ast \)-bracket \([a \ast b]\) introduced by Beilinson and Drinfeld [BD], the relation between the two brackets being given by letting \( \lambda = -\partial \otimes 1 \).

It is natural to introduce the general notion of a conformal algebra as a \( \mathbb{C}[\partial] \)-module \( L \) endowed with a \( \mathbb{C} \)-linear map \( L \otimes L \to \mathbb{C}[\lambda] \otimes L, \ a \otimes b \mapsto a_\lambda b \) satisfying the sesquilinearity property:

\[
(\partial a)_\lambda b = -\lambda (a_\lambda b), \quad a_\lambda (\partial b) = (\partial + \lambda)(a_\lambda b).
\]

Such a conformal algebra is called associative (respectively commutative) if

\[
a_\lambda (b_\mu c) = (a_\lambda b)_{\lambda + \mu} c \quad \text{(respectively } b_\lambda a = a_{\lambda + \partial} b),
\]

and the \( \lambda \)-product of an associative conformal algebra defines a \( \lambda \)-bracket

\[
[a_\lambda b] = a_\lambda b - b_{-\lambda - \partial} a,
\]

making it a Lie conformal algebra [K4, DK].

As above, we have the equivalent notion of a \( \ast \)-product on an \( H = \mathbb{C}[\partial] \)-module \( L \), which is an \( H \)-bilinear map

\[
L \otimes L \to (H \otimes H) \otimes_H L, \quad a \otimes b \mapsto a \ast b.
\]

(1.4)

Now it is clear that the notion of a \( \ast \)-product can be defined by (1.4) for any Hopf algebra \( H \) by making use of the comultiplication \( \Delta: H \to H \otimes H \) to define \( (H \otimes H) \otimes_H L \). A pseudoalgebra is a (left) \( H \)-module \( L \) endowed with an \( H \)-bilinear map (1.4). The name is motivated by the fact that this is an algebra in a pseudotensor category (introduced in [L], [BD]). Accordingly, the \( \ast \)-product will be called a pseudooperation.
One is able to define a pseudoproduct as soon as a structure of bialgebra is given on \( H \). However, in order to generalize the equivalence of a pseudodeformed and an \( H \)-conformal algebra structure on an \( H \)-module \( L \), we need \( H \) to be a Hopf algebra. In this case any element of \( H \odot H \) can be uniquely written as a (finite) sum:

\[
\sum_i (h_i \odot 1) \Delta(f_i), \quad \text{where} \ h_i \ \text{are linearly independent.}
\]

Hence the pseudoproduct on \( L \) can be written in the form:

\[
a \ast b = \sum_i (h_i \odot 1) \odot_H c_i. \tag{1.5}
\]

The corresponding \( H \)-conformal algebra structure is then a \( \mathbb{C} \)-linear map \( L \odot L \to H \otimes L \) given by

\[
ab = \sum_i h_i \otimes c_i. \tag{1.6}
\]

Every element \( x \) of \( H^* \) then defines an \( x \)-product \( L \odot L \to L \):

\[
a \cdot x b = \sum_i \langle x, S(h_i) \rangle c_i, \tag{1.7}
\]

where \( S \) is the antipode of \( H \).

The \( H \)-bilinearity property of the pseudoproduct (1.5) is, of course, easily translated to certain sesquilinearity properties of the products (1.6) and (1.7). In particular, in the case \( H = \mathbb{C}[\delta] \), the product (1.6) is the \( \lambda \)-product if we let \( \lambda = -\delta \), and the product (1.7) for \( x = \delta \) is the \( j \)-product described above, where \( H^* \cong \mathbb{C}[t] \), \( \langle \delta, \delta \rangle = 1 \). The equivalence of these three structures (discussed in Section 9) is very useful in the study of pseudodeformeds.

In order to define associativity of a pseudoproduct, we extend it from \( L \odot L \to H \otimes H \) to \( (H \otimes H \odot L) \odot L \to H \otimes H \) and to \( L \odot (H \otimes H \odot L) \to H \otimes H \) by letting:

\[
(f \odot_H a) \ast b = \sum_i (f \otimes 1) (\Delta \circ \text{id})(g_i) \odot_H c_i,
\]

\[
a \ast (f \odot_H b) = \sum_i (1 \otimes f) (\text{id} \circ \Delta)(g_i) \odot_H c_i, \quad \text{where} \ a \ast b = \sum g_i \odot_H c_i.
\]

Then the associativity property is given by the usual equality (in \( H \otimes H \odot L \)):

\[
(a \ast b) \ast c = a \ast (b \ast c).
\]

The easiest example of a pseudodeformed is a current pseudodeformed, defined as follows. Let \( H' \) be a Hopf subalgebra of \( H \) and let \( A \) be an \( H' \)-pseudodeformed (for example, if \( H' = \mathbb{C} \), then \( A \) is an ordinary algebra over \( \mathbb{C} \)). Then the associated current \( H \)-pseudodeformed is \( \text{Cur} \ A = H \otimes_H A \) with the pseudoproduct

\[
(f \odot_H a) \ast (g \odot_H b) = ((f \odot g) \odot_H 1)(a \ast b).
\]

The \( H \)-pseudodeformed \( \text{Cur} \ A \) is associative iff the \( H' \)-pseudodeformed \( A \) is.

The most important example of an associative \( H \)-pseudodeformed is the pseudodeformed of all pseudolinear endomorphisms of a finitely generated \( H \)-module \( V \), which is denoted by \( \text{Cend} \ V \) (see Section 10). A pseudolinear endomorphism of \( V \) is a \( \mathbb{C} \)-linear map \( \phi : V \to (H \otimes H) \odot_H V \) such that

\[
\phi(hv) = ((1 \otimes h) \odot_H 1) \phi(v), \quad h \in H, v \in V.
\]

The space \( \text{Cend} \ V \) of all such \( \phi \) becomes a \( (\text{left}) \) \( H \)-module if we define

\[
(h \phi)(v) = ((h \otimes 1) \odot_H 1) \phi(v).
\]

The definition of a pseudoproduct on \( \text{Cend} \ V \) is especially simple when \( V \) is a free \( H \)-module, \( V = H \odot V_0 \), where \( V_0 \) is a finite-dimensional vector space over \( \mathbb{C} \) with a
trivial action of $H$. Then $\text{Cend}V$ is isomorphic to $H \odot H \odot \text{End} V_0$, with $H$ acting by left multiplication on the first factor, and with the following pseudoproduct:

$$(f \odot a \odot A) \ast (g \odot b \odot B) = \sum_i (f \odot g a'_i) \odot H (1 \odot b a''_i \odot AB),$$

where $\Delta(a) = \sum_i a'_i \odot a''_i$.

The main objects of our study are Lie pseudoalgebras. The corresponding pseudoproduct is conventionally called *pseudobracket* and denoted by $[a \ast b]$. Given an associative pseudoalgebra with pseudoproduct $a \ast b$ we may give it a structure of a Lie pseudoalgebra by defining the pseudobracket

$$[a \ast b] = a \ast b - (\sigma \odot_H \text{id}) b \ast a,$$

where $\sigma: H \odot H \to H \odot H$ is the permutation of factors. It is immediate to see that this pseudobracket satisfies the following skew-commutativity and Jacobi identity axioms:

\begin{align*}
(1.8) \quad & [b \ast a] = - (\sigma \odot_H \text{id}) [a \ast b], \\
(1.9) \quad & [a \ast [b \ast c]] = [[a \ast b] \ast c] + ((\sigma \odot \text{id}) \odot_H \text{id}) [b \ast [a \ast c]].
\end{align*}

It is important to point out here that the above pseudobracket and both identities are well defined, provided that the Hopf algebra $H$ is cocommutative. A pseudoalgebra with pseudoproduct $[a \ast b]$ satisfying identities (1.8) and (1.9) is called a *Lie pseudoalgebra*. We will always assume that $H$ is cocommutative when talking about Lie pseudoalgebras. Of course, the simplest examples of Lie pseudoalgebras are $\text{Cur} A$, where $A$ is a $H' (\subset H)$ Lie pseudoalgebra (= Lie algebra if $H' = \mathbb{C}$). Needless to say, in the case $H = \mathbb{C}[\partial]$, $\Delta(\partial) = \partial \circ 1 + 1 \circ \partial$, the $H$-conformal algebras associated to Lie pseudoalgebras are nothing else but the Lie conformal algebras discussed above.

We will explain now the connection of the notion of a Lie pseudoalgebra to the more classical notion of a differential Lie algebra studied in [R1]–[R4], [C], [NW] and many other papers (see Section 7). Let $Y$ be a commutative associative algebra over $\mathbb{C}$ with compatible left and right actions of the Hopf algebra $H$. Then, given a Lie pseudoalgebra $L$, we let $A_L = Y \odot_H L$ with the obvious left $H$-module structure and the following Lie algebra (over $\mathbb{C}$) structure:

$$[(x \odot_H a), (y \odot_H b)] = \sum_i (x f_i)(y g_i) \odot_H c_i \quad \text{if} \quad [a \ast b] = \sum_i (f_i \odot g_i) \odot_H c_i.$$

Provided that $L$ is a free $H$-module, the Lie algebra $A_L$ is a free $Y$-module, hence $A_L$ is a differential Lie algebra in the sense of [NW]. The most classical case is again $H = \mathbb{C}[\partial]$, when $Y$ is simply a commutative associative algebra with a (left and right) derivation $\partial$, and we get the differential Lie algebras of Ritt [R1]–[R4]. Thus, the notion of a Lie pseudoalgebra is reminiscent of the notion of a group scheme: each Lie pseudoalgebra $L$, which is free as an $H$-module, gives rise to a functor $A_L$ from the category of commutative associative algebras with compatible left and right actions of $H$ to the category of differential Lie algebras (= category of formal differential groups).

For example, the functor $A$ corresponding to the Virasoro pseudoalgebra associates to any commutative associative algebra $Y$ with a derivation $\partial$ the differential Lie algebra $Y$ with bracket $[u, v] = u \partial v - v \partial u$, called the substitutional Lie algebra by Ritt. The current pseudoalgebra $\text{Cur} \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, associates to $Y$ the obvious differential Lie algebra $Y \odot \mathfrak{g}$. Thus, a result of [DK] asserts that any simple finite differential Lie algebra with "constant coefficients" is
isomorphic either to the substitutional Lie algebra or to $Y \otimes g$ where $g$ is a simple finite-dimensional Lie algebra. In the rank 1 case, but without the constant coefficients assumption, this is the main result of [R1].

The main tool in the study of pseudoalgebras is the annihilation algebra $A_X L$, where $X = H^*$ is the associative algebra dual to the coalgebra $H$. We find it remarkable that the annihilation algebra of the associative pseudoalgebra $H = H \otimes H$ is nothing else but the Drinfeld double (with obvious comultiplication) of the Hopf algebra $H$. Note that in the associative case $Y$ need not be commutative in order to define the functor $A_Y$, but in the Lie algebra case it must be. So, in order to construct the annihilation Lie algebra we again use cocommutativity of $H$.

Recall that, by Kostant’s theorem, any cocommutative Hopf algebra $H$ is a smash product of a group algebra $\mathbb{C}[G]$ and the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In Sections 5 and 13.7 we show that the theory of pseudoalgebras over a smash product of $\mathbb{C}[G]$ and any Hopf algebra $H$ reduces to that over $H$. This allows us in many cases to assume, without loss of generality, that $H$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$.

However, for most of our results we have to assume that $\mathfrak{g}$ is finite-dimensional. In this case the algebra $H = U(\mathfrak{g})$ is Noetherian, and the annihilation algebra $A_X L$ is linearly compact, provided that $L$ is finite (i.e., finitely generated as an $H$-module). Recall that a topological Lie algebra is called linearly compact if its underlying topological space is isomorphic to the space of formal power series in a finite number of indeterminates with the formal topology.

In Section 11 we prove “reconstruction” theorems, which claim that, under some mild assumptions, a Lie pseudoalgebra is completely determined by its annihilation Lie algebra along with the action of $\mathfrak{g}$. This reduces the classification of finite simple Lie pseudoalgebras to the well developed structure theory of linearly compact Lie algebras, which goes back to E. Cartan (see [G1, G2]).

We turn now to examples of finite Lie pseudoalgebras beyond the rather obvious examples of current Lie pseudoalgebras. The first example is the generalization of the Virasoro pseudoalgebra, defined for $H = \mathbb{C}[\mathfrak{g}]$ (which is the universal enveloping algebra of the 1-dimensional Lie algebra), to the case $H = U(\mathfrak{g})$, where $\mathfrak{g}$ is any finite-dimensional Lie algebra. This is the Lie pseudoalgebra $W(\mathfrak{g}) = H \otimes \mathfrak{g}$ with pseudoalgebra
\[ [(1 \otimes a) \ast (1 \otimes b)] = (1 \otimes 1) \otimes_H (1 \otimes [a, b]) + (b \otimes 1) \otimes_H (1 \otimes a) - (1 \otimes a) \otimes_H (1 \otimes b). \]

Since the associated annihilation algebra $A_X W(\mathfrak{g}) \simeq X \otimes \mathfrak{g}$ is isomorphic to the Lie algebra of formal vector fields on the Lie group $D$ with Lie algebra $\mathfrak{g}$, it is natural to call $W(\mathfrak{g})$ the pseudoalgebra of all vector fields. In fact we develop (in Section 8) a formalism of pseudoderivatives similar to the usual formalism of differential forms, which may be viewed as the beginning of a “pseudo differential geometry”.

This allows us to define the remaining three series of finite simple Lie pseudoalgebras: $S(\mathfrak{g}, \chi)$, $H(\mathfrak{g}, \chi, \omega)$ and $K(\mathfrak{g}, \theta)$. The annihilation algebras of the simple Lie pseudoalgebras $W(\mathfrak{g})$, $S(\mathfrak{g}, \chi)$, $H(\mathfrak{g}, \chi, \omega)$ and $K(\mathfrak{g}, \theta)$ are isomorphic to the four series of Lie–Cartan linearly compact Lie algebras $W_N$, $S_N$, $P_N$ (which is an extension of $H_N$ by a 1-dimensional center) and $K_N$, where $N = \dim \mathfrak{g}$. However the Lie pseudoalgebra $S(\mathfrak{g}, \chi)$, $H(\mathfrak{g}, \chi, \omega)$ and $K(\mathfrak{g}, \theta)$ depend on certain parameters $\chi, \omega$ and $\theta$, due to inequivalent actions of $\mathfrak{g}$ on the annihilation algebra. The parameter $\chi$ is a 1-dimensional representation of $\mathfrak{g}$, i.e., $\chi \in \mathfrak{g}^*$ such that $\chi([\mathfrak{g}, \mathfrak{g}]) = 0$. The parameter $\omega$ is an element of $\mathfrak{g}^* \wedge \mathfrak{g}^*$ such that $\omega^{\wedge 2} \neq 0$ and $d\omega \wedge \chi \omega = 0$ in the case
$H(\mathfrak{g}, \chi, \omega)$, when $N$ is even. The parameter $\theta \in \mathfrak{g}^*$ is such that $\theta \wedge (d\theta)^{(N-1)/2} \neq 0$ in the case $K(\mathfrak{g}, \theta)$, when $N$ is odd. In the cases $H(\mathfrak{g}, \chi, \omega)$, $K(\mathfrak{g}, \theta)$, these parameters are in one-to-one correspondence with "nondegenerate" skew-symmetric solutions $\alpha = r + s \circ 1 - 1 \circ s$ ($r \in \mathfrak{g}^{\wedge} \mathfrak{g}$, $s \in \mathfrak{g}$) of a modification of the classical Yang–Baxter equation, which is a special case of the dynamical classical Yang–Baxter equation (see [Fe, ES]).

The central result of the paper is the classification of finite simple Lie pseudoalgebras over the Hopf algebra $H = U(\mathfrak{g})$. As usual, a Lie pseudoalgebra $L$ is called simple if it is nonabelian (i.e., $[L, L] \neq 0$) and its only ideals are 0 and $L$. Our Theorem 13.10 states that any such Lie pseudoalgebra is isomorphic either to a current pseudoalgebra $\text{Cur} \mathfrak{g} = \text{Cur}^H \mathfrak{g}$ over a simple finite-dimensional Lie algebra $\mathfrak{g}$, or to a current pseudoalgebra $\text{Cur}^H L'$ over one of the Lie pseudoalgebras $L' = W(\mathfrak{g}'')$, $S(\mathfrak{g}', \chi')$, $H(\mathfrak{g}'', \chi', \omega'')$ or $K(\mathfrak{g}'', \theta')$, where $\mathfrak{g}'' = U(\mathfrak{g}'')$ and $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$.

A Lie pseudoalgebra $L$ is called semisimple if it contains no nonzero abelian ideals. One also defines in the usual way the derived pseudoalgebra, solvable and nilpotent pseudoalgebras, and for a finite Lie pseudoalgebra $L$ one has the solvable radical $\text{Rad} L$ (so that $L / \text{Rad} L$ is semisimple).

Our Theorem 13.15 states that any finite semisimple Lie $U(\mathfrak{g})$-pseudoalgebra is a direct sum of finite simple Lie pseudoalgebras and of Lie pseudoalgebras of the form $A \otimes \text{Cur} \mathfrak{g}$, where $A$ is a subalgebra of $W(\mathfrak{g})$ and $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. In addition, in Theorem 13.18 we show that any subalgebra of $W(\mathfrak{g})$ is simple, and in Corollary 13.26 we give a complete list of all these subalgebras.

Note, however, that Levi’s theorem on $L$ being a semidirect sum of $\text{L} / \text{Rad} L$ and $\text{Rad} L$ is not true even in the case $\dim \mathfrak{g} = 1$. This stems from the fact that the cohomology of simple Lie pseudoalgebras with nontrivial coefficients is (highly) nontrivial (see Section 15 and [BK]), in a sharp contrast with the Lie algebra case. For example, it follows from [BK] that there are precisely five cases (three isolated examples and two families) of non-split extensions of Vir by $\text{Cur} \mathfrak{C}$. Translated into the language of differential Lie algebras, this result goes back to Ritt [R3].

Closely related to the present paper are the papers [Ki] and [NW], where (in our terminology) the annihilation algebras of rank 1 over $H$ Lie pseudoalgebras, and of simple Lie pseudoalgebras of arbitrary finite rank, respectively, are studied. In fact, our Theorems 13.10 and 13.15 provide a completed form of the classification results of [NW] (in the "constant coefficients" case).

The structural results of the present paper in the simplest case $\dim \mathfrak{g} = 1$ reproduce the results of [DK]. However, this case is much easier than the case $\dim \mathfrak{g} > 1$, mainly due to the fact that only in this case is any finite torsionless $H$-module free.

Note also the close connection of our work to Hamiltonian formalism in the theory of nonlinear evolution equations (see the review of [DN2], the book [Do] and references there, and also [GD], [DN1], [Z], [M], [X], and many other papers). In Section 16 we derive, as a corollary of Theorems 13.10 and 13.15, a classification of simple and semisimple linear Poisson brackets in any finite number of indeterminates.

In Section 14 we develop a representation theory of finite Lie pseudoalgebras. First, we prove an analogue of Lie’s Lemma that any weight space for an ideal of a Lie pseudoalgebra $L$ acting on a finite module is an $L$-submodule (Proposition 14.2). This implies an analogue of Lie’s Theorem that a solvable Lie pseudoalgebra has an eigenvector in any finite module (Theorem 14.3), and an analogue of Cartan–Jacobi Theorem that describes all finite Lie pseudoalgebras which have a finite faithful irreducible module (Theorem 14.5). Finally, we reduce the classification
and construction of finite irreducible modules over semisimple Lie pseudoalgebras to that of irreducible modules over linearly compact Lie algebras of the type studied by Rudakov [Ru1, Ru2] (the complete classification will appear in a future publication). Note that complete reducibility fails already in the simplest case of Lie pseudoalgebras with dim δ = 1 [CKW].

In Section 15 we define cohomology of Lie pseudoalgebras and show that it describes module extensions, abelian pseudoalgebra extensions, and pseudoalgebra deformations. We also relate this cohomology to the Gelfand–Fuchs cohomology [Fu]. These results generalize those of [BK] in the dim δ = 1 case.

Note that in the case dim δ = 1 Lie pseudoalgebras are closely related to vertex algebras in a way similar to the relation of Lie algebras to universal enveloping algebras [K2]. We expect that, under certain conditions, there is a similar relation of “multi-dimensional” Lie pseudoalgebras to “multi-dimensional” vertex algebras defined in [Bo2]. In the case of a commutative Lie algebra δ the Lie pseudoalgebras encode the OPE between ultralocal fields (as well as the linear Poisson brackets). However, it is not clear how Lie pseudoalgebras are related to the OPE of realistic quantum field theories.

In order to end the introduction on a more optimistic note, we would like to point out that in the definition of a Lie pseudoalgebra one may replace the permutation σ by the map f ⊗ g → (g ⊗ f)R where R is an R-matrix for H, hence one can take for H any quasi-triangular Hopf algebra (defined in [D]). This observation, the appearance of the classical Yang–Baxter equation, and the fact that the annihilation algebra of the associative pseudoalgebra Cend H is the Drinfeld double of H, lead us to believe that there should be a deep connection between the theories of pseudoalgebras and quantum groups.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over an algebraically closed field k of characteristic 0.

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2. Preliminaries on Hopf Algebras

2.1. Notation and basic identities. Let H be a Hopf algebra with a coproduct Δ, a counit ε, and an antipode S. We will use the following notation (cf. Sweedler’s book [Sw]):

\[
\Delta(h) = h_{(1)} \otimes h_{(2)},
\]

\[
(\Delta \otimes \text{id}) \Delta(h) = (\text{id} \otimes \Delta) \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)},
\]

\[
(S \otimes \text{id}) \Delta(h) = h_{(-1)} \otimes h_{(2)}, \quad \text{etc.}
\]

Note that notation (2.2) uses the coassociativity of Δ. The axioms of the antipode and the counit can be written as follows:

\[
h_{(-1)} h_{(2)} = h_{(1)} h_{(-2)} = \varepsilon(h),
\]

\[
\varepsilon(h_{(1)}) h_{(2)} = h_{(1)} \varepsilon(h_{(2)}) = h,
\]
while the fact that $\Delta$ is a homomorphism of algebras translates as:

\[(fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}.\]

Equations (2.4) and (2.5) imply the following useful relations:

\[(2.7) \quad h_{(-1)}h_{(2)} \otimes h_{(3)} = 1 \otimes h = h_{(1)}h_{(-1)} \otimes h_{(3)}.\]

Let $G(H)$ be the subset of group-like elements of $H$, i.e., $g \in H$ such that $\Delta(g) = g \otimes g$. Then $G(H)$ is a group, because $S(g)g = gS(g) = 1$ for $g \in G(H)$. Let $P(H)$ be the subspace of primitive elements of $H$, i.e., $p \in H$ such that $\Delta(p) = p \otimes 1 + 1 \otimes p$. This is a Lie algebra with respect to the commutator $[p, q] = pq - qp$. Note that $G(H)$ acts on $P(H)$ by inner automorphisms: $gpg^{-1} \in P(H)$ for $p \in P(H), g \in G(H)$.

The proof of the following theorem may be found in [Sw].

**Theorem 2.1 (Kostant).** Let $H$ be a cocommutative Hopf algebra over $k$ (an algebraically closed field of characteristic 0). Then $H$ is isomorphic (as a Hopf algebra) to the smash product of the universal enveloping algebra $U(P(H))$ and the group algebra $k[G(H)]$.

An associative algebra $A$ is called an $H$-differential algebra if it is also a left $H$-module such that the multiplication $A \otimes A \to A$ is a homomorphism of $H$-modules. In other words,

\[(2.8) \quad h(xy) = (h_{(1)}x)(h_{(2)}y)\]

for $h \in H, x, y \in A$. The smash product $A \sharp H$ of an $H$-differential algebra $A$ with $H$ is the tensor product $A \otimes H$ of vector spaces but with a new multiplication:

\[(2.9) \quad (a \sharp g)(b \sharp h) = a(g_{(1)}b) \sharp g_{(2)}h.\]

If both $A$ and $H$ are Hopf algebras, then $A \sharp H$ is a Hopf algebra if we consider it as a tensor product of coalgebras. In the theorem above, $U(P(H))$ is a $k[G(H)]$-differential algebra with respect to the adjoint action of $G(H)$ on $P(H)$.

It is worth mentioning that as a byproduct of Kostant’s Theorem, we obtain that the antipode of a cocommutative Hopf algebra is an involution, i.e., $S^2 = id$.

We will often be working with the Hopf algebra $H = U(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra. It is well known that this is a Noetherian domain, and any two nonzero elements $f, g \in H$ have a nonzero left (respectively right) common multiple. In particular, $H = U(\mathfrak{g})$ has a skew-field of fractions $K$.

**Lemma 2.2.** Let $H$ be a Noetherian domain which has a skew-field of fractions $K$, and let $L$ be a finite $H$-module. Then there is a homomorphism $i: L \to F$ from $L$ to a free $H$-module $F$, whose kernel is the torsion submodule of $L$. If $L$ is torsion-free, then the module $F$ can be chosen in such a way that $hF \subseteq i(L)$ for some nonzero $h \in H$ and $i(F)/hF$ is torsion.

**Proof.** The kernel of the natural map $i: L \to L_K := K \otimes_H L$ is the torsion of $L$. The image of $L$ under this map is contained inside a free $H$-module of $L_K$. In order to see this, let us consider a set of $H$-generators $\{t_1, \ldots, t_n\}$ of $L$, and a $K$-basis $\{v_1, \ldots, v_k\}$ of $L_K$. We can express the elements $i(t_j)$ as $K$-linear combinations of the $v_i$’s, and by rescaling elements of this basis by a common multiple of the denominators, we can assume the $i(t_j)$’s to be $H$-linear combinations of the $v_i$’s. Hence the image $i(L)$ is contained in the $H$-module $F$ spanned by the $v_i$’s, which is free by construction.
The fact that $F/L$ is torsion is clear because there exist nonzero elements $h_i \in H$ such that $h_i v_i \in L$. If $h$ is a common multiple of the $h_i$'s, then $hF$ is contained in $L$. On the other hand, the inclusion $L \subseteq F$ implies $hL \subseteq hF$, hence $h(L/hF) = 0$ and $L/hF$ is torsion.

2.2. **Filtration and topology.** We define an increasing sequence of subspaces of a Hopf algebra $H$ inductively by:

(2.10) $F^n H = 0$ for $n < 0$, $F^0 H = k[G(H)]$.

(2.11) $F^n H = \{ h \in H \mid \Delta(h) \in F^n H \otimes h + h \otimes F^n H + \sum_{i=1}^{n-1} F^i H \otimes F^{n-i} H \}$.

It has the following properties (which are immediate from definitions):

(2.12) $(F^n H)(F^n H) \subset F^{n+n} H$,

(2.13) $\Delta(F^n H) \subset \sum_{i=0}^{n} F^i H \otimes F^{n-i} H$,

(2.14) $S(F^n H) \subset F^n H$.

When $H$ is cocommutative, using Theorem 2.1, one can show that:

(2.15) $\bigcup_n F^n H = H$.

(This condition is also satisfied when $H$ is a quantum universal enveloping algebra.)

Provided that (2.15) holds, we say that a nonzero element $a \in H$ has degree $n$ if $a \in F^n H \setminus F^{n-1} H$.

When $H$ is a universal enveloping algebra, we get its canonical filtration. Later in some instances we will also impose the following finiteness condition on $H$:

(2.16) $\dim F^n H < \infty \quad \forall n$.

It is satisfied when $H$ is a universal enveloping algebra of a finite-dimensional Lie algebra, or its smash product with the group algebra of a finite group.

Now let $X = H^* := \text{Hom}_k(H, k)$ be the dual of $H$. Recall that $H$ acts on $X$ by the formula $(h, f \in H, x \in X)$:

(2.17) $\langle hx, f \rangle = \langle x, S(h)f \rangle$,

so that $X$ is an associative $H$-differential algebra (see (2.8)). Moreover, $X$ is commutative when $H$ is cocommutative. Similarly, one can define a right action of $H$ on $X$ by

(2.18) $\langle xh, f \rangle = \langle x, fS(h) \rangle$,

and then we have

(2.19) $(xy)h = (xh(1))(yh(2))$.

Associativity of $H$ implies that $X$ is an $H$-bimodule, i.e.

(2.20) $f(xg) = (fx)g$, $f, g \in H, x \in X$.

Let $X = F_{-1}X \supset F_{-1}X \supset \cdots$ be the decreasing sequence of subspaces of $X$ dual to $F^n H$: $F_n X = (F^n H)^\perp$. It has the following properties:

(2.21) $(F_m X)(F_n X) \subset F_{m+n} X$,

(2.22) $(F^m H)(F_n X) \subset F_{n-m} X$. 


and
\[
\bigcap_n F_n X = 0, \quad \text{provided that (2.15) holds.}
\]

We define a topology of \(X\) by considering \(\{F_n X\}\) as a fundamental system of neighborhoods of 0. We will always consider \(X\) with this topology, while \(H\) with the discrete topology. It follows from (2.23) that \(X\) is Hausdorff, provided that (2.15) holds. By (2.21) and (2.22), the multiplication of \(X\) and the action of \(H\) on it are continuous; in other words, \(X\) is a topological \(H\)-differential algebra.

We define an antipode \(S\) : \(X \to X\) as the dual of that of \(H\):
\[
\langle S(x), h \rangle = \langle x, S(h) \rangle.
\]

Then we have:
\[
S(ab) = S(b)S(a) \quad \text{for } a, b \in X \text{ or } H.
\]

We will also define a comultiplication \(\Delta : X \to X \hat{\otimes} X\) as the dual of the multiplication \(H \hat{\otimes} H \to H\), where \(X \hat{\otimes} X : = (H \hat{\otimes} H)^*\) is the completed tensor product. Formally, we will use the same notation for \(X\) as for \(H\) (see (2.1)–(2.3)), writing for example \(\Delta(x) = x_{(1)} \otimes x_{(2)}\) for \(x \in X\). By definition, for \(x, y \in X\), \(f, g \in H\), we have:
\[
\langle xy, f \rangle = \langle x, f(y) \Delta(f) \rangle = \langle x, f_{(1)} \rangle \langle y, f_{(2)} \rangle,
\]
\[
\langle x, fg \rangle = \langle \Delta(x), f \otimes g \rangle = \langle x_{(1)}, f \rangle \langle x_{(2)}, g \rangle.
\]

We have:
\[
\begin{align*}
S(F_n X) & \subset F_n X, \\
\Delta(F_n X) & \subset \sum_{i=0}^n F_i X \hat{\otimes} F_{n-i} X.
\end{align*}
\]

If \(H\) satisfies the finiteness condition (2.16), then the filtration of \(X\) satisfies
\[
\dim X / F_n X < \infty \quad \forall n,
\]

which implies that \(X\) is linearly compact (see Section 6 below).

By a basis of \(X\) we will always mean a topological basis \(\{x_i\}\) which tends to 0, i.e., such that for any \(n\) all but a finite number of \(x_i\) belong to \(F_n X\). Let \(\{h_i\}\) be a basis of \(H\) (as a vector space) compatible with the filtration. Then the set of elements \(\{x_i\}\) of \(X\) defined by \(\langle x_i, h_j \rangle = \delta_{ij}\) is called the dual basis of \(X\). If \(H\) satisfies (2.16), then \(\{x_i\}\) is a basis of \(X\) in the above sense, i.e., it tends to 0. We have for \(g \in H\), \(y \in X\):
\[
g = \sum_i \langle g, x_i \rangle h_i, \quad y = \sum_i \langle y, h_i \rangle x_i,
\]

where the first sum is finite, and the second one is convergent in \(X\).

**Example 2.3.** Let \(H = \mathcal{U}(\mathfrak{g})\) be the universal enveloping algebra of an \(N\)-dimensional Lie algebra \(\mathfrak{g}\). Fix a basis \(\{\partial_i\}\) of \(\mathfrak{g}\), and for \(I = (i_1, \ldots, i_N) \in \mathbb{Z}_+^N\) let \(\partial^{(I)} = \partial_1^{i_1} \cdots \partial_N^{i_N}/i_1! \cdots i_N!\). Then \(\{\partial^{(I)}\}\) is a basis of \(H\) (the Poincaré–Birkhoff–Witt basis). Moreover, it is easy to see that
\[
\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}.
\]

If \(\{t_I\}\) is the dual basis of \(X\), defined by \(\langle t_I, \partial^{(J)} \rangle = \delta_{I,J}\), then (2.31) implies \(t_{J+K} t_{J+K} = t_{J+K}\). Therefore, \(X\) can be identified with the ring \(\mathcal{O}_N = k[[t_1, \ldots, t_N]]\) of
formal power series in $N$ indeterminates. Then the action of $H$ on $C_N$ is given by
differential operators.

**Lemma 2.4.** If $\{h_i\}, \{x_i\}$ are dual bases in $H$ and $X$, then

$$\Delta(x) = \sum_i x_i S(h_i) \odot x_i = \sum_i x_i \odot S(h_i) x$$

for any $x \in X$.

**Proof.** For $f, g \in H$, we have:

$$\langle \sum_i x_i S(h_i) \odot x_i, f \odot g \rangle = \sum_i \langle x_i S(h_i), f \rangle \langle x_i, g \rangle = \langle x S(g), f \rangle = \langle x, fg \rangle = \langle \Delta(x), f \odot g \rangle,$$

which proves the first identity. The second one is proved in the same way. \qed

2.3. **Fourier transform.** For an arbitrary Hopf algebra $H$, we introduce a map $\mathcal{F}: H \otimes H \to H \otimes H$, called the *Fourier transform*, by the formula

$$\mathcal{F}(f \odot g) = (f \odot 1)(S \odot \text{id})\Delta(g) = fg_{(-1)} \odot g_{(2)}.$$  

It follows from (2.7) that $\mathcal{F}$ is a vector space isomorphism with an inverse given by

$$(2.34) \quad \mathcal{F}^{-1}(f \odot g) = (f \odot 1)\Delta(g) = fg_{(1)} \odot g_{(2)}.$$  

Indeed, using the coassociativity of $\Delta$ and (2.7), we compute

$$\mathcal{F}^{-1}(fg_{(-1)} \odot g_{(2)}) = fg_{(-1)}(g_{(1)}) \odot (g_{(2)})_{(2)} = fg_{(-1)}g_{(2)} \odot g_{(3)} = f \odot g.$$  

The significance of $\mathcal{F}$ is in the identity

$$f \odot g = \mathcal{F}^{-1}(f \odot g) = (fg_{(-1)} \odot 1)\Delta(g_{(2)}),$$

which, together with properties (2.12)-(2.14) of the filtration of $H$, implies the next result.

**Lemma 2.5.** (i) Every element of $H \otimes H$ can be uniquely represented in the form $\sum_i (h_i \otimes 1)\Delta(l_i)$, where $\{h_i\}$ is a fixed $k$-basis of $H$ and $l_i \in H$. In other words, $H \otimes H = (H \otimes k)\Delta(H)$.

(ii) We have:

$$\mathcal{F}^n (H \otimes k) \Delta(H) = (F^n H \otimes H) \Delta(H) = (k \otimes F^n H) \Delta(H),$$

where $F^n (H \otimes H) = \sum_{i+j=n} F_i H \otimes F_j H$.

In particular, for any $H$-module $W$, we have:

$$(2.37) \quad (F^n H \otimes k) \otimes H W = (F^n H \otimes H) \otimes W = (k \otimes F^n H) \otimes W.$$  

**Proof.** For $h \in H \otimes H$ we have:

$$h = \sum_i (h_i \otimes 1)\Delta(l_i) = \mathcal{F}^{-1}(\sum_i h_i \odot l_i) \iff \sum_i h_i \odot l_i = \mathcal{F}(h).$$

This proves (i).

To prove (2.36), it is enough to show that $F^n (H \otimes H) \subset (F^n H \otimes k) \Delta(H)$. This follows from the fact that $\mathcal{F}(F^n (H \otimes H)) \subset (F^n H \otimes k) \Delta(H)$.
The Fourier transform $\mathcal{F}$ has the following properties (which are easy to check using (2.4)–(2.6)):

\begin{align}
(2.38) \quad \mathcal{F}((f \odot g) \Delta(h)) &= \mathcal{F}(f \odot g)(1 \odot h), \\
(2.39) \quad \mathcal{F}(hf \odot g) &= (h \odot 1)\mathcal{F}(f \odot g), \\
(2.40) \quad \mathcal{F}(f \odot hg) &= (1 \odot h(2))\mathcal{F}(f \odot g)(h(\pi) \odot 1), \\
(2.41) \quad \mathcal{F}_{12}\mathcal{F}_{13} = \mathcal{F}_{23} \mathcal{F}_{12}.
\end{align}

Here in (2.41), we use the standard notation $\mathcal{F}_{12} = \mathcal{F} \circ \text{id}$ acting on $H \otimes H \otimes H$.

### 3. Pseudotensor Categories and Pseudoalgebras

In this section, we review some definitions of Beilinson and Drinfeld [BD]; we also use the exposition in [BK, Section 12].

The theory of conformal algebras [K2] is in many ways analogous to the theory of Lie algebras. The reason is that in fact conformal algebras can be considered as Lie algebras in a certain “pseudotensor” category, instead of the category of vector spaces. A pseudotensor category [BD] is a category equipped with “polynomial maps” and a way to compose them (such categories were first introduced by Lambek [L] under the name multicategories). This is enough to define the notions of Lie algebras, representations, cohomology, etc.

As an example, consider first the category $\mathcal{V}ec$ of vector spaces (over $k$). For a finite nonempty set $I$ and a collection of vector spaces $\{L_i\}_{i \in I}$, $M$, we can define the space of polynomial maps from $\{L_i\}_{i \in I}$ to $M$ as

$$\text{Lin}(\{L_i\}_{i \in I}, M) = \text{Hom}(\otimes_{i \in I} L_i, M).$$

The symmetric group $S_I$ acts among these spaces by permuting the factors in $\otimes_{i \in I} L_i$.

For any surjection of finite sets $\pi: J \rightarrow I$ and a collection $\{N_j\}_{j \in J}$, we have the obvious compositions of polynomial maps

\begin{align}
(3.1) \quad \text{Lin}(\{L_i\}_{i \in I}, M) \otimes \bigotimes_{i \in I} \text{Lin}(\{N_j\}_{j \in J_i}, L_i) &\rightarrow \text{Lin}(\{N_j\}_{j \in J_I}, M), \\
(3.2) \quad \phi \times \{\psi_i\}_{i \in I} \mapsto \phi \circ \otimes \{\psi_i\}_{i \in I} &\equiv \phi(\{\psi_i\}_{i \in I}),
\end{align}

where $J_i = \pi^{-1}(i)$ for $i \in I$.

The compositions have the following properties:

**Associativity:** If $K \rightarrow J$, $\{P_k\}_{k \in K}$ is a family of objects and $\chi_j \in \text{Lin}(\{P_k\}_{k \in K_j}, N_j)$, then $\phi(\{\psi_i((\chi_i)_{i \in J_i})\}_{i \in I}) = (\phi(\{\psi_i\}_{i \in I}))((\chi_i)_{i \in J}) \in \text{Lin}(\{P_k\}_{k \in K}, M).

**Unit:** For any object $M$ there is an element $\text{id}_M \in \text{Lin}(\{M\}, M)$ such that for any $\phi \in \text{Lin}(\{L_i\}_{i \in I}, M)$ one has $\text{id}_M(\phi) = \phi((\text{id}_{L_i})_{i \in I}) = \phi$.

**Equivalence:** The compositions (3.1) are equivariant with respect to the natural action of the symmetric group.

**Definition 3.1** ([BD]). A *pseudotensor category* is a class of objects $\mathcal{M}$ together with vector spaces $\text{Lin}(\{L_i\}_{i \in I}, M)$, equipped with actions of the symmetric groups $S_I$ among them and composition maps (3.1), satisfying the above three properties.

**Remark 3.2.** For a pseudotensor category $\mathcal{M}$ and objects $L, M \in \mathcal{M}$, let $\text{Hom}(L, M) = \text{Lin}(\{L\}, M)$. This gives a structure of an ordinary (additive) category on $\mathcal{M}$ and all $\text{Lin}$ are functors $(\mathcal{M}^\text{op})^I \times \mathcal{M} \rightarrow \text{Vec}$, where $\mathcal{M}^\text{op}$ is the dual category of $\mathcal{M}$.
Remark 3.3. The notion of pseudotensor category is a straightforward generalization of the notion of operad. By definition, an operad is a pseudotensor category with only one object.

Definition 3.4. A Lie algebra in a pseudotensor category \( M \) is an object \( L \) equipped with \( \beta \in \text{Lin}(\{L, L\}, L) \) satisfying the following properties.

- **Skew-commutativity:** \( \beta = -\sigma_{12} \beta \), where \( \sigma_{12} = (12) \in S_2 \).
- **Jacobi identity:** \( \beta(\beta(\cdot, \cdot), \cdot) = \beta(\beta(\cdot, \cdot), \cdot) - \sigma_{12} \beta(\cdot, \beta(\cdot, \cdot)) \), where now \( \sigma_{12} = (12) \) is viewed as an element of \( S_3 \).

![Figure 1](image1.png)

**Figure 1.** A polylinear map from \( \{L_i\}_{i=1}^n \) to \( M \).

It is instructive to think of a polylinear map \( \phi \in \text{Lin}(\{L_i\}_{i=1}^n, M) \) as an operation with \( n \) inputs and 1 output, as depicted in Figure 1. The skew-commutativity and Jacobi identity for a Lie algebra \( (L, \beta) \) are represented pictorially in Figures 2 and 3 below.

![Figure 2](image2.png)

**Figure 2.** Skew-commutativity.

Definition 3.5. A representation of a Lie algebra \( (L, \beta) \) is an object \( M \) together with \( \rho \in \text{Lin}(\{L, M\}, M) \) satisfying

\[
\rho(\beta(\cdot, \cdot), \cdot) = \rho(\cdot, \rho(\cdot, \cdot)) - \sigma_{12} \rho(\cdot, \rho(\cdot, \cdot)).
\]

Similarly, one can define cohomology of a Lie algebra \( (L, \beta) \) with coefficients in a module \( (M, \rho) \) (cf. [BKV]).

Definition 3.6. An \( n \)-cochain of a Lie algebra \( (L, \beta) \), with coefficients in a module \( (M, \rho) \) over it, is a polylinear operation \( \gamma \in \text{Lin}(\{L, \ldots, L\}, M) \) which is skew-symmetric, i.e., satisfying for all \( i = 1, \ldots, n - 1 \) the identity shown in Figure 4.
The differential $d \gamma$ of a cochain $\gamma$ is defined by Figure 5. The same computation as in the ordinary Lie algebra case shows that $d^2 = 0$. The cohomology of the resulting complex is called the **cohomology of $L$ with coefficients in $M$** and is denoted by $H^*(L, M)$.

**Example 3.7.** A Lie algebra in the category of vector spaces $\text{Vec}$ is just an ordinary Lie algebra. The same is true for representations and cohomology.

**Example 3.8.** Let $H$ be a cocommutative bialgebra with a comultiplication $\Delta$ and a counit $\varepsilon$. Then the category $\mathcal{M}'(H)$ of left $H$-modules is a symmetric tensor category. Hence, $\mathcal{M}'(H)$ is a pseudotensor category with polylinear maps

$$\text{Lin}(\{L_i\}_{i \in I}, M) = \text{Hom}_H(\bigotimes_{i \in I} L_i, M).$$

The composition of polylinear maps is given by (3.2). An algebra (e.g., Lie or associative) in the category $\mathcal{M}'(H)$ will be called an $H$-**differential algebra**: this is an ordinary algebra which is also a left $H$-module and such that the product (or the bracket) is a homomorphism of $H$-modules, see (2.8).
\[
\gamma_i - \gamma_{i+1} = \sum_{1 \leq i \leq n+1} (-1)^{i+1} \gamma_i + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \beta_{ij}
\]

**Figure 5.** Differential of a cochain.
One can also define the notions of associative algebra or commutative algebra in a pseudotensor category, their representations and analogues of the Hochschild, cyclic, or Harrison cohomology.

**Definition 3.9.** An associative algebra in a pseudotensor category $\mathcal{M}$ is an object $A$ and a product $\mu \in \text{Lin}(\{A, A\}, A)$ satisfying

**Associativity:** $\mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot))$, see Figure 6 below. The algebra $(A, \mu)$ is called commutative if, in addition, $\mu$ satisfies

**Commutativity:** $\mu = \sigma_{12} \mu$, where $\sigma_{12} = (12) \in S_2$.

![Figure 6. Associativity.](image)

**Remark 3.10.** In order to define the notion of an associative algebra in a pseudotensor category, one does not use the actions of the symmetric groups among the spaces of polylinear maps. One can relax the definition of a pseudotensor category by forgetting these actions. Then what we call a “pseudotensor category” should be termed a “symmetric pseudotensor category”, while there is a more general notion of a “braided” one (cf. [So]).

**Proposition 3.11.** Let $(A, \mu)$ be an associative algebra in a pseudotensor category $\mathcal{M}$. Define $\beta \in \text{Lin}(\{A, A\}, A)$ as the commutator $\beta := \mu - \sigma_{12} \mu$, see Figure 7. Then $(A, \beta)$ is a Lie algebra in $\mathcal{M}$.

![Figure 7. Commutator.](image)
Let $H$ be a cocommutative bialgebra with a comultiplication $\Delta$. We introduce a pseudotensor category $\mathcal{M}^\ast (H)$ with the same objects as $\mathcal{M}^\ast (H)$ (i.e., left $H$-modules) but with another pseudotensor structure [BD]:

$$\text{Lin}(\{\langle L_i \rangle_i \} \in I, M) = \text{Hom}_{H^\otimes I}(\mathbb{E}_{i \in I} L_i, H^\otimes I \otimes_H M).$$

Here $\mathbb{E}_{i \in I}$ is the tensor product functor $\mathcal{M}^\ast (H)^I \rightarrow \mathcal{M}^\ast (H^\otimes I)$. For a surjection $\pi: J \rightarrow I$, the composition of polylinear maps is defined as follows:

$$\phi(\{\psi_i\}_{i \in I}) = \Delta(\pi)(\phi) \circ (\mathbb{E}_{\pi(i)} \psi_i).$$

Here $\Delta(\pi)$ is the functor $\mathcal{M}^\ast (H^\otimes I) \rightarrow \mathcal{M}^\ast (H^\otimes J)$, $M \rightarrow H^\otimes J \otimes_H M$, where $H^\otimes J$ acts on $H^\otimes I$ via the iterated comultiplication determined by $\pi$.

Explicitly, let $n_j \in N_j$ ($j \in J$), and write

$$\psi_i(\otimes_j \otimes_j n_j) = \sum g'_{i} \otimes_H l'_i, \quad g'_i, l'_i \in L_i,$$

where, as before, $J_i = \pi^{-1}(i)$ for $i \in I$. Let

$$\phi(\otimes_i l'_i) = \sum f^n \otimes_H m^n, \quad f^n, m^n \in M.$$ 

Then, by definition,

$$\phi(\{\psi_i\}_{i \in I}) \otimes_j n_j = \sum_{r,k} (\psi_{i \in I} g_k) \Delta(\pi)(f^n) \otimes_H m^n,$$

where $\Delta(\pi): H^\otimes I \rightarrow H^\otimes J$ is the iterated comultiplication determined by $\pi$. For example, if $\pi: \{1, 2, 3\} \rightarrow \{1, 2\}$ is given by $\pi(1) = \pi(2) = 1, \pi(3) = 2$, then $\Delta(\pi) = \Delta \circ \text{id}$; if $\pi(1) = 1, \pi(2) = \pi(3) = 2$, then $\Delta(\pi) = \text{id} \circ \Delta$.

The symmetric group $S_I$ acts among the spaces $\text{Lin}(\{L_i\}_{i \in I}, M)$ by simultaneously permuting the factors in $\mathbb{E}_{i \in I} L_i$ and $H^\otimes I$. This is the only place where we need the cocommutativity of $H$; for example, the permutation $\sigma_{12} = (12) \in S_2$ acts on $(H \otimes H) \otimes_H M$ by

$$\sigma_{12}((g \otimes f) \otimes_H m) = (g \otimes f) \otimes_H m,$$

and this is well defined only when $H$ is cocommutative.

One can generalize the above construction for (quasi)triangular bialgebras as follows.

**Remark 3.12.** Let $H$ be a triangular bialgebra with a universal R-matrix $R$. Recall that $R$ is an invertible element of $H \otimes H$ satisfying the following equations:

$$\sigma(R) = R^{-1},$$

$$\sigma(\Delta(h)) R = R \Delta(h), \quad \forall h \in H,$$

$$\text{id} \circ \Delta) R = R_{13} R_{12},$$

$$\Delta \circ \text{id}) R = R_{13} R_{23},$$

where $\sigma$ is the permutation $\sigma(f \otimes g) = g \otimes f$, and we use the standard notation $R_{12} = R \otimes 1 \in H \otimes H \otimes_H H$, etc. Then we define a pseudotensor category $\mathcal{M}^\ast (H)$ as above but with a modified action of the symmetric groups. It is easy to describe the action of the transposition $\sigma_{12} = (12) \in S_2$ on $(H \otimes H) \otimes_H M$; it is given by

$$\sigma_{12}((f \otimes g) \otimes_H m) = (g \otimes f) R \otimes_H m.$$
This is well defined because of (3.10), and $\sigma_2^2 = \text{id}$ because of (3.9). Since any permutation is a product of transpositions, this can be extended to an action of the symmetric group among the spaces of polylinear maps; due to (3.11), (3.12), this action is compatible with compositions.

If $H$ is quasitriangular, i.e., if we drop relation (3.9), we will get an action of the braid group instead of the symmetric one and a “braided” pseudotensor category (cf. Remark 3.10).

The following notion will be the main object of our study.

**Definition 3.13.** A Lie $H$-pseudoalgebra (or just a Lie pseudoalgebra) is a Lie algebra $(I, \beta)$ in the pseudotensor category $\mathcal{M}^*(H)$ as defined above.

Examples of Lie pseudoalgebras will be given in Sections 4 and 8 below. One can also define associative $H$-pseudoalgebras as associative algebras $(A, \mu)$ in the pseudotensor category $\mathcal{M}^*(H)$. It is convenient to define the general notion of an algebra in $\mathcal{M}^*(H)$ as follows.

**Definition 3.14.** An $H$-pseudoalgebra (or just a pseudoalgebra) is a left $H$-module $A$ together with an operation $\mu \in \text{Hom}_H(H \otimes (H \otimes H) \otimes_H A)$, called the pseudoproduct.

We will denote the pseudoproduct $\mu(a \otimes b) \in (H \otimes H) \otimes_H A$ of two elements $a, b \in A$ by $a * b$. It has the following defining property:

**$H$-bilinearity:** For $a, b \in A$, $f, g \in H$, one has

$$f a * g b = ((f \otimes g) \otimes_H 1)(a * b).$$

(3.13)

Explicitly, if

$$a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i,$$

then $f a * g b = \sum_i (f f_i \otimes g g_i) \otimes_H e_i$.

To describe explicitly the associativity condition for a pseudoproduct $\mu$, we need to compute the compositions $\mu(\mu(\cdot, \cdot), \cdot)$ and $\mu(\cdot, \mu(\cdot, \cdot))$ in $\mathcal{M}^*(H)$.

Let $a * b$ be given by (3.14), and let

$$e_i * c = \sum_{i, j} (f_{ij} \otimes g_{ij}) \otimes_H e_{ij}.$$

Then $(a * b) * c \equiv \mu(\mu(a \otimes b), c)$ is the following element of $H^3 \otimes_H A$ (cf. (3.8)):

$$\sum_{i, j} (f_{ij} f_{ij} f_{ij}(1) \otimes g_{ij} g_{ij}) \otimes_H e_{ij}.$$

(3.16)

Similarly, if we write

$$b * c = \sum_i (h_i \otimes l_i) \otimes_H d_i,$$

(3.17)

$$a * d_i = \sum_{i, j} (h_{ij} \otimes l_{ij}) \otimes_H d_{ij},$$

(3.18)

then

$$a * (b * c) = \sum_{i, j} (h_{ij} h_{ij}(1) \otimes l_{ij}(2)) \otimes_H d_{ij}.$$

(3.19)

Now a pseudoproduct $a * b$ is associative iff it satisfies

**Associativity:**

$$a * (b * c) = (a * b) * c$$

in $H^3 \otimes_H A$, where the compositions $(a * b) * c$ and $a * (b * c)$ are given by the above formulas.
The pseudoproduct $a \ast b$ is commutative iff it satisfies

**Commutativity:**

\[ b \ast a = (\sigma \otimes \text{id})(a \ast b), \]

where $\sigma: H \otimes H \to H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$. Explicitly,

\[ b \ast a = \sum_i (g_i \otimes f_i) \otimes_H e_i, \]

if $a \ast b$ is given by (3.14). Note that the right hand side of (3.21) is well defined due to the cocommutativity of $H$.

In the case of a Lie pseudoalgebra $(L, \beta)$, we will call the pseudoproduct $\beta$ a *pseudo*bracket, and we will denote it by $[a \ast b]$. Let us spell out its properties $(a, b, c \in L, f, g \in H)$:

**$H$-bilinearity:**

\[ [fa \ast gb] = ((f \otimes g) \otimes_H 1) [a \ast b]. \]

**Skew-commutativity:**

\[ [b \ast a] = -(\sigma \otimes_H \text{id}) [a \ast b]. \]

**Jacobi identity:**

\[ [a \ast [b \ast c]] - ([\sigma \otimes \text{id}) \otimes_H \text{id}) [b \ast [a \ast c]] = [[a \ast b] \ast c] \]

in $H \otimes_H L$, where the compositions $[[a \ast b] \ast c]$ and $[a \ast [b \ast c]]$ are defined as above.

**Proposition 3.15.** Let $(A, \mu)$ be an associative $H$-pseudoalgebra. Define a pseudo*bracket* $\beta$ as the commutator $[a \ast b] = a \ast b - (\sigma \otimes_H \text{id})(b \ast a)$. Then $(A, \beta)$ is a Lie $H$-pseudoalgebra (cf. Proposition 3.11).

The definitions of representations of Lie pseudoalgebras or associative pseudoalgebras are obvious modifications of the above.

**Definition 3.16.** A representation of an associative $H$-pseudoalgebra $A$ is a left $H$-module $M$ together with an operation $\rho \in \text{Lin} \{A, M\}, M)$, written as $a \ast c \equiv \rho(a \otimes c) \in (H \otimes H) \otimes_H M$, which satisfies (3.26) for $a, b \in A, c \in M$.

**Definition 3.17.** A representation of a Lie $H$-pseudoalgebra $L$ is a left $H$-module $M$ together with an operation $\rho \in \text{Lin} \{L, M\}, M)$, written as $a \ast c \equiv \rho(a \otimes c)$, which satisfies

\[ a \ast (b \ast c) - ((\sigma \otimes \text{id}) \otimes_H \text{id}) (b \ast (a \ast c)) = [a \ast b] \ast c \]

for $a, b \in L, c \in M$.

4. Some Examples of Lie Pseudoalgebras

In this section we give some examples of Lie pseudoalgebras, and discuss their relationship with previously known objects. Other important examples — the pseudoalgebras of vector fields — are treated in detail in Section 8.
4.1. **Conformal algebras.** The (Lie) conformal algebras introduced by Kac [K2] are exactly the (Lie) $k[\partial]$-pseudoalgebras, where $k[\partial]$ is the Hopf algebra of polynomials in one variable $\partial$. The explicit relation between the $\lambda$-bracket of [DK] and the pseudobracket of Section 3 is:

$$[a, b] = \sum_i p_i(\lambda) e_i \quad \iff \quad [a \ast b] = \sum_i (p_i(-\partial) \otimes 1) \otimes_{k[\partial]} e_i.$$ 

This correspondence has been explained in detail in the introduction.

Similarly, for $H = k[\partial_1, \ldots, \partial_N]$ we get conformal algebras in $N$ indeterminates, see [BKV, Section 10]. We may say that for $N = 0$, $H$ is $k$; then a $k$-conformal algebra is the same as a Lie algebra, cf. Example 3.7.

On the other hand, when $H = k[\Gamma]$ is the group algebra of a group $\Gamma$, one obtains the $\Gamma$-conformal algebras studied in [GK]. This is a special case of a more general construction described in Section 5 below.

4.2. **Current pseudoalgebras.** Let $H'$ be a Hopf subalgebra of $H$, and let $A$ be an $H'$-pseudoalgebra. Then we define the current $H$-pseudoalgebra $\text{Cur}_H^H A \equiv \text{Cur}_{H'} A$ as $H \otimes_{H'} A$ by extending the pseudoproduct $a \ast b$ of $A$ using the $H$-bilinearity. Explicitly, for $a, b \in A$, we define

$$((f \otimes_{H'} a) \ast (g \otimes_{H'} b)) = ((f \otimes g) \otimes H) (a \ast b) = \sum_i (f f_i \otimes g g_i) \otimes H (1 \otimes_{H'} e_i),$$

if $a \ast b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$. Then $\text{Cur}_H^H A$ is an $H$-pseudoalgebra which is Lie or associative when $A$ is so.

An important special case is when $H' = k$: given a Lie algebra $g$, let $\text{Cur}_H g = H \otimes g$ with the following pseudobracket:

$$[(f \otimes a) \ast (g \otimes b)] = (f \otimes g) \otimes H (1 \otimes [a, b]).$$

Then $\text{Cur}_H g$ is a Lie $H$-pseudoalgebra.

4.3. **$H$-pseudoalgebras of rank 1.** Let $L = He$ be a Lie pseudoalgebra which is a free $H$-module of rank 1. Then, by $H$-bilinearity, the pseudobracket on $L$ is determined by $[e \ast e]$, or equivalently, by an $a \in H \otimes H$ such that $[e \ast e] = a \otimes_H e$.

**Proposition 4.1.** $L = He$ with the pseudobracket $[e \ast e] = a \otimes_H e$ is a Lie $H$-pseudoalgebra iff $a$ satisfies the following equations:

\begin{align*}
(4.1) & \quad a = -\sigma(a), \\
(4.2) & \quad (a \otimes 1) (\Delta \otimes \text{id})(a) = (1 \otimes a) (\text{id} \otimes \Delta)(a) - (\sigma \otimes \text{id}) ((1 \otimes a) (\text{id} \otimes \Delta)(a)).
\end{align*}

Similarly, $A = Ha$ with a pseudoproduct $a \ast a = a \otimes_H a$ is an associative $H$-pseudo-algebra iff $a \in H \otimes H$ satisfies

$$(a \otimes 1) (\Delta \otimes \text{id})(a) = (1 \otimes a) (\text{id} \otimes \Delta)(a).$$

**Proof.** Follows immediately from definitions. Indeed, if $[e \ast e] = a \otimes_H e$, then:

$$[[e \ast e] \ast e] = (a \otimes 1) (\Delta \otimes \text{id})(a) \otimes H e,$$

$$[e \ast [e \ast e]] = (1 \otimes a) (\text{id} \otimes \Delta)(a) \otimes H e.$$
Lemma 4.2. Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. Then any solution $\alpha \in H \circ H$ of equations (4.1), (4.2) is of the form $\alpha = r + s \circ 1 = 1 \circ s$, where $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$.

In this case (4.2) is equivalent to the following system of equations:

\begin{align*}
[r, \Delta(s)] &= 0, \\
([p_{12}, r_{13}] + r_{12} s_3) + \text{cyclic} &= 0.
\end{align*}

As usual, $r_{12} = r \circ 1$, $s_3 = 1 \circ 1 \circ s$, etc., and “cyclic” here and further means applying the two nontrivial cyclic permutations on $H \circ H \circ H$.

Proof. Using an argument similar to that of [Ki], we will show that if $\alpha$ satisfies (4.2) then $\alpha \in H \circ (\mathfrak{d} + \mathfrak{k})$. Then (4.1) will imply the first claim, that $\alpha \in (\mathfrak{d} + \mathfrak{k}) \circ (\mathfrak{d} + \mathfrak{k})$.

Let $\{\partial_1, \ldots, \partial_N\}$ be a basis of $\mathfrak{d}$, and let us consider the corresponding Poincaré–Birkhoff–Witt basis of $H = U(\mathfrak{d})$ given by elements $\partial^{(I)} := \partial_{i_1} \cdots \partial_{i_N} / i_1! \cdots i_N!$, where $I = (i_1, \ldots, i_N) \in \mathbb{Z}_+^N$. In this basis the comultiplication takes the simple form (2.31). We can write $\alpha = \sum_I a_I \partial^{(I)}$, $a_I \in H$. Equation (4.2) then becomes:

\begin{equation}
\sum_I a_I \Delta(a_I) \otimes \partial^{(I)} = \sum_{I, J, K} (a_{J+K} \otimes a_I \partial^{(J)} - a_I \partial^{(J)} \otimes a_{J+K}) \otimes \partial^{(K)}.
\end{equation}

Let $p$ be the maximal value of $|I| = i_1 + \cdots + i_N$ for $I$ such that $a_I \neq 0$. We want to show that $p \leq 1$. Among all $I$ such that $|I| = p$ there will be some (nonzero) $a_I$ of maximal degree $d$. Then without loss of generality we can change the basis $\partial_1, \ldots, \partial_N$ and assume that the coefficient $a_{p_1, \ldots, p_N}$ is nonzero and of degree $d$. If $p > 1$, then no nonzero term in the left hand side of (4.5) has a third tensor factor of degree $2p$ or $2p-1$ since $2p-1 > p$. Hence, terms from the right hand side of degree $2p$ (respectively $2p-1$) in the third tensor factor must cancel against each other.

Terms having degree $2p$ in the third tensor factor cancel, since they give the following sum:

\begin{equation}
\sum_{|I| = |K| = p} a_K \circ a_I \circ [\partial^{(I)}, \partial^{(K)}],
\end{equation}

which in the third tensor factor has degree $2p-1$ and lower. Note also that their coefficients have total degree at most $2d$.

Terms having third tensor factors of degree $2p-1$, besides (4.6), arise when we choose $|I + K| = 2p-1$. Those with $|I| = p-1, |K| = p$ can be expressed in terms of commutators as above, and hence only contribute to lower degree. So, we only need to account for terms with $|I| = p, |K| = p-1$.

Let us focus on such terms having a third tensor factor proportional to $\partial_1^{2p-1}$, whose coefficient must be zero. They occur in (4.5) only when $I = (p, 0, \ldots, 0)$, $K = (p-1, 0, \ldots, 0)$. When $J = 0$, things cancel as above. The only other nonzero terms are the following:

\[ \sum_j (a_{K+\varepsilon_j} \circ a_I \partial_j - a_I \partial_j \circ a_{K+\varepsilon_j}) \otimes \partial^{(I)} \partial^{(K)}], \]

where $\{\varepsilon_j\}$ is the standard basis of $\mathbb{Z}_+^N$.

We have seen that all other contributions have coefficients of degree at most $2d$, so the sum $\sum_j (a_{K+\varepsilon_j} \circ a_I \partial_j - a_I \partial_j \circ a_{K+\varepsilon_j})$ must lie inside $\mathbb{F}^2 d(H \circ H)$. Hence
\[ \sum_j \alpha_{K+\varepsilon_j} \otimes \alpha_j \partial_j = (1 \otimes \alpha_f) \sum_j \alpha_{K+\varepsilon_j} \otimes \partial_j \] must lie there too. But this means that \( \alpha_{K+\varepsilon_j} \in F^{d-1}H \) for all \( j \), so in particular \( \alpha_f \in F^{d-1}H \), which is a contradiction.

This proves that \( \alpha \in (\mathcal{D} + \mathbb{k}) \otimes (\mathcal{D} + \mathbb{k}) \). Now if \( \alpha = r + s_1 - s_2 \), \( r \in \mathcal{D} \land \mathcal{D}, s \in \mathcal{D} \), then we have:

\[ (\Delta \otimes \text{id})(\alpha) = r_{13} + r_{23} + s_1 + s_2 - s_3, \]

and (4.2) becomes

(4.7) \[ ([r_{12}, r_{13} + s_1 + s_2] + r_{13}s_3] + \text{cyclic} = 0. \]
Comparing the terms in \( \mathcal{D} \otimes \mathcal{D} \otimes \mathbb{k} \), we see that (4.7) is equivalent to the system (4.3, 4.4).

Note that when \( \alpha = r \in \mathcal{D} \land \mathcal{D}, s = 0 \), (4.4) is exactly the classical Yang-Baxter equation

(4.8) \[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \]
Eq. (4.4) is a special case of the dynamical classical Yang-Baxter equation (see [Fe, ES]).

5. (\( H \otimes \mathbb{k}[[\Gamma]] \))-Pseudoalgebras

Let again \( H \) be a cocommutative Hopf algebra. Let \( \Gamma \) be a group acting on \( H \) by automorphisms, and let \( \hat{H} = H \otimes \mathbb{k}[\Gamma] \) be the smash product of \( H \) with the group algebra of \( \Gamma \). As an associative algebra this is the semidirect product of \( H \) with \( \mathbb{k}[\Gamma] \), while as a coalgebra it is the tensor product of coalgebras.

We will denote the action of \( \Gamma \) on \( H \) by \( g \cdot f \) for \( g \in \Gamma, f \in H \); then \( g \cdot f = gfg^{-1} \). Then a left \( \hat{H} \)-module \( L \) is the same as an \( H \)-module together with an action of \( \Gamma \) on it which is compatible with that of \( H \), i.e., such that \( (g \cdot f)l = g(f(g^{-1}l)) \) for \( g \in \Gamma, f \in H, l \in L \).

In this section we will study the relationship between the pseudotensor categories \( \mathcal{M}(\hat{H}) \) and \( \mathcal{M}(H) \). In particular, we will show that an \( \hat{H} \)-pseudoalgebra is the same as an \( H \)-pseudoalgebra on which the group \( \Gamma \) acts by preserving the pseudoproduct.

Let us start by defining maps \( \delta_I : \hat{H}^{\mathcal{O}I} \to H^{\mathcal{O}I} \otimes_{\hat{H}} \hat{H} \) for each finite nonempty set \( I \). It is enough to define \( \delta_I \) on elements of the form \( 1 \otimes l, f_i g_i \) where \( f_i \in H, g_i \in \Gamma \), in which case we let

\[ \delta_I(1 \otimes l, f_i g_i) = \begin{cases} (1 \otimes l, f_i) \otimes_{\hat{H}} g_i, & \text{if all } g_i \text{ are equal to some } g, \\ 0, & \text{if some of } g_i \text{ are different}. \end{cases} \]

It is easy to see that \( \delta_I \) is a homomorphism of both left \( H^{\mathcal{O}I} \)-modules and of right \( \hat{H} \)-modules.

This allows us to define a pseudotensor functor \( \delta : \mathcal{M}(\hat{H}) \to \mathcal{M}(H) \) as follows. For an object \( L \) (a left \( \hat{H} \)-module), we let \( \delta(L) \equiv L \) be the left \( H \)-module obtained by restricting the action of \( \hat{H} \) to \( H \subset \hat{H} \). For a polynilpotent map \( \phi \in \text{Lin}(\{L_i\}_{i \in I}, M) \) in \( \mathcal{M}(\hat{H}) \), i.e., for a homomorphism of left \( \hat{H}^{\mathcal{O}I} \)-modules \( \phi : \mathcal{O}_{i \in I} L_i \to \hat{H}^{\mathcal{O}I} \otimes_{\hat{H}} M \),

we let \( \delta(\phi) \) be the composition

\[ \delta(\phi) : \mathcal{O}_{i \in I} L_i \xrightarrow{\phi} \hat{H}^{\mathcal{O}I} \otimes_{\hat{H}} M \xrightarrow{\delta_{\mathcal{O}} \otimes \text{id}} (H^{\mathcal{O}I} \otimes_{H} \hat{H}) \otimes_{H} M \cong H^{\mathcal{O}I} \otimes_{H} M. \]
This is a homomorphism of left $H^\otimes I$-modules, i.e., a polylinear map in $\mathcal{M}^*(H)$. Moreover, since the maps $\delta_i$ are compatible with the actions of the symmetric groups and with the comultiplication of $\hat{H}$, it follows that $\delta$ is compatible with the actions of the symmetric groups and with compositions of polylinear maps, i.e., it is a pseudotensor functor.

As usual, the action of $\Gamma$ on $H$ can be extended to an action of $\Gamma$ on $H^\otimes I$ by using the comultiplication $\Delta^{(I)}(g) = \bigotimes_{i \in I} g_i$. Hence, $\Gamma$ also acts on $H^\otimes I \otimes_{\hat{H}} M$ by the formula

$$g \cdot ((\bigotimes_{i \in I} f_i) \otimes_{\hat{H}} m) = (\bigotimes_{i \in I} g \cdot f_i) \otimes_{\hat{H}} gm, \quad g \in \Gamma, \quad f_i \in H, \quad m \in M.$$

Then it is easy to see that $\psi = \delta(\phi)$ has the following property:

$$\psi(\bigotimes_{i \in I} g_i h_i) = g \cdot \psi(\bigotimes_{i \in I} h_i), \quad g \in \Gamma, \quad h_i \in L_i,$$

in other words, it commutes with the action of $\Gamma$.

We let $\mathcal{M}^*_\Gamma(H)$ be the subcategory of $\mathcal{M}^*(H)$ with objects left $\hat{H}$-modules, and with polylinear maps those polylinear maps $\psi$ of $\mathcal{M}^*(H)$ that commute with the action of $\Gamma$ (see (5.1)). This is a pseudotensor category, and $\delta$ is a pseudotensor functor from $\mathcal{M}^*(\hat{H})$ to $\mathcal{M}^*_\Gamma(H)$.

**Theorem 5.1.** If $\Gamma$ is a finite group, the functor $\delta: \mathcal{M}^*(\hat{H}) \to \mathcal{M}^*_\Gamma(H)$ constructed above is an equivalence of pseudotensor categories.

**Proof.** We will construct a pseudotensor functor $\Sigma$ from $\mathcal{M}^*_\Gamma(H)$ to $\mathcal{M}^*(\hat{H})$. On objects $L$ we let $\Sigma(L) = L$. In order to define it on polylinear maps, we need to find out how $\phi$ can be recovered from $\delta(\phi)$ and the action of $\Gamma$.

Denote by $i$ the embedding $H \hookrightarrow \hat{H}$, and let $\pi_i$ be the composition

$$\pi_i: H^\otimes I \otimes_{\hat{H}} H \overset{\epsilon ^\otimes \otimes_{\hat{H}}}\longrightarrow \hat{H}^\otimes I \otimes_{\hat{H}} \hat{H} \overset{\cong}\rightarrow \hat{H}^\otimes I \otimes_{\hat{H}} \hat{H} \overset{\pi_i}\rightarrow \hat{H}^\otimes I.$$

Explicitly, $\pi_i$ is given by the formula

$$\pi_i((\bigotimes_{i \in I} f_i) \otimes_{\hat{H}} g) = \bigotimes_{i \in I} f_i g_i, \quad f_i \in H, \quad g \in \Gamma.$$

This is a homomorphism of both left $H^\otimes I$-modules and of right $\hat{H}$-modules. Moreover, for $f_i \in H$, $g_i \in \Gamma$, we have:

$$\pi_{i} \delta_{i}(\bigotimes_{i \in I} f_{i} g_{i}) = \begin{cases} \bigotimes_{i \in I} f_{i} g_{i}, & \text{if all } g_i \text{ are equal,} \\ 0, & \text{otherwise.} \end{cases}$$

The crucial observation, which will allow us to invert $\delta_i$, is that for any $h_i \in \hat{H}$, $g_i \in \Gamma$, we have:

$$\sum_{(g_i) \in \Gamma^I/\Gamma} (\bigotimes_{i \in I} g_{i}) (\pi_{i} \delta_{i}) (\bigotimes_{i \in I} g_{i}^{-1} h_{i}) = \bigotimes_{i \in I} h_{i}. \quad (5.2)$$

Here $\Gamma$ acts diagonally on $\Gamma^I$ from the right; the left hand side of (5.2) is invariant under $(g_i) \mapsto (g_i g)$.

Given a polylinear map $\psi \in \operatorname{Lin}(\{L_i\}_{i \in I}, M)$ in $\mathcal{M}^*_\Gamma(H)$, we can extend it to a map

$$\tilde{\psi}: \bigotimes_{i \in I} L_i \overset{\psi}{\longrightarrow} H^\otimes I \otimes_{\hat{H}} M \overset{\cong}\rightarrow (H^\otimes I \otimes_{\hat{H}} \hat{H}) \otimes_{\hat{H}} M \overset{\pi_i \otimes_{\hat{H}} I}{\longrightarrow} \hat{H}^\otimes I \otimes_{\hat{H}} M.$$
(Note, however, that \( \hat{\psi} \) is not \( \hat{H} \otimes L \)-linear.) Now we define \( \Sigma \psi : \bigoplus e_i L_i \rightarrow \hat{H} \otimes L \otimes \hat{H} \) by the formula:

\[
(\Sigma \psi) (\odot e_i g_i) = \sum_{(g_i) \in \Gamma^L / \Gamma} ((\odot e_i g_i) \otimes \hat{H}) \hat{\psi} (\odot e_i g_i^{-1} L_i).
\]

It is easy to check that \( \Sigma \psi \) is \( \hat{H} \otimes L \)-linear, so it is a polynellar map in \( M^*(\hat{H}) \). Moreover, \( \partial \Sigma \psi = \hat{\psi} \). For a polynellar map \( \hat{\phi} \in \text{Lin}(\{L_i\}_{i \in I} \otimes M) \) in \( M^*(\hat{H}) \), it is immediate from (5.2) and the \( \hat{H} \otimes L \)-linearity of \( \hat{\psi} \) that \( \partial \hat{\phi} = \hat{\phi} \). Therefore, \( \Sigma : M^*_\Gamma (H) \rightarrow M^*(\hat{H}) \) is a pseudotensor functor inverse to \( \partial \).

\textbf{Remark 5.2.} The above theorem holds also for infinite groups \( \Gamma \) if we restrict ourselves to polynellar maps \( \psi \) of \( M^*_\Gamma (H) \) satisfying the following finiteness condition:

\[
\psi (\odot e_i g_i L_i) \neq 0 \quad \text{for only a finite number of } (g_i) \in \Gamma \setminus \Gamma^L
\]

for any fixed \( L_i \in L \). (Note that, by (5.1), this condition does not depend on the choice of representatives \( (g_i) \).) Indeed, the only place in the proof where we used the finiteness of \( \Gamma \) was to insure that the right hand side of (5.3) is a finite sum.

If \( \psi = \partial (\hat{\phi}) \) comes from a polynellar map \( \hat{\phi} \) of \( M^*(\hat{H}) \), then it satisfies (5.4), because \( \hat{\phi} \) is \( \hat{H} \otimes L \)-linear and for any element \( h \in \hat{H} \otimes L \) one has \( \partial h ((\odot e_i g_i) h) \neq 0 \) for only a finite number of \( (g_i) \in \Gamma \setminus \Gamma^L \).

Therefore, \( \partial : M^*(\hat{H}) \rightarrow M^*_\Gamma \text{fin} (H) \) is an equivalence of pseudotensor categories, where \( M^*_\Gamma \text{fin} (H) \) is the subcategory of \( M^*_\Gamma (H) \) consisting of polynellar maps \( \psi \) satisfying (5.4).

\textbf{Corollary 5.3.} A Lie \( \hat{H} = (H \hat{k}[\Gamma]) \)-pseudogebra \( L \) is the same as a Lie \( H \)-pseudogebra \( L \) on which the group \( \Gamma \) acts in a way compatible with the action of \( H \), by preserving the \( H \)-pseudobracket:

\[
[ga \ast gb] = g \cdot [a \ast b] \quad \text{for } g \in \Gamma, a, b \in L,
\]

and satisfying the following finiteness condition:

\[
geven \ a, b \in L, [ga \ast b] \text{ is nonzero for only a finite number of } g \in \Gamma.
\]

The \( H \)-pseudobracket of \( L \) is given by the formula:

\[
[a \hat{\ast} b] = \sum_{g \in \Gamma} ((g^{-1} \odot 1) \otimes \hat{H}) \hat{\psi} (ga \ast b), \quad a, b \in L.
\]

A similar statement holds for representations, as well as for associative pseudogeralgebras.

This result, combined with Kostant’s Theorem 2.1, will allow us in many cases to reduce the study of \( H \)-pseudogeralgebras to the case when \( H \) is a universal enveloping algebra (see Section 13.7).

\textbf{Example 5.4.} Let \( \Gamma \) be a subgroup of \( \hat{k}^* \) and let

\[
H = \hat{k} [\hat{\partial}] \hat{\otimes} \hat{k} [\Gamma] = \bigoplus_{m \geq 0, \alpha \in \Gamma} \hat{k} \partial^m T_\alpha
\]

with multiplication \( T_\alpha T_\beta = T_{\alpha \beta} \), \( T_1 = 1 \), \( T_\alpha \partial T_\beta = \alpha \partial \) and comultiplication \( \Delta (\partial) = \partial \otimes 1 + 1 \otimes \partial \), \( \Delta (T_\alpha) = T_\alpha \otimes T_\alpha \). Then the notion of a Lie \( H \)-pseudogebra is equivalent to that of a \( \Gamma \)-conformal algebra (cf. [K4]).
Example 5.5. Let now $H = k[\partial] \times F(\Gamma)$, where $F(\Gamma)$ is the function algebra of a finite abelian group $\Gamma$. In other words, $H = \bigoplus_{m \in \mathbb{Z}, \alpha \in \Gamma} k \, \partial^m \pi_\alpha$ with multiplication $\pi_\alpha \pi_\beta = \delta_{\alpha,\beta} \pi_\alpha$, $\partial \pi_\alpha = \pi_\alpha \partial$ and comultiplication $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\Delta(\pi_\alpha) = \sum_{\gamma \in \Gamma} \pi_{\alpha \gamma^{-1}} \otimes \pi_\gamma$. Then one gets the notion of a $\Gamma$-twisted conformal algebra (cf. [K4]).

6. A Digression to Linearly Compact Lie Algebras

We will view the base field $k$ as a topological field with discrete topology. A topological vector space $L$ over $k$ is called linearly compact if it is the space of all linear functionals on a vector space $V$ with discrete topology, with the topology on $L$ defined by taking all subspaces $\{U^\perp \subset L \mid U \subset V, \dim U < \infty\}$ as a fundamental system of neighborhoods of 0 in $L$. Here, as usual, $U^\perp$ denotes the subspace of $L$ consisting of all linear functionals vanishing on $U$.

In general, given a topological vector space $W$, we define a topology on $W^*$ by taking for the fundamental system of neighborhoods of 0 the subspaces $U^\perp$ where $U$ is a linearly compact subspace of $W$.

Several equivalent definitions of linear-compactness are provided by the next proposition.

Proposition 6.1. For a topological vector space $L$ over the topological field $k$ the following statements are equivalent:

1. $L$ is the dual of a discrete vector space.
2. The topological dual $L^*$ of $L$ is a discrete topological space.
3. $L$ is the topological product of finite-dimensional discrete vector spaces.
4. $L$ is the projective limit of finite-dimensional discrete vector spaces.
5. $L$ has a collection of finite-codimensional open subspaces whose intersection is $\{0\}$, with respect to which it is complete.

Proof. Can be found in [G1].

Remark 6.2. For both discrete and linearly compact vector spaces, the canonical map from $L$ to $L^{**}$ is an isomorphism.

A linearly compact (associative or Lie) algebra is a topological (associative or Lie) algebra for which the underlying topological space is linearly compact.

The basic example of a linearly compact associative algebra is the algebra $O_N = k[[t_1, \ldots, t_N]]$ of formal power series over $k$ in $N \geq 1$ indeterminates $t_1, \ldots, t_N$, with the usual formal topology for which $(t_1, \ldots, t_N)^j$, the powers of the ideal $(t_1, \ldots, t_N)$, form a fundamental system of neighborhoods of $O_N$.

Remark 6.3. The topological vector spaces $O_N$ are isomorphic and characterized among linearly compact vector spaces by each of the following properties:

1. $O_N^*$ is countable-dimensional.
2. $O_N$ has a filtration by open subspaces.

Remark 6.4. (i) One defines a completed tensor product of two linearly compact vector spaces $V, W$ by $V \hat{\otimes} W = (V^* \otimes W^*)^*$ where we put the discrete topology on $V^* \otimes W^*$. Then $V \hat{\otimes} W$ is linearly compact.

(ii) With this definition, $O_{M+N} \simeq O_M \hat{\otimes} O_N$ as topological algebras.

(iii) Given a commutative associative linearly compact algebra $O$ and a linearly compact Lie algebra $L$, their completed tensor product $O \hat{\otimes} L$ is again a linearly compact Lie algebra.
The basic example of a linearly compact Lie algebra is the Lie algebra $W_N$ of continuous derivations of the topological algebra $O_N$. The filtration
\[ F_j O_N = (t_1, \ldots, t_N)^{j+1}, \quad j = -1, 0, 1, \ldots \]
of $O_N$ induces the canonical filtration $F_j W_N$ of $W_N$, where
\[ F_j W_N = \{ D \in W_N \mid D(F_i O_N) \subset F_{i+j} O_N \quad \forall i \}, \quad j = -1, 0, 1, \ldots. \]

It is clear that $W_N$ consists of all linear differential operators of the form
\[ D = \sum_{i=1}^N P_i(t) \frac{\partial}{\partial t_i}, \quad \text{where} \quad P_i(t) \in O_N, \]
and that $F_j W_N$ ($j \geq -1$) consists of those $D$ for which all $P_i(t)$ lie in $F_j O_N$.

Let $E = \sum_{i=1}^N t_i \frac{\partial}{\partial t_i}$ be the Euler operator. The spectrum of $\text{ad} E$ consists of all integers $j \geq -1$, and, denoting by $W_{N,j}$ the $j$-th eigenspace of $\text{ad} E$ we obtain the canonical $\mathbb{Z}$-gradation:
\[ W_N = \prod_{j \geq -1} W_{N,j}, \quad [W_{N,j}, W_{N,j}] \subset W_{N,j+1}. \]

The following fact is well known.

**Lemma 6.5.** $W_{N,0} \cong \mathfrak{gl}_N(k)$ and one has the following isomorphism of $\mathfrak{gl}_N(k)$-modules:
\[ W_{N,j} \cong k^N \otimes (S^j k^N)^*. \]

Furthermore, one has a decomposition into a direct sum of irreducible submodules:
\[ W_{N,j} = W_{N,j}^0 + W_{N,j}^1, \quad \text{where} \quad W_{N,j}^0 \cong (S^j k^N)^* \quad (= 0 \text{ if } j = -1) \quad \text{and} \quad W_{N,j}^1 \cong \text{the highest component of } k^N \otimes (S^{j+1} k^N)^*. \]
The subspace $p = W_{N,-1} + W_{N,0} + W_{N,1}$ is a subalgebra of $W_N$ isomorphic to $\mathfrak{sl}_{N+1}(k)$.

Let $\Omega_N = \bigoplus_{j=-1}^N \Omega_{N,j}$ denote the algebra of differential forms over $O_N$. The defining representation of $W_N$ on $O_N$ extends uniquely to a representation on $\Omega_N$ commuting with the differential $d$.

Recall that a volume form is a differential $N$-form $\nu = f(t_1, \ldots, t_N) dt_1 \wedge \cdots \wedge dt_N$ such that $f(0) \neq 0$, a symplectic form is a closed 2-form $s = \sum_{i,j=1}^N s_{ij}(t_1, \ldots, t_N) dt_i \wedge dt_j$ such that $\det(s_{ij}(0)) \neq 0$, and a contact form is a 1-form $c$ such that $c \wedge (dc)^{(N-1)/2}$ is a volume form. The following facts are well known.

**Lemma 6.6.** (i) Any volume form can be transformed by an automorphism of $O_N$ to the standard volume form $\nu_0 = dt_1 \wedge \cdots \wedge dt_N$.

(ii) A symplectic form exists iff $N$ is even, $N = 2n$, and by an automorphism of $O_N$ it can be transformed to the standard symplectic form $s_0 = \sum_{i=1}^n dt_i \wedge dt_{n+i}$.

(iii) A contact form exists iff $N$ is odd, $N = 2n+1$, and by an automorphism of $O_N$ it can be brought to the standard contact form $c_0 = dt_N + \sum_{i=1}^n dt_i$.

Consider the following (closed) subalgebras of the Lie algebra $W_N$:
\[ S_N(\nu) = \{ D \in W_N \mid D\nu = 0 \} \quad (N \geq 2), \]
\[ H_N(s) = \{ D \in W_N \mid Ds = 0 \} \quad (N \text{ even } \geq 2), \]
\[ K_N(c) = \{ D \in W_N \mid Dc = fc \quad \text{for some } f \in O_N \} \quad (N \text{ odd } \geq 3). \]

Let also $S_N = S_N(\nu_0)$, $H_N = H_N(s_0)$, $K_N = K_N(c_0)$. Lemma 6.6 implies isomorphisms: $S_N(\nu) \cong S_N$, $H_N(s) \cong H_N$, $K_N(c) \cong K_N$, $S_2 \cong H_2$. 
The canonical filtration of $W_N$ induces canonical filtrations $F_j S_N(v) := F_j W_N \cap S_N(v)$, etc. Note that $\dim W_N / F_{-1} W_N = N$. A Lie subalgebra $\mathcal{L}$ of $W_N$ is called transitive if $\dim \mathcal{L} / (\mathcal{L} \cap F_{-1} W_N) = N$. It is known that the Lie algebras $W_N$, $S_N$, $H_N$ and $K_N$ are transitive. In addition, the canonical filtrations $F_j \mathcal{L}$ of these Lie algebras have the following transitivity property:

(6.1) \[ F_{j+1} \mathcal{L} = \{ a \in F_j \mathcal{L} \mid [a, \mathcal{L}] \subseteq F_j \mathcal{L} \}. \]

Noting that $E v_0 = N v_0$ and $E s_0 = 2 s_0$, we conclude that $ad E$ is an (outer) derivation of $S_N$ and $H_N$, hence the canonical gradation of $W_N$ induces canonical $\mathbb{Z}$-gradations $S_N = \bigoplus_{j \geq 1} S_{N,j}$ and $H_N = \bigoplus_{j \geq 1} H_{N,j}$.

Let $E' = 2 t_N \frac{d}{dt_N} + \sum_{j=1}^{N-1} \lambda_j \frac{d}{d\lambda_j}$. Then $E' c_0 = 2 c_0$, hence $E' \in K_0$ and the eigenspace decomposition of $ad E'$ defines the canonical $\mathbb{Z}$-gradation $K_N = \bigoplus_{j \geq -2} K_{N,j}$.

The following facts are well known.

**Lemma 6.7.** (i) $S_{N,0} \cong sl_N(k)$, $H_{N,0} \cong sp_N(k)$, $K_{N,0} \cong sp_{N-1}(k) \cong \mathfrak{so}_{N-1}(k)$. (ii) The $S_{N,0}$-module $S_{N,j}$ is isomorphic to the highest component of the $sl_N(k)$-module $k^N \otimes (S^{j+1} k^N)^*$. (iii) The $H_{N,0}$-module $H_{N,j}$ is isomorphic to the (irreducible) $sp_N(k)$-module $S^{j+1} k^N$. (iv) $K_{N,0} = sp_{N-1}(k) \oplus k E'$ and the $sp_{N-1}(k)$-module $K_{N,j}$ decomposes into the following direct sum of irreducible modules:

\[ K_{N,j} = \bigoplus_{i=0}^{\lfloor \frac{j}{2} \rfloor + 1} K_{N,j,i} \text{ where } K_{N,j,i} \cong S^{j+2-2i} k^{N-1}. \]

The subspace $\mathfrak{p} = K_{N,-2} + K_{N,-1} + K_{N,0} + K_{N,1} + K_{N,2}$ is a subalgebra of $K_N$ isomorphic to $sp_{N+1}(k)$.

The following celebrated theorem goes back to E. Cartan (see [G2] for a relatively simple proof).

**Theorem 6.8.** Any infinite-dimensional simple linearly compact Lie algebra is isomorphic to one of the topological Lie algebras $W_N$, $S_N$, $H_N$ or $K_N$.

Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be its subalgebra of codimension $N$. Then $F = \text{Hom}_F(\mathfrak{h})(U(\mathfrak{g}), \mathfrak{k})$, with the product $(f_1 f_2)(u) = f_1(u(1)) f_2(u(2)))$, is (canonically) isomorphic to the algebra of formal power series on $(\mathfrak{g}/\mathfrak{h})^* [B2]$, which is (non-canonically) isomorphic to the linearly compact algebra $\mathcal{O}_N$. The Lie algebra $D$ of continuous derivations of $F$ is then isomorphic to $W_N$. $F$ has a canonical $\mathfrak{g}$-action induced by the left-multiplication $\mathfrak{g}$-action on $U(\mathfrak{g})$, which gives us a homomorphism $\gamma$ of $\mathfrak{g}$ to $W_N$. (This is non-canonical since the identification of $F$ with $\mathcal{O}_N$ is not canonical.)

We will use in the sequel the following theorem of Guillemin and Sternberg [GS] (see [B2] for a simple proof).

**Proposition 6.9.** Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be its subalgebra of codimension $N$. Provided that $\mathfrak{h}$ contains no nonzero ideals of $\mathfrak{g}$, the above defined $\gamma$ is a Lie algebra isomorphism of $\mathfrak{g}$ with a subalgebra of $W_N$ which maps $\mathfrak{h}$ into $F_0 W_N$.

(Conversely, if $\mathfrak{g} \rightarrow W_N$ maps $\mathfrak{h}$ into $F_0 W_N$, then $\mathfrak{h}$ doesn't contain nonzero ideals of $\mathfrak{g}$.) Every Lie algebra homomorphism of $\mathfrak{g}$ to $W_N$, which coincides with $\gamma$ modulo $F_0 W_N$, is conjugated to $\gamma$ via a unique automorphism of $\mathcal{O}_N$. 


We have the following important property of the filtrations on $H$ and $X = H^*$, defined in Section 2.2.

**Lemma 6.10.** Let $H = U(\mathfrak{g}) \overline{\otimes} k[\Gamma]$ be a cocommutative Hopf algebra, and $X = H^*$. If $h \in F^n U(\mathfrak{g}) \subseteq H$ but $h \notin F^{n+1} U(\mathfrak{g})$, then $h F_n X = F_n X$. In particular, for any $h \in \mathfrak{g} \setminus \{0\}$ and for every open subspace $U \subseteq X$, there is some $n$ such that $h^n U = X$. Similar statements hold for the right action of $h$ as well.

**Proof.** By the construction of the filtrations it is evident that we can assume $H = U(\mathfrak{g})$. Then $X \simeq \mathcal{O}_N (N = \dim \mathfrak{g})$, and $\mathfrak{g} \to W_N$ acts on it by linear differential operators. The rest of the proof is clear. □

The following result from [G1, G2] will be essential for our purposes.

**Proposition 6.11.** (i) A linearly compact Lie algebra $\mathcal{L}$ satisfies the descending chain condition on closed ideals if and only if it has a fundamental subalgebra, i.e., an open subalgebra containing no ideals of $\mathcal{L}$.

(ii) When either of the assumptions of (i) holds, the noncommutative minimal closed ideals of $\mathcal{L}$ are of the form $\mathcal{O}_r \otimes S$ where $S$ is a simple linearly compact Lie algebra and $r \in \mathbb{Z}_+$.

We will also need the following examples of non-simple linearly compact Lie algebras:

$$CS_N(v) = \{ D \in W_N \mid Dv = av, \ a \in k \},$$

$$CH_N(s) = \{ D \in W_N \midDs = as, \ a \in k \}.$$  

As before, we have isomorphisms $CS_N(v) \simeq CS_N(\mathfrak{g})$ and $CH_N(s) \simeq CH_N(\mathfrak{g})$. Note also that $CS_N = kE \ltimes S_N$ and $CH_N = kE \ltimes H_N$. Another important example of a non-simple linearly compact Lie algebra is the Poisson algebra $P_N$, which is $O_N (N = 2n)$ endowed with the Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{n+i}} - \frac{\partial f}{\partial t_{n+i}} \frac{\partial g}{\partial t_i}.$$  

It is a nontrivial central extension of $H_N$:

$$0 \to k \to P_N \xrightarrow{\phi} H_N \to 0,$$

where $\phi(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{n+i}} - \frac{\partial f}{\partial t_{n+i}} \frac{\partial g}{\partial t_i}$.

We can describe also $K_N$ in a more explicit way, similar to the above description of $H_N$. For $f, g \in O_N$, define

$$\{f, g\}' = \{f, g\} + \frac{\partial f}{\partial t_{2n+1}}(E_{2n}g - 2g) - (E_{2n}f - 2f)\frac{\partial g}{\partial t_{2n+1}},$$

where $\{f, g\} + (E_{2n}g - 2g)$ is the Poisson bracket taken with respect to the variables $t_1, \ldots, t_{2n}$ and $E_{2n}$ is the Euler operator $\sum_{i=1}^{2n} t_i \frac{\partial}{\partial t_i}$. If we define

$$\psi(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{n+i}} - \frac{\partial f}{\partial t_{n+i}} \frac{\partial g}{\partial t_i} + \frac{\partial f}{\partial t_{2n+1}}(E_{2n}f - 2f)\frac{\partial g}{\partial t_{2n+1}},$$

then we have $\psi(f) g = \{f, g\}' + 2 \frac{\partial f}{\partial t_{2n+1}} g$. It is easy to see that $[\psi(f), \psi(g)] = \{f, g\}'$ and $\psi(f)c_0 = 2 \frac{\partial f}{\partial t_{2n+1}} c_0$. Thus $K_N$ is isomorphic to $O_N$ with the bracket $\{ , \}'$. 


For a linearly compact Lie algebra \( \mathcal{L} \) denote by \( \text{Der} \mathcal{L} \) the Lie algebra of its continuous derivations and by \( \hat{\mathcal{L}} \) the universal central extension of \( \mathcal{L} \). Then we have:

**Proposition 6.12.** (i) \( \text{Der} W_N = W_N \), \( \text{Der} S_N = CS_N \), \( \text{Der} H_N = CH_N \), \( \text{Der} K_N = K_N \).
(ii) \( \text{Der}(\mathcal{C}_r \widehat{\mathcal{L}}) = W_r \odot 1 + \mathcal{C}_r \widehat{\text{Der}} \mathcal{L} \) for any simple linearly compact Lie algebra \( \mathcal{L} \).
(iii) The Lie algebras \( \mathcal{C}_r \widehat{W}_N \), \( \mathcal{C}_r \widehat{S}_N \) (for \( N > 2 \)) and \( \mathcal{C}_r \widehat{K}_N \) have no non-trivial central extensions. The universal central extension of \( \mathcal{C}_r \widehat{H}_N \) is \( \mathcal{C}_r \widehat{P}_N \).
(iv) If \( \mathfrak{g} \) is a simple finite-dimensional Lie algebra, then \( \mathcal{C}_r \widehat{(\mathcal{C}_r \odot \mathfrak{g})} = \mathcal{C}_r \odot \mathfrak{g} + (\mathcal{C}_r \odot \mathfrak{g}) / \text{ad} \mathcal{O}_r \) with the bracket

\[
[f \odot a, g \odot b] = fg \odot [a, b] + (a \odot b) f dg \mod \text{ad} \mathcal{O}_r,
\]

where \( (a \odot b) \) is the Killing form on \( \mathfrak{g} \).

**Proof.** For a proof of (iv) see [Ka].

In order to prove (ii), notice that if \( d \) is a derivation of \( \mathcal{C}_r \widehat{\mathcal{L}} \), then its action on \( 1 \odot L \) is given by \( d_1(1 \odot x) = \sum a_i \odot d_i(x) \) for all \( x \in \mathcal{L} \), where the \( a_i \) form a topological basis of \( \mathcal{C}_r \) and the \( d_i \) are continuous derivations of \( \mathcal{L} \). Subtracting \( \sum a_i \odot d_i \) from \( d \), we get a derivation \( \tilde{d} \) acting trivially on \( 1 \odot \mathcal{L} \). We are going to show that if \( \mathcal{L} \) is simple then \( \tilde{d} \) is of the form \( D \odot 1 \) where \( D \in \text{Der} \mathcal{O}_r = W_r \).

Let us fix \( P \in \mathcal{O}_r \). Then \( \tilde{d}(P \odot x) \) can be written as \( \sum_i a_i \odot f_i(x) \), where \( f_i \) are continuous \( k \)-endomorphisms of \( \mathcal{L} \). From \( \tilde{d}(\{[P \odot x, 1 \odot y] \} = \{[a \odot (P \odot x), 1 \odot y] \} \) we see that \( [f_i(x), y] = f_i([x, y]) \) for every \( x, y \in \mathcal{L} \). This means that \( f_i \) commutes with \( \text{ad} \) for all \( y \in \mathcal{L} \). By a Schur's lemma argument [G1, Proposition 4.4] and simplicity of \( \mathcal{L} \), we conclude that the \( f_i \) are multiples of the identity map, hence \( \tilde{d}(P \odot x) = a_P \odot x \) for some \( a_P \in \mathcal{C}_r \) and all \( x \in \mathcal{L} \). It is now immediate to check that the mapping \( D : P \mapsto a_P \) is indeed a derivation of \( \mathcal{C}_r \), proving (ii).

In order to prove the rest of the statements, denote by \( \mathfrak{a} \) the 0th component of the canonical \( Z \)-gradation of \( \mathcal{L} = W_N, S_N, H_N \) or \( K_N \). This is a reductive subalgebra of \( \mathcal{L} \), hence \( \text{Der} \mathcal{L} = V \oplus \mathcal{L} \), where \( [a, V] \subset V \). But \( [V, \mathcal{L}] \subset \mathcal{L} \), hence \( [a, V] = 0 \), i.e., any element \( D \in V \) defines an endomorphism of \( \mathcal{L} \) viewed as an \( \mathfrak{a} \)-module. Since \( E \in W_N \) and \( E' \in K_N \), we conclude that \( D \) also preserves the canonical gradation of these Lie algebras and we may assume that \( D \) acts trivially on the \((-1)\)st component. Using the transitivity of \( W_N \) and \( K_N \), we conclude that \( D = 0 \). By Lemma 6.7, all components of the canonical \( Z \)-gradation of \( S_N \) and \( H_N \) are inequivalent \( \mathfrak{a} \)-modules, hence \( D \) preserves this gradation in this case as well. Subtracting from \( D \) a multiple of \( E \), we may assume that \( D \) acts trivially on the \((-1)\)st component and, using transitivity, we conclude that \( D \) is a multiple of \( E \). Thus (ii) is proved.

Since \( \mathfrak{a} \) acts completely reducibly on the space \( Z^2 \) of 2-cocycles on \( \mathcal{C}_r \widehat{\mathcal{L}} \) with values in \( k \), and since \( \mathfrak{a} \) acts trivially on cohomology, we may choose a subspace \( U \) of \( Z^2 \), complementary to the space of trivial 2-cocycles, on which \( \mathfrak{a} \) acts trivially. Hence for any 2-cocycle \( a \in \mathcal{L} \) we have: \( a(a, b) = 0 \) if \( a \in M_1, b \in M_2 \) and \( M_i \) are irreducible non-contragredient \( \mathfrak{a} \)-submodules of \( \mathcal{L} \). Let \( \mathcal{L}_j = \mathcal{C}_r \odot \mathcal{L}_j \) for short, where \( \mathcal{L}_j \) is the \( j \)th component of the canonical gradation.

It follows from Lemma 6.7(ii) that all pairs of \( \mathfrak{a} \)-submodules in \( \mathcal{L} = \mathcal{C}_r \widehat{S}_N \) are non-contragredient, except for the adjacent \( \mathfrak{a} \)-submodules in \( \mathcal{L}_0 = \mathcal{C}_r \odot S_N \). Thus,
we have:

\[ \alpha(a, b) = 0 \text{ if } a \in \Omega_{N,i}, b \in \Omega_{N,j}, \quad i \neq 0 \text{ or } j \neq 0. \]

Taking now \( a \in \Omega_{N,-1}, b \in \Omega_{n,1} \) and \( c \in \Omega_{N,0} \), the cocycle condition

\[ \alpha([a, b], c) + \alpha([b, c], a) + \alpha([c, a], b) = 0 \]

gives \( \alpha([a, b], c) = 0 \). Since \( \Omega_{N,0} = \Omega_{N,-1} + \Omega_{N,1} \), we conclude that \( \alpha = 0 \). Hence all central extensions of \( \Omega_N \) are trivial.

Likewise, \( \alpha \) is zero on any pair of subspaces \( \Omega_{N,i}, \Omega_{N,j} \), unless \( i + j = 0 \), and on the pair \( \Omega_{N,-1}, \Omega'_{n,1} \) (see Lemma 6.5). Choosing \( a \in \Omega_{N,-1}, b \in \Omega'_{n,1}, c \in \Omega_{N,0} \), we obtain, as above, from the cocycle condition, that \( \alpha \) is zero on the pair \( \Omega_{N,0}, [\Omega_{N,0}, \Omega_{N,0}] \). It follows from (iv) applied to the subalgebra \( \mathcal{O}_r \otimes \mathfrak{sl}(N+1) \) of \( \Omega_N \) (see Lemma 6.5) that \( \alpha \) is zero on this subalgebra if \( N > 1 \). Thus any cocycle on \( \Omega_N (N > 1) \) is trivial. In the case of \( \Omega_1 \) the cocycle \( \alpha \) is trivial. The case of \( \Omega_N \) is similar.

In the remaining case of \( \Omega_N \) we show, as above, that the cocycle \( \alpha \) is trivial on any pair \( \Omega_{N,i}, \Omega_{N,j} \) if \( i \neq j \). Using the cocycle condition for \( a \in \Omega_{N,k}, b \in \Omega'_{N,k+1} \) and \( c \in \Omega_{N,-1} \), and the fact that \( \Omega_{N,k} = [\Omega_{N,k+1}, \Omega_{N,-1}] \), we conclude that \( \alpha \) is trivial on any pair \( \Omega_{N,i}, \Omega_{N,j} \) as well, unless \( i = -1 \). It is easy to see that this implies that \( \Omega_N = \Omega_N^* \).

\[ \square \]

7. \( H \)-Pseudoalgebras and \( H \)-Differential Algebras

In this section, \( H \) will be a cocommutative Hopf algebra, and as before, \( X = H^* \).

7.1. The Annihilation Algebra. Let \( Y \) be an \( H \)-bimodule which is a commutative associative \( H \)-differential algebra both for the left and for the right action of \( H \) (see (2.8), (2.19)); for example, \( Y = X := H^* \).

For a left \( H \)-module \( L \), let \( \mathbb{A}_Y L = Y \otimes_H L \). We define a left action of \( H \) on \( \mathbb{A}_Y L \) in the obvious way:

\[ (1) \quad h(x \otimes_H a) = hx \otimes_H a, \quad h \in H, \; x \in Y, \; a \in L. \]

If, in addition, \( L \) is an \( H \)-pseudoalgebra with a pseudo-product \( a \ast b \), we can define a product on \( \mathbb{A}_Y L \) by the formula:

\[ (2) \quad (x \otimes_H a)(y \otimes_H b) = \sum_i (x f_i)(y g_i) \otimes_H e_i, \]

\[ \text{if} \quad a \ast b = \sum_i (f_i \ast g_i) \otimes_H e_i. \]

By (2.19) and the \( H \)-bilinearity (3.23) of the pseudo-product, it is clear that (7.2) is well defined.

**Proposition 7.1.** If \( L \) is a Lie \( H \)-pseudoalgebra, then \( \mathbb{A}_Y L \) is a Lie \( H \)-differential algebra, i.e., a Lie algebra which is also a left \( H \)-module so that

\[ (3) \quad h[\alpha, \beta] = [h(\alpha), h(\beta)], \quad \text{for } h \in H, \; \alpha, \beta \in \mathbb{A}_Y L. \]

Similarly, if \( L \) is an associative \( H \)-pseudoalgebra, then \( \mathbb{A}_Y L \) is an associative \( H \)-differential algebra. A similar statement holds for modules as well: if \( M \) is an \( L \)-module, then \( \mathbb{A}_Y M \) is an \( \mathbb{A}_Y \)-module with a compatible \( H \)-action so that

\[ (4) \quad h(am) = (h(a) \ast h(m)) \quad \text{for } h \in H, \; a \in \mathbb{A}_Y L, \; m \in \mathbb{A}_Y M. \]
Proof. Equation (7.3) follows from (2.8). The skew-commutativity of the bracket (7.2) follows immediately from that of $[a \ast b]$. The proof of the Jacobi identity is straightforward by using (3.25). Let us check for example that the associativity of $L$ is equivalent to that of $\mathcal{A}_Y L$; the case of the Jacobi identity is similar.

We will use the notation from (3.14)-(3.19), and we will write $a_x \equiv x \ominus_H a$ for $a \in L, x \in Y$. Then we want to compute the products $a_x(b_y c_z)$ and $(a_x b_y) c_z$. By definition, if we have (3.17) and (3.18), then

\[ b_y c_z = \sum_i (y h_i)(z l_i) \otimes_H d_i \]

and

\[ a_x(b_y c_z) = \sum_{i,j} (x h_{ij}) \left( ((y h_i)(z l_i))l_{ij} \right) \otimes_H d_i \]
\[ = \sum_{i,j} (x h_{ij})(y h_i l_{ij(I)})l_{ij} z g_{ij} \otimes_H d_i. \]

Similarly, if we have (3.14) and (3.15), then

\[ (a_x b_y) c_z = \sum_{i,j} (x f_i f_{ij(2)}) (y g_i f_{ij(1)}) (z h_{ij}) \otimes_H c_i. \]

Now recalling (3.16) and (3.19), we see that the associativity of $L$ is equivalent to that of $\mathcal{A}_Y L$. \hfill \square

**Definition 7.2.** The $H$-differential algebra $\mathcal{A}(L) \equiv \mathcal{A}_X L := X \ominus_H L$ is called the **annihilation algebra** of the $\mathcal{P}$-algebra $L$. We will write $a_x \equiv x \ominus_H a$ for $a \in L, x \in X$.

**Remark 7.3.** When $L$ is an associative $H$-algebra, one does not need the cocommutativity of $H$ or the commutativity of $Y$ in order to define $\mathcal{A}_Y L$ (cf. Remark 3.10).

**Lemma 7.4.** Let $H = U(\mathfrak{g}) \otimes \mathbb{K} [\Gamma]$, and let $M$ be a left $H$-module. If an element $a \in M$ is $U(\mathfrak{g})$-torsion, i.e., if $ha = 0$ for some $h \in U(\mathfrak{g}) \setminus \{0\}$, then $x \ominus_H a = 0$. In particular, for $H = U(\mathfrak{g})$, we have: $\mathcal{A}(M) \simeq M/\text{Tor}_H M$, where $\text{Tor}_H M$ is the torsion submodule of $M$.

**Proof.** We have $0 = x \ominus_H ha = x h \ominus_H a$ for every $x \in X$. Since the right action of $h$ on $X$ is surjective (see Lemma 6.10), it follows that $x \ominus_H a = 0$ for any $x \in X$. \hfill \square

**7.2. The functor $\mathcal{A}_Y$.** Analyzing the proof of Proposition 7.1, one can notice that the definition of $\mathcal{A}_Y$ is a special case of a more general construction which we describe below.

First, recall that a commutative associative $H$-differential algebra $Y$ is the same as a commutative associative algebra in the pseudotensor category $\mathcal{M}_H^0$ from Example 3.8. We denote by $\mathcal{M}_H^0$ the category of $H$-bimodules, provided with a pseudotensor structure given by (3.3), but with Hom$_H$ there replaced by Hom$_{H \otimes H}$ which means homomorphisms of $H$-bimodules. The composition of polylinear maps in $\mathcal{M}_H^0$ is given again by (3.2). Then $H$-differential algebras $Y$ considered above are exactly the commutative associative algebras in $\mathcal{M}_H^0$.

Instead of one $H$-bimodule $Y$ one can use several: for any collections of objects $Y_i \in \mathcal{M}_H^0$ and $L_i \in \mathcal{M}_H^0$ ($i \in I$) we can consider the left $H$-modules $\mathcal{A}_Y L_i = Y_i \ominus_H L_i$ as objects of $\mathcal{M}_H^0$. Assume we are given polylinear maps $f \in \text{Lin}\{Y_i\}_{i \in I}, Z \in \mathcal{M}_H^0$ and $\phi \in \text{Lin}\{L_i\}_{i \in I}, M \in \mathcal{M}_H^0$. Then we define a polylinear map $f \ominus_H \phi \in \text{Lin}\{Y_i \ominus_H L_i\}_{i \in I}, Z \ominus_H M \in \mathcal{M}_H^0$ as the
following composition:

\[
\oplus_{i \in I} (Y_i \otimes_H L_i) \xrightarrow{\sim} (\oplus_{i \in I} Y_i) \otimes_{H^I} (\oplus_{i \in I} L_i)
\]

\[
\xrightarrow{id \otimes \phi} (\oplus_{i \in I} Y_i) \otimes_{H^I} (H^I \otimes_H M) \xrightarrow{\sim} (\oplus_{i \in I} Y_i) \otimes_H M \xrightarrow{f \otimes id} Z \otimes_H M.
\]

**Proposition 7.5.** The above definition is compatible with compositions of polyno-

ear maps in \(\mathcal{M}^\alpha(H)\), \(\mathcal{M}^\bullet(H)\), and \(\mathcal{M}^I(H)\):

\[
f((\{y_i\}_i) \otimes_H \phi((\{\psi_i\}_i)) = (f \otimes_H \phi)(\{y_i \otimes_H \psi_i\}_i).
\]

The proof of this proposition is straightforward and is left to the reader.

**Corollary 7.6.** Let \((Y, \nu)\) be a commutative associative algebra in \(\mathcal{M}^\alpha(H)\). For

a finite nonempty set \(I\), let \(\nu^{(I)} : Y^{\otimes I} \to Y\) be the iterated multiplication \(\nu \circ id \cdots (\nu \circ id \circ \cdots \circ id)\). Recall that for an object \(L\) in \(\mathcal{M}^\alpha(H)\), we define \(A_Y(L) := Y \otimes_H L\). For a polynomial \(\phi \in \text{Lin}(\{L_i\}_i, M)\) in \(\mathcal{M}^\alpha(H)\), let \(A_Y(\phi) := \nu^{(I)} \otimes_H \phi\). Then \(A_Y\) is a pseudotensor functor from \(\mathcal{M}^\alpha(H)\) to \(\mathcal{M}^\alpha(H)\).

As a special case of this corollary, we obtain Proposition 7.1.

Let us give another application of Proposition 7.5. An instance of an \(H\)-bimodule

is \(H\) itself (however, \(H\) is not \(H\)-differential algebra). The coproduct \(\Delta : H \to H \otimes H\), the evaluation map \(\text{ev} : X \otimes H \to k\), and the isomorphism \(k \otimes H \xrightarrow{\sim} H\) are

all homomorphisms of \(H\)-bimodules, so the composition

\[
\eta : X \otimes H \xrightarrow{id \otimes \Delta} X \otimes H \otimes H \xrightarrow{\text{ev} \otimes \text{id}} k \otimes H \xrightarrow{\sim} H
\]

is a polynomial map in \(\mathcal{M}^\alpha(H)\). Let again \(L\) be a (Lie) pseudoalgebra and \((M, \rho)\) be an \(L\)-module, where \(\rho \in \text{Lin}(\{L_i\}_i, M)\) in \(\mathcal{M}^\alpha(H)\). Then \(\eta \otimes_H \rho \in \text{Lin}(\{X \otimes_H L, H \otimes_H M\}, H \otimes_H M)\) is a polynomial map in \(\mathcal{M}^\alpha(H)\). In other words, we get a homomorphism of \(H\)-modules \(\eta \otimes_H \rho : A(L) \otimes M \to M\). Proposition 7.5 now implies:

**Corollary 7.7.** The above map \(\eta \otimes_H \rho\) provides \(M\) with the structure of an \(A(L)\)-module, and this structure is compatible with that of an \(H\)-module (cf. (7.4)).

For \(a \in L, x \in X\), the action of \(a_x \equiv x \otimes_H a\) on an element \(m \in M\) will be denoted by \(a_x \cdot m\). This defines \(x\)-products \(a_x \cdot m := a_x \cdot m \in M\). When \(M = L\) is the Lie pseudoalgebra with the adjoint action, these will be called \(x\)-brackets and denoted as \([a, b]\). Then all the axioms of (Lie or associative) pseudoalgebras, representations, etc., can be reformulated in terms of the properties of the \(x\)-brackets or products — this will be done in Section 9. Although this may seem a mere tautology, it is more explicit and convenient in some cases.

Finally, let us give two more constructions.

**Example 7.8.** The base field \(k\), with the action \(h \cdot 1 = \varepsilon(h) (h \in H)\), is a commutative associative \(H\)-differential algebra. Then for any \(H\)-pseudoalgebra \(L\), \(A_k \cdot L = k \otimes_H L\) is a Lie \(H\)-differential algebra. Explicitly, \(A_k \cdot L \simeq L / H_+ L\), where \(H_+ = \{h \in H | \varepsilon(h) = 0\}\) is the augmentation ideal. The Lie bracket in \(L / H_+ L\) is given by (cf. (7.2)):

\[
[a \mod H_+ L, b \mod H_+ L] = \sum \varepsilon(f_i) \varepsilon(g_i) \varepsilon \mod H_+ L,
\]

if

\[
[a \ast b] = \sum (f_i \ast g_i \otimes_H \varepsilon_i).
\]
In the case when $\mathfrak{d} = k\mathfrak{d}$ is 1-dimensional, we recover the usual construction $L \mapsto L/\partial L$ that assigns a Lie algebra to any Lie conformal algebra [K2].

Remark 7.9. Let $Y$ be a commutative associative $H$-differential algebra with a right action of $H$, and let $L$ be a Lie $H$-pseudoalgebra. We provide $Y \otimes L$ with the following structure of a left $H$-module:

\begin{equation}
(7.8) \quad h(x \otimes a) = x h_{(-1)} \otimes h_{(0)} a, \quad h \in H, \; x \in Y, \; a \in L.
\end{equation}

Then define a Lie pseudobracket on $Y \otimes L$ by the formula:

\begin{equation}
(7.9) \quad [(x \otimes a) \ast (y \otimes b)] = \sum_i (f_i(1) \otimes g_i(1)) \otimes h ((x f_i(2))(y g_i(2)) \otimes e_i),
\end{equation}

if $[a \ast b]$ is given by (7.7). It is easy to check that (7.9) is well defined and provides $Y \otimes L$ with the structure of a Lie $H$-pseudoalgebra. Moreover, $A_Y L \simeq (Y \otimes L)/H \phi(Y \otimes L)$ as a Lie algebra (cf. Example 7.8).

In the case $\mathfrak{d} = k\mathfrak{d}$, the Lie $k[\mathfrak{d}]$-pseudoalgebra (= conformal algebra) $Y \otimes L$ is known as an affinehization of the conformal algebra $L$ [K2].

7.3. Relation to differential Lie algebras. Fix two positive integers $N, r$ and let $\mathcal{O}_N = k[[t_1, \ldots, t_N]]$, $\mathcal{L} = \mathcal{O}_N \otimes k^r$. A structure of a Lie algebra on $\mathcal{L}$ is called local (and $\mathcal{L}$ is called a local Lie algebra [Ki]) if the Lie bracket is given by matrix bi-differential operators. More explicitly, let $\{e^i\}$ be a basis of $k^r$. Then for any $x, y \in \mathcal{O}_N$, the bracket in $\mathcal{L}$ is given by:

\begin{equation}
(7.10) \quad [x \otimes e^i, y \otimes e^j] = \sum_{k,l} (P^{ij}_{kl} \cdot x)(Q^{ij}_{kl} \cdot y) \otimes e^k,
\end{equation}

where $P^{ij}_{kl}, Q^{ij}_{kl}$ are differential operators with coefficients in $\mathcal{O}_N$. The number $r$ is called the rank of $\mathcal{L}$.

A related notion is that of a differential Lie algebra [R1]-[R4] (see also [C]). This is a Lie algebra structure on $\mathcal{L} = Y \otimes k^r$, where $Y$ is any commutative associative $H = k[\partial_1, \ldots, \partial_N]$-differential algebra, given by (7.10) for $x, y \in Y$. $P^{ij}_{kl}, Q^{ij}_{kl} \in Y \otimes H$. One can allow a universal enveloping algebra $H = U(\mathfrak{d})$ (dim$\mathfrak{d} = N$) in place of $k[\partial_1, \ldots, \partial_N]$, cf. [NW].

Recall that for $H = U(\mathfrak{d})$, $X = H^*$ is a commutative associative $H$-differential algebra that can be identified with $\mathcal{O}_N$ for $N = \dim \mathfrak{d}$. Moreover, the action of $H$ (and of $X \otimes H$) on $X$ is given by differential operators in this identification. Therefore a differential Lie algebra for $Y = X$ is the same as a local Lie algebra.

Then the results of Section 7.1 immediately imply:

Proposition 7.10. Let $L = H \otimes k^r$ be a Lie pseudoalgebra which is a free $H$-module of rank $r$. Let $Y$ be an $H$-bimodule which is a commutative associative $H$-differential algebra both for the left and for the right action of $H$ (see (2.8), (2.19)). Then $A_Y L \simeq Y \otimes k^r$ is a differential Lie algebra. In particular, $A_L(L) = A_X L$ is a local Lie algebra.

Note that the differential Lie algebras $A_Y L$ that we get are with “constant coefficients”: in (7.10) all $P^{ij}_{kl}, Q^{ij}_{kl} \in H$.

7.4. Topology on the annihilation algebra. Now let us discuss the problem of defining a topology on $A(M) = X \otimes_H M$ where $M$ is any finite $H$-module. Recall that $X$ has a decreasing filtration $X = F_{-1} X \supseteq F_0 X \supseteq \cdots$ defined in Section 2. We can use this filtration to construct an induced filtration on $A(M)$ as follows.
Choose a finite-dimensional (over \( k \)) subspace \( M_0 \) of \( M \) which generates \( M \) over \( H \), and set:

\[
F_i \mathcal{A}(M) = \{ x \otimes_H m \mid x \in F_i X, \ m \in M_0 \}.
\]

Note that since \( H \) is cocommutative, its filtration satisfies (2.15), hence \( \bigcap F_i X = 0 \). This implies:

\[
\bigcap_i F_i \mathcal{A}(M) = 0.
\]

The filtration (7.11) will in general depend on the choice of \( M_0 \), but the topology induced by it will not, as any two such filtrations are equivalent by the next lemma.

**Lemma 7.11.** Let \( M_0 \) and \( M'_0 \) be two finite-dimensional subspaces of \( M \) generating it over \( H \), and let \{\( F_i \mathcal{A}(M) \), \( F'_i \mathcal{A}(M) \)\} be the corresponding filtrations on \( \mathcal{A}(M) \). Then there exist integers \( a, b \) such that \( F_{i+a} \mathcal{A}(M) \subset F'_i \mathcal{A}(M) \subset F_{i+b} \mathcal{A}(M) \) for all values of \( i \).

**Proof.** Let us choose bases of \( M_0 \) and \( M'_0 \), and let us fix expressions of elements from the first basis as \( H \)-linear combinations of elements from the second basis. Denote by \( a \) the highest degree of the coefficients of all these expressions. Using (2.22), we see that \( F_{i} \mathcal{A}(M) \subset F_{i-a} \mathcal{A}(M) \) for all \( i \). Repeating the same reasoning after switching the roles of \( M_0 \) and \( M'_0 \), we get \( F'_i \mathcal{A}(M) \subset F_{i-b} \mathcal{A}(M) \) for some \( b \) and all \( i \).

**Proposition 7.12.** Let \( H \) be a cocommutative Hopf algebra which satisfies (2.16).

(i) If \( M \) is a finite \( H \)-module, then \( \mathcal{A}(M) \) is a linearly compact topological vector space when provided with the filtration (7.11). The action of \( H \) on \( \mathcal{A}(M) \) is continuous if we endow \( H \) with the discrete topology.

(ii) If \( L \) is a finite Lie \( H \)-pseudoalgebra, then its annihilation algebra \( \mathcal{A}(L) \) is a linearly compact Lie \( H \)-differential algebra, i.e., it is a linearly compact topological vector space and both the Lie bracket and the action of \( H \) are continuous.

A similar statement holds for representations and for associative pseudoalgebras as well.

**Proof.** (i) The linear-compactness follows from Proposition 6.1, since (7.11) a filtration by finite-codimensional subspaces with trivial intersection and \( \mathcal{A}(M) \) is complete with respect to this filtration. The continuity of the \( H \)-action follows from (2.22):

\[
F^i H \cdot F_j \mathcal{A}(M) \subset F_{j-i} \mathcal{A}(M) \quad \text{for all } i, j.
\]

(ii) It only remains to check that the Lie bracket of \( \mathcal{A}(L) \) is continuous. Let \( L_0 \) be a finite-dimensional (over \( k \)) subspace of \( L \) which generates it over \( H \). For \( a, b \in L_0 \), we can write

\[
[a \ast b] = \sum_i (f_i \otimes g_i) \otimes_H \epsilon_i
\]

for some \( f_i, g_i \in H \) and \( \epsilon_i \in L_0 \). Then the Lie bracket in \( \mathcal{A}(L) \), for \( x, y \in X \), is given by:

\[
[x \otimes_H a, y \otimes_H b] = \sum_i (xf_i) (yg_i) \otimes_H \epsilon_i.
\]

We can find a number \( p \) such that all coefficients \( f_i, g_i \in H \) occurring in pseudobrackets of any elements \( a, b \in L_0 \) belong to \( F^p H \). Then equations (2.21), (2.22)
imply:
\[(7.14) \quad [F_i \mathcal{A}(L), F_j \mathcal{A}(L)] \subseteq F_{i+j-s} \mathcal{A}(L) \quad \text{for all } i, j,
\]
where \(s = 2p\). This shows that the Lie bracket is continuous. \(\blacksquare\)

**Lemma 7.13.** Let \(H = U(\mathfrak{d}) \otimes k[\Gamma]\). Then for any nonzero \(h \in \mathfrak{d}\) and for every open subspace \(U\) of \(\mathcal{A}(M)\) there is some \(n\) such that \(h^n U = \mathcal{A}(M)\). In particular, each such \(h\) acts surjectively on \(\mathcal{A}(M)\).

**Proof.** Follows immediately from Lemma 6.10. \(\blacksquare\)

### 7.5. Growth of the annihilation algebra

Let \(M\) be a finite \(H\)-module. Then any choice of a finite-dimensional subspace \(M_0\) generating \(M\) over \(H\) provides \(\mathcal{M} = \mathcal{A}(M)\) with a filtration \(\mathcal{M}_n := \mathfrak{m}_n \otimes_H M_0\).

**Definition 7.14.** For a filtered vector space \(\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots\) we define its *growth* \(gw\mathcal{M}\) to be \(d\) if the function \(n \mapsto \dim \mathcal{M}/\mathcal{M}_n\) can be bounded from above and below by polynomials of degree \(d\).

By Lemma 7.11, a different choice of \(M_0\) would give a uniformly equivalent filtration of the same growth as \(\{\mathcal{M}_n\}\). Hence, we can speak of the growth of \(\mathcal{A}(M)\) independently of the choice of \(M_0\).

**Proposition 7.15.** Let \(H = U(\mathfrak{d})\) be the universal enveloping algebra of a finite-dimensional Lie algebra \(\mathfrak{d}\), and \(M\) be a finitely generated \(H\)-module. Then the growth of \(\mathcal{A}(M)\) is equal to the dimension of \(\mathfrak{d}\).

**Proof.** First of all, notice that we can assume \(M\) is torsion-free, since by Lemma 7.4, \(\mathcal{A}(M) \cong \mathcal{A}(M/\text{Tor} M)\) where \(\text{Tor} M\) is the torsion submodule of \(M\). The proof of the proposition is then based on Lemma 2.2 and the following two lemmas.

**Lemma 7.16.** The map \(A(f) : A(M) \to A(F)\) induced by the inclusion \(f : M \hookrightarrow F\) constructed in Lemma 2.2 is uniformly continuous, i.e., for every \(i\) we have:
\[
F_{i-a} A(F) \subseteq A(f)(F_{i-\mathcal{M}_i}) \subseteq F_{i+b} A(F),
\]
where \(a\) and \(b\) are independent of \(i\).

The same is true for \(A(g) : A(F) \to A(M)\) where \(g\) is the embedding \(g : hF \hookrightarrow M\) from Lemma 2.2.

**Proof.** Let us choose finite-dimensional vector subspaces \(F_0\) of \(F\) generating \(F\) over \(H\), and \(M_0\) of \(M\) generating \(M\) over \(H\) and containing \(hF_0\). Let us also choose a second finite-dimensional vector subspace \(F'_0\) of \(F\) containing \(M_0\) and generating \(F\) over \(H\). We will denote the filtrations induced by these subspaces by \(\{F_i\}\), \(\{\mathcal{M}_i\}\), and \(\{F'_i\}\), respectively.

Up to identifying \(hF\) with \(F\), we have constructed injective maps \(F \xrightarrow{g} M \xrightarrow{f} F\) such that the composition \(fg\) is a multiplication by \(h\). These maps induce maps \(A(F) \xrightarrow{A(g)} A(M) \xrightarrow{A(f)} A(F)\) which are surjective, as one can see by tensoring by \(X\) and using that \(A(T) = 0\) if \(T\) is a torsion \(H\)-module (see Lemma 7.4).

The above maps are also continuous with respect to the common topology defined by any of the above constructed filtrations. In fact, by construction, one has
\[
A(g)(F_i) \subseteq \mathcal{M}_i \quad \text{and} \quad A(f)(\mathcal{M}_i) \subseteq F'_i.
\]

The second inclusion proves that \(A(f)(\mathcal{M}_i) \subseteq F_{i+b}\) for some \(b\) independent of \(i\), because the filtrations \(\{F_i\}\) and \(\{F'_i\}\) are uniformly equivalent by Lemma 7.11.
Applying $A(f)$ to the first inclusion, we get $A(f)A(g)(F_{i}) \subseteq A(f)(M_{i})$. On the other hand, $A(f)A(g) = A(fg) = h \otimes_{H} \text{id}_{F}$, and Lemma 6.10 implies that $A(f)A(g)(F_{i}) = F_{i-a}$ where $a$ is such that $h \in F^{a-1} H$ but $h \notin F^{a-1} H$. Therefore $F_{i-a} \subseteq A(f)(M_{i})$ for all $i$.

A similar argument works for $A(g)$.

\textbf{Lemma 7.17.} If $\varphi: M \to N$ is a surjective uniformly continuous map of filtered modules, then $\text{gw} M \geq \text{gw} N$.

\textit{Proof.} By assumption $\varphi(M_{i}) \subseteq N_{i+b}$ for all $i$ and some $b$ independent of $i$. The induced map $M/M_{i} \to N/N_{i+b}$ is a surjective map of finite-dimensional vector spaces. Hence $\text{gw} M \geq \text{gw} N$. \hfill \square

Using the above lemmas, now we can complete the proof of Proposition 7.15. We have constructed an embedding $f: M \hookrightarrow F$ of $M$ into a free $H$-module $F$, and we have shown that the induced map $A(f): A(M) \to A(F)$ is surjective and uniformly continuous. This implies $\text{gw} A(M) \geq \text{gw} A(F)$. Similarly, the inclusion $hF \hookrightarrow M$ gives us $\text{gw} A(F) = \text{gw} A(hF) \geq \text{gw} A(M)$. Therefore, $\text{gw} A(M) = \text{gw} A(F)$. It remains to note that, for any (nonzero) free $H$-module $F$ of finite rank, one has $\text{gw} A(F) = \dim \mathfrak{d}$. This follows from the fact that $\text{gw} X = N = \dim \mathfrak{d}$ because $X \simeq \mathcal{O}_{N}$.

\textbf{Remark 7.18.} The growth of a linearly compact Lie algebra $\mathcal{L}$ satisfying the descending chain condition can be defined as follows. Take a fundamental subalgebra $A \subseteq \mathcal{L}$, and build a filtration of $\mathcal{L}$ by:

$$L_{0}^{A} = A, \quad L_{i+1}^{A} = \{ x \in L_{i}^{A} \mid [x, \mathcal{L}] \subseteq L_{i}^{A} \}, \quad i \geq 0.$$ 

Taking $A' = L_{k}^{A}$, we have: $L_{i}^{A'} = L_{i+k}$, hence replacing $A$ by $A'$ does not change the growth. Therefore, by the Chevalley principle [G1], the growth of this filtration does not depend on the choice of $A$. We will denote this common growth by $\text{gw} \mathcal{L}$.

Notice that all simple linearly compact Lie algebras satisfy the descending chain condition, and therefore have a well defined growth which equals $N$ for $W_{N}, S_{N}, H_{N}$ and $K_{N}$, and $0$ for finite-dimensional Lie algebras.

\section{8. Primitive Pseudoealgebras of Vector Fields}

In this section, $\mathfrak{d}$ will be a (finite-dimensional) Lie algebra and $H = U(\mathfrak{d})$ will be its universal enveloping algebra. As usual, we will identify $\mathfrak{d}$ with its image in $H$.

Then $X := H^{*}$ is the algebra of formal power series on $\mathfrak{d}^{*}$, which is isomorphic as a topological algebra to $\mathcal{O}_{N}$ for $N = \dim \mathfrak{d}$. In this section we are going to define $H$-pseudoealgebra analogs of the primitive linearly compact Lie algebras $W_{N}, S_{N}, H_{N}, K_{N}$, which will be called \textit{primitive pseudoealgebras of vector fields}.

\subsection{8.1. $W(\mathfrak{d})$.}

Let $Y$ be a commutative associative algebra on which $\mathfrak{d}$ acts by derivations from the right (i.e., $Y$ is an $H$-differential algebra). One can define a left action of $Y \otimes \mathfrak{d}$ on $Y$ using the right action of $\mathfrak{d}$ on $Y$:

$$\begin{align*}
(x \otimes a)z &= -(za), \\
x, z \in Y, \quad a \in \mathfrak{d}.
\end{align*}$$

This will define a representation of $Y \otimes \mathfrak{d}$ in $Y$ if the Lie bracket of $\mathfrak{d}$ is extended to $Y \otimes \mathfrak{d}$ by the formula

$$\begin{align*}
[x \otimes a, y \otimes b] &= xy \otimes [a, b] - x(ya) \otimes b + (xb)y \otimes a.
\end{align*}$$
In particular, for \( Y = X = H^* \), this gives the Lie algebra of all vector fields on \( X \), which is isomorphic to \( W_N \) for \( N = \dim \mathfrak{d} \).

Comparing (8.2) with (7.2), we are led to define the pseudoalgebra \( W(\mathfrak{d}) = H \odot \mathfrak{d} \) with pseudobracket

\[
[(f \odot a) * (g \odot b)] = (f \odot g) \odot_H (1 \odot [a, b]) \\
= (f \odot ga) \odot_H (1 \odot b) + (fb \odot g) \odot_H (1 \odot a).
\]

It is easy to check that \( W(\mathfrak{d}) \) is indeed a Lie pseudoalgebra, and that the Lie algebra \( \mathcal{A}_Y W(\mathfrak{d}) \) defined in Section 7 is isomorphic to \( Y \odot \mathfrak{d} \) with bracket defined by (8.2).

In a similar fashion, the module \( Y \) over \( Y \odot \mathfrak{d} \), defined by (8.1), leads to a structure of a \( W(\mathfrak{d}) \)-module on \( H 

\[
(f \odot a) * g = -(f \odot ga) \odot_H 1.
\]

8.2. Differential forms. We can think of \( X = H^* \) as the space of functions on \( \mathfrak{d} \), and of the elements of \( X \odot \mathfrak{d} \) as vector fields. Then the space of \( n \)-forms \( (n = 0, 1, \ldots) \) is

\[
\Omega^n_X := \text{Hom}_k(\bigwedge^n \mathfrak{d}, X) \simeq X \odot \bigwedge^n \mathfrak{d}^*.
\]

It is convenient to extend the elements \( \omega \in \Omega^n_X \) to functions from \( \bigwedge^n (X \odot \mathfrak{d}) \) to \( X \), polynomial over \( X 

\[
\omega(x_1 \odot a_1 \wedge \cdots \wedge x_n \odot a_n) = x_1 \cdots x_n \omega(a_1 \wedge \cdots \wedge a_n),
\]

so that

\[
\Omega^n_X = \text{Hom}_X(\bigwedge^n (X \odot \mathfrak{d}), X).
\]

We view \( X \) as a left \( (X \odot \mathfrak{d}) \)-module using the right action of \( \mathfrak{d} \), see (8.1). There is a differential \( d: \Omega^n_X \to \Omega^{n+1}_X \) satisfying \( d^2 = 0 \); this is just the usual differential for the cohomology of \( \mathfrak{d} \) with coefficients in \( X \) where \( X \) is viewed as a right \( \mathfrak{d} \)-module:

\[
|d\omega|(a_1 \wedge \cdots \wedge a_n) = \sum_{i<j}(-1)^{i+j}\omega([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge a_n)
\]

\[
+ \sum_i(-1)^i\omega(a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge a_n) \cdot a_i.
\]

The following analogue of the Poincaré Lemma is very useful.

**Lemma 8.1.** The cohomology of the complex \( (\Omega^*_X, d) \) is trivial.

**Proof.** Follows from the fact that the homology of \( \mathfrak{d} \) with coefficients in \( U(\mathfrak{d}) \) is trivial — see e.g. [Fu]. \( \square \)

For a vector field \( A \in X \odot \mathfrak{d} \), we have the contraction operator \( \iota_A: \Omega^n_X \to \Omega^{n-1}_X \) given by:

\[
(\iota_A \omega)(a_1 \wedge \cdots \wedge a_{n-1}) = \omega(A \wedge a_1 \wedge \cdots \wedge a_{n-1}).
\]

We define the Lie derivative \( L_A: \Omega^n_X \to \Omega^n_X \) by Cartan's formula \( L_A = d\iota_A + \iota_A d \).

Explicitly, for \( x \odot a \in X \odot \mathfrak{d} \), we have:

\[
(L_{x \odot a} \omega)(a_1 \wedge \cdots \wedge a_n) = -x(\omega(a_1 \wedge \cdots \wedge a_n) \cdot a)
\]

\[
+ \sum (-1)^i(x \cdot a_i) \omega(a_1 \wedge a_i \wedge \cdots \wedge a_n)
\]

\[
+ \sum (-1)^i x \omega([a_i, a_i] \wedge a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge a_n).
\]
The Lie derivative provides each $\Omega^*_X$ with the structure of a module over the Lie algebra of vector fields $X \odot \mathfrak{d}$.

For $n = 0$, $\Omega^0_X = X$ and this is the usual action (8.1) of $X \odot \mathfrak{d}$ on $X$. When $n = N = \dim \mathfrak{d}$, we have $\Omega^N_X = X v_0$ where $v_0 \in \bigwedge^N \mathfrak{d}^*$, $v_0 \neq 0$ is a volume form. An easy calculation shows that

$$L_{x \odot \mathfrak{d}}(y v_0) = -((x y) (a + \text{tr ad } a)) v_0, \quad x, y \in X, \quad a \in \mathfrak{d}. \tag{8.6}$$

8.3. **Pseudoforms.** The module $\Omega^*_X$ over the Lie algebra $X \odot \mathfrak{d}$ leads to a module $\Omega^n(\mathfrak{d})$ over the Lie pseudoalgebra $W(\mathfrak{d})$ which we now define. We let

$$\Omega^n(\mathfrak{d}) = H \odot \bigwedge^n \mathfrak{d}^*, \quad \Omega(\mathfrak{d}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathfrak{d}) \quad (N = \dim \mathfrak{d}).$$

The elements of $\Omega(\mathfrak{d})$ are called *pseudoforms*.

$\Omega^n(\mathfrak{d})$ is a free $H$-module, so that $A(\Omega^n(\mathfrak{d})) = X \otimes_H \Omega^n(\mathfrak{d}) \cong X \otimes \bigwedge^n \mathfrak{d}^* = \Omega^n_X$.

The action of $W(\mathfrak{d}) = H \odot \mathfrak{d}$ on $\Omega^n(\mathfrak{d})$ is obtained by comparing (7.2) with (8.5). To write an explicit formula, we identify $\Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n \mathfrak{d}$ to $H$, and $(H \otimes \odot) \otimes_H \Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n \mathfrak{d}$ to $H \otimes H$. Then for $f \odot a \in W(\mathfrak{d})$, $w \in \Omega^n(\mathfrak{d})$, and $a_i \in \mathfrak{d}$, we have:

$$(f \odot a) \ast w(a_1 \wedge \cdots \wedge a_n) = -f \odot w(a_1 \wedge \cdots \wedge a_n) a + \sum_i (-1)^i f a_i \odot w(a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n)$$

$$+ \sum_i (-1)^i f \odot w([a, a_i] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n). \tag{8.7}$$

When $n = 0$, $\Omega^0(\mathfrak{d}) = H$, and we recover (8.4). In the other extreme case, when $n = N := \dim \mathfrak{d}$, $\Omega^N(\mathfrak{d}) = H v_0$ is again a free $H$-module of rank one, where $v_0 \in \bigwedge^N \mathfrak{d}^*$, $v_0 \neq 0$ is any volume form on $\mathfrak{d}$. We have (cf. (8.6)):

$$(f \odot a) \ast v_0 = -(f(a + \text{tr ad } a) \odot 1 + f \odot a) \odot_H v_0. \tag{8.8}$$

Define polylinear maps $s_i \in \text{Lin}(\{W(\mathfrak{d}), \Omega^n(\mathfrak{d})\}, \Omega^{n-1}(\mathfrak{d}))$ by

$$(f \odot a) \ast s_i w(a_1 \wedge \cdots \wedge a_{n-1}) = f \odot w(a \wedge a_1 \wedge \cdots \wedge a_{n-1}). \tag{8.9}$$

Also define a differential $d: \Omega^n(\mathfrak{d}) \to \Omega^{n+1}(\mathfrak{d})$ by

$$(dw)(a_1 \wedge \cdots \wedge a_{n+1}) = \sum_{i < j} (-1)^{i+j} w([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_{n+1})$$

$$+ \sum_i (-1)^i w(a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_{n+1}) a_i \quad \text{if } n > 1,$$

$$(dw)(a) = -wa \quad \text{if } n = 0,$$

so that $d$ is $H$-linear and $d^2 = 0$. For any pseudoform $w$ and $x \in X$, we have a differential form $x \odot_H w$ and the relation $d(x \odot w) = x \odot_H dw$.

Then we have the following analogue of Cartan’s formula for the action of $W(\mathfrak{d})$ on $\Omega(\mathfrak{d})$:

$$(8.10) \quad \alpha \ast w = ((\text{id} \otimes \text{id}) \otimes_H d)(\alpha \ast w) + \alpha \ast (dw) \in (H \otimes H) \otimes_H \Omega(\mathfrak{d}).$$

This implies that the action of $W(\mathfrak{d})$ commutes with $d$:

$$(8.11) \quad \alpha \ast d(w) = ((\text{id} \otimes \text{id}) \otimes_H d)(\alpha \ast w).$$

We note that the maps $\alpha \ast$, anticommute with each other:

$$(8.12) \quad \alpha \ast (\beta \ast w) + ((\sigma \otimes \text{id}) \otimes_H \text{id}) \beta \ast (\alpha \ast w) = 0$$
for $\alpha, \beta \in W(\mathfrak{d})$, $w \in \Omega(\mathfrak{d})$.

The wedge product on $\bigwedge^* \mathfrak{d}^*$ can be extended to a pseudoproduct $*$ on $\Omega(\mathfrak{d}) = H \otimes \bigwedge^* \mathfrak{d}^*$, so that it becomes a current pseudogealgebra. Then it is easy to check that for $\alpha \in W(\mathfrak{d})$, $v \in \bigwedge^m \mathfrak{d}^*$, $w \in \bigwedge^n \mathfrak{d}^*$, one has:

\begin{equation}
(8.13) \quad \alpha * (v \wedge w) = (\alpha * v) \wedge w + ((\sigma \otimes \text{id}) \otimes_H \text{id}) v \wedge (\alpha * w),
\end{equation}

and similarly,

\begin{equation}
(8.14) \quad \alpha *_s (v \wedge w) = (\alpha *_s v) \wedge w + (-1)^m((\sigma \otimes \text{id}) \otimes_H \text{id}) v \wedge (\alpha *_s w).
\end{equation}

This can be interpreted as saying that $\alpha *$ and $\alpha *_s$ are superderivations of $\Omega(\mathfrak{d})$, see Example 10.10 below.

8.4. $S(\mathfrak{d}, \chi)$. The divergence of a vector field $\sum x_i \otimes a_i \in X \otimes \mathfrak{d}$ is defined by $\text{div}(\sum x_i \otimes a_i) = \sum x_i a_i \in X$. Then one easily checks

\begin{equation}
(8.15) \quad \text{div}([A, B]) = A \cdot \text{div}(B) - B \cdot \text{div}(A), \quad A, B \in X \otimes \mathfrak{d},
\end{equation}

so that the divergence zero vector fields form a Lie subalgebra $S_X$ of $W_X$. Let $\chi$ be a trace form on $\mathfrak{d}$, i.e., a linear functional from $\mathfrak{d}$ to $k$ which vanishes on $[\mathfrak{d}, \mathfrak{d}]$. Then we can define

$$\text{div}^X(\sum x_i \otimes a_i) := \sum x_i (a_i + \chi(a_i)),$$

which still satisfies (8.15).

Remark 8.2. Let $\chi$ be as above, and let $\psi = \chi - \text{tr} \text{ad}$, which is again a trace form on $\mathfrak{d}$. We can consider $\psi$ as an element of $\Omega^1_X = X \otimes \mathfrak{d}^*$; then $d\psi = 0$ and by Lemma 8.1 we have $\psi = -dz$ for some $z \in X$. This means that $\psi(a) = za$ for all $a \in \mathfrak{d}$. Let $y = e^z$; then $ya = y\psi(a)$ for $a \in \mathfrak{d}$. Consider the volume form $v = yv_0$, where $v_0 \in \bigwedge^N \mathfrak{d}^*$, $v_0 \neq 0$. Equation (8.6) gives

\begin{equation}
(8.16) \quad LA v = -\text{div}^X(A)v \quad \text{for} \quad A \in X \otimes \mathfrak{d}.
\end{equation}

Therefore, the Lie algebra of vector fields $A$ with $\text{div}^X(A) = 0$ coincides with the Lie algebra $S_X(v)$ of vector fields annihilating the volume form $v$.

Using the notation $a_x \equiv x \otimes_H a \in A(W(\mathfrak{d})) \simeq X \otimes \mathfrak{d}$ for $a \in W(\mathfrak{d})$, $x \in X$, we find for $a = h \otimes a$:

$$a_x = x \otimes_H (h \otimes a) = xh \otimes_H (1 \otimes a) \equiv xh \otimes a \in X \otimes \mathfrak{d},$$

hence, $\text{div}^X(a_x) = xh(a + \chi(a))$. Define the divergence operator $\text{div}^X: W(\mathfrak{d}) \to H$ by the formula:

\begin{equation}
(8.17) \quad \text{div}^X(\sum h_i \otimes a_i) = \sum h_i (a_i + \chi(a_i)).
\end{equation}

Then we have:

\begin{equation}
(8.18) \quad \text{div}^X(a_x) = x \cdot \text{div}^X a \quad \text{for} \quad a \in W(\mathfrak{d}), x \in X.
\end{equation}

Since $\text{div}^X$ is $H$-linear, we can define

$$\text{div}^X: (H \otimes H) \otimes_H W(\mathfrak{d}) \xrightarrow{\text{id} \otimes_H \text{div}^X} (H \otimes H) \otimes_H H \xrightarrow{\sigma} H \otimes H.$$

Similarly to (8.15), one has:

\begin{equation}
(8.19) \quad \text{div}^X([\alpha \wedge \beta]) = (\text{div}^X \alpha \wedge 1)\sigma(\beta) - (1 \wedge \text{div}^X \beta)\alpha, \quad \alpha, \beta \in W(\mathfrak{d}),
\end{equation}

where $\sigma: H \otimes \mathfrak{d} \to \mathfrak{d} \otimes H$ is the transposition.
Equation (8.19) implies that
\[(8.20) \quad S(\mathfrak{d}, \chi) := \{ a \in W(\mathfrak{d}) \mid \text{div}^\chi a = 0 \}\]
is a subalgebra of the Lie pseudoalgebra \(W(\mathfrak{d})\). By Eq. (8.18) and Remark 8.2, its annihilation algebra
\[(8.21) \quad \mathcal{A}(S(\mathfrak{d}, \chi)) = \{ A \in W_N \mid \text{div}^\chi A = 0 \} \simeq S_N.\]
The rank of \(S(\mathfrak{d}, \chi)\) as an \(H\)-module is \(N - 1\); however, it is free only for \(N = 2\).

**Proposition 8.3.** \(S(\mathfrak{d}, \chi)\) is generated over \(H\) by elements
\[(8.22) \quad e_{ab} := (a + \chi(a)) \otimes b - (b + \chi(b)) \otimes a - 1 \otimes [a, b] \quad \text{for} \quad a, b \in \mathfrak{d}.\]
These elements satisfy \(e_{ab} = -e_{ba}\) and the relations (for \(\chi = 0\)):
\[(8.23) \quad a e_{ca} + b e_{ea} + c e_{ab} = e_{[a, b], c} + e_{[b, c], a} + e_{[c, a], b}.\]
For \(\chi = 0\), their pseudobrackets are given by:
\[(8.24) \quad [e_{ab} \ast e_{cd}] = (a \otimes c) \otimes_H e_{cd} + b \otimes_H (c \otimes_H e_{ad} - (a \otimes c) \otimes_H e_{cd} - (b \otimes d) \otimes_H e_{ac}) + (a \otimes 1) \otimes_H e_{[c, d]} - (b \otimes 1) \otimes_H e_{a, [c, d]} - (1 \otimes c) \otimes_H e_{d, [c, b]} + (1 \otimes d) \otimes_H e_{c, [b, d]} - (1 \otimes 1) \otimes_H e_{[a, b], [c, d]}.\]
For arbitrary \(\chi\), replace everywhere in (8.23), (8.24) all \(h \in \mathfrak{d}\) with \(h + \chi(h)\).

**Remark 8.4.** Equation (8.24) implies that for \(\chi = 0\):
\[(8.25) \quad [e_{ab} \ast e_{ab}] = (b \otimes a - a \otimes b) \otimes_H e_{ab} + (1 \otimes b - b \otimes 1) \otimes_H e_{a, [b, d]} + (a \otimes 1 - 1 \otimes a) \otimes_H e_{b, [a, d]}.\]
(Again, for any \(\chi\), replace \(a, b\) with \(a + \chi(a), b + \chi(b)\).) In particular, when the elements \(a, b\) span a Lie algebra, \(H e_{ab}\) is a Lie pseudoalgebra.

In the proof of Proposition 8.3 we are going to use the following lemma.

**Lemma 8.5.** Let \(H = U(\mathfrak{d})\), and let \(\{\partial_1, \ldots, \partial_N\}\) be a basis of \(\mathfrak{d}\). If elements \(h_i \in F^d H\) are such that \(\sum_i h_i \partial_i \in F^d H\), then there exist \(f_{ij} \in F^{d-1} H\) such that
\[
\sum_i h_i \otimes \partial_i = \sum_{i, j} (f_{ij} \otimes 1)(\partial_i \otimes \partial_j - \delta_j \otimes \partial_i) \mod F^{d-1} H \otimes \mathfrak{d}.
\]

**Proof.** The proof is by induction on the number of \(h_i\) not contained in \(F^{d-1} H\), the basis of induction being trivial. Consider \(\sum_{i=1}^n h_i \partial_i \in F^d H\), with all \(h_i \notin F^{d-1} H\).
We can write \(h_i = f_i \partial_1 + k_i\) so that \(k_i \in F^d H\) is a linear combination of Poincaré-Birkhoff-Witt basis elements of \(H\) not containing \(\partial_1\) in their expression, and \(f_i \in F^{d-1} H\). Then:
\[
\sum_{i=1}^n h_i \partial_i = h_1 \partial_1 + \sum_{i=2}^n (f_i \partial_1 \partial_1 + k_i \partial_i)
= (h_1 + \sum_{i=1}^{n-1} f_i \partial_i) \partial_1 + \sum_{i=2}^n k_i \partial_i + \sum_{i=1}^n f_i [\partial_1, \partial_i].
\]
Since the third summand in the right hand side belongs to $F^d H$, it follows that the first and second summands lie in $F^d H$ too. This implies: $$h_1 + \sum_{i=2}^{n} f_i \partial_i \in F^{d-1} H.$$ Hence
\[\sum_{i=1}^{n} h_i \odot \partial_i = \sum_{i=1}^{n} \left( f_i \partial_i \odot \partial_i - f_i \partial_i \odot \partial_i \right) + \sum_{i=2}^{n} k_i \odot \partial_i \mod F^{d-1} H \odot \mathfrak{a},\]
and we can apply the inductive assumption.

**Proof of Proposition 8.3.** First of all, it is easy to check that the elements (8.22) indeed belong to $S(\mathfrak{a}, \chi)$. Equation (8.23) is easy, and the computation of the pseudobrackets is straightforward using (8.3), reformulated as
\[(1 \odot a) * (1 \odot b) = \left( (a + \chi(a)) \odot 1 \right) \odot_H (1 \odot b) - (1 \odot (b + \chi(b))) \odot_H (1 \odot a) - (1 \odot 1) \odot_H e_{\mathfrak{a}}.\]

Now let us consider an element $a = \sum_i h_i \odot \partial_i \in S(\mathfrak{a}, \chi)$, $h_i \in H$. We will prove that $a$ can be expressed as $H$-linear combination of the above elements (8.22) by induction on the maximal degree $d$ of the $h_i$. Since $a \in S(\mathfrak{a}, \chi)$, then $\sum_i h_i(\partial_i + \chi(\partial_i)) = 0$, hence $\sum_i h_i \partial_i \in F^d H$.

By Lemma 8.5, we can find elements $f_{i,j} \in F^{d-1} H$ such that
\[a = \sum_{i,j} (f_{i,j} \odot 1)(\partial_i \odot \partial_j - \partial_j \odot \partial_i) \mod F^{d-1} H \odot \mathfrak{a}.\]
Therefore the difference
\[a - \sum_{i,j} (f_{i,j} \odot 1)(\partial_i + \chi(\partial_i)) \odot \partial_j - (\partial_j + \chi(\partial_j)) \odot \partial_i - 1 \odot [\partial_i, \partial_j])\]
still lies in $S(\mathfrak{a}, \chi)$ and its first tensor factor terms have degree strictly less than $d$. By inductive assumption, we are done.

**Remark 8.6.** (i) Let, as before, $\chi \in \mathfrak{a}^*$ be such that $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$. For any $\lambda \in k$, let $V_{\lambda, \chi} = H v$ be a free $H$-module of rank 1 with the following action of $W(\mathfrak{a})$ on it:
\[(8.27)\]
\[a \ast v = (\lambda \text{div}^H a \odot 1 - a) \odot_H v.\]
Using (8.19), it is easy to check that this indeed defines a representation of $W(\mathfrak{a})$.

For $\lambda = 0$ we get the action (8.4), while for $\lambda = -1$, $\chi = \text{tr} \text{ad}$ we get (8.8). One can show that all representation of $W(\mathfrak{a})$ on a free $H$-module of rank 1 are given by
\[(8.28)\]
\[1 \odot a \ast v = ((\lambda a + \chi'(a)) \odot 1 - 1 \odot a) \odot_H v,\]
where $a \in \mathfrak{a}$, $\lambda \in k$ and $\chi'$ is a trace form on $\mathfrak{a}$. This can be rewritten as in (8.27), for $\chi = \chi' / \lambda$ whenever $\lambda \neq 0$.

(ii) More generally, let $M$ be any $W(\mathfrak{a})$-module, equipped with a compatible action of $H = \text{Cur} k$. Here $H = \text{Cur} k$ is the associative pseudoalgebra with a pseudoproduct $f \ast g = (f \odot g) \odot_H 1$, and compatibility of the actions of $W(\mathfrak{a})$ and $H$ means that
\[(8.29)\]
\[a \ast (h \ast m) - (\sigma \odot \text{id}) \odot_H (a \ast m) = (a \ast h) \ast m\]
for $a \in W(\mathfrak{a})$, $h \in H$, $m \in M$, where $a \ast h = -(1 \odot h)a \odot_H 1$ is the action (8.4) of $W(\mathfrak{a})$ on $H$. 

Then, for any $\lambda, \chi$ as above,

$$a \ast_{\lambda, \chi} m = \lambda(\text{div} \, \alpha) \ast m + \alpha \ast m$$

is an action of $W(\mathfrak{D})$ on $M$.

8.5. **Pseudoalgebras of rank** 1. All Lie pseudoalgebras that are free of rank one over $H$ were described by Proposition 4.1 and Lemma 4.2. The next lemma implies that all of them are subalgebras of $W(\mathfrak{D})$.

**Lemma 8.7.** Let $\alpha \in H \otimes H$ be a solution of equations (4.1), (4.2). Write $\alpha = r + s \otimes 1 - 1 \otimes s$ with a skew-symmetric $r \in \mathfrak{D} \otimes \mathfrak{D}$ and $s \in \mathfrak{D}$, as in Lemma 4.2. Consider $\iota = -r + 1 \otimes s \in H \otimes \mathfrak{D}$ as an element of $W(\mathfrak{D})$. Then $[\iota * \iota] = \alpha \otimes_H \iota$ in $W(\mathfrak{D})$.

**Proof.** Straightforward computation, using the definition (8.3) and equations (4.3), (4.4).

Let us study equations (4.3, 4.4) in more detail. We can write

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i)$$

for some linearly independent $a_i, b_i \in \mathfrak{D}$. Denote by $\mathfrak{D}_1$ their linear span, and let $\mathfrak{D}_0 = \mathfrak{D}_1 + \mathfrak{K}_s$.

**Lemma 8.8.** $\mathfrak{D}_0$ is a Lie subalgebra of $\mathfrak{D}$, and $\mathfrak{D}_1$ is ad-s-invariant. Moreover, $[a_i, a_j], [b_i, b_j], [a_i, b_j]$ and $[a_i, b_i] + s$ belong to $\mathfrak{D}_1$ for $i \neq j$.

**Proof.** Similar to that of Proposition 2.2.6 in [CP]. If $\mathfrak{D}_1 = \mathfrak{D}$, there is nothing to prove. Let $\{c_j\}$ be elements that complement $\{a_i, b_i\}$ to a basis of $\mathfrak{D}$. If $s$ is not in $\mathfrak{D}_0$ we take it to be one of the $c_j$'s.

Write out (4.3) as

$$\sum_i (a_i \otimes s) \otimes b_i - [b_i, s] \otimes a_i + a_i \otimes [b_i, s] - b_i \otimes [a_i, s] = 0.$$  

Now, if $[a_i, s]$ involves some $c_j$'s, there is no way to cancel out the terms $c_j \otimes b_i$. This proves that $[s, \mathfrak{D}_1] \subset \mathfrak{D}_1$.

Similarly, (4.4) reads

$$\sum_{i,j} ([a_i, a_j] \otimes b_i \otimes b_j - [b_i, a_j] \otimes a_i \otimes b_j + [b_i, b_j] \otimes a_i \otimes a_j - [a_i, b_j] \otimes b_i \otimes a_j + \text{cyclic})$$

$$+ \sum_i (a_i \otimes b_i \otimes s - b_i \otimes a_i \otimes s + \text{cyclic}) = 0.$$  

If, for example, $[a_i, a_j]$ involves $c_k$'s, then the terms $c_k \otimes b_i \otimes b_j$ cannot be cancelled. Therefore $[a_i, a_j] \in \mathfrak{D}_1$. If $[a_i, b_j]$ involves $c_k$'s, then the terms $c_k \otimes b_i \otimes a_j$ can be cancelled only with terms coming from $s \otimes r$. This shows that $[a_i, b_j] + \delta_{ij}s \in \mathfrak{D}_1$.

The universal enveloping algebra $H_0 = U(\mathfrak{D}_0)$ is a Hopf subalgebra of $H = U(H)$. Since $\alpha \in H_0 \otimes H_0$, we can consider the Lie pseudoalgebra $H_0 \mathfrak{e}$ with pseudobracket $[\iota \ast \iota] = \alpha \otimes_H \iota$. Then our pseudoalgebra $H \mathfrak{e}$ is a current pseudoalgebra over $H_0 \mathfrak{e}$.

Clearly, $\mathfrak{D}_1$ is even dimensional. There are two cases which are treated in detail in the next two subsections: when $\mathfrak{D}_0 = \mathfrak{D}_1$ and when $\mathfrak{D}_0 = \mathfrak{D}_1 \oplus \mathfrak{K}_s$. They give rise to Lie pseudoalgebras $H(\mathfrak{D}, \chi, \omega), K(\mathfrak{D}, \theta)$ whose annihilation Lie algebras are of hamiltonian and contact type, respectively. The following theorem summarizes some of the results of Sections 4.3 and 8.5–8.7.

**Theorem 8.6.** Any Lie pseudoalgebra which is free of rank one is either abelian or isomorphic to a current pseudoalgebra over one of the Lie pseudoalgebras $H(\mathfrak{D}, \chi, \omega), K(\mathfrak{D}, \theta)$ defined in Sections 8.6, 8.7, respectively.
8.6. $H(\mathfrak{d}, \chi, \omega)$. This is defined as a Lie $H$-pseudoalgebra of rank 1 (see Section 8.5) corresponding to a solution $(r, s)$ of equations (4.3), (4.4) with a nondegenerate $r \in \mathfrak{d} \wedge \mathfrak{d}$ (i.e., $\mathfrak{d}_1 = \mathfrak{d}$), in which case $N = \dim \mathfrak{d}$ is even. The parameters $\chi$ and $\omega$ are defined as follows.

Since $r$ is nondegenerate, the linear map $\mathfrak{d}^* \to \mathfrak{d}$ induced by it is invertible; its inverse gives rise to a 2-form $\omega \in \bigwedge^2 \mathfrak{d}^*$. Explicitly, if $r = \sum r^i j \mathfrak{d}_j$ where \{$\mathfrak{d}_i$\} is a basis of $\mathfrak{d}$, then $\omega(\mathfrak{d}_i \wedge \mathfrak{d}_j) = \omega_{ij}$ is the matrix inverse to $r^{ij}$. We also define a 1-form $\chi := i_r \omega \in \mathfrak{d}^*$.

Conversely, given a nondegenerate skew-symmetric 2-form $\omega$ and a 1-form $\chi$, we can define uniquely $r \in \mathfrak{d} \wedge \mathfrak{d}$ as the dual to $\omega$ and $s \in \mathfrak{d}$ so that $\chi = i_r \omega$.

**Lemma 8.10.** When $r \in \mathfrak{d} \wedge \mathfrak{d}$ is nondegenerate, equations (4.3), (4.4) are equivalent to the following identities for the above defined $\omega$, $\chi$:

\begin{align}
(8.32) & \quad \mathrm{d}\omega + \chi \wedge \omega = 0, \\
(8.33) & \quad \mathrm{d}\chi = 0,
\end{align}

which simply mean that $\omega$ is a 2-cocycle for $\mathfrak{g}$ in the one dimensional $\mathfrak{g}$-module defined by $\chi$. This establishes a one-to-one correspondence between solutions $(r, s)$ of (4.3), (4.4) with nondegenerate $r$ and solutions $(\omega, \chi)$ of (8.32), (8.33) with nondegenerate $\omega$.

**Proof.** Let us write $[\mathfrak{d}_i, \mathfrak{d}_j] = \sum \delta^k_{ij} \mathfrak{d}_k$ and $s = \sum s^k \mathfrak{d}_k$ (summation over repeated indices). Then (4.4) is equivalent to

\begin{equation}
(8.34) \quad \left( \sum r^{ik} r^{mj} e^{ik} + r^{mj} s^l \right) + \text{cyclic} = 0,
\end{equation}

where “cyclic” means summing over cyclic permutations of the indices $m, j, l$. Multiply this equation by $\omega_{jn} \omega_{pq} \omega_{mq}$ and sum over $m, j, l$. Using that $\sum r^{ij} \omega_{jn} = \delta^k_{ij}$, we get

\begin{equation}
(8.35) \quad \left( \sum r^{ij} \omega_{pq} \right) + \text{cyclic} = 0,
\end{equation}

where now the cyclic permutations are over $n, p, q$. This is exactly Eq. (8.32).

Conversely, multiplying (8.35) by $r^{in} r^{ij} r^{jk}$ and summing over $n, p, q$, we get (8.34).

Similarly, since $[s, \mathfrak{d}_i] \subset \mathfrak{d}_i$, we can write $[s, \mathfrak{d}_i] = \sum \delta^k_{ij} \mathfrak{d}_k$. Then (4.3) is equivalent to

\begin{equation}
(8.36) \quad \sum r^{ij} c^k_i + \sum r^{kl} c^k_j = 0,
\end{equation}

which after multiplying by $\omega_{jm} \omega_{kn}$ and summing over $j, k$ becomes

\begin{equation}
(8.37) \quad \sum r^k_m \omega_{km} + \sum c^k_l \omega_{mj} = 0,
\end{equation}

or $L_r \omega = 0$. Conversely, (8.37) gives (8.36) after multiplying by $r^{im} r^{in}$ and summing over $m, n$.

Now start with a solution $(r, s)$ of (4.3), (4.4). Above we have deduced (8.32) and $L_r \omega = 0$. On the other hand, since $i_r \chi = 0$, we have $i_r (\chi \wedge \omega) = 0$, and (8.32) implies $i_r \mathrm{d}\omega = 0$. Together with $L_r \omega = 0$ this gives $\mathrm{d}\omega = 0$, which is (8.33).

If we start with a solution $(\omega, \chi)$ of (8.32), (8.33), the above arguments can be inverted to show that $L_r \omega = 0$, and we get (4.3, 4.4).

In the basis $\{a_i, b_i\}$ of $\mathfrak{d}$ we have (8.31) and $\omega(a_i \wedge b_i) = -\omega(b_i \wedge a_i) = -1$, all other values of $\omega$ are zero. For $e = -r + 1 \otimes s$ and any $x \in X$, the element $e_x := x \otimes_H e$ of
the annihilation algebra $\mathcal{A}(W(\mathfrak{d})) \cong X \otimes \mathfrak{d}$ is equal to $- \sum(xa_i \otimes b_i - xb_i \otimes a_i) + x \otimes s$, and it is easy to check that

$$(8.38) \quad \omega(\epsilon_x \wedge a) = x(-a + \chi(a)), \quad a \in \mathfrak{d}.$$ 

Since $d\chi = 0$, Lemma 8.1 implies that $\chi = dy$ for some $y \in \Omega_X^0 = X$, i.e., $\chi(a) = -ya$. Then $\hat{\omega} := e^y \omega$ satisfies $\hat{\omega}(\epsilon_x \wedge a) = -(xe^y)a$ for any $x \in X$, $a \in \mathfrak{d}$. This is equivalent to $\iota_{e_x} \hat{\omega} = d(xe^y)$. Moreover, (8.32) implies $d\hat{\omega} = 0$. Therefore, $L_{e_x} \hat{\omega} = 0$, and we have the following proposition.

**Proposition 8.11.** Let $H(\mathfrak{d}, \chi, \omega) := H \mathfrak{d}$ be a Lie $H$-pseudoalgebra of rank 1 corresponding to a solution $(r, s)$ of equations (4.3), (4.4) with a nondegenerate $r \in \mathfrak{d} \otimes \mathfrak{d}$. Define the 2-form $\hat{\omega}$ as above. Then $\hat{\omega}$ is a symplectic form, and the subalgebra $X \otimes_H H(\mathfrak{d}, \chi, \omega)$ of $X \otimes_H W(\mathfrak{d}) \cong X \otimes \mathfrak{d}$ is the Lie algebra $H_N(\hat{\omega})$ of vector fields annihilating $\hat{\omega}$ (which is isomorphic to $H_N$).

**Proof.** It remains to show that, conversely, any vector field that preserves the 2-form $\hat{\omega}$ is equal to $\epsilon_x$ for some $x \in X$. Indeed, let $A \in X \otimes \mathfrak{d}$ be such that $L_A \hat{\omega} = 0$. Since $d\hat{\omega} = 0$ and $\hat{\omega} = e^y \omega$, this is equivalent to $d(e^y L_A \omega) = 0$ which implies $e^y L_A \omega = dz$ for some $z \in X$. In other words, $e^y \omega(A \wedge a) = -za$ for any $a \in \mathfrak{d}$. Using $\chi(a) = -ya$, we get $\omega(A \wedge a) = x(-a + \chi(a))$. This, together with (8.38), implies $A = \epsilon_x$ since the 2-form $\omega$ is nondegenerate. \hfill \Box

**Remark 8.12.** Let $r \in \mathfrak{d} \otimes \mathfrak{d}$ be given by (8.31), and let $x = \sum_i [a_i, b_i]$. Then $H(\mathfrak{d}, \chi, \omega, -r)$ is a free vector space spanned by $[a_i, b_i]$ of rank $n$. (The fact that $d_x \omega = 0$ is equivalent to $L_x \omega = 0$, or, in the proof of Lemma 8.10, to $[\Delta(x), r] = 0$, which is easy to check.)

**Example 8.13.** Let the Lie algebra $\mathfrak{d}$ be 2-dimensional with basis $\{a, b\}$ and commutation relations $[a, b] = \lambda b$. Then up to multiplication by a scalar, all nondegenerate solutions $(r, s)$ of (4.3) are given by: $r = a \otimes b - b \otimes a$, any $s$ in case $\lambda = 0$, and by the same $r$, and $s \in k \lambda b$ when $\lambda \neq 0$. It is immediate to see that in both cases $s$ can be written as $-\phi(a)b + \phi(b)a + [a, b]$ for some trace form $\phi \in \mathfrak{d}^*$. Then $r - 1 \otimes s = \epsilon_{ab}$ is a free generator of $S(\mathfrak{d}, \phi)$, since $\dim \mathfrak{d} = 2$ (see Proposition 8.3). This shows that the above pairs $(r, s)$ also satisfy (4.4). We have: $H(\mathfrak{d}, \chi, \omega, -r) = S(\mathfrak{d}, \phi)$, where $\chi = \iota_s \omega = \phi - \text{tr} \phi$. (Note that $\text{tr} \phi = \iota_s \omega$ for $x = [a, b] = \lambda b$.)

**Example 8.14.** When $\mathfrak{d}$ is abelian of dimension $N = 2n > 2$, then (8.32) and (8.33) become $\chi \wedge \omega = 0$, hence $\chi = 0$ and $\omega$ is any nondegenerate skew-symmetric 2-form. In this case all solutions of (4.3), (4.4) are: $s = 0$ and $r$ given by (8.31) in some basis $\{a_i, b_i\}$ of $\mathfrak{d}$.

**Example 8.15.** When the Lie algebra $\mathfrak{d}$ is simple, there are no solutions $(\omega, \chi)$ of (8.32), (8.33) with a nondegenerate $\omega$. Indeed, since $[\mathfrak{d}, \mathfrak{d}] = \mathfrak{d}$, we have $\chi = 0$, and $\omega$ is a 2-cocycle: $d\omega = 0$. Any 2-cocycle $\omega \in \Lambda^2 \mathfrak{d}^*$ for a simple Lie algebra $\mathfrak{d}$ is degenerate, since $\omega = \alpha \delta$ for some $\alpha \in \mathfrak{d}^*$ and the stabilizer $\mathfrak{d}_\alpha$ of $\alpha$ is always non-zero.

**8.7. $K(\mathfrak{d}, \theta)$.** This is defined as a Lie $H$-pseudoalgebra of rank 1 (see Section 8.5) corresponding to a solution $(r, s)$ of equations (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \otimes k s$ and nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$; in this case $N = \dim \mathfrak{d}$ is odd. The parameter $\theta$ is defined below.
Let $\{\partial_i\}$ be a basis of $\mathfrak{d}_1$, and $r = \sum r^i j \partial_i \otimes \partial_j$. As before, we define a 2-form $\omega$ on $\mathfrak{d}_1$ by $\omega(\partial_i \wedge \partial_j) = \omega_{ij}$, where $\{\omega_{ij}\}$ is the matrix inverse to $(r^i j)$. Let us write 

$[\partial_i, \partial_j] = \sum c^k_{ij} \partial_k + c_{ij} s$ and $[s, \partial_j] = \sum c^k_j \partial_k$. Then we have:

**Lemma 8.16.** With the above notation, equations (4.3), (4.4) are equivalent to the following identities:

\begin{align}
\omega &= 0 & \text{(8.39)}
\end{align}

\begin{align}
\omega_{ij} &= \omega_{ij} & \text{(8.40)}
\end{align}

\begin{align}
L_s \omega &= 0 & \text{(8.41)}
\end{align}

If we extend $\omega$ to a 2-form on $\mathfrak{d}$ by defining $\iota_s \omega = 0$, then $\omega$ is closed: $d\omega = 0$.

**Proof.** The proof is very similar to that of Lemma 8.10. There we showed that $L_s \omega = 0$ is equivalent to (4.3), and the same argument applies here. Similarly, (4.4) is equivalent to (8.39, 8.40). Now if $\iota_s \omega = 0$, then $L_s \omega = 0$ implies $\iota_s d\omega = 0$, which together with (8.39) leads to $d\omega = 0$. \(\square\)

Let $\omega$ be extended to a 2-form on $\mathfrak{d}$ by defining $\iota_s \omega = 0$, so that $d\omega = 0$. We define a 1-form $\theta \in \mathfrak{d}^*$ by $\theta(s) := -1, \theta(\partial_i) := 0$. Then we have $d\theta = \omega$; indeed:

\begin{align}
(d\theta)(\partial_i \wedge \partial_j) &= -\theta([\partial_i, \partial_j]) = c_{ij} = \omega_{ij} = \omega(\partial_i \wedge \partial_j),
\end{align}

\begin{align}
(d\theta)(s \wedge \partial_j) &= -\theta([s, \partial_j]) = 0 = \omega(s \wedge \partial_j),
\end{align}

using (8.40) and the fact that $[s, \partial_1] \subset \mathfrak{d}_1$.

**Lemma 8.17.** There is a one-to-one correspondence between contact forms $\theta$, i.e. 1-forms $\theta \in \mathfrak{d}^*$ such that $\theta \wedge (d\theta)^{(N-1)/2} \neq 0 \ (N = \dim \mathfrak{d})$, and solutions $(r, s)$ of (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \oplus k s$ and nondegenerate $r \in \mathfrak{d}_1 \otimes \mathfrak{d}_1$.

**Proof.** Given $(r, s)$, above we have defined the 1-form $\theta$ such that $\theta(s) = -1, \theta(\partial_i) = 0$ and $d\theta = \omega$. Since $\omega \in \bigwedge^2 \mathfrak{d}_1^*$ is nondegenerate, we have $\theta \wedge (d\theta)^{(N-1)/2} \neq 0$. Conversely, starting with a contact 1-form $\theta \in \mathfrak{d}^*$, we can define $s$ and $\omega$ satisfying (8.39)-(8.41). \(\square\)

**Example 8.18.** When $\mathfrak{d}$ is the Heisenberg Lie algebra with a basis $\{a_i, b_i, c\}$ and the only nonzero commutation relations $[a_i, b_i] = c \ (1 \leq i \leq n, N = 2n + 1)$, then

\begin{align}
r = \sum_{i=1}^n (a_i \otimes b_i - b_i \otimes a_i), & \quad s = -c
\end{align}

is a solution of (4.3), (4.4).

**Example 8.19.** When $\mathfrak{d}$ is abelian and $\dim \mathfrak{d} = 2n + 1 > 1$, then equations (4.3), (4.4) have no solutions $(r, s)$ with $\mathfrak{d} = \mathfrak{d}_1 \oplus k s$ and a nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$, because $d\theta = 0$ and therefore there are no contact forms.

**Example 8.20.** When the Lie algebra $\mathfrak{d}$ is simple, a solution $(r, s)$ of (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \oplus k s$ and a nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$ exists iff $\mathfrak{d} = \mathfrak{sl}_2$, and it is as follows:

\begin{align}
r = c \wedge f := c \otimes f - f \otimes c, & \quad s = -h
\end{align}

Only $\mathfrak{d} = \mathfrak{sl}_2$ is possible since the dimension of the stabilizer of $\theta$ equals 1.

Now let us compute $L_{\iota_s} \theta$. Recall that, as in Section 8.6, for any $x \in X$ we identify $e_x := x \otimes H \in \mathfrak{h}$ with $-\sum [x a_i \otimes b_i - a_i \otimes b_i] + x \otimes s$. Similarly to (8.38), it is easy to see that $\omega(e_x \wedge a) = -xa$ for $a \in \mathfrak{d}_1$ (in this case $\chi = \iota_s \omega := 0$). On
the other hand, $t_\varepsilon \theta = \theta(\varepsilon_r) = -x$, and hence $(dt_\varepsilon \theta)(a) = -(dx)(a) = xa$ for any $a \in \mathfrak{d}$. Therefore $(I_{t_\varepsilon} \theta)(a) = 0$ for $a \in \mathfrak{d}_1$, and $(I_{t_\varepsilon} \theta)(s) = xs$. In other words,

$$I_{t_\varepsilon} \theta = -(xs)\theta,$$

and we have the following proposition.

**Proposition 8.21.** Let $K(\mathfrak{d}, \theta) := \{x \in \text{Lie H-pseudoalgebra of rank 1 corresponding to a solution } (r, s) \text{ of equations (4.3), (4.4)} \}$ with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathfrak{k}s$ and a non-degenerate $r \in \mathfrak{d}_1 \otimes \mathfrak{d}_1$, where the 1-form $\theta \in \mathfrak{d}^*$ is defined by $\theta(s) = -1$, $\theta|_{\mathfrak{s}} = 0$. Then $\theta$ is a contact form, and the subalgebra $X \otimes_H K(\mathfrak{d}, \theta)$ of $X \otimes_H W(\mathfrak{d}) \simeq X \otimes \mathfrak{d}$ is the Lie algebra $K_N(\theta)$ of vector fields that preserve $\theta$ up to a multiplication by a function (which is isomorphic to $K_N$).

**Proof.** It remains to show that, conversely, any vector field from $K_N(\theta)$ is equal to $e_r$ for some $x \in X$. Indeed, let $A \in X \otimes \mathfrak{d}$ be such that $L_A \theta = f\theta$ for some $f \in X$. Let us write $A = \sum_i (x_i \otimes a_i + y_i \otimes b_i) + x \otimes s$ for some $x_i, y_i, x \in X$. Then $\omega(A \wedge a_i) = y_i$ and $\omega(A \wedge b_i) = -x_i$, while $\theta(A) = -x$. Therefore $(L_A \theta)(a) = \omega(A \wedge a) + xa$, which implies $y_i + xa_i = 0$, $-x_i + xb_i = 0$, and $xs = -f$. \hfill \square

**Remark 8.22.** To any $H$-type Lie pseudoalgebra, i.e., to any triple $(\mathfrak{d}, \omega, \chi)$ where $\mathfrak{d}$ is a finite-dimensional Lie algebra, $\omega \in \Lambda^2 \mathfrak{d}^*$ is a non-degenerate 2-form and $\chi \in \mathfrak{d}^*$ satisfying (8.32) and (8.33), we can associate a $K$-type Lie pseudoalgebra as follows. Set on the vector space $\mathfrak{d}' = \mathfrak{d} \oplus \mathfrak{k}c$ the Lie bracket $[\cdot, \cdot]'$ defined as:

$$[g, h]' = [g, h] + \omega(g, h)c, \quad [g, c]' = \chi(g)c,$$

for $g, h \in \mathfrak{d}$. Then $c + s \in \mathfrak{d}'$ stabilizes $\mathfrak{d}$, where $s \in \mathfrak{d}$ is the unique element such that $\chi = t_s \omega$; indeed,

$$[g, s + c]' = [g, s] + \omega(g, s)c + \chi(g)c = [g, s] \in \mathfrak{d}.$$

Define $\theta \in (\mathfrak{d}')^*$ as the unique element restricting to 0 on $\mathfrak{d}$ such that $\theta(c) = 1$.

Note that not all $K$-type data are obtained in this way, since the Lie algebra $\mathfrak{d}'$ just constructed always has a one dimensional ideal $\mathfrak{k}c$, and this fails in Example 8.20.

### 8.8. Annihilation algebras of pseudoalgebras of vector fields

To conclude this section, we determine the annihilation algebras of the primitive pseudoalgebras of vector fields defined above, and of current pseudoalgebras over them.

**Theorem 8.23.** (i) If $L$ is one of the Lie $H = U(\mathfrak{d})$-pseudoalgebras $W(\mathfrak{d}), S(\mathfrak{d}, \chi), H(\mathfrak{d}, \chi, \omega), K(\mathfrak{d}, \theta)$, then its annihilation algebra $A(L)$ is isomorphic to $W_N$, $S_N$, $P_N$ or $K_N$, respectively.

(ii) If $L = \text{Cur } L'$ is a current pseudoalgebra over the Lie $H'$-pseudoalgebra $L'$, then its annihilation algebra $A(L)$ is isomorphic to a current Lie algebra $C_c \otimes A(L')$ over $A(L')$, where $H' = U(\mathfrak{d}')$ and $\mathfrak{d}'$ is a codimension $r$ subalgebra of $\mathfrak{d}$.

**Proof.** (i) has already been established in this section. The only subtlety is that in the cases $L = S(\mathfrak{d}, \chi), H(\mathfrak{d}, \chi, \omega)$ or $K(\mathfrak{d}, \theta)$ we determined the subalgebra $X \otimes_H L \subset A(W(\mathfrak{d})) \simeq X \otimes \mathfrak{d}$, instead of the Lie algebra $A(L) := X \otimes_H L$. When $L$ is a free $H$-module of rank one, $A(L)$ is isomorphic to $X$ as a topological $H$-module. For $L = H_N$, however, inside $X \otimes_H L \subset X \otimes \mathfrak{d}$ the constants are lost. That is why we get $A(L) \simeq P_N$ instead of $H_N$.

In the case $L = K(\mathfrak{d}, \theta)$ it is immediate to check that no element from $A(L)$ maps to zero inside $X \otimes \mathfrak{d}$. When $L = S(\mathfrak{d}, \chi), \dim \mathfrak{d} > 2$, then a direct but lengthy
inspection using (8.24) and (7.4) shows that whenever an element of $\mathcal{A}(L)$ maps to zero inside $X \odot \mathfrak{d}$, then it is indeed central inside $\mathcal{A}(L)$. So $\mathcal{A}(L)$ is a central extension of $S_N$, but as $L$ equals its derived algebra, the same must be true of $\mathcal{A}(L)$, which is therefore an irreducible central extension. By Proposition 6.12(iii) all central extensions of $S_N$ ($N \geq 2$) are trivial, and we obtain that $\mathcal{A}(L)$ is isomorphic to $S_N$.

(ii) Note that $X = H^*$ maps surjectively to $X' = (H')^*$ with kernel isomorphic to $O_r$. Moreover $X \cong O_N, X' \cong O_{N'}$ ($N' = N - r$), and hence $X \cong O_r \odot X'$. We have: $\mathcal{A}(L') := X' \odot H', L' \mathfrak{d}$ and $\mathcal{A}(L) := X \odot H L = X \odot H (H \odot H', L') \cong X \odot H, L' \cong (O_r \odot X') \odot H, L' \cong O_r \odot X'$.

\[ \square \]

Remark 8.24. Let us assume that the base field $k = \mathbb{C}$, and let $L$ be as in Theorem 8.23(i). Then the action of $\mathfrak{d}$ on $\mathcal{A}(L)$ can be constructed via the embedding of $\mathfrak{d}$ in $W_N$ as follows.

(i) Any $N$-dimensional Lie algebra $\mathfrak{d}$ can be embedded in $W_N$: every $a \in \mathfrak{d}$ defines a left-invariant vector field on the connected simply-connected Lie group $D$ with Lie algebra $\mathfrak{d}$, and we take the corresponding formal vector field in the formal neighborhood of the identity element. (See also Proposition 6.9.)

(ii) If we have a homomorphism of Lie algebras $\chi: \mathfrak{d} \to \mathbb{C}$, it defines a homomorphism $\tilde{\chi}$ of $D$ to $\mathbb{C}^X$. Consider a volume form $\nu$ on $D$ defined, up to a constant factor, by the property $g \cdot \nu = \tilde{\chi}(g) \nu_0$, where $\nu_0$ is the value of $\nu$ at the identity element. Then we get an embedding of $\mathfrak{d}$ in $CS_N(\nu) = \text{Der} S_N(\nu) \cong \text{Der} S_N$.

(iii) Given $\chi$ and $\omega \in \Lambda^2 \mathfrak{d}^*$ such that $d \omega + \chi \wedge \omega = 0$, consider a 2-form $s$ on $D$ whose value at the identity element is $\omega$ and such that $g \cdot s = \tilde{\chi}(g)s$. Then $s$ is a symplectic form on $D$, and we get an embedding of $\mathfrak{d}$ in $CH_N(s) = \text{Der} H_N(s) \cong \text{Der} P_N$.

(iv) Given a contact form $\theta \in \mathfrak{d}^*$, consider the left-invariant 1-form $\theta$ on $D$ with the value $\theta$ at the identity element. Then $\theta$ is a contact form on $D$, and we get an embedding of $\mathfrak{d}$ in $K_N(\theta) \cong K_N$.

9. $H$-Conformal Algebras

The goal of this section is to reformulate the definition of a Lie (or associative) $H$-pseudoalgebra in terms of the properties of the $x$-brackets (or products) introduced in Section 7.2. The resulting notion, equivalent to that of an $H$-pseudoalgebra, will be called an $H$-conformal algebra.

Let us start by deriving explicit formulas for the $x$-brackets. We will use the notation of Section 7.2. Let $(L, \beta)$ be a Lie $H$-pseudoalgebra with a pseudobracket

\[ (9.1) \quad [a \ast b] \equiv \beta(a \odot b) = \sum_i (f_i \odot g_i) \odot \mu \epsilon_i. \]

Then for $a \in X, h \in H$ we have $\eta(x \odot h) = \langle x, h_{(1)} \rangle h_{(2)}$ (see (7.5)), and

\[ (\eta \odot \mu \beta)((x \odot h a) \odot (h \odot h b)) = \sum_i \eta(x f_i \odot h g_i) \odot \mu \epsilon_i \]
\[ = \sum_i \langle x f_i, (h g_i)_{(1)} \rangle (h g_i)_{(2)} \odot \mu \epsilon_i. \]

Taking $h = 1$, we get the following expression for the $x$-bracket in $L$:

\[ (9.2) \quad [a \ast b] = \sum_i \langle S(x), f_i g_i(\mu) \rangle g_i(2) \epsilon_i, \quad \text{if} \quad [a \ast b] = \sum_i (f_i \odot g_i) \odot \mu \epsilon_i. \]

Here we can recognize the Fourier transform $\mathcal{F}$, defined by (2.33):

\[ \mathcal{F}(f \odot g) = fg_{(-1)} \odot g(2). \]
The identity (2.35):

\[ f \otimes g = (fg_{(-1)} \otimes 1) \Delta(g_{(2)}), \]

implies

\[ [a \ast b] = \sum_i (f_i g_{(-1)} \otimes 1) \circ H g_{(2)} e_i. \]

Hence, \([a \ast b]\) can be written uniquely in the form \(\sum_i (h_i \otimes 1) \circ H e_i\), where \(\{h_i\}\) is a fixed \(k\)-basis of \(H\) (cf. Lemma 2.5).

We introduce another bracket \([a, b] \in H \otimes L\) as the Fourier transform of \([a \ast b]\):

\[ [a, b] = \sum_i \mathcal{F}(f_i \otimes g_i) (1 \otimes e_i) = \sum_i f_i g_{(-1)} \otimes g_{(2)} e_i. \]

In other words,

\[ [a, b] = \sum_i h_i \otimes e_i \quad \text{if} \quad [a \ast b] = \sum_i (h_i \otimes 1) \circ H e_i. \]

Then we have:

\[ [a, b] = (\langle S(x), \cdot \rangle \circ \text{id}) [a, b] = \sum_i \langle S(x), h_i \rangle e_i. \]

Using properties (2.38)–(2.41) of the Fourier transform, it is straightforward to derive the properties of the bracket (9.5). Then the definition of a Lie pseudoalgebra can be equivalently reformulated as follows.

**Definition 9.1.** A Lie \(H\)-conformal algebra is a left \(H\)-module \(L\) equipped with a bracket \([\cdot, \cdot] : L \otimes L \to H \otimes L\), satisfying the following properties \((a, b, c \in L, h \in H)\):

**\(H\)-sesqui-linearity:**

\[ [h a, b] = (h \otimes 1) [a, b], \]

\[ [a, h b] = (1 \otimes h_{(2)}) [a, b] (h_{(-1)} \otimes 1). \]

**Skew-commutativity:** If \([a, b]\) is given by (9.5), then

\[ [b, a] = -\sum_i h_{i(-1)} \otimes h_{i(2)} e_i. \]

**Jacobi identity:**

\[ [a, [b, c]] = \sigma(\text{id} \otimes \text{id}) [b, [a, c]] = (\mathcal{F}^{-1} \otimes \text{id}) [[a, b], c] \]

in \(H \otimes H \otimes L\), where \(\sigma : H \otimes H \to H \otimes H\) is the permutation \(\sigma(f \otimes g) = g \otimes f\), and

\[ [a, [b, c]] = (\sigma \otimes \text{id})(\text{id} \otimes [a, c])[b, c], \]

\[ [[a, b], c] = (\text{id} \otimes [c, \cdot])[a, b]. \]

**Examples 9.2.** (i) For the current Lie pseudoalgebra \(\text{Cur} \mathfrak{g} = H \otimes \mathfrak{g}\) with the pseudobracket (4.2), the bracket (9.5) is given by:

\[ [f \otimes a, g \otimes b] = f g_{(-1)} \otimes (g_{(2)} \otimes [a, b]). \]

(ii) For the Lie pseudoalgebra \(W(\mathfrak{d}) = H \otimes \mathfrak{d}\) with pseudobracket defined by (8.5), the bracket (9.5) is given by:

\[ [1 \otimes a, 1 \otimes b] = 1 \otimes (1 \otimes [a, b]) + a \otimes (1 \otimes b) + b \otimes (1 \otimes a) - 1 \otimes (a \otimes b). \]

One can also reformulate Definition 9.1 in terms of the \(x\)-brackets (9.6).

**Definition 9.3.** A Lie \(H\)-conformal algebra is a left \(H\)-module \(L\) equipped with \(x\)-brackets \([a, x] \in L\) for \(a, b \in L, x \in X\), satisfying the following properties:
Locality:

\[(9.13) \quad \text{codim}\{x \in X \mid [a_x b] = 0\} < \infty \quad \text{for any } a, b \in L.\]

Equivalently, for any basis \(\{x_i\}\) of \(X\),

\[(9.14) \quad [a_{x_i} b] \neq 0 \quad \text{for only a finite number of } i.\]

**H-sesquilinearity:**

\[(9.15) \quad [ha_x b] = [a_x b],\]

\[(9.16) \quad [a_x bb] = h(\omega) [a_{h(-1)x} b].\]

**Skew-commutativity:** Choose dual bases \(\{h_i\}, \{x_i\}\) in \(H\) and \(X\). Then:

\[(9.17) \quad [a_x b] = -\sum_i \langle x, h_i(-1) \rangle h_i(-2) [b_{x_i} a].\]

**Jacobi identity:**

\[(9.18) \quad [a_x [b_{y} c]] - [b_{y} [a_x c]] = [[a_{x(x)} b]_{x(1)} y c].\]

Lemma 2.4 implies that (9.18) can be rewritten as follows:

\[(9.19) \quad [a_x [b_{y} c]] - [b_{y} [a_x c]] = \sum_i [[a_{x(x)} b]_{x(1)} y c].\]

In particular, the right hand side of (9.18) is well defined: the sum is finite because of (9.14).

The definitions of an **associative H-conformal algebra** or of representations of \(H\)-conformal algebras are obvious modifications of the above. For example, in terms of \(x\)-products, the associativity looks as follows (cf. (9.18)):

\[(9.20) \quad a_x (b_y c) = (a_{x(x)} b)_{x(1)} y c.\]

The same argument as the one used for \(F\) shows that the map \(x \otimes y \mapsto x(\omega) \otimes x(1) y\) has an inverse given by \(x \otimes y \mapsto x(\omega) \otimes x(1) y\). Therefore, (9.20) is equivalent to the following equation:

\[(9.21) \quad a_{x(x)} (b_{x(1)} y c) = (a_x b)_{y(c)}.\]

We also note that there is a simple relationship between the \(x\)-bracket (9.6) of a Lie \(H\)-conformal algebra (or equivalently, pseudoalgebra) \(L\) and the commutator in its annihilation Lie algebra \(A(L)\) defined in Section 7. Let \(\{h_i, \{x_i\}\) again be dual bases in \(H\) and \(X\). Then in (9.5) one has \(c_i = [a_{s(-1)x_i} b]\); therefore

\[(9.22) \quad [a, b] = \sum_i S(h_i) \otimes [a_{x_i} b] \quad \text{and} \quad [a \ast b] = \sum (S(h_i) \otimes 1) \otimes_H [a_{x_i} b].\]

Recall that we denote the element \(x \otimes_H a\) of \(A(L) := X \otimes_H L\) by \(a_x\). Then the definition (7.2) and (9.22) imply

\[(9.23) \quad [a_x b_x] = \sum_i [a_{x(x)} b(x(1)) y] = [a_{x(x)} b]_{x(1) y},\]

using (2.32). This is also equivalent to:

\[(9.24) \quad [a_x b]_y = [a_{x(x)} b_{x(1)} y] = \sum_i [a_{x_i} b_{(x_i y)}].\]

Comparing these formulas with Eq. (9.18), we obtain the following important result.
Proposition 9.4. Any module $M$ over a Lie pseudoalgebra $L$ has a natural structure of an $\mathcal{A}(L)$-module, given by $(x \otimes_H a) \cdot m = a_x m$ ($a \in L, x \in X, m \in M$). This action is compatible with the action of $H$ (see (7.4)) and satisfies the locality condition:

$$\text{codim}\{x \in X \mid a_x m = 0\} < \infty, \quad a \in L, \; m \in M;$$

or equivalently, for any basis $\{x_i\}$ of $X$,

$$a_{x_i} m \neq 0 \quad \text{for only a finite number of } i.$$ (The above conditions on $M$ mean that, when endowed with the discrete topology, $M$ is a topological $\mathcal{A}(L)$-module in the category $\mathcal{M}(H)$.)

Conversely, any $\mathcal{A}(L)$-module $M$ satisfying the above conditions has a natural structure of an $L$-module, given by

$$a \cdot m = \sum_i (S(h_i) \otimes 1) a_{x_i} \cdot m,$$

where $\{h_i\}, \{x_i\}$ are dual bases in $H$ and $X$, and we use the notation $a_x \equiv x \otimes_H a$.

This proposition provides the main tool for constructing modules over Lie pseudoalgebras. Of course, there is an analogous result in the case of associative algebras as well.

Finally, let us give two more expressions for the bracket in $\mathcal{A}(L)$ which will be useful later. Recall that, by Proposition 9.4, we have an action of $\mathcal{A}(L)$ on $L$ given by $a_x \cdot b = [a_x, b]$. Recall also that the action of $H$ on $\mathcal{A}(L)$ is defined by $h(a_x) = a_{h \cdot x}$. Let $a \in \mathcal{A}(L), b \in L, y \in X$. Then:

$$[a \cdot b]_y = \sum [h_i a, b_{x,y}],$$

$$[a, b] = \sum (\langle (S(h_i)a) \cdot b \rangle)_{x,y}.$$

Note that the infinite sums on the right hand sides make sense since they are convergent in the complete topology of $\mathcal{A}(L)$. It is enough to prove both statements for $a$ of the form $a_x = x \otimes_H a$ since such elements span $\mathcal{A}(L)$. Equation (9.29) then follows from (9.23) and (2.32). Analogously, (9.28) derives from (9.24) by noticing that $x_{(-1)} \otimes x_{(2)} = \sum_i x_i \otimes h_i x$.

10. $H$-Pseudolinear Algebra

The definition of a module over a pseudoalgebra motivates the following definition of a pseudolinear map.

**Definition 10.1.** Let $V$ and $W$ be two $H$-modules. An $H$-pseudolinear map from $V$ to $W$ is a $k$-linear map $\phi: V \to (H \otimes H) \otimes_H W$ such that

$$\phi(hv) = ((1 \otimes h) \otimes_H 1) \phi(v), \quad h \in H, \; v \in V.$$

We denote the space of all such $\phi$ by $\text{Chom}(V,W)$. We define a left action of $H$ on $\text{Chom}(V,W)$ by:

$$\phi(hv) = ((h \otimes 1) \otimes_H 1) \phi(v).$$

When $V = W$, we set $\text{Cend} V = \text{Chom}(V,V)$.

**Example 10.2.** Let $A$ be an $H$-pseudoalgebra and $V$ be an $A$-module. Then for any $a \in A$ the map $m_a: V \to (H \otimes H) \otimes_H V$ defined by $m_a(v) = a \cdot v$ is an $H$-pseudolinear map. Moreover, we have $hm_a = m_{ha}$ for $h \in H$. 
Consider the map \( \rho: \text{Chom}(V, W) \odot V \to (H \odot H) \odot_H W \) given by \( \rho(\phi \odot v) = \phi(v) \). By definition it is \( H \)-bilinear, so it is a polylinear map in \( \mathcal{M}^*(H) \). We will also use the notation \( \phi \ast v := \phi(v) \) and consider this as a pseudoproduct (or rather action, see Proposition 10.5 below).

The corresponding \( x \)-products are called Fourier coefficients of \( \phi \) and are given by a formula analogous to (9.2):

\[
\phi_x v = \sum_i \langle S(x), f_i g_i(-1) \rangle g_i(2) w_i, \quad \text{if} \quad \phi(v) = \sum_i (f_i \otimes g_i) \otimes_H w_i.
\]

They satisfy a locality relation and an \( H \)-sesqui-linearity relation similar to (9.13) and (9.16):

\[
\text{codim} \{ x \in X \mid \phi_x v = 0 \} < \infty \quad \text{for any} \quad v \in V,
\]

\[
\phi_x (h v) = h(2)_v \phi_{h(-1)x} v.
\]

Conversely, any collection of maps \( \phi_x \in \text{Hom}(V, W) \), \( x \in X \), satisfying relations (10.4), (10.5) comes from an \( H \)-pseudolinear map \( \phi \in \text{Chom}(V, W) \). Explicitly (cf. (9.27)):

\[
\phi(v) = \sum_i (S(h_i) \otimes 1) \otimes_H \phi_{x_i} v,
\]

where \( \{ h_i \} \), \( \{ x_i \} \) are dual bases in \( H \) and \( X \).

**Remark 10.3.** It follows from (10.5) that for \( \phi \in \text{Chom}(V, W) \), the map \( \phi_1 : V \to W \) is \( H \)-linear, where \( 1 \in X \) is the unit element. This establishes an isomorphism \( \text{Hom}_H(V, W) \simeq \mathbb{k} \otimes_H \text{Chom}(V, W) \simeq \text{Chom}(V, W)/H_+ \text{Chom}(V, W) \), where \( H_+ = \{ h \in H \mid \varepsilon(h) = 0 \} \) is the augmentation ideal.

**Lemma 10.4.** Let \( U, V, W \) be three \( H \)-modules, and assume that \( U \) is finite. Then there is a unique polylinear map

\[
\mu \in \text{Lin}(\{ \text{Chom}(V, W), \text{Chom}(U, V) \}, \text{Chom}(U, W))
\]

in \( \mathcal{M}^*(H) \), denoted as \( \mu(\phi \otimes \psi) = \phi \ast \psi \), such that

\[
\phi \ast (\psi \ast u) = (\phi \ast \psi) \ast u
\]

in \( H \otimes_H W \) for \( \phi \in \text{Chom}(V, W) \), \( \psi \in \text{Chom}(V, U) \), \( u \in U \).

**Proof.** We define \( \phi \ast \psi \) in terms of its Fourier coefficients — the \( x \)-products \( \phi_x \psi \). We have already seen, when we discussed associativity, that (10.7) is equivalent to the following equation (cf. (9.26)):

\[
(\phi_x \psi)_y u = (\phi_x (\psi_x)_y) u.
\]

This can be inverted to find (cf. (9.21)):

\[
(\phi_x \psi)_y u = \phi_x (\psi_x)_y u = \sum_i \phi_x (\psi_x)_i (\psi_{h_i} s(x))_y u.
\]

In order that \( \phi \ast \psi \) be well defined, we need to check that \( \phi_x \psi \) satisfies locality, i.e., that \( \phi_x \psi = 0 \) when \( x \in F_n X \) with \( n \gg 0 \). By properties (2.21), (2.28), (2.29) of the filtration \( \{ F_n X \} \), and locality of \( \phi \) and \( \psi \), it follows that for each \( u \in U \) there is an \( n \) such that \( (\phi_x \psi)_y u = 0 \) for \( x \in F_n X \) and all \( y \in X \). Since \( U \) is finite, we can choose an \( n \) that works for all \( u \) belonging to a set of generators of \( U \) over \( H \). The \( H \)-sesqui-linearity of \( (\phi_x \psi)_y u \) with respect to \( y \) (for fixed \( x \)) follows from (10.7), but can also be checked by a direct calculation. It implies that there is \( n \) for which \( (\phi_x \psi)_y u = 0 \) for all \( y \) and \( u \). A similar argument to the above shows that for each fixed \( x \) there is \( n \) such that \( (\phi_x \psi)_y u = 0 \) for \( y \in F_n X \) and all \( u \in U \). Therefore, all \( \phi_x \psi \) belong to \( \text{Chom}(U, W) \). \( \square \)
Specifying to the case $U = V = W$, we obtain a pseudoproduct $\mu$ on $\text{Cend} V$, and an action $\rho$ of $\text{Cend} V$ on $V$.

**Proposition 10.5.** (i) For any finite $H$-module $V$, the above pseudoproduct provides $\text{Cend} V$ with the structure of an associative $H$-pseudoalgebra. $V$ has a natural structure of a $\text{Cend} V$-module given by $\phi \ast v \equiv \phi(v)$.

(ii) For an associative $H$-pseudoalgebra $A$, giving a structure of an $A$-module on $V$ is equivalent to giving a homomorphism of associative $H$-pseudoalgebras from $A$ to $\text{Cend} V$.

**Proof.** Part (i) is an immediate consequence of Lemma 10.4. Indeed, the only thing that remains to be checked is the associativity of $\text{Cend} V$, and it follows from (10.7):

\[
((\phi \ast \psi) \ast \chi) \ast v = (\phi \ast (\psi \ast (\chi \ast v)) = \phi \ast (\psi \ast (\chi \ast v))
\]

\[
\phi \ast ((\psi \ast \chi) \ast v) = (\phi \ast (\psi \ast \chi)) \ast v.
\]

To prove part (ii), we associate with each $a \in A$ the $H$-pseudolinear map $m_a \in \text{Cend} V$ given by $m_a(v) = a \ast v$. Then

\[
(m_a \ast m_b) \ast v = m_a \ast (m_b \ast v) = a \ast (b \ast v) = (a \ast b) \ast v = m_{ab} \ast v,
\]

which shows that $m_a \ast m_b = m_{ab}$.

We denote by $gc\ V$ the Lie $H$-pseudoalgebra obtained from the associative one $\text{Cend} V$ by the construction of Proposition 3.15. Then $V$ is a $gc\ V$-module, and one has a statement analogous to part (ii) above.

**Proposition 10.6.** Let $V$ be a finite $H$-module. Then, for a Lie $H$-pseudoalgebra $L$, giving a structure of an $L$-module on $V$ is equivalent to giving a homomorphism of Lie $H$-pseudoalgebras from $L$ to $gc\ V$.

**Remark 10.7.** Let $L$ be a Lie $H$-pseudoalgebra, and $U, V$ be finite $L$-modules. Then the formula $(a \in L, u \in U, \phi \in \text{Chom}(U, V))$

\[
(a \ast \phi)(u) = a \ast (\phi \ast u) - ((\sigma \circ \text{id}) \circ_H \text{id}) \phi \ast (a \ast u)
\]

provides $\text{Chom}(U, V)$ with the structure of an $L$-module.

**Definition 10.8.** (i) Let $A$ be an associative $H$-pseudoalgebra. A derivation of $A$ is an $H$-pseudolinear map $\phi \in gc\ A$ which satisfies

\[
\phi \ast (a \ast b) = (\phi \ast a) \ast b + ((\sigma \circ \text{id}) \circ_H \text{id}) a \ast (\phi \ast b), \quad a, b \in A.
\]

We denote the space of all such $\phi$ by $\text{Der} A$.

(ii) Similarly, for a Lie $H$-pseudoalgebra $L$, let $\text{Der} L$ be the space of all $\phi \in gc\ L$ satisfying

\[
\phi \ast [a \ast b] = [(\phi \ast a) \ast b] + [(\sigma \circ \text{id}) \circ_H \text{id}] [a \ast (\phi \ast b)], \quad a, b \in L.
\]

The next result follows easily from definitions.

**Lemma 10.9.** (i) For any $H$-pseudoalgebra $A$, $\text{Der} A$ is a subalgebra of $gc\ A$.

(ii) When $A$ is associative (respectively Lie), we have a homomorphism of pseudoa-

\[
i: A \to \text{Der} A \text{ given by } i(a)(b) = a \ast b - (\sigma \circ_H \text{id}) b \ast a \text{ (respectively } i(a)(b) = [a \ast b], \quad a, b \in A,
\]

whose kernel is the center of $A$.

(iii) For any $x \in X$ and $\phi \in \text{Der} A$, $\phi_x$ is a derivation of the annihilation algebra

\[
of A.
\]

(iv) Let $A$ be an associative $H$-pseudoalgebra and $L$ be the corresponding Lie

pseudoalgebra with pseudobracket given by commutator. Then $\text{Der} A \subset \text{Der} L$. 
Example 10.10. Recall that in Section 8.3 we defined the $W(\mathfrak{d})$-module of pseudoforms $\Omega(\mathfrak{d}) = H \otimes \wedge^* \mathfrak{d}^*$. Since $\wedge^* \mathfrak{d}^*$ is an associative algebra with respect to the wedge product, we can consider $\Omega(\mathfrak{d})$ as an associative pseudoalgebra: the current pseudoalgebra over $\wedge^* \mathfrak{d}^*$. Then, as in the case of usual differential forms, for any $\alpha \in W(\mathfrak{d})$, $\alpha$ and $\alpha^\ast$, are superderivations of $\Omega(\mathfrak{d})$, see (8.13, 8.14).

In the case when $V$ is a free $H$-module, one can give an explicit description of $\text{Cend} V$, and hence of $\text{gc} V$, as follows.

Proposition 10.11. Let $V = H \otimes V_0$, where $H$ acts trivially on $V_0$ and $\dim V_0 < \infty$. Then $\text{Cend} V$ is isomorphic to $H \otimes H \otimes \text{End} V_0$, with $H$ acting by a left multiplication on the first factor, and with the following pseudoproduct:

\[(f \circ a \circ A) \ast (g \circ b \circ B) = (f \circ ga_{(1)}) \otimes_H (1 \circ ba_{(2)} \otimes AB).\]

The action of $\text{Cend} V$ on $V$ is given by

\[ (f \circ a \circ A) \ast (h \otimes v) = (f \circ ha) \otimes_H (1 \circ Av). \]

The pseudobracket in $\text{gc} V$ is given by

\[ [f \circ a \circ A] \ast (g \circ b \circ B) = (f \circ ga_{(1)}) \otimes_H (1 \circ ba_{(2)} \otimes AB) - (fb_{(1)} \circ g) \otimes_H (1 \circ ab_{(2)} \otimes BA). \]

Proof. Since $(H \otimes H) \otimes V_0 \cong H \otimes H \otimes \text{End} V_0$, we can identify $\text{Cend} V$ with $H \otimes H \otimes \text{End} V_0$ so that its action on $V$ is given by (10.12). To prove (10.11), we use (10.7) and the explicit definition of associativity from Section 3. Due to $H$-bilinearity, we can assume that $f = g = h = 1$. Then:

\[
(1 \circ a \circ A) \ast ((1 \circ b \circ B) \ast (1 \circ v)) = (1 \circ a \circ A) \ast ((1 \circ b) \otimes_H (1 \circ Bv))
= (1 \circ a_{(1)} \otimes ba_{(2)}) \otimes_H (1 \circ ABv).
\]

On the other hand, we have:

\[
((1 \circ a_{(1)}) \otimes_H (1 \circ ba_{(2)} \otimes AB)) \ast (1 \circ v) = (1 \circ a_{(1)} \otimes ba_{(2)}) \otimes_H (1 \circ ABv).
\]

Now (10.11) follows from the uniqueness from Lemma 10.4.

Remark 10.12. Let $V = H \otimes V_0$, where $H$ acts trivially on $V_0$ and $\dim V_0 < \infty$. Then $\text{Cur End} V_0$ can be identified with $H \otimes 1 \otimes \text{End} V_0 \subset \text{Cend} V$. Similarly, $\text{Cur gc} V_0$ is a subalgebra of $\text{gc} V$.

When $V = H \otimes k^n$, we will denote $\text{Cend} V$ by $\text{Cend}_n$, and $\text{gc} V$ by $\text{gc}_n$. Of course, the essential case is when $V = H$ is of rank one. Let us describe the associative algebra $A_V \text{Cend}_1$, where $A_V$ is as in Section 7. As an $H$-module it is isomorphic to $Y \otimes H \text{Cend}_1 \cong Y \otimes H$ with $H$ acting on the first factor. We have $a_x = x \otimes_H (1 \circ a) \equiv x \circ a$ for $x \in Y$, $a \in H$. Comparing (7.2) with (10.11), we see that the product in $Y \otimes H$ is given by:

\[ (x \circ a)(y \circ b) = x(ya_{(1)}) \otimes ba_{(2)}. \]

Hence $A_V \text{Cend}_1$ is isomorphic to the smash product $Y \# H$ (see Section 2). The annihilation algebra $A_0(\text{Cend}_1) \equiv A_Y \text{Cend}_1 \cong X \# H$ is isomorphic as an associative algebra to the Drinfeld double of $H$ (see [D]). For $H = U(\mathfrak{d})$, $A_0(\text{Cend}_1)$ can be identified with the associative algebra of all differential operators on $X$, while $A_0(\text{gc}_1)$ with the corresponding Lie algebra.
Example 10.13. Let $H = U(\mathfrak{h})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{h}$. We identify $\mathfrak{h}$ with its image in $H$, so that $\mathfrak{g}^1 = H \otimes H$ contains $H \otimes \mathfrak{h}$. We claim that $f \otimes a \mapsto -f \otimes a$ ($f \in H$, $a \in \mathfrak{h}$) is an embedding of Lie pseudoalgebras $W(\mathfrak{h}) \hookrightarrow \mathfrak{g}^1$, compatible with their actions on $H$. This is immediate by comparing (10.13) with (8.3) and (10.12) with (8.4).

Consider $H$ as Our $k$, i.e., as an associative $H$-pseudoalgebra with a pseudoproduct $f \ast g = (f \otimes g) \circ_H 1$. Then $W(\mathfrak{h}) = \text{Der} H \subset \mathfrak{g}^1$.

Example 10.14. Let $H = k[\Gamma]$ be the group algebra of a group $\Gamma$. Then for $V = H$ and $f, g, a, b \in \Gamma$, (10.11) takes the form:

$$ (f \ast g) \ast (g \ast b) = (f \otimes g a) \circ_H (1 \otimes b a). $$

We end this section with two lemmas that will be useful in representation theory.

Lemma 10.15. For $\phi \in \text{Chom}(V, W)$, let

$$ \ker_n \phi = \{ v \in V \mid \phi_x v = 0 \quad \forall x \in F_n X \}, $$

so that, for example, $\ker_{-1} \phi = \ker \phi$. If $V$ is a finite $H$-module and $F^n H$ is finite-dimensional, then $\ker_n \phi / \ker \phi$ is finite-dimensional.

Proof. Since $\ker \phi$ is an $H$-submodule of $V$, after replacing $V$ with $V / \ker \phi$, we can assume that $\ker \phi = 0$.

By definition, $\ker_n \phi = \phi^{-1} ((F^n H \otimes k) \otimes_H W)$. Since, by Lemma 2.37, $(F^n H \otimes k) \otimes_H W = (k \otimes F^n H) \otimes_H W$, we have $\phi(\ker_n \phi) \subset (k \otimes F^n H) \otimes_H W$.

On the other hand, since $V$ is finite over $H$ and $\phi$ satisfies (10.1), there exists a finite-dimensional subspace $W'$ of $W$ such that $\phi(\ker_n \phi) \subset (k \otimes H) \otimes_H W'$. It follows that $\phi(\ker_n \phi) \subset (k \otimes F^n H) \otimes_H W'$, which is finite-dimensional. Since $\phi$ is injective, $\ker_n \phi$ is finite-dimensional. \hfill $\Box$

Lemma 10.16. Let $\phi \in \text{Chom}(V, W)$ and $h \in H$. If $h$ is not a divisor of zero, then:

(i) $h\phi = 0$ implies $\phi = 0$;

(ii) $ht \in \ker \phi$ implies $t \in \ker \phi$.

Proof. Part (i) follows from Eq. (10.2): if $\phi(v) = \sum_i (f_i \otimes 1) \otimes_H w_i$, then $(h\phi)(v) = \sum_i (h f_i \otimes 1) \otimes_H w_i$, which implies $f_i = 0$, which implies $f_i = 0$. Similarly, part (ii) follows from (10.1), since we can write $\phi(v)$ uniquely in the form $\sum_i (1 \otimes g_i) \otimes_H w_i$. \hfill $\Box$

Corollary 10.17. Let $L$ be a pseudoalgebra, and $M$ be an $L$-module. Then any torsion element from $L$ acts trivially on $M$, and any torsion element from $M$ is acted on trivially by $L$. In particular, the torsion of a Lie pseudoalgebra is central.

11. Reconstruction of an $H$-Pseudoalgebra from an $H$-Differential Algebra

11.1. The reconstruction functor $\mathcal{C}$. Let, as before, $H$ be a cocommutative Hopf algebra and $X = H^*$. Given a topological left $H$-module $\mathcal{L}$ (where $H$ is endowed with the discrete topology), let

$$(11.1) \qquad \mathcal{C}(\mathcal{L}) = \text{Hom}_H^{\text{cont}}(X, \mathcal{L})$$
be the space of continuous $H$-homomorphisms. We define a structure of a left $H$-module on $\mathcal{C}(\mathcal{L})$ by

\[(ha)(x) = a(xh).\]

Then $\mathcal{C}$ is a covariant functor from the category of topological $H$-modules to the category of $H$-modules.

**Lemma 11.1.** (i) The functor $\mathcal{C}$ is left exact: $\mathcal{C}(i)$ is injective if $i$ is injective.

(ii) The functor $\mathcal{C}$ preserves direct sums: $\mathcal{C}(\mathcal{L}_1 \oplus \mathcal{L}_2) = \mathcal{C}(\mathcal{L}_1) \oplus \mathcal{C}(\mathcal{L}_2)$.

(iii) Assume that the Hopf algebra $H$ contains nonzero primitive elements. If $\mathcal{L}$ is finite-dimensional over $k$ with discrete topology, then $\mathcal{C}(\mathcal{L}) = 0$.

(iv) If $H = U(\mathfrak{d})$, then $\mathcal{C}(\mathcal{L})$ has no torsion as an $H$-module.

**Proof.** Parts (i) and (ii) are obvious.

(iii) By Kostant’s Theorem 2.1, $H = U(\mathfrak{d}) \cong k[\Gamma]$ with $\mathfrak{d} \neq 0$. If $\mathcal{L}$ is finite-dimensional, any continuous homomorphism $\alpha : X \to \mathcal{L}$ must contain some $F_n X$ in its kernel. Let $h \in F^n U(\mathfrak{d})$ but $h \notin F^{n-2} U(\mathfrak{d})$. Then, by Lemma 6.10, $h F_n X = X$ so that for each $x \in X$, $x = hy$ for some $y \in F_n X$. This implies $a(x) = a(hy) = h(a(y)) = 0$, since $a(y) = 0$, proving part (iii).

Similarly, part (iv) follows from the fact that $Xh = X$ for any nonzero $h \in U(\mathfrak{d})$. \qed

If, in addition, $\mathcal{L}$ is a topological Lie $H$-differential algebra, we define $x$-brackets in $\mathcal{C}(\mathcal{L})$ by the formula (cf. (9.24)):

\[(a, b)(y) = [\alpha (x_2), \beta (x_{-1})y] = \sum_i [\alpha (x_i), \beta (h_i S(x)) y].\]

This is well defined because the infinite sum in the right hand side converges in $\mathcal{L}$. Equation (11.3) is also equivalent to (cf. (9.23)):

\[(a, b)(y) = \sum_i [\alpha (x_i), \beta (x_{-i})(y)] = \sum_i [\alpha (x_i), \beta (x S(h_i)) y].\]

**Proposition 11.2.** For any topological Lie $H$-differential algebra $\mathcal{L}$, $\mathcal{C}(\mathcal{L})$ satisfies properties (9.15)-(9.18).

**Proof.** This can be verified by straightforward but rather tedious computations. To illustrate them, let us check (9.15). By definition, we have:

\[ [ha, b](y) = \sum_i ((ha)(x_i), \beta (h_i S(x)) y) = \sum_i [a, b](x_i, \beta (h_i S(x)) y),\]

while

\[ [ar, b](y) = \sum_i [a, b](x_i, \beta (h_i S(x)) y).\]

Hence (9.15) is a consequence of the following identity:

\[ \sum_i x_i h \otimes h_i = \sum_i x_i \otimes h_i S(h),\]

which can be checked by pairing both sides with $f \otimes z \in H \otimes X$. Indeed,

\[ \sum_i (x_i h f)(h_i z) = (zh f) = (z, f S(h)) = \sum_i (x_i, f)(h_i S(h), z).\]

A more conceptual proof can be given by noticing that formula (11.3) is the same as the formula for the commutator of $H$-pseudolinear maps. For $a \in \mathcal{C}(\mathcal{L})$ consider the family $a(x) \in \text{Hom}(\mathcal{L}, \mathcal{L})$ indexed by $x \in X$. It is easy to see that it satisfies (10.5). So, if it also satisfies (10.4), it would give an $H$-pseudolinear map from $\mathcal{L}$ to itself. Although this is not true in general, the argument still works because all infinite series that appear will be convergent. (In other words, we embed $\mathcal{C}(\mathcal{L})$ in a certain completion of $\mathcal{C}(\mathcal{L})$.) \qed
In order to have the locality (9.13), one has to impose some restrictions on \( L \). In particular, the condition that \( L \) be a linearly compact topological Lie algebra will often suffice to guarantee locality of \( \mathcal{C}(L) \).

In what follows, we explain how to reconstruct an \( H \)-pseudoalgebra \( L \) which is finitely generated over \( H \) from its annihilation Lie algebra \( \mathcal{A}(L) \). Recall that \( \mathcal{A}(L) \) is a linearly compact topological Lie algebra (Proposition 7.12). In many of the proofs we never exploit the algebra structure on \( \mathcal{A}(L) \), so the corresponding statements hold for finite \( H \)-modules in general.

We let \( \hat{L} = \mathcal{C}(L) := \text{Hom}^\text{cont}_H(X, X \circ_H L) \). There is an obvious map

\[
\Phi: L \to \hat{L}, \quad a \mapsto a(x) = x \circ_H a.
\]

It is clear by definitions that \( \Phi \) is a homomorphism of \( H \)-pseudoalgebras (or \( H \)-modules if \( L \) is only an \( H \)-module).

11.2. The case of free modules. Let \( L \) be a Lie \( H \)-pseudoalgebra which is free as an \( H \)-module: \( L = H \otimes L_0 \) with the trivial action of \( H \) on \( L_0 \). Then \( \mathcal{L} = \mathcal{A}(L) := X \circ_H (H \otimes L_0) \approx X \otimes L_0 \) as an \( H \)-module.

**Proposition 11.3.** When \( L \) is a Lie \( H \)-pseudoalgebra that is a free \( H \)-module, the map \( \Phi \) defined by (11.6) is an isomorphism of Lie \( H \)-pseudoalgebras.

**Proof.** To construct the inverse of \( \Phi \), identify \( \mathcal{L} \) with \( X \otimes L_0 \) and consider

\[
\Psi: \hat{L} \to L, \quad a \mapsto \sum_i S(h_i) \circ (x_i) a(x_i).
\]

Here, as before, \( \{h_i\}, \{x_i\} \) are dual bases in \( H \) and \( X \), and \( \varepsilon(x) = (1, x) \) for \( x \in X \). This is well defined, i.e., the sum is finite, because \( a(x_i) \in F, X \otimes L_0 \) for all but a finite number of \( x_i \) and \( \varepsilon(F, X) = 0 \). Using identity (11.5), it is easy to see that \( \Psi \) is \( H \)-linear. Next, we have for \( a \in L_0 \):

\[
\Psi \Phi(1 \circ a) = \sum_i S(h_i) \circ (1, x_i) a = S(1) \circ a = 1 \circ a,
\]

showing that \( \Psi \Phi = \text{id} \). In particular, \( \Psi \) is surjective.

Assume that \( \Psi(a) = 0 \) for some \( a \in \hat{L} \). This means that \( \langle 1, \cdot \rangle \circ \text{id} a(x) = 0 \) for any \( x \in X \). But then for any \( h \in H \), we have:

\[
\langle (S(h), \cdot) \circ \text{id} \rangle a(x) = \langle (1, \cdot) \circ \text{id} \rangle((h \circ 1) a(x)) = \langle (1, \cdot) \circ \text{id} \rangle h(a(x)) = \langle (1, \cdot) \circ \text{id} \rangle h(x) = 0,
\]

which implies \( a = 0 \). Hence \( \Psi \) is injective.

**Remark 11.4.** If \( L \) is only a free \( H \)-module, then \( \Phi \) is an isomorphism of \( H \)-modules. Analogous results hold for representations, or for associative \( H \)-pseudoalgebras.

11.3. Reconstruction of a non-free module. Throughout this subsection \( L \) will be a (possibly non-free) finitely generated \( H \)-module, and \( H \) will be the universal enveloping algebra \( U(\mathfrak{g}) \) of a finite-dimensional Lie algebra \( \mathfrak{g} \).

The natural map \( \Phi: L \to \hat{L} \) (see (11.6)) is in general neither injective nor surjective. However, the induced map \( \varphi = \mathcal{A}(\Phi): \mathcal{A}(L) \to \mathcal{A}(\hat{L}) \) has a left inverse \( \psi: x \circ_H a \mapsto a(x) \). This shows that \( \mathcal{A}(\Phi) \) is injective, and that \( \psi \) is surjective.

We want to figure out to what extent injectivity and surjectivity of \( \Phi \) fail. First of all let us remark that, by Lemma 7.4, every torsion element \( a \in L \) has all zero Fourier coefficients, i.e., it belongs to the kernel of \( \Phi \). In fact, we have:

**Proposition 11.5.** The kernel of \( \Phi: L \to \hat{L} \) equals the torsion of \( L \).
Proof. It remains to show that a non-torsion element \( a \in L \) does not lie in the kernel of \( \Phi \). Consider the map \( i: L \to F \) constructed in Lemma 2.2. Then \( i(a) \neq 0 \). The map \( A(i) \) induced by \( i \) maps the Fourier coefficient \( x \otimes_H a \) of \( a \) to the corresponding Fourier coefficient \( x \otimes_H i(a) \) of a nonzero element in the free \( H \)-module \( F \). Now, it is clear from Proposition 11.3 that \( x \otimes_H i(a) \neq 0 \) for some \( x \in X \), hence \( x \otimes_H a \) must be nonzero too.

Corollary 11.6. A finite \( H \)-module \( L \) is torsion iff \( X \otimes_H L = 0 \).

Corollary 11.7. Let \( M, N \) be finite \( H \)-modules, \( f: M \to N \) be an \( H \)-linear map, and assume \( N \) to be torsionless. Then \( \mathcal{A}(f) = 0 \) if and only if \( f = 0 \).

\[ \begin{align*}
\text{Proof.} & \quad \mathcal{A}(f) = 0 \text{ means that } X \otimes_H f(M) = 0, \text{ hence } f(M) \subset N \text{ is torsion.} \\
\text{Remark 11.8.} & \quad \text{If } \Phi \text{ is an isomorphism, then } \Phi^{-1} \text{ induces } \psi, \text{ i.e., } \psi = \mathcal{A}(\Phi^{-1}).
\end{align*} \]

Remark 11.9. If \( \Phi \) is an isomorphism, then \( \Phi^{-1} \) induces \( \psi \), i.e., \( \psi = \mathcal{A}(\Phi^{-1}) \). Corollary 11.7 tells us that if \( L \) is torsionless and \( \psi \) is induced by some map \( \Psi \), then \( \Phi \) is an isomorphism and \( \Psi = \Phi^{-1} \).

Proposition 11.10. For any map of \( H \)-modules \( f: M \to N \), the following conditions are equivalent:

1. \( N/f(M) \) is torsion.
2. \( A(f): A(M) \to A(N) \) is surjective.
3. \( \text{gcd} (\text{ker} A(f)) < \text{dim} \frak{a} \).

\[ \begin{align*}
\text{Proof.} & \quad \text{To show the equivalence of 1. and 2., it is enough to tensor the exact sequence } \\
& \quad M \xrightarrow{f} N \to N/f(M) \to 0 \text{ with } X, \text{ getting the exact sequence } A(M) \to A(N) \to \\
& \quad X \otimes_H N/f(M) \to 0, \text{ and apply Corollary 11.6.} \\
& \quad \text{Assume that 3. holds but } N/f(M) \text{ is not torsion. Then it contains a nonzero element } a \text{ which generates a free } H \text{-module. Then } X \otimes_H a \simeq X \text{ has growth } \text{dim} \frak{a}, \text{ which is a contradiction.} 
\end{align*} \]

11.4. Reconstruction of a Lie pseudoalgebra. Now let \( L \) be a Lie \( H \)-pseudoalgebra which is finite as an \( H \)-module. Again, \( H = U(\frak{g}) \) will be the universal enveloping algebra of a finite-dimensional Lie algebra \( \frak{g} \). Let \( \mathcal{L} = \mathcal{A}(L) \) be the annihilation Lie algebra of \( L \), and \( \hat{L} = C(\mathcal{L}) \), as before.

By Proposition 11.2, \( \hat{L} \) satisfies all the properties of a Lie \( H \)-pseudoalgebra except the locality (9.13). An indirect way to establish the locality property for \( \hat{L} \) is by embedding it in the bigger (local) Lie pseudoalgebra \( gc_L \). In order to map \( \hat{L} \) to \( gc_L \), we need to assign to each element of \( \hat{L} \) a pseudolinear map from \( L \) to itself.

This can be done as follows. Recall that \( \mathcal{L} \) acts on \( L \) by \( \langle x \otimes_H a \rangle \cdot b = [a, b] \) for \( a, b \in L, x \in X \) (see Proposition 9.4). Now in terms of \( x \)-products the action of \( \hat{L} \) on \( L \) is given by \( \hat{a} \cdot b = a(x) \cdot b \). The locality condition \( a(x) = 0 \) for \( x \in F_n X, n \gg 0 \) is satisfied because \( a \) is continuous and \( L \) is a discrete topological \( \mathcal{L} \)-module (see Proposition 9.4). All the other axioms of a Lie pseudoalgebra representation follow easily from definitions.

We now need to find conditions for \( j: \hat{L} \to gc_L \) to be injective. Then \( \hat{L} \) would embed into \( gc_L \), which will show locality.

Lemma 11.11. If \( L \) is torsionless, the kernel of the above defined \( j: \hat{L} \to gc_L \) consists of all elements \( a \) such that \( a(X) \) is contained in the center of \( \mathcal{L} \).
Proof. Since $L$ is torsionless, $\Phi$ is injective by Proposition 11.5. Hence, for $a, b \in L$, $x \in X$, one has $[a, b]_y = 0$ iff $[a, b]_y = 0$ for all $y \in Y$. By (9.23), (9.24), this is equivalent to $[a, b]_y = 0$. Hence, for $l \in L$, $l \cdot b = 0$ for all $b$ iff $l$ lies in the center of $L$. Now $a \in \hat{L}$ is in the kernel of $j$ iff $a(x) \cdot b = 0$ for all $x$ and $b$, which means that $a(x)$ is central for all $x$. \hfill \Box

Lemma 11.12. If $L$ is torsionless and $L = A(L)$ has a finite-dimensional center, then $j: \hat{L} \to gcL$ is injective.

Proof. Let $a \in \hat{L}$ be in the kernel of $j$; then by the previous lemma $a(X)$ is contained in the center of $L$. The latter is finite-dimensional by assumption, so the kernel $N$ of $a$ is of finite codimension in $X$. This implies that $N$ is open in $X$, and it contains $F_i X$ for some $i$. Let $h \in F^{i+1}H$ but $h \notin F^i H$; then by Lemma 6.10, $h F_i X = F_{i-1} X = X$. Since $a$ is $H$-linear, $N$ is an $H$-submodule of $X$. Then $X = h F_i X \subset h N \subset N$, therefore $N = X$ and $a = 0$. \hfill \Box

Proposition 11.13. Let $L$ be a Lie $H$-pseudoalgebra which is finite and torsionless as an $H$-module. If its annihilating Lie algebra $L = A(L)$ has a finite-dimensional center, then $\hat{L} = C(L)$ is a Lie $H$-pseudoalgebra containing $L$ as an ideal.

Proof. The only thing that remains to be checked is the locality property for $\hat{L}$. It follows from that of $gcL$, since in this case $j: \hat{L} \to gcL$ is injective. \hfill \Box

For any topological Lie $H$-differential algebra $L$, we have a natural homomorphism $\psi: AC(L) \to L$, given by $x \circ_H a \mapsto a(x)$ for $a \in C(L)$, $x \in X$. The map $\psi$ does not need to be surjective, but we have a good control on injectivity, which can sometime prove useful.

Proposition 11.14. The kernel of $\psi: AC(L) \to L$ lies in the center of $AC(L)$.

Proof. Follows easily from (9.28) and (9.29). Say that $a$ lies in the kernel of $\psi$. Since $\psi$ is a homomorphism of Lie $H$-algebras, its kernel is an $H$-stable ideal of $AC(L)$. Then by (9.28), $(a \cdot b)_y \in \ker \psi$ for all $b \in C(L)$, $y \in X$, because in the right hand side all elements $h \cdot a$ lie in $\ker \psi$. This means that $(a \cdot b)(y) = 0$ for all $y \in X$, hence $a \cdot b = 0$ for every $b \in C(L)$. Now, use this in (9.29) to obtain that $a$ is central. \hfill \Box

12. Reconstruction of Pseudoalgebras of Vector Fields

In this section, we show that the reconstruction procedure of Section 11, when applied to the primitive Lie algebras of vector fields (or current algebras over them), gives the primitive pseudoalgebras of vector fields defined in Section 8 (or current pseudoalgebras over them).

As before, $\mathfrak{d}$ will be an $N$-dimensional Lie algebra, and $H = U(\mathfrak{d})$ its universal enveloping algebra. $L$ will be a Lie algebra provided with an action of $\mathfrak{d}$ and a filtration by subspaces $L = L_{-1} \supset L_{-2} \supset \cdots$. When $L$ is a subalgebra of $W_N$, it will always be considered with the filtration induced by the canonical filtration of $W_N$.

The Lie algebra $\text{Der } L$ of derivations of $L$ has the induced filtration:

$$(\text{Der } L)_k := \{d \in \text{Der } L \mid d(L_j) \subset L_{i+j} \quad \forall j\}.$$
The action of \( \mathfrak{g} \) is called \textit{transitive} if the composition of the homomorphism \( \mathfrak{g} \to \text{Der} \mathcal{L} \) and the projection \( \text{Der} \mathcal{L} \to \text{Gr}_{\mathcal{L}}(\text{Der} \mathcal{L}) := \text{Der} \mathcal{L}/(\text{Der} \mathcal{L})_0 \) is a linear isomorphism. This is equivalent to the following two conditions: \( \mathfrak{g} \hookrightarrow \text{Der} \mathcal{L} \) intersects \( (\text{Der} \mathcal{L})_0 \) trivially and \( \dim \text{Gr}_{\mathcal{L}}(\text{Der} \mathcal{L}) = N \).

**Lemma 12.1.** Let \( \mathcal{L} \) be a current Lie \( H \)-pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields. Then the action of \( \mathfrak{g} \) on its annihilation Lie algebra \( \mathcal{L} = \mathcal{A}(L) \) is transitive.

**Proof.** By Theorem 8.23, \( \mathcal{L} = \mathcal{O}_r \mathring{\mathcal{L}}' \) is a current Lie algebra over \( \mathcal{L}' \), where \( \mathcal{L}' \) is either a finite-dimensional simple Lie algebra \( \mathfrak{g} \) (for \( r = N = \dim \mathfrak{g} \)), or one of the Lie algebras of vector fields \( W_{N'}, S_{N'}, P_{N} \), or \( K_{N'} \) \((N' = N - r)\). In particular, we know that \( \dim \text{Gr}_{\mathcal{L}}(\text{Der} \mathcal{L}) = N \). By Lemma 7.13, a sufficiently high power of any nonzero element \( a \in \mathfrak{g} \) maps any given open subspace of \( \mathcal{L} \) surjectively onto \( \mathcal{L} \). This cannot hold if \( a \) belongs to \( (\text{Der} \mathcal{L})_0 \), therefore \( \mathfrak{g} \hookrightarrow \text{Der} \mathcal{L} \) is injective and the image of \( \mathfrak{g} \) intersects \( (\text{Der} \mathcal{L})_0 \) trivially. Comparing the dimensions, we get that \( \mathfrak{g} \to \text{Gr}_{\mathcal{L}}(\text{Der} \mathcal{L}) \) is an isomorphism. \( \Box \)

The main results of this section can be summarized by Theorem 12.2 below. Its proof follows from Sections 12.1–12.7.

**Theorem 12.2.** Let \( \mathcal{L} = \mathcal{O}_r \mathring{\mathcal{L}}' \) be a current Lie algebra over \( \mathcal{L}' \), where \( \mathcal{L}' \) is a simple linearly compact Lie algebra of growth \( N' = N - r \). Assume that \( \mathfrak{g} \) acts transitively on \( \mathcal{L} \). Then there is a codimension \( r \) subalgebra \( \mathfrak{g}' \) of \( \mathfrak{g} \), acting transitively on \( \mathcal{L}' \), such that the \( H \)-pseudoalgebra \( \mathcal{C}(\mathcal{L}) \) is isomorphic to a current pseudoalgebra over the \( H' \)-pseudoalgebra \( \mathcal{C}(\mathcal{L}') \), where \( H' = H(\mathfrak{g}') \). Moreover, \( \mathcal{C}(\mathcal{L}') \) is either a finite-dimensional simple Lie algebra (and \( \mathfrak{g}' = 0 \)) or one of the primitive \( H' \)-pseudoalgebras of vector fields \( W(\mathfrak{g}') \), \( S(\mathfrak{g}', \chi') \), \( f(\mathfrak{g}', \chi', \omega') \), or \( K(\mathfrak{g}', \theta') \).

12.1. **Reconstruction from \( W_N \).** Recall that \( X = H^* \) can be identified with \( \mathcal{O}_X = k[\{t_1, \ldots, t_N\}] \). The action of \( \mathfrak{g} \) on \( X \) gives an action on \( \mathcal{O}_X \) in terms of linear differential operators, i.e., an embedding \( \mathfrak{g} \hookrightarrow W_N = \text{Der} \mathcal{O}_X \) which we call the \textit{canonical embedding} of \( \mathfrak{g} \) in \( W_N \). Note that this embedding is transitive, i.e., \( \mathfrak{g} \subset W_N \) is complementary to \( \mathfrak{f} \cap W_N \).

A structure of an \( H \)-differential algebra on \( W_N \) is equivalent to a transitive action of \( \mathfrak{g} \) on \( W_N \) by derivations. Since \( \text{Der} W_N = W_N \), this is the same as a transitive embedding of \( \mathfrak{g} \) in \( W_N \). By Proposition 6.9, any two such embeddings are equivalent, i.e., conjugate by an automorphism of \( W_N \). With the canonical action of \( \mathfrak{g} \), \( W_N \) becomes isomorphic to the annihilation algebra of the \( H \)-pseudoalgebra \( W(\mathfrak{g}) \) defined in Section 8.1. Since \( W(\mathfrak{g}) \) is a free \( H \)-module, Proposition 11.3 shows that the reconstruction of \( W_N \) is \( W(\mathfrak{g}) \), i.e., \( \mathcal{C}(W_N) = W(\mathfrak{g}) \).

12.2. **Reconstruction from subalgebras of \( W_N \).** Let \( \mathcal{L} \) be a linearly compact Lie subalgebra of \( W_N \), with the induced filtration and with a transitive action of \( \mathfrak{g} \) on it. After an automorphism of \( W_N \), we can assume that the action of \( \mathfrak{g} \) is the canonical one. Then \( \mathcal{C}(\mathcal{L}) \) is a subalgebra of \( W(\mathfrak{g}) = \mathcal{C}(W_N) \), because the functor \( \mathcal{C} \) is left exact. Below we will be concerned with the case when \( \mathcal{L} \) is the subalgebra consisting of vector fields annihilating some differential form.

Let \( \omega \in \Omega^0(\mathfrak{g}) \) be a pseudoform, and \( I \subset H \) be a right ideal. We denote by \( W(\mathfrak{g}, \omega, I) \) the set of all elements \( \alpha \in W(\mathfrak{g}) = H \circ \mathfrak{g} \) such that

\[
\alpha \ast \omega \in (H \circ \omega) \circ \Omega^0(\mathfrak{g}).
\]
It is easy to check that $W(\mathfrak{d}, w, I)$ is a subalgebra of $W(\mathfrak{d})$.

**Lemma 12.3.** Let $\omega \in \Omega^n_X$ be a differential form, and $W_N(\omega)$ be the Lie subalgebra of $W_N$ consisting of vector fields annihilating $\omega$. If $\omega = y \otimes H w$ for some $y \in \mathcal{X}$, $w \in \Omega^n(\mathfrak{d})$, then $\mathcal{C}(W_N(\omega))$ is isomorphic to the Lie pseudoalgebra $W(\mathfrak{d}, w, I)$ where $I = \{ h \in H \mid y \cdot h = 0 \}$.

**Proof.** As was already remarked, $\mathcal{C}(W_N(\omega))$ is a subalgebra of $W(\mathfrak{d})$. Since $\Omega^n(\mathfrak{d}) = H \otimes \bigwedge^n \mathfrak{d}^*$ is a free $H$-module, we have $(H \otimes H) \otimes H \Omega^n(\mathfrak{d}) \simeq H \otimes H \otimes \bigwedge^n \mathfrak{d}^*$. For $\alpha \in W(\mathfrak{d})$, write $\alpha \ast w = \sum_i (f_i \otimes g_i) \otimes H w_i$ with $f_i, g_i \in H$ and linearly independent $w_i \in \bigwedge^n \mathfrak{d}^*$. Then for any $x \in X$ we have (cf. (7.2)):

$$L_{x \otimes H} \omega = \sum_i (x \cdot f_i)(y \cdot g_i) w_i.$$ 

This is zero for any $x$ iff $y \cdot g_i = 0$ for all $i$, which means $g_i \in I$. \(\square\)

12.3. **Reconstruction from current algebras over $W_N$.** Let now $\mathcal{L} = \mathcal{O}_r \hat{\otimes} W_N^1$ be a current algebra over $W_N^1$, and $\mathfrak{d}$ be an $N = N' + r$ dimensional Lie algebra acting transitively on $\mathcal{L}$. Then, by Proposition 6.12, $\mathfrak{d} \hookrightarrow \operatorname{Der} \mathcal{L} = W_r \hat{\otimes} 1 + \mathcal{O}_r \hat{\otimes} W_N^1 \subset W_N$. The Lie algebra $\mathcal{L}$ is described as the subalgebra of $W_N$ consisting of vector fields annihilating the functions $t_{N+1}, \ldots , t_N$, hence it is an intersection of algebras of the form $W_N(f_i)$ ($f_i \in \Omega^n_X = X$), see Section 12.2.

After an automorphism of $W_N$, we can assume that the action of $\mathfrak{d}$ on it is the canonical one. Then $\mathcal{L}$ becomes the intersection of $W_N(f_i)$ ($i = N' + 1, \ldots , N$) where $f_i$ is the image of $t_i$. Now Lemma 12.3 implies that $\mathcal{C}(\mathcal{L}) = W(\mathfrak{d}, 1, I)$ where $1 \in \Omega_0(\mathfrak{d}) = H$ and $I = \{ h \in H \mid f_i \cdot h = 0 \ (i = N' + 1, \ldots , N) \}$.

Recall that for $\alpha \in W(\mathfrak{d}, 1, I)$, its action on $1 \in H$ is given by $\alpha \ast 1 = -\alpha \otimes H 1 \equiv -\alpha$. Therefore $\alpha \in W(\mathfrak{d}, 1, I)$ iff $\alpha$ belongs to $(H \otimes \mathfrak{d}) \cap (H \otimes I) = H \otimes \mathfrak{d}'$, where the intersection $\mathfrak{d}' = \mathfrak{d} \cap I$ is a Lie subalgebra of $\mathfrak{d}$. Then $H' = U(\mathfrak{d}')$ is a Hopf subalgebra of $H$, and $H \otimes \mathfrak{d}' \simeq H \otimes \mathfrak{d}'$, $(H' \otimes \mathfrak{d}')$ is a current pseudoalgebra over $H' \otimes \mathfrak{d}' = W(\mathfrak{d}')$. We have thus proved the following lemma.

**Lemma 12.4.** The reconstruction of a current Lie algebra over $W_N^1$, provided with a transitive action of a Lie algebra $\mathfrak{d}$, is a current Lie $H$-pseudoalgebra over $W(\mathfrak{d}')$ where $\mathfrak{d}'$ is an $N'$-dimensional Lie subalgebra of $\mathfrak{d}$.

This result is a special case of Lemma 12.8 below.

12.4. **Solving compatible systems of linear differential equations.** Let $A$ be any associative $k$-algebra, and let $\mathcal{O}_r = k[[t_1, \ldots , t_r]]$, $W_r = \operatorname{Der} \mathcal{O}_r$, as before. For fixed $n \geq 0$, let $f_i(t) \in A[[t_1, \ldots , t_r]]$ ($i = 1, \ldots , r + n$) be formal power series with coefficients in $A$, where $t = (t_1, \ldots , t_r)$. Note that $W_r$ acts on $A[[t_1, \ldots , t_r]]$ by derivations.

Given $r + n$ linear differential operators $D_1, \ldots , D_{r + n} \in W_r$, consider the following system of differential equations for an unknown $y(t) \in A[[t_1, \ldots , t_r]]$:

$$D_i(y(t)) = y(t)f_i(t), \quad i = 1, \ldots , r + n. \tag{12.2}$$

We assume that the operators $D_i$ satisfy

$$[D_i, D_j] = \sum_k c_{ij}^k(t) D_k \quad \text{with} \quad c_{ij}^k(t) \in \mathcal{O}_r, \tag{12.3}$$

in other words, the space of all operators of the form $\sum_i p_i(t) D_i$ with $p_i(t) \in \mathcal{O}_r$ is a Lie algebra.
Suppose we have found a solution to the system (12.2). Combining equations (12.2) and (12.3), we get:
\[
[D_i, D_j](y) = D_i D_j(y) - D_j D_i(y) = D_i(yf_j) - D_j(yf_i) = yf_i f_j + yD_i(f_j) - yf_j - yD_j(f_i).
\]
and
\[
[D_i, D_j](y) = \sum_k c^k_{ij} D_k(y) = \sum_k c^k_{ij} yf_k.
\]
The system (12.2) is called compatible if
\begin{equation}
[f_i(t), f_j(t)] + D_i(f_j(t)) - D_j(f_i(t)) = \sum_k c^k_{ij}(t) f_k(t)
\end{equation}
for all \(i, j\).
When \(y(t)\) is not a divisor of zero in \(A[[t_1, \ldots, t_r]]\) the compatibility of the system is a necessary condition for having a solution. The compatibility (12.4) is equivalent to saying that \(\sum p_k(t) D_i \rightarrow \sum p_k(t) (D_i + f_i(t))\) is a homomorphism of Lie algebras.

We will be interested in solving a more general system of equations than (12.2). Before formulating it, let us note that the above remarks have obvious analogues for systems of the form
\begin{equation}
D_i(z(t)) = -h_i(t) z(t), \quad i = 1, \ldots, r + n
\end{equation}
with \(z(t), h_i(t) \in A[[t_1, \ldots, t_r]]\). The compatibility of (12.5) is equivalent to (12.4) with \(f_i\) replaced by \(h_i\).

Now consider the system
\begin{equation}
D_i(g(t)) = g(t) f_i(t) - h_i(t) g(t), \quad i = 1, \ldots, r + n
\end{equation}
for an unknown \(g(t) \in A[[t_1, \ldots, t_r]]\). We will show it has a solution, provided that both (12.2) and (12.5) are compatible and some initial conditions at \(t = 0\) are satisfied. (The compatibility of (12.2) and (12.5) implies the compatibility of (12.6).)

Proposition 12.5. In the above notation, let the operators \(D_i \in W_r\) satisfy (12.3) and
\begin{equation}
D_{i|t=0} = \begin{cases} \partial_{t_i}, & 1 \leq i \leq r, \\ 0, & r + 1 \leq i \leq r + n. \end{cases}
\end{equation}
Assume that the systems (12.2) and (12.5) are compatible (cf. (12.4)), and that
\begin{equation}
f_i(0) = h_i(0), \quad r + 1 \leq i \leq r + n.
\end{equation}
Then the system (12.6) has a unique solution \(g(t) \in A[[t_1, \ldots, t_r]]\) for any given initial condition \(g(0) \in A\) which commutes with \(f_i(0)\) \((r + 1 \leq i \leq r + n)\).

Proof. For \(r = 0\), both sides of (12.6) are trivial. For \(r \geq 1\), we will proceed by induction on \(r\).

First of all, note that the compatibility or solvability of the systems (12.2), (12.5) or (12.6) does not change when we apply an automorphism of \(\mathcal{O}_r\). The same is true when we make an elementary transformation: multiply one equation by a function (an element of \(\mathcal{O}_r\)) and add it to another equation. For example, we can replace all \(D_i, (i \neq r)\) by \(D_i - p_i(t) D_r\), and correspondingly \(f_i(t)\) by \(f_i(t) - p_i(t) f_r(t)\) and \(h_i(t)\) by \(h_i(t) - p_i(t) h_r(t)\), as long as we do not violate (12.7, 12.8).

Any vector field \(D_r \in W_r\) satisfying \(D_{i|t=0} = \partial_{t_r}\) can be brought to \(\partial_{t_r}\) after an automorphism of \(\mathcal{O}_r\), so we will assume that \(D_r = \partial_{t_r}\). Replacing \(D_i\) \((i \neq r)\) by \(D_i - D_i(t_r) D_r\), we can assume in addition that \(D_i(t_r) = 0\) for \(i \neq r\).
Now it makes sense to put \( t_r = 0 \) in the equations with \( i \neq r \) in (12.6). Let us denote \( \bar{D}_i = D_i|_{s=0}, f_i(\bar{f}) = f_i(t_1, \ldots, t_{r-1}, 0), \bar{h}_i(\bar{f}) = h_i(t_1, \ldots, t_{r-1}, 0), \bar{f} = (t_1, \ldots, t_{r-1}). \) Consider the reduced system

\[
(12.9) \quad \bar{D}_i(\bar{g}(\bar{f})) = \bar{g}(\bar{f}) \bar{f}_i(\bar{f}) - \bar{h}_i(\bar{f}) \bar{g}(\bar{f}), \quad i = 1, \ldots, r - 1, r + 1, \ldots, r + n
\]

for an unknown \( \bar{g}(\bar{f}) \in A[[t_1, \ldots, t_{r-1}]]. \) Note that, since \( D_i(t_r) = \delta_{ir}, \) we have:

\[ [D_i, D_j](t_r) = 0 \]

for any \( i, j \); hence \( [D_i, D_j] \) does not contain \( D_r. \) In particular, putting \( t_r = 0 \) we see that the operators \( \bar{D}_i \) satisfy (12.3). The other assumptions of the proposition are also easy to check, so by induction the system (12.9) has a solution \( \bar{g}(\bar{f}). \)

The equation

\[
(12.10) \quad \partial_r g(t) = g(t) f_r(t) - h_r(t) g(t)
\]

has a unique solution \( g(t) \) satisfying the initial condition

\[
(12.11) \quad g(t_1, \ldots, t_{r-1}, 0) = \bar{g}(\bar{f}).
\]

We claim that this \( g(t) \) is then a solution of the system (12.6). Indeed, it satisfies (12.6) for \( t_r = 0. \) Next, we compute for \( i \neq r \) (using (12.9), (12.10), and the compatibility of (12.2), (12.5));

\[
D_r D_i(g)|_{s=0} = [D_r, D_i](g)|_{s=0} + D_i D_r(g)|_{s=0} =
\]

\[
= \sum_i c_i \bar{D}_i g + \bar{D}_i(\bar{g} \bar{f}_i - \bar{h}_i \bar{g})
\]

\[
= \sum_i c_i (\bar{g} \bar{f}_i \bar{h}_i - \bar{h}_i \bar{g}) + (\bar{g} \bar{f}_i \bar{h}_i - \bar{h}_i \bar{g}) \bar{f}_i + \bar{g} \bar{D}_i(\bar{f}_i) - \bar{D}_i(\bar{h}_i) \bar{g} - \bar{h}_i(\bar{g} \bar{f}_i - \bar{h}_i \bar{g})
\]

\[
= g(D_r(f_i) + f_i f_r)|_{s=0} - (D_r(h_i) - h_i h_r) g|_{s=0} - \bar{h}_i g \bar{f}_i - \bar{h}_i \bar{g} \bar{f}_i
\]

\[
= D_r(g f_i - h_i g)|_{s=0}.
\]

This shows that \( \partial_r (D_i(g) - g f_i + h_i g)|_{s=0} = 0. \) We can apply the same argument with \([D_r, D_i] \) instead of \( D_i, \) and so on, to show that all derivatives with respect to \( t_r \) vanish at \( t_r = 0. \)

**Remark 12.6.** Any solution \( g(t) \) of the system (12.6), such that \( g(0) \) is invertible in \( A, \) is invertible in \( A[[t_1, \ldots, t_r]]. \) Its inverse \( g(t)^{-1} \) satisfies (12.6) with \( f_i \leftrightarrow -h_i. \)

### 12.5. Reconstruction from a Current Lie algebra.

Let \( L' \) be a simple linearly compact Lie algebra, and let \( L = \hat{O}_r \circ L' \) be a current algebra over \( L'. \) The filtration by subspaces \( L' = L'_{-1} \supset L'_{-2} \supset \cdots \) and the canonical filtration of \( \hat{O}_r \) give rise to the product filtration of \( L. \) Assume that \( \hat{O} \) acts on \( L' \) transitively by derivations.

By Proposition 6.12, we have Der \( L = W_r \oplus 1 + \hat{O}_r \circ \text{Der} L'. \)

Denote by \( j \) the embedding \( \hat{O} \hookrightarrow \text{Der} L, \) and by \( p \) the projection \( \text{Der} L \twoheadrightarrow W_r. \) The preimage \( \hat{O}' := (pj)^{-1}(F_0 W_r) \) is a Lie subalgebra of \( \hat{O} \) of codimension \( r. \) We have \( \hat{O}' \hookrightarrow F_0 W_r \oplus 1 + \hat{O}_r \circ \text{Der} L'. \) The latter contains \( F_0 W_r \oplus 1 + F_0 \hat{O}_r \circ \text{Der} L' \) as an ideal, hence we get a Lie algebra homomorphism \( j' : \hat{O}' \to \text{Der} L'. \) It leads to a transitive action of \( \hat{O}' \) on \( L' \), because the the action of \( \hat{O} \) on \( L \) is transitive.

**Lemma 12.7.** Any two transitive embeddings \( j : \hat{O} \hookrightarrow \text{Der} L, \) that induce the same subalgebra \( \hat{O}' \) and the same \( j' : \hat{O}' \to \text{Der} L' \), are equivalent up to an automorphism of \( \text{Der} L. \)

**Proof.** Let us choose a basis \( \{ \delta_i \} \) of \( \hat{O} \) and write \( j(\delta_i) = D_i + f_i(t) \) (i = 1, ..., N = \( r + n = \dim \hat{O} \)) where \( D_i \in W_r \) and \( f_i(t) \in \hat{O}_r \circ \text{Der} L, \) \( t = (t_1, \ldots, t_r). \) Note that \( D_i = (pj)(\delta_i) \) and \( p_j : \hat{O} \to W_r \) is a Lie algebra homomorphism. We can choose
the basis \( \{ \partial_i \} \) in such a way that \( D_i|_{x=0} = \partial_i \) for \( 1 \leq i \leq r \), and \( D_i|_{x=0} = 0 \) for \( r+1 \leq i \leq r+n \). Then \( \{ \partial_i \}_{i=r+1, \ldots, r+n} \) is a basis of \( \mathcal{O} \). Moreover, note that \( j' : \mathcal{O}' \to \text{Der} \mathcal{C}' \) is given by \( j'(\partial_i) = f_i(0) \).

Let \( \tilde{j} \) be another transitive embedding of \( \mathcal{O} \) into \( \text{Der} \mathcal{L} \). Since, by Proposition 6.9, the homomorphism \( p_j \) is uniquely determined by the choice of \( \mathcal{O}' \), we can assume that \( p_j(\partial_i) = D_i \). Then \( \tilde{j}(\partial_i) = D_i + h_i(t) \) for some \( h_i(t) \in \mathcal{O}_r \hat{\otimes} \text{Der} \mathcal{L}' \). By assumption, \( j' = \tilde{j}' \), hence \( f_i(0) = h_i(0) \) for \( r+1 \leq i \leq r+n \).

Now we want to find an automorphism \( g(t) \in \mathcal{O}_r \hat{\otimes} \text{Aut} \mathcal{L}' \) such that \( g(0) = \text{id} \) and \( g(t) \circ (D_i + f_i(t)) = (D_i + h_i(t)) \circ g(t) \). This equation is equivalent to (12.6), and it is easy to see that all conditions of Proposition 12.5 are satisfied: for example, the system (12.2) is compatible because \( j' \) is a homomorphism. This completes the proof.

Now given the embedding \( j' : \mathcal{O}' \to \text{Der} \mathcal{L}' \) we can consider the reconstruction \( \mathcal{L}' := \mathcal{C}(\mathcal{L}') \) which is a Lie \( H' \)-pseudoalgebra, where \( H' = U \langle \mathcal{O}' \rangle \). Given \( L' \) we can take the current \( H' \)-pseudoalgebra \( L := \text{Cur} L' = H \circ_H L' \). Since its annihilation Lie algebra \( \mathcal{A}(L) \) is isomorphic to \( \mathcal{L} \), we get an embedding \( j : \mathcal{O} \to \text{Der} \mathcal{L} \). It induces the same embedding \( j' \) as our initial \( j \), so by the previous lemma \( j' \) and \( \tilde{j} \) are equivalent. But then the reconstruction \( \mathcal{C}(\mathcal{L}) \) of \( \mathcal{L} \) provided with \( j \) is isomorphic to the reconstruction of \( \mathcal{L} \) provided with \( j' \) which is \( L \). This can be summarized as follows.

**Lemma 12.8.** The reconstruction \( \mathcal{C}(\mathcal{L}) \) of a current Lie algebra \( \mathcal{L} = \mathcal{O}_r \hat{\otimes} \mathcal{L}' \) over a simple linearly compact Lie algebra \( \mathcal{L}' \), provided with a transitive action of a Lie algebra \( \mathcal{O} \), is a current Lie \( H \)-pseudoalgebra over the \( H' \)-pseudoalgebra \( \mathcal{C}(\mathcal{L}') \), where \( H' = U \langle \mathcal{O}' \rangle \) and \( \mathcal{O}' \) is a Lie subalgebra of \( \mathcal{O} \) of codimension \( r \).

12.6. Reconstruction from \( S_N \). Now consider \( S_N \) with a transitive action of \( \mathcal{O} \) on it. Since \( \text{Der} S_N = C S_N \subset W_N \), we have \( \mathcal{O} \to W_N \). After an automorphism of \( W_N \), we can assume \( \mathcal{O} \to W_N \) is the canonical embedding, while \( S_N \) becomes \( W_N(\omega) (\equiv S_N(\omega)) \) where \( \omega \in \Omega^N_X \) is a volume form. We can write \( \omega = y \circ_H v \) with \( y \in X \) and \( v \in \Lambda^N \mathcal{O}^* \). Then, by Lemma 12.3, the reconstruction of \( W_N(\omega) \) is \( W(\mathcal{O}, v, l) \) where \( l = \{ h \in H \mid y \cdot h = 0 \} \) is as before.

The action of \( W(\mathcal{O}) \) on \( v \) is given by (8.8). In the notation of Section 8.4, we have for \( a \in W(\mathcal{O}) \):

\[
\alpha \ast v = -(\text{div}^{\text{trad}}(\alpha) \circ 1 + \alpha) \circ_H v.
\]

This shows that \( a \in W(\mathcal{O}, v, l) \) iff \( \text{div}^{\text{trad}}(\alpha) \circ 1 + \alpha \in H \circ_H l \).

Note that, since \( \omega \neq 0 \), we have \( l \cap k = 0 \). The intersection \( l \cap (\mathcal{O} + k) \) is a Lie algebra. The projection \( \pi : (\mathcal{O} + k) \to \mathcal{O} \) is a Lie algebra homomorphism, which maps \( l \cap (\mathcal{O} + k) \) isomorphically onto a subalgebra \( \mathcal{O}' \) of \( \mathcal{O} \). The inverse isomorphism \( \mathcal{O}' \to l \cap (\mathcal{O} + k) \) is given by \( a \mapsto a + \chi(a) \) for some linear functional \( \chi : \mathcal{O}' \to k \) which vanishes on \( \mathcal{O} \). Conversely, any such \( \chi \) gives rise to an isomorphism as above.

For \( \beta \in H \circ_H (\mathcal{O} + k) \), the equation \( \beta \in H \circ_H l \) is equivalent to the following two conditions: \( (\text{id} \circ \pi)(\beta) \in H \circ \mathcal{O}' \) and \( (\text{id} \circ \pi + \text{id} \circ \chi)(\pi)(\beta) = \beta \). Applying this for \( \beta = \text{div}^{\text{trad}}(\alpha) \circ 1 + \alpha \), we get \( (\text{id} \circ \pi)(\alpha) = \alpha \in H \circ \mathcal{O}' \) and \( (\text{id} \circ \pi + \text{id} \circ \chi)(\pi)(\beta) = \alpha + (\text{id} \circ \chi)(\alpha) = \beta \). The latter equation is equivalent to \( (\text{id} \circ \chi)(\alpha) = \text{div}^{\text{trad}}(\alpha) \circ 1 \), i.e. to \( \text{div}^{\text{trad}}(\chi) = 0 \). We have proved:

**Lemma 12.9.** The reconstruction of the Lie algebra \( S_N \), provided with a transitive action of a Lie algebra \( \mathcal{O} \), is a current Lie \( H \)-pseudoalgebra over \( S(\mathcal{O}', \chi') \) where \( \mathcal{O}' \)
is a Lie subalgebra of $\mathfrak{d}$ and $\chi'$ is a linear functional $\mathfrak{g}' \to \mathfrak{k}$ which vanishes on $[\mathfrak{g}', \mathfrak{g}']$.

In fact, one can show that in this case $\mathfrak{g}' = \mathfrak{d}$, i.e., $\dim I \cap (\mathfrak{d} + \mathfrak{k}) = N$, but the above statement is sufficient for our purposes.

12.7. **Reconstruction from $K_N$ and $H_N$.** Now let $\mathcal{L}$ be one of the Lie algebras $K_N$ or $P_N$, together with a transitive action of $\mathfrak{d}$ on it. We know from Section 6 that as a topological vector space $\mathcal{L}$ is homeomorphic to $X$. Since, by Proposition 6.9, all transitive actions of $\mathfrak{d}$ on $X$ are equivalent, $\mathcal{L}$ is isomorphic to the “canonical” $H$-module $X$, i.e., we may assume that the embedding $\mathfrak{d} \hookrightarrow W_N = \text{Der } X$ is the canonical one (Section 12.1). Then, by Proposition 11.3, the reconstruction of $\mathcal{L}$ is isomorphic to $H$ as an $H$-module. In other words, $\mathfrak{c}(\mathcal{L})$ is a free $H$-module of rank one.

**Lemma 12.10.** The reconstruction of the Lie algebras $K_N$ and $H_N$, provided with a transitive action of a Lie algebra $\mathfrak{d}$, is a free $H$-module of rank one.

**Proof.** It is enough to show that the reconstruction functor $\mathfrak{c}$ gives the same result on the topological $H$-modules $X = P_N$ and $X/\mathfrak{k} = H_N$. In order to do so, we must show that every $H$-linear continuous homomorphism of $X$ to $X/\mathfrak{k}$ can be obtained from a unique $H$-linear continuous homomorphism of $X$ to itself by composing with the canonical projection $X \to X/\mathfrak{k}$.

Since $X/\mathfrak{k}$ is linearly compact, by Remark 6.2 there is a bijection between $\text{Hom}_H^\text{cont} [X, X/\mathfrak{k}]$ and $\text{Hom}_H ((X/\mathfrak{k})^*, H)$. The space $(X/\mathfrak{k})^*$ is nothing but the augmentation ideal $H_+ = \ker \varepsilon \subset H$. Therefore we are reduced to show that every $H$-linear map $\phi : H_+ \to H$ is a restriction of a unique $H$-linear map $H \to H$.

An $H$-linear $\phi : H_+ \to H$ is determined by its value on $\mathfrak{d} \subset H_+$. If $a, b \in \mathfrak{d}$, then $ab - ba = [a, b]$, hence $a\phi(b) - b\phi(a) = \phi([a, b])$. Let $d$ be the maximal degree of $\phi(a)$. Then $a\phi(b) = b\phi(a)$ modulo $F^d H$. This means that there exists some $a \in F^d H$ such that for every $a \in \mathfrak{d}$, $\phi(a) = aa$ modulo $F^{d-1} H$. Then the difference between $\phi$ and right multiplication by $a$ is still $H$-linear, and its maximal degree on elements from $\mathfrak{d}$ is strictly less than $d$. The proof now follows by induction.

All Lie pseudoalgebras that are free $H$-module of rank one are classified in Theorem 8.9: they are isomorphic to current pseudoalgebras over $K(\mathfrak{g}', \mathfrak{g}')$ or $H(\mathfrak{g}', \chi', \omega')$.

13. **Structure Theory of Lie Pseudoalgebras**

13.1. **Structural correspondence between a Lie pseudoalgebra and its annihilation algebra.** Recall that a Lie $H$-pseudoalgebra $L$ is called finite if it is finitely generated as an $H$-module. If $H$ is Noetherian (e.g., $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$) and $L$ is finite, then $L$ is a Noetherian $H$-module, i.e., every increasing sequence of $H$-submodules of $L$ stabilizes.

For any two subspaces $A$ and $B$ of $L$, let

$$[A, B] = \{[a, b] \mid a \in A, b \in B, x \in X\}.$$  

Define the derived series of $L$ by $L^{(0)} = L$, $L^{(1)} = [L, L]$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. A Lie pseudoalgebra $L$ is called solvable if $L^{(n)} = 0$ for some $n$. Similarly, define the central series of $L$ by $L^{(0)} = L$, $L^1 = [L, L]$, $L^{n+1} = [L^n, L]$. The Lie pseudoalgebra $L$ is called nilpotent if $L^n = 0$ for some $n$. 


A Lie pseudoalgebra $L$ is called simple if it contains no nontrivial ideals and is not abelian. Note that $[L, L]$ is an ideal of $L$, so in particular, $[L, L] = L$ if $L$ is simple. $L$ is called semisimple if it contains no nonzero abelian ideals.

We will show that, as in the Lie algebra case, $L$ is semisimple if and only if its radical is zero. Provided that it exists, we define the radical of $L$, $\text{Rad } L$, to be its maximal solvable ideal. When $H$ is Noetherian and $L$ is finite, $\text{Rad } L$ exists because of the Noetherianity of $L$ and part (ii) of the next lemma.

**Lemma 13.1.** (i) If $S$ is a solvable ideal in $L$ and $L/S$ is solvable, then $L$ is solvable.

(ii) If $S_1, S_2$ are solvable ideals in $L$, then their sum $S_1 + S_2$ is a solvable ideal.

(iii) $L/\text{Rad } L$ is semisimple. $L$ is semisimple iff $\text{Rad } L = 0$.

**Proof.** (i) is standard.

(ii) follows from (i) and the fact that $(S_1 + S_2)/S_1 \simeq S_2/(S_1 \cap S_2)$.

(iii) If $L/\text{Rad } L$ has an abelian ideal $I$, then the preimage of $I$ under the natural projection $L \to L/\text{Rad } L$ must be solvable and strictly bigger than $\text{Rad } L$, which is a contradiction. □

It is easy to see, using (9.23, 9.24), that for any two subspaces $A, B \subseteq L$, we have:

$$[X \odot_H A, X \odot_H B] = X \odot_H [A, B]$$

as subspaces of $\mathcal{A}(L) = X \odot_H L$. In particular, if $I$ is an ideal of $L$, then $X \odot_H I$ is an ideal of $\mathcal{A}(L)$. We will call an ideal of $\mathcal{A}(L)$ regular if it is of the form $X \odot_H I$ for some ideal $I$ of $L$.

**Lemma 13.2.** Let $L$ be a Lie $H$-pseudoalgebra and $I \subseteq L$ be an ideal. Then:

(i) $X \odot_H I = 0$ only if $I$ is central.

(ii) $X \odot_H I = \mathcal{A}(L)$ only if $[L, L] \subseteq I$.

**Proof.** (i) has already been proved, when $L$ is finite, in Corollary 11.6 and Remark 11.8. In the general case, it can be deduced from Proposition 9.4. Let $a \in I$, then $a_x \equiv x \odot_H a = 0$ for any $x \in X$. Hence the action of $a_x$ on $L$ is trivial, and by (9.22), $[a \ast b] = 0$ for any $b \in L$.

In order to prove (ii), notice that $X \odot_H L/I = 0$. Then build a Lie $H$-pseudoalgebra structure on $\tilde{L} = L \oplus L/I$ by letting $L$ act on the abelian ideal $L/I$ via the adjoint action. Then $L/I$ is central in $\tilde{L}$, hence $L$ acts trivially on $L/I$. This means $[L, L] \subseteq I$. □

Using this lemma and (13.2), it is easy to prove the next two results.

**Proposition 13.3.** A Lie pseudoalgebra $L$ is solvable (respectively nilpotent) if and only if its annihilation Lie algebra $\mathcal{A}(L)$ is.

**Proposition 13.4.** Let $L$ be a centerless Lie $H$-pseudoalgebra which is equal to its derived subalgebra $[L, L]$. Then $L$ is simple if either of the following conditions holds:

(i) $\mathcal{A}(L)$ has no nontrivial $H$-invariant ideals.

(ii) $L$ is finite or free, and $\mathcal{A}(L)$ has no non-central $H$-invariant ideals.

**Proof.** (i) is immediate from Lemma 13.2.

Assume that (ii) holds but $L$ is not simple. Then $\mathcal{A}(L)$ has a nontrivial central regular ideal. If $a_x \equiv x \odot_H a$ is central in $\mathcal{A}(L)$ for every $x \in X$, then by (9.24)
\[ [a, b]_y = 0 \text{ for every } b \in L, \ x, y \in X. \] When \( L \) is either finite or free, \( l_y = 0 \) for all \( y \in X \) if and only if \( l = 0 \) (cf. Corollary 11.6). Therefore \( [a, b] = 0 \) for all \( b \in L, \ x \in X \), and by (9.22) we get \( [a \ast b] = 0 \) for any \( b \in L \). Hence \( a = 0 \).

As an immediate consequence we obtain:

**Corollary 13.5.** Let \( L \) be a current Lie \( H \)-pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields. Then \( L \) is simple.

**Proof.** It is easy to check that \( L \) satisfies the assumptions of Proposition 13.4 (see Theorem 8.23).

The following proposition will play an important role in the classification of finite simple Lie pseudoalgebras.

**Proposition 13.6.** For any Lie \( H \)-pseudoalgebra \( L \), any non-central \( H \)-invariant ideal \( J \) of \( A(L) \) contains a nonzero regular ideal.

**Proof.** Let \( a \in J \) be non-central. Assume that \( X \otimes_H a \cdot l = 0 \) for all \( l \in L \). Note that by Proposition 9.4, we have: \( h(a \cdot l) = (h(a) \cdot (h(l))) \) for \( h \in H \). This implies: \( (h(a)) \cdot l = \sum_i (h(a) \cdot (h(l))) \), which gives \( X \otimes_H (h(a)) \cdot l = 0 \) for any \( h \in H, \ l \in L \). Then we can use (9.29) to show that \( a \) is central in \( A(L) \), which is a contradiction.

Therefore, there is some \( l \in L \) such that \( a \cdot l = a \) has a nonzero Fourier coefficient, i.e., \( X \otimes_H a \neq 0 \). Since \( a_y = (a \cdot l)_y = \sum_i [h_i a, l_{x_i}] \), and \( J \) is \( H \)-stable, we see that all Fourier coefficients of \( a \) lie in \( J \). Then, due to (9.24), all elements in the ideal \( (a) \) of \( L \) generated by \( a \) have all of their Fourier coefficients in \( J \), i.e., \( 0 \neq X \otimes_H (a) \subset J \).

13.2. **Annihilation algebras of finite simple Lie \( U(\mathfrak{d}) \)-pseudoalgebras.** We will now approach the problem of classification of all finite simple Lie \( H \)-pseudoalgebras. In view of Kostant’s Theorem 2.1 and the results of Section 5, we will first restrict ourselves to the case when \( H \) is the universal enveloping algebra of a Lie algebra \( \mathfrak{d} \). Moreover, we will assume that \( \mathfrak{d} \) is finite-dimensional; in this case \( H = U(\mathfrak{d}) \) is filtered by finite-dimensional subspaces. The classification is done in two steps: the first one (done in this subsection) is classifying all Lie algebras that can arise as \( A(L) \) for some finite simple Lie \( H \)-pseudoalgebra \( L \), the second step (done in the next subsection) involves a reconstruction of \( L \) from its annihilation Lie algebra \( A(L) \) and the action of \( H \) on it.

**Theorem 13.7.** If \( L \) is a finite simple Lie \( H = U(\mathfrak{d}) \)-pseudoalgebra, then its annihilation Lie algebra \( A(L) \) is isomorphic (as a topological Lie algebra) to an irreducible central extension of a current Lie algebra \( \mathfrak{c} \otimes \mathfrak{s} \) where \( \mathfrak{s} \) is a simple linearly compact Lie algebra of growth \( gw(s) = \dim(\mathfrak{d}) - r \).

**Proof.** First of all, we observe that \( \mathcal{L} = A(L) \) is a linearly compact Lie algebra with respect to the topology defined in Section 7.4, see Proposition 7.12(ii). Consider the extended annihilation algebra \( \mathcal{L}' = \mathfrak{d} \otimes \mathcal{L} \), obtained by letting \( \mathfrak{d} \) act on \( \mathcal{L} = A(L) \) according to its \( H = U(\mathfrak{d}) \)-module structure.

**Lemma 13.8.** \( \mathcal{L}' = \mathfrak{d} \otimes \mathcal{L} \) is a linearly compact Lie algebra possessing a fundamental subalgebra, i.e., an open subalgebra containing no ideals of \( \mathcal{L}' \).
Proof. Indeed, if \( L_0 \) is a finite-dimensional subspace of \( L \) generating it over \( H \), then because of (7.14), \( L_i = F_i X \otimes_H L_0 \) is a subalgebra of \( L \) for \( i \geq s \). None of \( L_i \) contains ideals of \( \mathfrak{d} \triangleleft \mathcal{L} \), since every such ideal is stable under the action of \( H \) and \( H \cdot F_i X = X \), which implies \( H \cdot L_i \triangleleft \mathcal{L} \).

The center \( Z \) of \( \mathcal{L} \) is an \( H \)-stable closed ideal. The quotient \( \mathcal{L}^c / Z = \mathfrak{d} \triangleleft (\mathcal{L} / Z) \) is a linearly compact Lie algebra possessing a fundamental subalgebra \( \mathcal{L}_r/(Z \cap \mathcal{L}_s) \). Theorem 13.7 will be deduced from Proposition 6.11 applied for \( \mathcal{L}^c := \mathcal{L} / Z \).

By Proposition 13.6, any nonzero \( H \)-stable ideal of \( \mathcal{L}^c := \mathcal{L} / Z \) contains the image of a nonzero regular ideal of \( \mathcal{L} \). Since \( L \) is simple, this means that the only nonzero \( H \)-stable ideal of \( \mathcal{L} \) is the whole \( \mathcal{L} \). Then every nonzero ideal of \( \mathcal{L}^c \) contained in \( \mathcal{L} \) must equal \( \mathcal{L} \). Hence \( \mathcal{L} \) is a minimal closed ideal of a linearly compact Lie algebra satisfying the assumptions of Proposition 6.11(i), and is therefore of the form stated in part (ii) of this proposition.

Therefore, \( \mathcal{L} \) is a central extension of a current Lie algebra over a simple linearly compact Lie algebra. Moreover, \( \mathcal{L} \) equals its derived subalgebra (otherwise we would have a proper nonabelian subideal of \( \mathcal{L} \)). Hence it is an irreducible central extension.

Consider the canonical filtration \( F_n(\mathcal{L}_r \hat{\triangleleft} s) := \sum_{i \geq 0} F_{n-i} \mathcal{L}_r \hat{\triangleleft} F_i s \), where \( F_i s \) is the canonical filtration of \( s \) defined in Section 6 (if \( \text{dim} s < \infty \) we put \( F_i s = 0 \) for \( i \geq 0 \)). Then the growth of \( \mathcal{L}_r \hat{\triangleleft} s \) (with respect to this filtration) equals \( \text{gw} \mathcal{L}_r + \text{gw} s = r + \text{gw} s \). It is clear from Proposition 6.12 that any irreducible central extension of \( \mathcal{L}_r \hat{\triangleleft} s \) has the same growth. On the other hand, with respect to the filtration defined by (7.11), the growth of \( \mathcal{L} \) is equal to \( N = \text{dim} \mathfrak{d} \) (see Proposition 7.15). We have to show that the two different filtrations give the same growth.

Recall that by Lemma 7.13, a sufficiently high power of any nonzero element \( a \in \mathfrak{d} \) maps any given open subspace of \( \mathcal{L} \) surjectively onto \( \mathcal{L} \). Then the same argument as in the proof of Lemma 12.1 shows that \( \mathfrak{d} \rightarrow \text{Der} \mathcal{L} \) intersects \( F_n(\text{Der} \mathcal{L}) \) trivially, where \( F_n(\text{Der} \mathcal{L}) \) is induced by the canonical filtration on \( \mathcal{L}_r \hat{\triangleleft} s \). This implies \( N \leq r + \text{gw} s \).

To show the inverse inequality, note that since \( F_n(\mathcal{L}_r \hat{\triangleleft} s) \) is open in \( \mathcal{L} \cong \mathcal{L} / Z \cong \mathcal{L}_r \hat{\triangleleft} s \), it contains some \( \mathcal{L}_m := \mathcal{L}_m / (Z \cap \mathcal{L}_m) \). Now (7.14) implies \( [\mathcal{L}_i, \mathcal{L}] \subset \mathcal{L}_{i-1} \), which together with (6.1) leads to \( \mathcal{L}_{m+n(i+1)} \subset F_n(\mathcal{L}_r \hat{\triangleleft} s) \) for all \( n \geq 0 \). This implies \( N \geq r + \text{gw} s \).

This completes the proof of Theorem 13.7. \( \square \)

In fact, the above arguments can be used to prove a stronger statement than Theorem 13.7.

**Corollary 13.9.** Let \( L \) be a finite Lie \( H \)-pseudoalgebra and \( M \) be a minimal nonabelian ideal of \( L \). Then the annihilation algebra of \( M \) is one of the Lie algebras described in Theorem 13.7.

**Proof.** The only place in the proof of Theorem 13.7 where we used the simplicity of \( L \) was where we deduced that any nonzero regular ideal of \( \mathcal{A}(L) \) must equal the whole \( \mathcal{A}(L) \). This argument is modified as follows. Let \( J = X \otimes_H I \) be a nonabelian regular ideal of \( \mathcal{A}(L) \) contained in \( \mathcal{A}(M) \). Then the minimality of \( M \) implies that \( I = M \) and \( J = \mathcal{A}(M) \). The proof is concluded again by applying Proposition 6.11. \( \square \)
13.3. Classification of finite simple Lie $U(\mathfrak{d})$-pseudoalgebras. We will call a pseudoalgebra of vector fields any subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$. As in Section 8, a pseudoalgebra of vector fields is called primitive if it is one of the following: $W(\mathfrak{d}), S(\mathfrak{d}, \chi), H(\mathfrak{d}, \chi, \omega)$ or $K(\mathfrak{d}, \theta)$ (then its annihilation algebra $\mathcal{A}(L)$ is isomorphic to one of the primitive Lie algebras $W_N$, $S_N$, $P_N$ or $K_N$).

The following is the main theorem of this section.

**Theorem 13.10.** Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Then any finite simple Lie $H$-pseudoalgebra $L$ is isomorphic to a current pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields.

Explicitly, $L \simeq \text{Cur}_H^{\mathfrak{d}'}$, $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$, and $L'$ is one of the following:
(a) $L'$ is a finite-dimensional simple Lie algebra and $\mathfrak{d}' = 0$;
(b) $L' = W(\mathfrak{d}')$, $\mathfrak{d}'$ is arbitrary;
(c) $L' = S(\mathfrak{d}', \chi')$, where $\mathfrak{d}'$ is arbitrary and $\chi' \in (\mathfrak{d}')^*$ is such that $\chi'(\mathfrak{d}', \mathfrak{d}') = 0$;
(d) $L' = H(\mathfrak{d}', \chi', \omega')$, where $N' = \dim \mathfrak{d}'$ is even, $\chi'$ is as in (c), and $\omega' \in \bigwedge^2 (\mathfrak{d}')^*$ such that $(\omega')^{N'/2} \neq 0$ and $d\omega' + \chi' \wedge \omega' = 0$;
(e) $L' = K(\mathfrak{d}', \theta')$, where $N' = \dim \mathfrak{d}'$ is odd and $\theta' \in (\mathfrak{d}')^*$ is such that $\theta' \wedge (\theta')^{(N'-1)/2} \neq 0$.

**Proof.** By Theorem 13.7, the annihilation algebra $\mathcal{L}$ of $L$ is an irreducible central extension of a current Lie algebra $\widehat{\mathcal{L}} = \mathcal{O}_r \widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a simple linearly compact Lie algebra of growth $N' = N - r$. We have surjective maps

$$\mathcal{O}_r \widehat{\mathfrak{g}} \rightarrow \mathcal{L} \rightarrow \mathcal{O}_r \widehat{\mathfrak{g}}, \tag{13.3}$$

where $\mathfrak{s}$ is the universal central extension of $\mathfrak{g}$. By Theorem 6.8, $\mathfrak{g}$ is either finite-dimensional (when $N' = 0$) or one of the Lie algebras $W_N$, $S_N$, $H_N$, or $K_N$. By Proposition 6.12, we have $\mathfrak{s} = \mathfrak{g}$ in all cases, except $\mathfrak{g} = H_N$ in which case the center of $\mathfrak{s} = P_N$ is 1-dimensional.

Note that $\text{Der} \mathfrak{s} = \text{Der} \mathfrak{g}$, and therefore, by Proposition 6.12, we have $\text{Der}(\mathcal{O}_r \widehat{\mathfrak{g}}) = \text{Der}(\mathcal{O}_r \widehat{\mathfrak{g}}) = \text{Der}(\mathcal{L} \rightarrow \text{Der}(\mathcal{L} \widehat{\mathfrak{g}})).$ Then the action of $\mathfrak{d}$ on $\mathcal{L}$ induces actions on $\mathcal{O}_r \widehat{\mathfrak{g}}$ and $\mathcal{L} \widehat{\mathfrak{g}}$. The argument from the proof of Lemma 12.1 shows that these actions are transitive.

Now, let us apply the reconstruction functor $\mathcal{C}$ to the maps in (13.3). By Theorem 12.2, $\mathcal{C}(\mathcal{O}_r \widehat{\mathfrak{g}}) \simeq \text{Cur}_H^{\mathfrak{d}'}$, $\mathcal{C}(\mathfrak{s})$, and $\mathcal{S} := \mathcal{C}(\mathfrak{s})$ is one of the Lie pseudoalgebras described in (a)–(e) above. Moreover, by Lemma 11.1, we have $\mathcal{C}(\mathfrak{g}) \simeq \mathcal{C}(\mathfrak{s}) = \mathcal{S}$, and hence $\mathcal{C}(\widehat{\mathfrak{g}}) \simeq \text{Cur}_H^{\mathfrak{d}'}$. We therefore obtain $H$-linear maps $\text{Cur}_H^{\mathfrak{d}'} \rightarrow \mathcal{L} \rightarrow \text{Cur}_H^{\mathfrak{d}'}$, $S$ whose composition is the identity. Hence $\mathfrak{L} := \mathcal{C}(\mathcal{L})$ is isomorphic to $\text{Cur}_H^{\mathfrak{d}'}$, which is a simple Lie pseudoalgebra (Corollary 13.5).

The homomorphism $\Phi : L \rightarrow \mathcal{L}$ given by (11.6) is injective because $L$ is centerless (Remark 11.8). The action of $\mathfrak{L}$ on $L$ built in Section 11.4 shows that the image of $\Phi$ is an ideal of $\mathfrak{L}$. Since $\mathfrak{L}$ is simple, it follows that $\Phi$ is an isomorphism.

Corollary 13.9 and the above proof imply the following result.

**Corollary 13.11.** Let $L$ be a finite Lie pseudoalgebra and $M$ be a minimal nonabelian ideal of $L$. Then $M$ is a simple Lie pseudoalgebra.

**Lemma 13.12.** If $L$ is a centerless Lie pseudoalgebra, then any nonzero finite ideal of $L$ contains a nonzero minimal ideal.
Proof. By Zorn’s Lemma, it is enough to show that \( \bigcap I_\alpha \neq 0 \) for any collection of finite ideals \( \{I_\alpha\}_{\alpha \in A} \) of \( L \) such that \( I_\alpha \subset I_\beta \) for \( \alpha < \beta \), where \( A \) is a totally ordered index set. Assume that \( \bigcap I_\alpha = 0 \). Then there is a chain of ideals \( \{I_\alpha\}_{\alpha \in A'} \) (\( A' \subset A \)) of the same rank whose intersection is zero. Fix some \( \alpha_0 \in A' \). Then for any \( \beta \in A' \), \( \beta < \alpha_0 \), the module \( I_{\alpha_0}/I_\beta \) is torsion, so by Corollary 10.17, \( L \) acts trivially on it. This implies \( [L, I_{\alpha_0}] \subset I_\beta \) for each such \( \beta \), hence \( I_{\alpha_0} \) is central. \( \square \)

13.4. Derivations of finite simple \( U(\mathfrak{g}) \)-pseudoalgebras. We will determine all derivations of a finite simple \( U(\mathfrak{g}) \)-pseudoalgebra \( L \) (see Definition 10.8).

First let us consider the case when \( L = \text{Cur} \mathfrak{g} := H \circ \mathfrak{g} \) is a current pseudoalgebra over a finite-dimensional Lie algebra \( \mathfrak{g} \). The Lie pseudoalgebra \( W(\mathfrak{g}) \) acts on \( L \) by just acting on the first factor in \( H \circ \mathfrak{g} \) (cf. (8.4)):

\[
(13.4) \quad (f \circ a) \ast (g \circ b) = -(f \circ ga) \circ h (1 \circ b), \quad f, g \in H, \ a \in \mathfrak{g}, \ b \in \mathfrak{g}.
\]

We also have an embedding \( \text{Cur} \mathfrak{g} \subset \text{Der} L \). The image of \( \text{Cur} \mathfrak{g} \) in \( \text{Der} L \) is normalized by that of \( W(\mathfrak{g}) \), and the two forms a semidirect sum \( W(\mathfrak{g}) \ltimes \text{Cur} \mathfrak{g} \) which as an \( H \)-module is isomorphic to \( H \circ (\mathfrak{g} \ltimes \text{Cur} \mathfrak{g}) \).

Lemma 13.13. For any simple finite-dimensional Lie algebra \( \mathfrak{g} \), we have \( \text{Der} \text{Cur} \mathfrak{g} = W(\mathfrak{g}) \ltimes \text{Cur} \mathfrak{g} \).

Proof. By Lemma 10.9(iii), the annihilation algebra \( \mathcal{A}((\text{Der} \text{Cur} \mathfrak{g})) \subset \mathcal{A} \text{Cur} \mathfrak{g} = \text{Der} (X \circ \mathfrak{g}) \). By Proposition 6.12(iii), the latter is isomorphic to \( W_N \circ 1 + \mathcal{O}_N \circ \mathfrak{g} \), since \( X \simeq \mathcal{O}_N \). Now Theorem 12.2 shows that \( \text{Der} \text{Cur} \mathfrak{g} \subset W(\mathfrak{g}) \circ \text{Cur} \mathfrak{g} \). \( \square \)

The same argument as in the proof of the lemma shows that \( \text{Der} L = L \) when \( L \) is one of the primitive pseudoalgebras of vector fields \( W(\mathfrak{g}) \), \( S(\mathfrak{g}, \chi) \), \( H(\mathfrak{g}, \chi, \omega) \) or \( K(\mathfrak{g}, \delta) \). In fact, the same holds when \( L \) is a current pseudoalgebra over one of them.

Lemma 13.14. Let \( L \) be a simple pseudoalgebra of vector fields (i.e., \( L \) is a current pseudoalgebra over one of the primitive ones). Then \( \text{Der} L = L \).

Proof. Let \( L \) be a current pseudoalgebra over \( L' \), and \( L' \subset W(\mathfrak{g}') \) be one of the primitive pseudoalgebras of vector fields, where \( \mathfrak{g}' \) is a Lie subalgebra of \( \mathfrak{g} \). Then, by Theorem 8.23, the annihilation algebra \( \mathcal{L} = \mathcal{A}(L) \) is a current Lie algebra over \( \mathcal{L}' = \mathcal{A}(L') \): \( \mathcal{L} = \mathcal{O}_r \mathcal{L}' \), and \( \mathcal{L}' \) is isomorphic to \( W_N' \), \( S_N' \), \( P_N' \) or \( K_N \), where \( N' = \dim \mathfrak{g}' = N - r, N = \dim \mathfrak{g} \).

By Lemma 10.9(iii), for each \( \phi \in \text{Der} L \) and \( x \in X \), \( \phi_x \) is a derivation of \( \mathcal{L} \). By Proposition 6.12, we have: \( \text{Der} L = W_r \circ 1 + \mathcal{O}_r \mathcal{L}' \) and in all cases \( \mathcal{L}' \) is a Lie subalgebra of \( W_N \). In particular, we see that \( \text{Der} L \subset W_N \), and hence \( \text{Der} L \) is a subalgebra of \( W(\mathfrak{g}) \).

Assume that \( \phi \in W(\mathfrak{g}) \) is in \( \text{Der} L \) but not in \( \text{Cur} W(\mathfrak{g}') = H \circ \mathfrak{g}' \subset W(\mathfrak{g}) \). Let us choose a basis \( \{\partial_i\}_{i=1, \ldots, N} \) of \( \mathfrak{g} \) so that it contains a basis \( \{\partial_i\}_{i=r+1, \ldots, N} \) of \( \mathfrak{g}' \). We can write \( \phi = \sum f_i \circ \partial_i \) for some \( f_i \in H \) so that, say, \( f_1 \neq 0 \). As usual, for \( x \in X \), the element \( \phi_x \) of \( \mathcal{A}(W(\mathfrak{g})) \simeq X \circ \mathfrak{g} \) is identified with \( \sum x f_i \circ \partial_i \). Clearly, \( x f_i \circ \partial_i \notin \mathcal{O}_r \mathcal{L}' \) if \( \partial_i \notin \mathfrak{g}' \). We can always find \( x \in X \) such that all \( x f_i \) are either zero or contain other variables than those in \( W_r \). But then \( \sum x f_i \circ \partial_i \) cannot lie in \( W_r \circ 1 + \mathcal{O}_r \mathcal{L}' \).

This contradiction shows that \( \text{Der} L \subset \text{Cur} W(\mathfrak{g}') \). Now since \( L' \subset W(\mathfrak{g}') \) is such that \( \text{Der} L' = L' \), we get \( \text{Der} L = L \). \( \square \)
13.5. **Finite semisimple Lie $U(\mathfrak{g})$-pseudoalgebras.** Recall that a Lie $H$-pseudoalgebra $L$ is called *semisimple* if it contains no nonzero abelian ideals. Let $H = U(\mathfrak{g})$, for a finite-dimensional Lie algebra $\mathfrak{g}$.

If $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra, then by Lemma 13.13, we have $\text{Der} \text{Cur} \mathfrak{g} = W(\mathfrak{g}) \ltimes \text{Cur} \mathfrak{g}$. It is easy to see that for any subalgebra $A$ of the Lie pseudoalgebra $W(\mathfrak{g})$, the Lie pseudoalgebra $A \ltimes \text{Cur} \mathfrak{g}$ is semisimple. Indeed, assume that $I \subset A \ltimes \text{Cur} \mathfrak{g}$ is an abelian ideal. Since $\text{Cur} \mathfrak{g}$ is an ideal, their intersection $I \cap \text{Cur} \mathfrak{g}$ is an ideal in $\text{Cur} \mathfrak{g}$. All ideals in $\text{Cur} \mathfrak{g}$ are of the form $\text{Cur} h$, where $h$ is an ideal in the (semisimple) Lie algebra $\mathfrak{g}$. This implies $I \cap \text{Cur} \mathfrak{g} = 0$. But this is impossible unless $I = 0$ because the pseudobracket of any element from $(W(\mathfrak{g}) + \text{Cur} \mathfrak{g}) \setminus \text{Cur} \mathfrak{g}$ with elements from $\text{Cur} \mathfrak{g}$ gives nonzero elements from $\text{Cur} \mathfrak{g}$ (see (13.4)).

Now we can classify all finite semisimple Lie $U(\mathfrak{g})$-pseudoalgebras.

**Theorem 13.15.** Any finite semisimple Lie $U(\mathfrak{g})$-pseudoalgebra $L$ is a direct sum of finite simple Lie pseudoalgebras (described by Theorem 13.10) and of pseudoalgebras of the form $A \ltimes \text{Cur} \mathfrak{g}$, where $A$ is a subalgebra of $W(\mathfrak{g})$ and $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.

**Proof.** Consider the set $\{M_i\}$ of all minimal nonzero ideals of $L$. This set is nonempty by Lemma 13.12, and finite because $L$ is a Noetherian $H$-module. The adjoint action of $L$ on $M_i$ gives a homomorphism of Lie pseudoalgebras $L \to \text{Der} M_i$, cf. Lemma 10.9(ii).

We claim that the direct sum of these homomorphisms is an injective map. Indeed, let $N \subset L$ be the set of all elements that act trivially on all $M_i$. This set is an ideal of $L$. If it is nonzero it must contain some minimal ideal $M_i$. But then this $M_i$ is abelian, which contradicts the semisimplicity of $L$.

Therefore we have embeddings $\bigoplus M_i \subset L \subset \bigoplus \text{Der} M_i$. By Corollary 13.11 all $M_i$ are simple Lie pseudoalgebras. If $M_i$ is not a current pseudoalgebra over a finite-dimensional Lie algebra, then by Lemma 13.14, $\text{Der} M_i = M_i$. For $M_i = \text{Cur} \mathfrak{g}$, we have $\text{Der} \mathfrak{g} = W(\mathfrak{g}) \ltimes \text{Cur} \mathfrak{g}$. Any subalgebra of $W(\mathfrak{g}) \ltimes \text{Cur} \mathfrak{g}$ containing $\text{Cur} \mathfrak{g}$ is of the form $A \ltimes \text{Cur} \mathfrak{g}$, where $A$ is a subalgebra of $W(\mathfrak{g})$.

Recall that a pseudoalgebra of vector fields is any subalgebra of the Lie pseudoalgebra $W(\mathfrak{g})$.

**Proposition 13.16.** For any two nonzero elements $a, b \in W(\mathfrak{g})$, we have $[a \ast b] \neq 0$. In particular, $W(\mathfrak{g})$ does not contain nonzero abelian elements, i.e., elements $a$ such that $[a \ast a] = 0$.

**Proof.** Let us write
\[
a = \sum_i h_i \otimes \partial_i, \quad b = \sum_j k_j \otimes \partial_j,
\]
where $h_i, k_j \in H$ and $\{\partial_i\}$ is a basis of $\mathfrak{g}$. Denote by $m$ (respectively $n$) the maximal degree of the $h_i$ (respectively $k_j$). We have (cf. (8.3)):
\[
[a \ast b] = \sum_{i,j} (h_i \otimes k_j) \otimes H (1 \otimes [\partial_i, \partial_j]) - \sum_{i,j} (h_i \otimes k_j) \otimes H (1 \otimes \partial_j) + \sum_{i,j} (h_i \partial_j \otimes k_j) \otimes H (1 \otimes \partial_i).
\]
Let $[a \ast b] = 0$. Then only the second summand contains coefficients from $H \otimes H$ of degree $(m, n + 1)$, hence it must be zero modulo $F^m H \otimes F^n H$. Since the $\partial_j$ are linearly independent, the same is true for each term $\sum_{i} h_i \otimes k_j \partial_i = (1 \otimes k_j)a$. If we choose $j$ such that $k_j$ is of degree exactly $n$, we get a contradiction.
Corollary 13.17. A finite Lie $U(\mathfrak{d})$-pseudoalgebra $L$ contains no nonzero abelian elements if it is a direct sum of pseudoalgebras of vector fields.

Proof. Assume that $L$ is not a direct sum of pseudoalgebras of vector fields. If $L$ is not semisimple, then $\text{Rad} L$ contains nonzero abelian elements. If $L$ is semisimple, Theorem 13.15 implies that $L$ contains a subalgebra of the form $A \ltimes \text{Cur} \mathfrak{g}$ with $\mathfrak{g} \neq 0$, and therefore contains nonzero abelian elements (for example $1 \otimes a$ for any $a \in \mathfrak{g}$).

The converse statement follows from Proposition 13.16. □

Theorem 13.18. Any pseudoalgebra of vector fields is simple.

Proof. By Proposition 13.16, a pseudoalgebra $L$ of vector fields does not contain nonzero abelian elements, and hence is semisimple. Then, by Theorem 13.15, $L$ is a direct sum of finite simple Lie pseudoalgebras and of pseudoalgebras of the form $A \ltimes \text{Cur} \mathfrak{g}$. In fact, there is only one summand, as $[a, b] 
eq 0$ for any two nonzero elements $a, b \in W(\mathfrak{d})$. Furthermore, $L$ cannot be of the form $A \ltimes \text{Cur} \mathfrak{g}$ with $\mathfrak{g} \neq 0$, because $\text{Cur} \mathfrak{g}$ contains nonzero abelian elements.

Corollary 13.19. Any finite semisimple Lie $U(\mathfrak{d})$-pseudoalgebra $L$ is a direct sum of pseudoalgebras of the form $A \ltimes \text{Cur} \mathfrak{g}$, where $A$ is either 0 or one of the simple pseudoalgebras of vector fields (described by Theorem 13.10), and $\mathfrak{g}$ is either 0 or a simple finite-dimensional Lie algebra.

13.6. Homomorphisms between finite simple Lie $U(\mathfrak{d})$-pseudoalgebras. In this subsection, $H = U(\mathfrak{d})$ is again the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$.

Theorem 13.20. For any finite-dimensional Lie algebra $\mathfrak{g}$ and any pseudoalgebra of vector fields $L$, there are no nontrivial homomorphisms between $L$ and $\text{Cur} \mathfrak{g}$.

Proof. Any homomorphism $\text{Cur} \mathfrak{g} \to L$ leads to abelian elements in $L$, and therefore is zero (see Proposition 13.16).

Let $f$ be a homomorphism from $L$ to $\text{Cur} \mathfrak{g}$. Then $f$ induces a homomorphism of Lie algebras $A(f) : A(L) \to A(\text{Cur} \mathfrak{g})$. By Theorem 13.18, $L$ is simple, so $L = \text{Cur}^H_{H'} L'$ where $L'$ is a primitive $H'$-pseudoalgebra of vector fields ($H' = U(\mathfrak{d}')$ and $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$). By Theorem 8.23, the annihiliation algebra $L = A(L)$ is isomorphic to a current Lie algebra $O_{\mathfrak{d}} \otimes L'$ over $L' = A(L')$. Moreover, the quotient of $L'$ by its center is infinite-dimensional and simple. On the other hand, the annihiliation algebra $A(\text{Cur} \mathfrak{g}) \cong X \otimes \mathfrak{g}$ is a current Lie algebra over $\mathfrak{g}$, which is a projective limit of finite-dimensional Lie algebras $(X/F_n X) \otimes \mathfrak{g}$. Hence the adjoint action of $L' \equiv 1 \otimes L'$ on $L$ maps trivially to each of them via $A(f)$. But since $[L', L] = L$, this implies that each $L \to (X/F_n X) \otimes \mathfrak{g}$ is trivial. Therefore $A(f) = 0$, and by Corollary 11.7, we get $f = 0$. □

Theorem 13.21. Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional simple Lie algebras. Then any isomorphism $f : \text{Cur} \mathfrak{g} \to \text{Cur} \mathfrak{h}$ maps $1 \otimes \mathfrak{g}$ onto $1 \otimes \mathfrak{h}$, and thus is induced by some isomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$. In particular, $\text{Aut} \text{Cur} \mathfrak{g} \cong \text{Aut} \mathfrak{g}$.

Recall that $A(\text{Cur} \mathfrak{g}) \cong X \otimes \mathfrak{g}$ is a current Lie algebra. In the proof of the theorem we are going to use the following lemma.

Lemma 13.22. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $R$ be a commutative associative algebra. Then all ideals of $R \otimes \mathfrak{g}$ are of the form $I \otimes \mathfrak{g}$ where $I$ is an ideal of $R$. 
Proof. As a \(g\)-module, \(R \otimes g\) is isomorphic to a direct sum of several copies of \(g\). Any ideal \(J\) of \(R \otimes g\) is in particular a \(g\)-module, hence it is spanned over \(k\) by elements of the form \(r \otimes a\) where \(r \in R\) and \(a\) is a root vector in \(g\). If \(r \neq 0\) is such that \(r \otimes a \in J\) for some nonzero \(a \in g\), then \(r \otimes g \subseteq J\), since \(\{a \in g \mid r \otimes a \in J\}\) is an ideal of \(g\) and \(g\) is simple. Setting \(I = \{r \in R \mid r \otimes g \subseteq J\}\), we see that \(I\) is an ideal of \(R\) and \(J = I \otimes g\).

Proof of Theorem 13.21. Define a map \(\rho : \mathcal{A}(Cur \mathfrak{h}) \to \mathfrak{h}\) by the formula:

\[
\rho(x \otimes_H (1 \otimes a)) = \langle x, 1 \rangle a, \quad x \in X, \ a \in \mathfrak{h}.
\]

For \(a = \sum \alpha_i \otimes a_i \in \text{Cur } \mathfrak{h} = H \otimes \mathfrak{h}\), we have:

\[
\rho(x \otimes_H a) = \sum \langle S(x), h_i \rangle a_i
\]

It is easy to see that \(\rho\) is a surjective Lie algebra homomorphism.

Any isomorphism \(f : \mathfrak{g} \isom \text{Cur } \mathfrak{h}\) induces an isomorphism of Lie algebras \(\varphi = \mathcal{A}(f) : \mathcal{A}(\text{Cur } \mathfrak{g}) \isom \mathcal{A}(\text{Cur } \mathfrak{h})\). By Lemma 13.22, \(\ker \varphi = I \otimes \mathfrak{g}\) for some proper ideal \(I\) of \(X\). Recall that \(X\) is isomorphic to a topological algebra to \(\mathcal{O}_X = k[[t_1, \ldots, t_N]] (N = \dim \mathfrak{d})\), and \(\mathcal{O}_X\) has a unique maximal ideal \(M = \{s \mid t \text{ is an infinite prime}\}\).

Noting that \(M_0\) corresponds to \(F_0 X = \{x \in X \mid \langle x, 1 \rangle = 0\}\) via the isomorphism \(X \cong \mathcal{O}_X\), we deduce that \(I \subseteq F_0 X\). If \(I \neq F_0 X\), then \((F_0 X)/I \otimes \mathfrak{g}\) is a nontrivial ideal of \((X/I) \otimes \mathfrak{g} \cong \mathfrak{h}\), which is impossible because \(\mathfrak{h}\) is simple. It follows that \(\ker \rho = F_0 X \otimes \mathfrak{g}\).

Now fix \(a \in \mathfrak{g}\) and write \(f(a) = \sum \alpha_i \otimes a_i\) for some \(\alpha_i \in \mathfrak{h}\) and linearly independent \(a_i \in \mathfrak{h}\). Assume that, say, \(\alpha_1 \not\in k = F^k h\). Then we can find \(x \in F_0 X\) such that \(\langle S(x), \alpha_1 \rangle \neq 0\). Then, by (13.6), the element \(x \otimes a \in F_0 X \otimes \mathfrak{g}\) is mapped by \(\rho \varphi\) to \(\sum \langle S(x), \alpha_i \rangle \otimes a_i \neq 0\), which is a contradiction. This shows that \(f(a) \in 1 \otimes \mathfrak{h}\), completing the proof.

We turn now to the description of subalgebras of \(W(\mathfrak{d})\). Recall that the Lie pseudoalgebra \(W(\mathfrak{d})\) acts on \(H = U(\mathfrak{d})\) by (8.4). Hence any homomorphism of Lie pseudoalgebras \(L \to W(\mathfrak{d})\) gives rise to a structure of an \(L\)-module on \(H\).

Let us first consider the case when \(L\) is a free \(H\)-module of rank one: \(L = H e\) with a pseudobracket \([e * e] = a \otimes_H e, a \in H \otimes H\). Let \(M = H m\) be an \(L\)-module, with action \(e * m = \beta \otimes_H m, \beta \in H \otimes H\). We already know (Lemma 4.2) that \(\alpha\) must be of the form \(\alpha = s \otimes_1 1 - 1 \otimes s\) where \(e \in \mathfrak{d} \land \mathfrak{d}, s \in \mathfrak{d}\). Moreover, \(r\) and \(s\) satisfy equations (4.3) and (4.4). Furthermore, \(\beta\) defines a representation of \(L\) if and only if it satisfies the following equation in \(H \otimes H \otimes H\) (cf. Proposition 4.1):

\[
(1 \otimes \beta)(\text{id} \otimes \Delta)(\beta) - (\sigma \otimes \text{id})( (1 \otimes \beta)(\text{id} \otimes \Delta)(\beta) ) = (\alpha \otimes 1)(\Delta \otimes \text{id})(\beta).
\]

Proposition 13.23. If \(L = H e\) with \([e * e] = a \otimes_H e, a \in r + s \otimes_1 1 - 1 \otimes s\), then the only nonzero homomorphism \(L \to W(\mathfrak{d})\) is given by \(e \mapsto -r + 1 \otimes s\).

Proof. The statement of the proposition is equivalent to saying that all solutions \(\beta\) of (13.7) with \(\beta \in H \otimes \mathfrak{d}\) are either trivial or of the form \(\beta = r - 1 \otimes s\). It is easy to check that the latter is indeed a solution (cf. Lemma 8.7).

Let us choose a basis \(\{\partial_i\}\) of \(\mathfrak{d}\), and write \(\beta = \sum h^i \otimes \partial_i\) and \(r = \sum_{i,j} r^{ij} \partial_i \otimes \partial_j\) for some \(h^i \in H, r^{ij} \in \mathfrak{d}\). We will assume throughout the proof that \(\beta \neq 0\), and denote by \(d\) the maximal degree of the \(h^i\). Substituting the above expressions for
\[ \sum_{i,j} h^i \circ h^j \otimes [\partial_i, \partial_j] + \sum_{i,j} (h^i \circ h^j \circ \partial_j - h^i \partial_j \circ h^j) \otimes \partial_i \]
\[ = \sum_{i,j,k} r^{ij}_{jk} h^k \partial_i \circ \partial_j \otimes h^j \partial_k + \sum_k (s h^k \otimes h^k - h^k \otimes s h^k) \otimes \partial_k. \]

If \( d > 1 \), expressing all \( h^k \) in the Poincaré–Birkhoff–Witt basis relative to the basis \( \{ \partial_i \} \), we see that \( H \otimes H \)-coefficients of degree \( 2d+1 \) in the second summation in the left hand side cannot cancel with terms from other summations, which only contribute lower degree terms. Therefore
\[ \sum_{i,j} (h^i \circ h^j \partial_j - h^j \partial_j \circ h^i) \otimes \partial_i = 0 \mod F^{2d}(H \otimes H) \otimes \mathcal{D}. \]

This implies that \( \sum_j h^j \circ h^j \partial_j = \sum_j h^j \partial_j \circ h^j \mod F^{2d}(H \otimes H) \) for every \( i \), which gives a contradiction, since we can choose \( h^i \) to have degree exactly \( d \). So \( d \leq 1 \), and we can write \( \beta = \sum_{i,j} \beta^{ij} \partial_i \circ \partial_j + 1 \otimes t \) with \( \beta^{ij} \in \mathcal{k} \), \( t \in \mathcal{D} \).

Substituting this into (13.7) and comparing degree four terms we get \( \beta^{ij} \beta^{kl} = r^{ij}_{kl} \) for all \( i, j, k, l \). Since \( \beta \neq 0 \) we conclude that \( \beta^{ij} = r^{ij} \) for all \( i, j \). We are only left with showing that \( t = -s \). Substitute \( \alpha = r + s_1 - s_2 \) and \( \beta = r + t_2 \) into (13.7), and then use (4.4) to obtain:
\[ r_{12}(s_3 + t_3) + [l_3, r_{13} - r_{23}] + r_3 s_2 - r_3 s_1 - s_1 r_{13} + s_2 r_{23} \]
\[ = t_3(t_1 - t_2 + s_1 - s_2). \]

Notice that \( r_{12}(s_3 + t_3) \) is the only term lying in \( \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \) and everything else belongs to \( H \otimes H \otimes \mathcal{k} + \mathcal{K} \otimes H + H \otimes H \otimes H \). Hence \( r_{12}(s_3 + t_3) = 0 \), which is only possible if \( r = 0 \) or \( s + t = 0 \). In the latter case we are done. In the former, the left hand side of (13.8) becomes zero, and \( t \neq 0 \) since \( \beta \neq 0 \). Thus \( t_1 + s_1 - t_2 - s_2 = 0 \) and \( t + s = 0 \).

Proposition 13.23 shows that nonabelian Lie pseudoalgebras that are free of rank one over \( H \) embed uniquely in \( W(\mathcal{D}) \). We will show that the other simple pseudoalgebras of vector fields are spanned as Lie pseudoalgebras by subalgebras of rank one, and therefore also embed uniquely in \( W(\mathcal{D}) \). Recall that any pseudoalgebra of vector fields is in fact simple (Theorem 13.18).

**Theorem 13.24.** (i) For any subalgebra \( L \) of \( W(\mathcal{D}) \), there is a unique nonzero homomorphism \( L \rightarrow W(\mathcal{D}) \).

(ii) The only automorphism of a pseudoalgebra of vector fields is the identity.

**Proof.** Part (ii) is an immediate consequence of (i).

By the above remarks, it remains to prove (i) in the cases when \( L \) is a current pseudoalgebra over either \( W(\mathcal{D}') \) or \( S(\mathcal{D}', \chi') \), where \( \mathcal{D}' \) is a subalgebra of \( \mathcal{D} \).

In the former case \( (L = \text{Cur} W(\mathcal{D}') := H \otimes \mathcal{D}', H' = U(\mathcal{D}') \), note that \( L \) is spanned over \( H = U(\mathcal{D}) \) by elements \( \tilde{a} = 1 \otimes h (1 \otimes a) \) for \( a \in \mathcal{D}' \). Then \( [\tilde{a} \ast \tilde{a}'] = (a \otimes 1 - 1 \otimes a) \otimes H \tilde{a} \), and by Proposition 13.23 we know that the only nonzero homomorphism of the Lie \( H \)-pseudoalgebra \( H \tilde{a} \) to \( W(\mathcal{D}) \) maps \( \tilde{a} \) to \( 1 \otimes a \) of \( W(\mathcal{D}) \). Hence any embedding of \( L \) in \( W(\mathcal{D}) \) maps each \( \tilde{a} \) to the corresponding element \( 1 \otimes a \) of \( W(\mathcal{D}) \).
Now let $L$ be a current pseudoequivalence over $S(\mathfrak{g}', \chi')$. We will give the proof in the case when $L = S(\mathfrak{g}, \chi)$, the case of currents being completely analogous. We are going to make use of the following lemma.

**Lemma 13.25.** If $\mathfrak{g}$ is a finite-dimensional Lie algebra of dimension $N > 1$, then there exist 2-dimensional subalgebras $\mathfrak{d}_i$ $(i = 1, \ldots, N - 1)$ such that $\dim \sum_{i=1}^N \mathfrak{d}_i = r + 1$ for every $r = 1, \ldots, N - 1$.

**Proof.** If $\mathfrak{g}$ has a semisimple element $h$, we complement it to a basis of ad $h$ eigenvectors $\{h, h_1, \ldots, h_{N-1}\}$. The subalgebras $\mathfrak{d}_i = k h + k h_i$ then satisfy the statement of the lemma.

If $\mathfrak{g}$ has no semisimple elements, then from Levi’s theorem we know that $\mathfrak{g}$ must be solvable. In this case it has a 1-dimensional ideal $k h$. Complementing $h$ to a basis $\{h, h_1, \ldots, h_{N-1}\}$, we conclude as before. \qed

Now consider a 2-dimensional subalgebra of $\mathfrak{g}$ with basis $\{a, b\}$. Then the element $c_{ab} \in S(\mathfrak{g}, \chi)$ from Proposition 8.3 depends on the choice of basis only up to multiplication by a nonzero element of $k$. Moreover, the $H$-span of this element is a (free) rank one subalgebra of $S(\mathfrak{g}, \chi)$, as can be easily checked (cf. Remark 8.4).

Let $S_i$ be the rank one subalgebras of $S(\mathfrak{g}, \chi)$ associated as above with the 2-dimensional subalgebras $\mathfrak{d}_i$ of $\mathfrak{g}$ constructed in Lemma 13.25. Then by comparing second tensor factors, we see that $S_{r+1} \cap \sum_{i=1}^r S_i = 0$ for each $r = 1, \ldots, N - 1$. Therefore the sum of all $S_i$ is a free $H$-submodule $F$ of $S(\mathfrak{g}, \chi)$ of rank $N - 1$. Since the rank of $S(\mathfrak{g}, \chi)$ is also $N - 1$, we see that $S(\mathfrak{g}, \chi)/F$ is a torsion $H$-module.

Denote by $S$ the subalgebra of $S(\mathfrak{g}, \chi)$ generated by $F$. Since $S(\mathfrak{g}, \chi)/S$ is a torsion $H$-module, we conclude, by Corollary 10.17, that $S$ is an ideal of $S(\mathfrak{g}, \chi)$. Hence $S(\mathfrak{g}, \chi) = S$ by simplicity of $S(\mathfrak{g}, \chi)$. Now by Proposition 13.23, each subalgebra $S_i$ embeds uniquely in $W(\mathfrak{g})$. Hence $S = S(\mathfrak{g}, \chi)$ embeds uniquely in $W(\mathfrak{g})$.

This completes the proof of Theorem 13.24. \qed

Combining a number of previous results, we get an explicit description of all subalgebras of $W(\mathfrak{g})$, and of all isomorphisms between the simple Lie pseudoequivalences listed in Theorem 13.10.

**Corollary 13.26.** A complete list of all subalgebras $L$ of $W(\mathfrak{g}) = H \odot \mathfrak{g}$ is:

(a) $L = H \odot \mathfrak{g}' = \text{Cur}^H_{\mathfrak{g}}(W(\mathfrak{g}'))$, where $\mathfrak{g}'$ is any subalgebra of $\mathfrak{g}$ and $H' = U(\mathfrak{g}')$;

(b) $L = \{ \sum_i h_i \otimes a_i \in H \otimes \mathfrak{g}' \mid \sum_i h_i (a_i + \chi'(a_i)) = 0 \} \simeq \text{Cur}^H_{\mathfrak{g}'}(S(\mathfrak{g}', \chi'))$, where $\mathfrak{g}'$ is any subalgebra of $\mathfrak{g}$ and $\chi' \in (\mathfrak{g}')^*$ is such that $\chi'(\mathfrak{g}', \mathfrak{g}') = 0$;

(c) $L = \{ (h \otimes 1)(r - 1 \otimes s) \mid h \in H \}$, where $r \in \mathfrak{g} \land \mathfrak{g}$ and $s \in \mathfrak{g}$ satisfy (4.3), (4.4). In this case, $L$ is isomorphic to a current pseudoequivalence over $H(\mathfrak{g}', \chi', \omega')$ or $K(\mathfrak{g}', \theta')$ (see Sections 8.6 and 8.7).

**Corollary 13.27.** The simple Lie $H = U(\mathfrak{g})$-pseudoequivalences listed in Theorem 13.10 are isomorphic only in the following cases ($H' = U(\mathfrak{g}')$):

(i) $\text{Cur}^H_{\mathfrak{g}'} \simeq \text{Cur}^H_{\mathfrak{g}''}$ when $\mathfrak{g}'$ and $\mathfrak{g}''$ are isomorphic Lie algebras.

(ii) $\text{Cur}^H_{\mathfrak{g}'} H(\mathfrak{g}', \chi', \omega') \simeq \text{Cur}^H_{\mathfrak{g}''} H(\mathfrak{g}', \chi', \omega'')$ when $\omega'' = c \omega'$ for some nonzero $c \in k$.

(iii) $\text{Cur}^H_{\mathfrak{g}'} K(\mathfrak{g}', \theta') \simeq \text{Cur}^H_{\mathfrak{g}''} K(\mathfrak{g}', \theta'')$ when $\theta'' = c \theta'$ for some nonzero $c \in k$.

(iv) $\text{Cur}^H_{\mathfrak{g}'} W(\mathfrak{g}') \simeq \text{Cur}^H_{\mathfrak{g}''} K(\mathfrak{g}', \theta')$ when $\dim \mathfrak{g}' = 1$.

(v) $\text{Cur}^H_{\mathfrak{g}'} H(\mathfrak{g}', \omega) \simeq \text{Cur}^H_{\mathfrak{g}''} S(\mathfrak{g}', \chi'')$ when $\dim \mathfrak{g}' = 2$ and $\chi'' = \chi' + \text{tr ad}$.
13.7. Finite simple and semisimple Lie \((U(\mathfrak{g}) \ast \mathbb{k}[\Gamma])\)-pseudoalgebras. Let, as before, \(H = U(\mathfrak{g})\) be the universal enveloping algebra of a finite-dimensional Lie algebra \(\mathfrak{g}\). Let \(\Gamma\) be a (not necessarily finite) group acting on \(\mathfrak{g}\) by automorphisms. The action of \(\Gamma\) on \(\mathfrak{g}\) can be extended to an action on \(H\) which we denote by \(g \cdot f\) for \(g \in \Gamma, f \in H\). Recall that the smash product \(\hat{H} = H \ast \mathbb{k}[\Gamma]\) is a Hopf algebra, with the product determined by \(g \cdot f = gf_{g^{-1}}\), and coproduct \(\Delta(fg) = \Delta(f)\Delta(g)\) \((g \in \Gamma, f \in H)\).

A left \(\hat{H}\)-module \(L\) is the same as an \(H\)-module together with an action of \(\Gamma\) on it which is compatible with that of \(H\). An \(\hat{H}\)-module \(L\) will be called finite if it is finite as an \(H\)-module.

Let \(\hat{L}\) be a Lie \(\hat{H}\)-pseudoalgebra with a pseudobracket denoted as \([a \ast b]\). By Corollary 5.3, \(\hat{L}\) is also a Lie \(H\)-pseudoalgebra, which we denote as \(L\) with a pseudobracket \([a \ast b]\). \(L\) is equipped with an action of \(\Gamma\), and \([a \ast b]\) is \(\Gamma\)-equivariant, see (5.5). As an \(\hat{H}\)-module, \(L = \hat{L}\). The relationship between the two pseudobrackets is given by (5.7).

Then the following statements are easy to check.

**Lemma 13.28.** (i) \([a \ast b] = 0\) iff \([ga \ast b] = 0\) for all \(g \in \Gamma\).

(ii) \(I \subset L = \hat{L}\) is an ideal of the Lie \(\hat{H}\)-pseudoalgebra \(\hat{L}\) iff it is a \(\Gamma\)-invariant ideal of the Lie \(H\)-pseudoalgebra \(L\).

(iii) If \(I\) is as in (ii), then its derived pseudoalgebra \([I, I]\) is the same with respect to both pseudobrackets \([a \ast b]\) and \([a \ast b]\).

(iv) \(\text{Rad} \hat{L} = \text{Rad} L\).

**Proof.** (i) If \([a \ast b] = 0\) then all its coefficients in front of \((gH \otimes \mathbb{k}) \otimes_H L\) are zero for different \(g \in \Gamma\). Since \([a \ast b] \in (H \otimes H) \otimes_H L\), it follows that all \([ga \ast b] = 0\).

(ii) and (iii) are clear by (5.7).

(iv) follows from (i)–(iii) and the fact that \(\text{Rad} L\) is \(\Gamma\)-invariant. (\(\text{Rad} L\) is \(\Gamma\)-invariant because \([a \ast b]\) is \(\Gamma\)-equivariant, see (5.5).)

**Proposition 13.29.** The Lie \(\hat{H}\)-pseudoalgebra \(\hat{L}\) is solvable (respectively semisimple) if and only if the Lie \(H\)-pseudoalgebra \(L\) is.

**Proof.** Follows from Lemmas 13.28(iv) and 13.1(iii).

**Proposition 13.30.** The Lie \(\hat{H}\)-pseudoalgebra \(\hat{L}\) is finite and simple if and only if the Lie \(H\)-pseudoalgebra \(L\) is a finite direct sum of isomorphic finite simple Lie \(H\)-pseudoalgebras and \(\Gamma\) acts on them transitively.

**Proof.** By Lemma 13.28, \(\hat{L}\) is simple iff \(L\) is not abelian and has no nontrivial \(\Gamma\)-invariant ideals. In particular, \(L\) is semisimple. Using Theorem 13.15 and the fact that \(\mathbb{k}[\Gamma]/I\) is an ideal of \(L\) iff \(I\) is an ideal, we see that \(\hat{L}\) is a direct sum of isomorphic finite simple Lie \(H\)-pseudoalgebras.

13.8. Examples of infinite simple subalgebras of \(g_{\mathfrak{g}}\). In this subsection, \(H\) is an arbitrary oocommutative Hopf algebra. Let us define a map \(\omega: H \otimes H \rightarrow H \otimes H\) by the formula:

\[
\omega(f \otimes a) = fa_{(\cdot - 1)} \otimes a_{(\cdot - 2)} = (f \otimes 1) \Delta(S(a)).
\]

It is easy to check that \(\omega^2 = \text{id}\); this also follows from the identities \(\omega = \mathcal{F}^{-1}(\text{id} \otimes S) = (\text{id} \otimes S)\mathcal{F}\) where \(\mathcal{F}\) is the Fourier transform defined by (2.33).
Lemma 13.31. The above $\omega$ is an anti-involution of $\text{Cend}_1 = H \odot H$, i.e., it is an $H$-linear map satisfying $\omega^2 = \text{id}$ and

\begin{equation}
\omega(a \ast b) = (\sigma \odot_H \omega)(b \ast a), \quad a, b \in \text{Cend}_1,
\end{equation}

where, as before, $\sigma : H \odot H \to H \odot H$ is the permutation of the factors.

Proof. It only remains to check (13.10), which is straightforward. \hfill \Box

When $H = U(\mathfrak{d})$, the annihilation algebra $\mathcal{A}(\text{Cend}_1)$ is isomorphic to the associative algebra of all differential operators on $X$, and $\omega$ induces its standard anti-involution $\ast$ (formal adjoint).

Let $\gamma : \text{End}(k^n) \to \text{End}(k^n)$ be an anti-involution, i.e., $\gamma^2 = \text{id}$ and $\gamma(A)\gamma(B) = \gamma(BA)$. Then we can define an anti-involution $\omega$ of $\text{Cend}_n = H \odot H \odot \text{End}(k^n)$ by the formula (cf. (13.9)),

\begin{equation}
\omega(f \odot a \odot A) = f a_{(-1)} \odot a_{(-2)} \odot \gamma(A).
\end{equation}

Let $\mathfrak{g}_n(\omega)$ be the set of all $a \in \text{Cend}_n$ such that $\omega(a) = -a$. This is a subalgebra of the Lie pseudoalgebra $\mathfrak{g}_n$. Indeed, it is an $H$-submodule because $\omega$ is $H$-linear.

If $\omega(a) = -a$, $\omega(b) = -b$, then:

\[(\text{id} \odot_H \omega)(a \ast b) = (\text{id} \odot_H \omega)(a \ast b - (\sigma \odot_H \text{id})(b \ast a)) = (\sigma \odot_H \text{id})\omega(b) - \omega(a) \ast \omega(b) = -[a \ast b].\]

Two important examples of Lie pseudoalgebras $\mathfrak{g}_n(\omega)$ are obtained when $k^n$ is endowed with a symmetric or skew-symmetric non-degenerate bilinear form, and $\gamma(A)$ is the adjoint of $A$ with respect to this form. In these cases, we denote $\mathfrak{g}_n(\omega)$ by $\mathfrak{o}_n$ and $\mathfrak{sp}_n$, respectively.

Proposition 13.32. Let $H = U(\mathfrak{d})$, $\mathfrak{d} \neq 0$. Then $\mathfrak{o}_n$ and $\mathfrak{sp}_n$ are infinite subalgebras of $\mathfrak{g}_n$ that act irreducibly on $H \odot k^n$. We have: $\mathfrak{o}_n \cap \text{Cur} \mathfrak{gl}_n = \text{Cur} \mathfrak{o}_n$ and $\mathfrak{sp}_n \cap \text{Cur} \mathfrak{gl}_n = \text{Cur} \mathfrak{sp}_n$.

Proof. The second statement is obvious by the definitions. Since $\mathfrak{o}_n$ ($n \geq 3$) and $\mathfrak{sp}_n$ ($n \geq 2$) act irreducibly on $k^n$, we only have to check that the action of $\mathfrak{o}_n$ on $H \odot k^n$ is irreducible for $n = 1, 2$. Using diagonal matrices, we see that it suffices to check that $\mathfrak{oc}_1$ acts irreducibly on $H$.

Recall that this action is given by (see (10.12)):

\[a \ast h = (1 \odot h)\alpha \odot_H 1 \quad \text{for} \quad \alpha \in \mathfrak{oc}_1 = H \odot H, \ h \in H.\]

For $a \in \mathfrak{d}$, let $a = 1 \odot a - \omega(1 \odot a) = 2 \odot a + a \odot 1 \in \mathfrak{oc}_1$. Then $a \ast h = (1 \odot ha) \odot_H 1 + (1 \odot h) \odot_H a$. If $M \subset H$ is an $\mathfrak{oc}_1$-submodule, and $h \in M$, $h \neq 0$, then the previous formula implies $1 \in M$. Therefore $M = H$. \hfill \Box

Remark 13.33. It follows from Theorem 14.5 below that in the case $H = U(\mathfrak{d})$, $\mathfrak{d} \neq 0$, the Lie pseudoalgebras $\mathfrak{g}_n$, $\mathfrak{o}_n$ and $\mathfrak{sp}_n$ are semisimple. In fact, one can show that in this case they are simple. In the case $H = k[\Gamma]$ with a finite group $\Gamma$, the Lie pseudoalgebra $\mathfrak{g}_n$ has a center that is a free $H$-module of rank 1, the quotient by which is simple.

If $I$ is a left ideal of the associative pseudoalgebra $\text{Cend}_n$ and $L$ is a subalgebra of the Lie pseudoalgebra $\mathfrak{g}_n$, then their intersection $I \cap L$ is again a subalgebra of $\mathfrak{g}_n$. All left ideals of $\text{Cend}_n$ are described in the next proposition.
Proposition 13.34. Any left ideal of the associative pseudoalgebra $\text{Cend}_n$ is a sum of ideals of the form $H \odot R \odot E$ where $R \subset H$ is a right ideal and $E \subset \text{End}(k^n)$ is a left ideal. More explicitly, all left ideals of $\text{Cend}_n$ are:

$$I(i_1, \ldots, i_n) := \bigoplus_{i=1}^n H \odot R_i \odot E_i,$$

where $R_i \subset H$ is a right ideal, and $E_i$ is the set of all matrices that have all columns zero except the $i$-th one.

Proof. Let $I \subset \text{Cend}_n$ be a left ideal, $a = 1 \odot a \odot A \in \text{Cend}_n$, and $\beta = \sum_i g_i \odot b_i \odot R_i \in I$ with linearly independent $g_i$. Then:

$$a \ast \beta = \sum_i (1 \odot g_i a_{i(1)}) \odot H (1 \odot b_i a_{i(2)} \odot AB_i) \in (H \odot H) \odot_H I.$$

Taking $a = 1$, we see that all $1 \odot b_i \odot AB_i \in I$. In particular, $1 \odot b_i \odot R_i \in I$, and hence each element from $I$ is an $H$-linear combination of elements of the form $\beta = 1 \odot b \odot B$. For such $\beta$, we have $a \ast \beta = (1 \odot a_{(1)}) \odot H (1 \odot ba_{(2)} \odot AB)$. For $a \in \mathfrak{d}$, $A = \text{Id}$, we get that $1 \odot ba \odot B \in I$. This proves the first statement of the proposition. The second one follows from the description of all left ideals in $\text{End}(k^n)$.

In a similar fashion one can describe the right ideals of $\text{Cend}_n$. It is easy to see that $\text{Cend}_n$ has no two-sided ideals, i.e., it is a simple associative pseudoalgebra.

14. Representation Theory of Lie Pseudoalgebras

14.1. Conformal version of the Lie Lemma. Let $L$ be a Lie $H$-pseudoalgebra and $V$ be an $L$-module. In this subsection, $H = U(\mathfrak{d})$ will be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. In particular, $H$ is a Noetherian ring with no divisors of zero.

Let $I \subset L$ be an ideal and $\varphi \in \text{Hom}_H(I, H)$ be such that

$$V_\varphi := \{v \in V \mid a \ast v = (\varphi(a) \odot 1) \odot_H v \quad \forall a \in I\}$$

is nonzero. We will call the elements of $V_\varphi$ eigenvectors for $I$ with an eigenvalue $\varphi$. Note that every $v \in V_\varphi$ is an eigenvector for the action of $X \odot H I \subset \mathcal{A}(L)$ on $V$. By abuse of notation, we will also write $a_x v = \varphi(a_x) v$ for $a \in I$, $x \in X$, $v \in V_\varphi$, where $\varphi(a_x) = \langle S(x), \varphi(a) \rangle$, cf. (9.6).

Clearly, if $\varphi = 0$, then $V_\varphi$ is an $L$-submodule of $V$.

Lemma 14.1. If $\varphi \neq 0$, then $HV_\varphi$ is a free $H$-module, isomorphic to $H \odot V_\varphi$ with $H$ acting on the first tensor factor.

Proof. Assume that

$$\sum_i f_i v_i = 0$$

for some $f_i \in H$, $v_i \in V_\varphi$. Let (14.2) be a relation of this form with $f_i \in F^n \ast H$ so that $\sum_i n_i$ is minimal. We call $\sum_i n_i$ the degree of the relation (14.2). Assume that $v_i$'s are linearly independent, so that the degree of (14.2) is positive.

We can find $a \in I$, $x \in X$ such that $\varphi(a_x) \neq 0$. Applying $a_x$ to (14.2) and using (9.16), we obtain

$$\sum_i f_i(a_x) \varphi(a_{(i)} x) v_i = 0.$$

Subtracting this from (14.2), we get a relation of lower degree than (14.2), because $\Delta(f) \in 1 \odot f + \sum_{i=1}^n F^i H \odot F^{n-i} H$ for $f \in F^n H$. 

\]
The following result is an analogue of Lie's Lemma.

**Proposition 14.2.** If $V$ is finite as an $H$-module, then:

\[
L \ast V_\varphi \subset \left( H \otimes k \right) \otimes_H (\mathfrak{d}V_\varphi + V_\varphi)
\]

In other words, for every $\beta \in \mathcal{A}(L)$, there exist $\delta_\beta \in \mathfrak{d}$ and $A_\beta \in \text{End}V_\varphi$ such that

\[
\beta v = (\delta_\beta + A_\beta)v \quad \text{for any } v \in V_\varphi.
\]

In particular, $HV_\varphi$ is an $L$-submodule of $V$.

**Proof.** Fix nonzero elements $w \in V_\varphi$, $\beta \in \mathcal{A}(L)$, and let $w_n = \beta^n w$. Let $W_n$ be the linear span of $w_0, \ldots, w_n$; we set $W_n = 0$ for $n < 0$. For $a \in L$, $x \in X$, we have:

\[
a_x w_n = \varphi(a_x)w_n + n\varphi([a_x, \beta])w_{n-1} + W_{n-2}.
\]

In particular, all $HW_n$ are $L$-modules.

Since $V$ is a Noetherian $H$-module, there exists $N \geq 0$ such that $HW_{N-1} \neq HW_N = HW_{N+1}$. In particular,

\[
w_{N+1} \in (N + 1)hw_N + HW_{N-1}
\]

for some $h \in H$.

Writing (14.5) for $n = N + 1$ and using (14.6), we get

\[
a_x w_{N+1} \in \varphi(a_x)(N + 1)hw_N + (N + 1)\varphi([a_x, \beta])w_N + HW_{N-1}.
\]

On the other hand, applying $a_x$ to both sides of (14.6) and using the $H$-sesquilinearity gives

\[
a_x w_{N+1} \in \varphi(a_{h_{(N+1)}})w_N + HW_{N-1}.
\]

Comparison of the last two equations gives

\[
f w_N \in HW_{N-1} \quad \text{for } f = \varphi(a_x)h + \varphi([a_x, \beta]) - \varphi(a_{h_{(N+1)}})h_{(N)}.
\]

If $f \neq 0$, then the module $HW_N/HW_{N-1}$ is torsion, hence $I$ acts on it as zero by Corollary 10.17. This gives $a_x w_N \in HW_{N-1}$ for all $a \in L$, $x \in X$. Then (14.5) implies $\varphi(a_x)w_N \in HW_{N-1}$. Since $HW_{N-1} \neq HW_N$, it follows that $\varphi = 0$, which contradicts the assumption $f \neq 0$.

Therefore $f = 0$. This is possible only when $h \in F^1 H = \mathfrak{d} + k$. Then for any $v \in V_\varphi$, one has:

\[
0 = f v = ha_x v + [a_x, \beta]v - h_{(N)}a_{h_{(M)}}v = [a_x, \beta - h]v.
\]

This implies that $(\beta - h)v \in V_\varphi$, proving (14.4). \( \square \)

### 14.2. Conformal version of the Lie Theorem.

**Theorem 14.3.** Let $H = U(\mathfrak{d}) \sharp k[\Gamma]$ with $\dim \mathfrak{d} < \infty$. Let $L$ be a solvable Lie $H$-pseudorialgebra and $V$ be an $L$-module which is finite over $U(\mathfrak{d})$. Then there exists an eigenvector for the action of $L$ on $V$, i.e., $v \in V \setminus \{0\}$ and $\varphi \in \text{Hom}_H(L, H)$ such that $a \ast v = (\varphi(a) \otimes 1) \ast_H v$ for all $a \in L$.

**Proof.** Using Corollary 5.3 and Proposition 13.29, we can assume that $H = U(\mathfrak{d})$. The proof will be by induction on the length of the derived series of $L$.

First consider the case when $L$ is abelian. By a Zorn’s Lemma argument, it is enough to find an eigenvector when $L = Ha$ is abelian generated by one element $a$.

In this case, the statement follows from Lemma 10.15 and the usual Lie Theorem, because all $\ker_n a$ are $L$-submodules.
Now let $L$ be nonabelian, $I = [L, L] \neq 0$. By the inductive assumption, $I$ has a space of eigenvectors $V_\varphi \neq 0$. If $\varphi = 0$, then $V_\varphi$ is an $L$-submodule of $V$ on which $I$ acts as zero. The abelian $H$-pseudoalgebra $L/I$ has an eigenvector in $V_\varphi$, which is also an eigenvector for $L$.

Now assume that $\varphi \neq 0$. By Proposition 14.2, we have for $\alpha, \beta \in \mathcal{A}(L), v \in V_\varphi$:

$$\alpha v = (\partial_\alpha + A_\alpha)v,$$
$$\beta v = (\partial_\beta + A_\beta)v,$$
$$[\alpha, \beta] v = \varphi([\alpha, \beta]) v.
$$

On the other hand, we can compute:

$$\alpha \beta v = \alpha (\partial_\beta + A_\beta)v = \partial_\alpha (\partial_\beta v) - (\partial_\beta \alpha)v + \alpha (A_\beta v)$$
$$= \partial_\beta (\partial_\alpha + A_\alpha)v - (\partial_\beta \alpha + \alpha A_\beta)v + (\partial_\alpha + A_\alpha)A_\beta v$$
$$= \partial_\beta \partial_\alpha v - \partial_\beta \alpha v + \partial_\alpha A_\beta v + \partial_\alpha A_\beta v - A_\beta \alpha v + A_\alpha A_\beta v.$$

It follows that

$$[\partial_\alpha, \partial_\beta] = \partial_\alpha \beta - \partial_\beta \alpha.$$

Assume that $\partial_\alpha x \neq 0$ for some $a \in L, x \in X$, and write $\partial_x = \partial_\alpha x$ for short. For $\alpha = a_x, \beta = a_y$, the above equation becomes:

$$[\partial_x, \partial_y] = \partial_\alpha y - \partial_\beta x.$$

(recall that $ha_x = a_{hx}$ for $h \in H$). Note that $\partial_y = 0$ if $y \in F_n X$ for sufficiently large $n$. Take the minimal such $n$, and let $x \in F_{n-1} X$ be such that $\partial_x \neq 0$. By Lemma 6.10, there exists $y \in F_n X$ such that $x = \partial_y y$. Then $\partial_y = 0$ and $\partial_x = \partial_\alpha y - \partial_\beta x = [\partial_x, \partial_y] = 0$, which is a contradiction.

It follows that all $\partial_\alpha x = 0$, hence $L$ preserves $V_\varphi$. By Lemma 14.1, $\dim V_\varphi < 0$, and therefore $L$ has an eigenvector by the usual Lie Theorem for $\mathcal{A}(L)$.

**Corollary 14.4.** Let $L$ be a solvable Lie $H$-pseudoalgebra and $V$ be a finite $L$-module (i.e., finite over $U(\mathfrak{d})$). Then $V$ has a filtration by $L$-submodules $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that for any $i$ the $L$-module $V_i / V_{i-1}$ is generated over $H$ by eigenvectors of some given eigenvalue $\varphi_i$ in $\text{Hom}_H(L, H)$.


**Theorem 14.5.** Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Let $L$ be a Lie $H$-pseudoalgebra acting faithfully and irreducibly on the finite $H$-module $V$. Then one of the following two possibilities holds:

(i) $L$ is semisimple, either finite or infinite.

(ii) $\text{Rad} L$ is a free $H$-module of rank one. In this case, there is a subspace $\hat{V}$ of $V$ such that $V \simeq H \otimes \hat{V}$ is a free $H$-module, and $L$ is a subalgebra of the Lie pseudoalgebra $W(\hat{V}) \ltimes \text{Cur gl} \hat{V}$. In particular, $L$ is finite, $\text{Rad} L$ is identified with $I \otimes \text{id}_\mathfrak{f} \subset \text{Cur gl} \hat{V}$ for some left ideal $I$ of $H$, $L_1 := L \cap (W(\mathfrak{d}) \ltimes \text{Cur gl} \hat{V})$ is semisimple, and $L = L_1 \ltimes \text{Rad} L$ is a semidirect sum of pseudoalgebras.

**Proof.** Assume that $L$ is not semisimple, i.e., it has a nonzero abelian ideal $A$. Then, by Theorem 14.3, $A$ has an eigenvector. If $\hat{V} = V_\varphi$ is the corresponding eigenspace of $V$, then, by Proposition 14.2, $H \hat{V}$ is an $L$-submodule of $V$. The irreducibility of $V$ implies that $V = H \hat{V}$. Now, by Lemma 14.1, $V \simeq H \otimes \hat{V}$ is a free $H$-module, since $\varphi \neq 0$. 


Proposition 14.2 and the faithfulness of $V$ also show that $L$ embeds into $W(\mathfrak{d}) \ltimes \text{Cur}\, {\mathfrak{gl}} \bar{V}$. Since the latter is finite, $L$ is finite. Then $\text{Rad}\, L$ exists, and we can assume that $\bar{V}$ is an eigenspace for $\text{Rad}\, L$. For each element $a \in \text{Rad}\, L$ and $v \in \bar{V}$ we have $a \ast v = (\phi(a) \otimes 1) \circ_H v$, which means that $\text{Rad}\, L$ is identified with $l \otimes {\text{id}}_F \subset \text{Cur}\, {\mathfrak{gl}} \bar{V}$ for $l = \phi(\text{Rad}\, L)$.

In order to show that $L = L_1 \ltimes \text{Rad}\, L$ is a semidirect sum, let us first remark that $W(\mathfrak{d}) \ltimes \text{Cur}\, {\mathfrak{gl}} \bar{V} = (W(\mathfrak{d}) \ltimes \text{Cur}\, \mathfrak{sl} \bar{V}) \ltimes (H \otimes \text{id}_F)$. Then lift $L / \text{Rad}\, L$ to some $H$-submodule $S$ of $L \subset (W(\mathfrak{d}) \ltimes \text{Cur}\, \mathfrak{sl} \bar{V})$. Every element in $IS$ is sum of an element from $W(\mathfrak{d}) \ltimes \text{Cur}\, \mathfrak{sl} \bar{V}$ and some element of $\text{Rad}\, L = l \otimes \text{id}_F$, so we can locate a subspace $\bar{S}$ spanned by the semisimple parts of elements in $IS$.

The Lie pseudoalgebra $\bar{L}$ generated by $\bar{S}$ will be contained in $L_1$, and will have the same rank as $L / \text{Rad}\, L$. Hence it is an ideal of $L_1$ and a complement in $L$ to $\text{Rad}\, L$. Since $L / \text{Rad}\, L$ is simple, this shows $\bar{L} \cong L_1 \cong L / \text{Rad}\, L$, and the semidirect sum decomposition.

14.4. Conformal version of Engel’s Theorem. As an application of the results of Section 14.2, we can prove a conformal analogue of Engel’s Theorem.

**Theorem 14.6.** Let $H = U(\mathfrak{d}) \ltimes \mathbb{K}[\Gamma]$ with dim $\mathfrak{d} < \infty$, and let $L$ be a finite Lie $H$-pseudoalgebra (i.e., finite over $U(\mathfrak{d})$). Assume that the action of any element $a \in \mathcal{A}(L)$ on $L$ is nilpotent. Then $L$ is a nilpotent Lie pseudoalgebra.

**Proof.** First of all, note that the property that any element $a$ of $\mathcal{A}(L)$ acts nilpotently on $L$ remains valid when we replace $L$ by any quotient of $L$ by an ideal. In particular, $L / \text{Rad}\, L$ will have that property. However, $L / \text{Rad}\, L$ is semisimple, and from the classification of finite semisimple Lie pseudoalgebras we see that this is impossible, unless $L / \text{Rad}\, L = 0$.

Therefore $L$ is solvable. The nilpotence of all $a \in \mathcal{A}(L)$ imply that all eigenvalues for $L$ are zero. Now Corollary 14.4 implies that $L$ is a nilpotent Lie pseudoalgebra.

14.5. Generalized weight decomposition for nilpotent Lie pseudoalgebras. Let $L$ be a (not necessarily finite) Lie $H$-pseudoalgebra, and $V$ be a finite $L$-module, where $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$.

Recall that for any $\phi \in \text{Hom}_H(L, H)$, the eigenspace $V_\phi$ of $V$ is defined by:

$$V_\phi = \{ v \in V \mid a \ast v = (\phi(a) \otimes 1) \circ_H v \ \forall a \in L \}.$$ (14.10)

Let $V_{\phi_1}^\prime = 0$ and set inductively

$$V_{\phi_{i+1}}^\prime = H \{ v \in V \mid a \ast v = (\phi(a) \otimes 1) \circ_H v \in (H \otimes H) \circ_H V_{\phi_i}^\prime \ \forall a \in L \}. $$ (14.11)

Then $V_{\phi}^\prime = HV_{\phi}$ and $V_{\phi_{i+1}}^\prime / V_{\phi_i}^\prime = H(V_{\phi_i}^\prime / V_{\phi_i}^\prime)_\phi$. The $V_{\phi_i}^\prime$ form an increasing sequence of $H$-submodules of $V$ which stabilizes (because of noetherianity) to some $H$-submodule of $V$ denoted $V_{\phi}^\prime$. If $V_{\phi_{n+1}}^\prime \neq V_{\phi}^\prime = V_{\phi}^\prime$, then we say the depth of $V_{\phi}^\prime$ to be $n$. We call $V_{\phi}^\prime$ the **generalized weight submodule** of $V$ relative to the weight $\phi$.

When $L$ is nilpotent, it is obviously solvable, and, by Corollary 14.4, any finite $L$-module $V$ has a filtration by $L$-submodules so that the successive quotients are generalized weight modules.

The main result of this subsection is the following theorem.

**Theorem 14.7.** Let $L$ be a nilpotent Lie $H$-pseudoalgebra and $V$ be a finite $L$-module. Then $V$ decomposes as a direct sum of generalized weight modules.
Proof. In order to prove the statement, it is enough to show that all \( L \)-module extensions between generalized weight modules relative to distinct weights are trivial.

The strategy is to consider first the case when \( L = (T) \) is the Lie pseudoalgebra generated by one element \( T \in g^* V \). Then in the general case, we show that the generalized weight spaces \( V^\psi \) relative to some element \( T \in L \) are \( L \)-invariant.

**Lemma 14.8.** Let \( V \) be a finite \( H \)-module, \( T \in g^* V \), and \( L = (T) \) be a nilpotent Lie pseudoalgebra. If \( V \) contains a \( T \)-generalized weight module \( V^\psi \) and \( V/V^\psi = W = W^\psi \) with \( \psi \neq \varphi \), then \( V \cong V^\psi \oplus W \) as \( L \)-modules.

Proof. Since \( W_{i+1}/W_i = H(W/W_i^\psi) \psi \) for any \( i \), it suffices to prove the statement when \( W = W^\psi = HW^\psi \).

Let us first consider the case when \( W = H \bar{T} \) is a cyclic \( H \)-module. In order to prove that the extension is trivial, we need to find a lifting \( v \in V \) of \( \bar{T} \) such that \( T \star v = (\psi \circ 1) \odot_H v \) and to show that \( H \bar{T} + V^\psi \) is a direct sum of \( H \)-modules (here and below, we write just \( \psi \) instead of \( \psi(T) \)). We will prove this by induction on the depth of \( V^\psi \), the basis of induction being trivial.

Let us state the statement be true for all \( T \)-generalized weight modules of depth \( \leq n \) and consider a module \( V^\psi \) of depth \( n + 1 \). Fix an arbitrary lifting \( v \in V \) of \( \bar{T} \); then:

\[
(14.12) \quad T \star v = (\psi \circ 1) \odot_H v \mod (H \odot (H) \odot_H V^\psi).
\]

Set

\[
T_1 = T, \quad T_{m+1} = [T_m \star T] \in H^\odot (m+1) \odot_H L \quad \text{for} \quad m \geq 1.
\]

Then we claim that for \( m \geq 1 \), \( T_{m+1} \star v \in H^\odot (m+2) \odot_H V^\psi \) implies \( T_m \star v \in H^\odot (m+1) \odot_H V^\psi \). We are going to show this first in the case when \( \varphi \neq 0 \), the proof for \( \varphi = 0 \) only requiring minor changes.

So, let \( \varphi \neq 0 \). Then \( V^\psi / V_n^\psi = V_{n+1}^\psi / V_n^\psi = H(V^\psi / V_n^\psi) \varphi \) is a free \( H \)-module, because it is generated by its \( \varphi \)-eigenspace and we can apply Lemma 14.1. We pick some \( H \)-basis \( \{w^j\} \) for \( V^\psi \) modulo \( V_n^\psi \). If \( \{h^i\} \) is some \( k \)-basis of \( H \) compatible with its filtration, we write

\[
(14.13) \quad T_m \star v = \sum_{i,j} (a^i_j \odot h^i) \odot_H w^j \mod H^\odot (m+1) \odot_H V_n^\psi,
\]

where \( a^i_j \in H^\odot m \).

Notice that for \( m > 1 \), \( T_m \) belongs to \( H^\odot m \odot_H [L, L] \) where \( [L, L] \) is the derived algebra of \( L \), hence all weights are zero on it. This means that

\[
(14.14) \quad T_m \star V^\psi \subset H^\odot (m+1) \odot_H V_n^\psi \quad \text{for} \quad m > 1.
\]

We have:

\[
T_{m+1} \star v = [T_m \star T] \star v = T_m \star (T \star v) - ((\sigma \circ \text{id}) \odot_H \text{id}) T \star (T_m \star v).
\]

We compute the right hand side, using (3.16), (3.19) and (14.12)-(14.14), and obtain:

\[
T_{m+1} \star v = \sum_{i,j} \left( a^i_j \odot (\psi h^i_1) \odot H^i_2 - a^i_j \odot \varphi h^i \right) \odot H w^j \mod H^\odot (m+2) \odot_H V_n^\psi.
\]

Now the assumption \( T_{m+1} \star v \in H^\odot (m+2) \odot_H V_n^\psi \) implies that coefficients of all \( w^j \) must be zero. Let us choose the highest degree \( d \) for which there is some \( h^i \) of degree \( d \) such that \( a^i_j \neq 0 \) for some \( j \). Then we get \( a^i_j \odot (\psi - \varphi) = 0 \) for all \( j \) and all \( h^i \) of degree \( d \), hence \( a^i_j = 0 \), giving a contradiction. This proves that all \( a^i_j = 0 \), and therefore \( T_m \star v \in H^\odot (m+1) \odot_H V_n^\psi \).
Now, because of nilpotence of $L$, $T_N = 0$ for $N \gg 0$, and obviously $0 \ast v \in H^{\otimes (N+1)} \otimes_H V^\varphi_v$. Thus we can pull the statement back to $m = 2$ to obtain that $T \ast v$ maps any lifting $v$ of $\bar{v}$ inside $H^{\otimes 2} \otimes_H V^{\varphi_v}$.

Now we can choose the lifting $v$ of $\bar{v}$ so that $T \ast v - (\psi \otimes 1) \otimes v \in H^{\otimes 2} \otimes_H V^{\varphi_v}$. Indeed, performing the same computation as above, using instead of (14.12)

$$T \ast v = (\psi \otimes 1) \otimes v + \sum_{i,j} (\alpha_i^j \otimes h^j) \otimes_H v^j \mod H^{\otimes 2} \otimes_H V^{\varphi_v}_v$$

for some $\alpha_i^j \in H$, we get $\alpha_i^j \otimes (\varphi - \psi) - (\varphi - \psi) \otimes \alpha_i^j = 0$. This shows that $\alpha_i^j = c_i^j (\varphi - \psi)$ for some choice of $c_i^j \in K$. Now choose $v$ to be the lifting of $\bar{v}$ minimizing the top degree $d$ of $h^j$ such that some $\alpha_i^j$ is nonzero. Then we can use $v' = v + \sum c_i^j h^j w^j$ in order to kill all degree $d$ terms, against minimality of $v$. This shows that $\alpha_i^j = 0$ for all $i, j$, and $T \ast v = (\psi \otimes 1) \otimes v$ modulo $H^{\otimes 2} \otimes_H V^{\varphi_v}_v$.

In order to show that the sum of $H$-modules $H v + V^{\varphi}_v$ is direct, observe that if $\bar{v}$ is a torsion element, i.e., if there is some nonzero $h \in H$ such that $h \bar{v} = 0$, then $h v$ must lie inside $V^{\varphi}_v$ and must be mapped to zero by our choice of the lifting $v$. Hence $h v$ is itself zero, thus proving that the sum of $H$-modules $H v + V^{\varphi}_v$ is direct. If instead $v$ is not torsion, then $H v$ is free, hence projective, and the above sum is still direct.

We have thus located an $L$-submodule $H v + V^{\varphi}_v$ of $V$. By inductive assumption, this extension of $L$-modules is trivial, and the sum is direct. Then $V/(H v + V^{\varphi}_v) \simeq V^{\varphi}/V^{\varphi}_v$, and we must only compute the extension of $H v + V^{\varphi}_v$ by $V^{\varphi}/V^{\varphi}_v$. But we know $V^{\varphi}$ is an $L$-submodule, hence the extension will have no contribution with respect to $H v$, while we know the extension of $V^{\varphi}_v$ by $V^{\varphi}/V^{\varphi}_v$ to be exactly $V^{\varphi}$. Hence $V = H v \oplus V^{\varphi}$. This concludes the proof in the case $\varphi \neq 0$.

If $\varphi = 0$, then we choose a $K$-basis of $V^{\varphi}$ modulo $V^{\varphi}_v$, and use in (14.13) coefficients of the form $\alpha_j \otimes 1$. The rest of the proof is the same.

Finally, consider the general case of a non-cyclic $H$-module $W$. We distinguish two cases. If $\psi \neq 0$, then $W = HW_\psi$ is free by Lemma 14.1, and it decomposes as a direct sum of cyclic modules to which we can apply the above argument independently. If $\psi = 0$, then we choose generators $\bar{v}^i$ of $W$ over $H$, lift them to elements $v^i$ of $V$ in such a way that each of them is mapped by $T$ to zero, and then argue that if $\sum h_i \bar{v}^i = 0$ then $\sum h_i v^i$ is an element of $V^{\varphi}_v$ killed by $T$, hence is zero. Therefore the extension of $H$-modules splits, and so does that of $L$-modules, by the above computation.

Now let $L$ be any nilpotent Lie $H$-pseudoalgebra, $V$ be a finite $L$-module, and $T \in L, T \neq 0$.

**Lemma 14.9.** Every $T$-generalized weight submodule of $V$ is stabilized by the action of $L$.

**Proof.** We set

$$(14.15)\quad L_{i,-1} = 0, \quad L_{i,1} = \{a \in L \mid [T \ast a] \in (H \otimes H) \otimes_H L(i)\}$$

and

$$(14.16)\quad V_{i,-1} = 0, \quad V_{i,1} = \{v \in V \mid T \ast v - (\varphi(T) \otimes 1) \otimes_H v \in (H \otimes H) \otimes_H V(i)\}.$$ 

Then the $L(i)$ are $H$-submodules of $L$ whose union is all of $L$ (since $L$ is nilpotent), and the $V(i)$ are vector subspaces of $V$ whose union is all of the $T$-generalized weight...
space $V^\phi$. It is easy to show by induction on $n = i + j$ that
\begin{equation}
L(i) \otimes V(j) \subseteq (H \otimes H) \otimes_H V(i+j).
\end{equation}
Indeed, the basis of induction (say $n = -1$) is trivial, and the inductive step follows from (14.15), (14.16) and the identity $[T \otimes a] \ast v = T \ast (a \ast v) = (\sigma \circ \text{id}) \otimes_H \text{id} \ast (T \ast v)$. Equation (14.17) implies that $L \otimes V^\phi \subseteq (H \otimes H) \otimes_H V^\phi$, as desired. \qed

We are now able to complete the proof of Theorem 14.7. Let $V = \bigoplus V_i$ be finest among all decompositions into direct sum of $L$-submodules of $V$ such that all of the $H$-torsion of $V$ is contained in one of the $V_i$. Note that such a finest decomposition always exists, because any decomposition defines a partition of rank $V$ into non-negative integers, and finer decompositions define finer partitions.

We claim that each $V_i$ is a generalized weight module for $L$. Otherwise, there must be some element $T \in L$ for which some of the $V_i$ is not a $T$-generalized weight module. But if so, then $V_i$ decomposes into a direct sum of its $T$-generalized weight submodules, and all torsion elements lie in the $T$-eigenspace of eigenvalue 0. Since all $T$-generalized weight submodules are $L$-invariant, we obtain a contradiction. Therefore $V$ is a direct sum of its generalized weight submodules. \qed

14.6. **Representations of a Lie pseudoalgebra and of its annihilation algebra.** Let $H = U(\mathfrak{g})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, and $L$ be a finite Lie $H$-pseudoalgebra.

Recall that the annihilation algebra $\mathcal{L} = \mathcal{A}(L)$ of $L$ possesses a filtration by subspaces $\mathcal{L} = \mathcal{L}_{-1} \supseteq \mathcal{L}_0 \supseteq \cdots$ satisfying (7.14):
\[
[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j-s}, \quad \text{for all } i, j \text{ and some fixed } s,
\]
that make $\mathcal{L}$ a linearly compact Lie algebra (Proposition 7.12). Moreover, $\mathcal{L}$ is an $H$-differential algebra, i.e., $\mathfrak{g}$ acts on it by derivations. The semidirect sum $\mathcal{L}^c := \mathfrak{g} \ltimes \mathcal{L}$ is called the extended annihilation algebra. Letting $\mathcal{L}_n = \mathcal{L}_n$ for all $n$ makes $\mathcal{L}^c$ a topological Lie algebra as well.

An $\mathcal{L}^c$-module (or $\mathcal{L}$-module) $V$ is called conformal if any $v \in V$ is killed by some $\mathcal{L}_n$; in other words, if $V$ is a topological $\mathcal{L}^c$-module when endowed with the discrete topology. Now Proposition 9.4 can be reformulated as follows.

**Proposition 14.10.** Any module $V$ over the Lie pseudoalgebra $L$ has a natural structure of a conformal $\mathcal{L}^c$-module, and vice versa. Moreover, $V$ is irreducible as an $L$-module if and only if it is irreducible as an $\mathcal{L}^c$-module.

Together with the next two lemmas, this proposition is an important tool in the study of representation theory of Lie pseudoalgebras.

**Lemma 14.11.** Let $L$ be a finite Lie pseudoalgebra and $V$ be a finite $L$-module. For $n \geq -1$, let
\[
\ker_n V = \{ v \in V \mid \mathcal{L}_n v = 0 \},
\]
so that, for example, $\ker_{-1} V = \ker V$ and $V = \bigcup_n \ker_n V$. Then all vector spaces $\ker_n V / \ker V$ are finite-dimensional.

**Proof.** The proof is an application of Lemma 10.15, using the following fact. Let $A$ be a vector space and $A_i \supseteq B_i$ ($i = 1, \ldots, k$) be subspaces of $A$ such that all $A_i / B_i$
are finite-dimensional, then $\bigcap A_i / \bigcap B_i$ is finite-dimensional. It is enough to show this for $k = 2$, in which case it follows from the isomorphism

$$\frac{(A_1 \cap A_2) / (B_1 \cap B_2)}{(A_1 \cap B_2) / (B_1 \cap B_2)} \cong \frac{A_1 \cap A_2}{A_1 \cap B_2}$$

Note that $[\mathcal{L}_s, \mathcal{L}_n] \subset \mathcal{L}_n$ for any $n$, and in particular $\mathcal{L}_s$ is a Lie algebra.

**Lemma 14.12.** Let $L$ be a finite Lie pseudoalgebra and $V$ be a finite $L$-module such that $\ker V = 0$. Then $V$ is locally finite as an $\mathcal{L}_s$-module, i.e., any vector $v \in V$ is contained in a finite-dimensional subspace invariant under $\mathcal{L}_s$.

**Proof.** Any $v \in V$ is contained in some $\ker_s V$, which is finite-dimensional by Lemma 14.11, and $\mathcal{L}_s$-invariant because $[\mathcal{L}_s, \mathcal{L}_n] \subset \mathcal{L}_n$.

Let $V$ be a finite irreducible $L$-module. Then $\ker V = 0$. Take some $n$ such that $\ker_s V \neq 0$. This space is finite-dimensional and $\mathcal{L}_s$-invariant; let $U$ be an irreducible $\mathcal{L}_s$-submodule of $\ker_s V$. The $\mathcal{L}_s$-submodule of $V$ generated by $U$ is a factor of the induced module $\text{Ind}^{\mathcal{L}_s}_{\mathcal{L}_n} U$. Therefore, $V$ is a factor module of $\text{Ind}^{\mathcal{L}_s}_{\mathcal{L}_n} U$.

In many cases $\mathcal{L}$ acts on $\mathcal{L}$ by inner derivations so that we have an injective homomorphism $\mathcal{L} \rightarrow \mathcal{L}$. In this case, $\mathcal{C}$ is isomorphic to the direct sum of $\mathcal{L}$ and $\mathcal{L}$, and we have $\text{Ind}^{\mathcal{C}}_{\mathcal{L}_n} U \cong H \otimes \text{Ind}^{\mathcal{L}_s}_{\mathcal{L}_n} U$.

The above results, combined with the results of Rudakov [Ru1, Ru2] and [Ko], will allow us to classify all finite irreducible representations of all finite semisimple Lie pseudoalgebras (work in progress).

### 15. Cohomology of Lie Pseudoalgebras

#### 15.1. The complexes $C^*(L, M)$ and $\bar{C}^*(L, M)$

Recall that in Section 3 we defined cohomology of a Lie algebra in any pseudotensor category (Definition 3.6). Now we will spell out this definition for the case of Lie $H$-pseudoalgebras, i.e., for the pseudotensor category $\mathcal{M}^*(H)$ (see (3.4)). As before, $H$ is a cocommutative Hopf algebra. Let $L$ be a Lie $H$-pseudoalgebra and $M$ be an $L$-module.

By definition, $C^n(L, M)$, $n \geq 1$, consists of all

$$\gamma \in \text{Lin}(\{L, \ldots, L\}, M) := \text{Hom}_{\mathcal{M}^*(H)}(L^\otimes_n, H^\otimes_n \otimes_H M)$$

that are skew-symmetric (see Figure 4). Explicitly, $\gamma$ has the following defining properties (cf. (3.23), (3.24)):

1. **$H$-polylinearity:**

$$\gamma(h_1 a_1 \otimes \cdots \otimes h_n a_n) = ((h_1 \otimes \cdots \otimes h_n) \otimes_H 1) \gamma(a_1 \otimes \cdots \otimes a_n)$$

for $h_i \in H$, $a_i \in L$.

2. **Skew-symmetry:**

$$\gamma(a_1 \otimes \cdots \otimes a_i+1 \otimes a_i \otimes \cdots \otimes a_n)$$

$$= -(\sigma_{i,i+1} \otimes_H \text{id}) \gamma(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n),$$

where $\sigma_{i,i+1} : H^\otimes_n \rightarrow H^\otimes_n$ is the transposition of the $i$th and $(i+1)$st factors.
For $n = 0$, we put $C^0(L, M) = k \otimes_H M \cong M / H^+_+ M$, where $H^+ = \{ h \in H \mid \varepsilon(h) = 0 \}$ is the augmentation ideal. The differential $d$: $C^0(L, M) = k \otimes_H M \to C^1(L, M) = \text{Hom}_H(M, M)$ is given by:

$$
(d(1 \otimes m))(a) = \sum_i (\id \otimes (h_i \varepsilon))(m_i) \in M
$$

\[(15.4) \quad \text{if} \quad a * m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M\]

for $a \in L$, $m \in M$.

For $n \geq 1$, the differential $d$: $C^n(L, M) \to C^{n+1}(L, M)$ is given by Figure 5. Explicitly:

$$
(d^x)(a_1 \otimes \cdots \otimes a_{n+1})
$$

$$
= \sum_{1 \leq i \leq n+1} (-1)^{i+1}(\sigma_{i \rightarrow i} \otimes_H \id) a_i \ast x(a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_{n+1})
$$

\[(15.5) \quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j}(\sigma_{i \rightarrow j, j \rightarrow j} \otimes_H \id)
\]

$$
\times (\{a_i \ast a_j\} \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_{n+1}),
$$

where $\sigma_{i \rightarrow i}$ is the permutation $h_i \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{i-1} \otimes h_i \otimes h_{i+1} \otimes \cdots \otimes h_{n+1}$, and $\sigma_{i \rightarrow j, j \rightarrow j}$ is the permutation $h_i \otimes h_j \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_j \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{i-1} \otimes h_i \otimes h_{i+1} \otimes \cdots \otimes h_{n+1}$.

In (15.5) we also use the following conventions. If $a \ast b = \sum_i f_i \otimes_H c_i \in H^{\otimes 2} \otimes_H M$ for $a \in L$, $b \in M$, then for any $f \in H^{\otimes n}$ we set:

$$
a \ast (f \otimes b) = \sum_i (1 \otimes f)(\id \otimes H^{\otimes (n-1)})(f_i) \otimes_H c_i \in H^{\otimes (n+1)} \otimes_H M,
$$

where $D^{\otimes (n-1)} = \id \otimes \cdots \otimes \id \otimes \Delta$ is the iterated comultiplication $(\Delta^{(0)} := \id)$. Similarly, if $\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes_H v_i \in H^{\otimes n} \otimes_H M$, then for $g \in H^{\otimes 2}$ we set:

$$
\gamma((g \otimes a_1) \otimes a_2 \otimes \cdots \otimes a_n)
$$

$$
= \sum_i (g \otimes 1^{(n-1)})(\Delta \otimes \id^{(n-1)})(g_i) \otimes_H v_i \in H^{\otimes (n+1)} \otimes_H M.
$$

These conventions reflect the compositions of polynomials maps in $M^*(H)$, see (3.8). Note that (15.5) holds also for $n = 0$ if we define $D^{(-1)} := \varepsilon$.

The fact that $d^2 = 0$ is most easily checked using Figure 5 and the same argument as in the usual Lie algebra case. The cohomology of the resulting complex $C^*(L, M)$ is called the cohomology of $L$ with coefficients in $M$ and is denoted by $H^*(L, M)$.

One can also modify the above definition by replacing everywhere $\otimes$ with $\otimes$. Let $\check{C}^n(L, M)$ consist of all skew-symmetric $\gamma \in \text{Hom}_{H^{\otimes n}}(L^{\otimes n}, H^{\otimes n} \otimes M)$, cf. (15.2), (15.3). Then we can define a differential $d$: $\check{C}^n(L, M) \to \check{C}^{n+1}(L, M)$ by (15.5) with $\otimes$ replaced everywhere by $\otimes$; then again $d^2 = 0$. (In fact, one can define a pseudotensor category $\check{M}^*(H)$ by replacing $\otimes$ with $\otimes$ everywhere in the definition of $M^*(H)$.) The corresponding cohomology $H^*(L, M)$ will be called the basic cohomology of $L$ with coefficients in $M$. In contrast, $H^*(L, M)$ is sometimes called the reduced cohomology (cf. [BK]).

15.2. Extensions and deformations. We will show that the cohomology theory of Lie pseudoalgebras defined in Section 15.1 describes extensions and deformations, just as any cohomology theory. This result is a straightforward generalization of Theorem 3.1 from [BK].
Theorem 15.1. (i) The isomorphism classes of $H$-split extensions
\[ 0 \to M \to E \to N \to 0 \]
of finite modules over a Lie $H$-pseudogebra $L$ are in one-to-one correspondence with elements of $H^1(L, \text{Chom}(N, M))$.
(ii) Let $C$ be an $L$-module, considered as a Lie $H$-pseudogebra with respect to the zero pseudobracket. Then the equivalence classes of $H$-split “abelian” extensions
\[ 0 \to C \to \hat{L} \to L \to 0 \]
of the Lie $H$-pseudogebra $L$ correspond bijectively to $H^2(L, C)$.
(iii) The equivalence classes of first-order deformations of a Lie $H$-pseudogebra $L$ (leaving the $H$-action intact) correspond bijectively to $H^3(L, L)$.

Proof. (i) Let
\[ 0 \to M \xrightarrow{i} E \xrightarrow{p} N \to 0 \]
be an extension of $L$-modules, which is split over $H$. Choose a splitting $E = M \oplus N = \{ m + n \mid m \in M, n \in N \}$ as $H$-modules. The fact that $i$ and $p$ are homomorphisms of $L$-modules implies ($a \in L, m \in M, n \in N$):
\begin{align}
(a \ast_E m) &= a \ast_M m, \\
(a \ast_E n - a \ast_N n) &= \gamma(a)(n) \in H^{\otimes 2} \otimes_H M.
\end{align}
It is clear that $\gamma(a) \in \text{Chom}(N, M)$ and $\gamma : L \to \text{Chom}(N, M)$ is $H$-linear; in other words, $\gamma \in C^1(L, \text{Chom}(N, M)) = \text{Hom}_H(L, \text{Chom}(N, M))$.
For $a, b \in L, n \in N$, we have (cf. (3.26));
\begin{align*}
[a \ast b] \ast_E n &= a \ast_E (b \ast_E n) - ((\sigma \otimes \text{id}) \otimes_H \text{id})(b \ast_E (a \ast_E n)), \\
[a \ast b] \ast_N n &= a \ast_N (b \ast_N n) - ((\sigma \otimes \text{id}) \otimes_H \text{id})(b \ast_N (a \ast_N n)).
\end{align*}
Subtracting these two equations and using (15.6), (15.7), we get:
\begin{align*}
\gamma([a \ast b])(n) &= a \ast_M \gamma(b)(n) - ((\sigma \otimes \text{id}) \otimes_H \text{id})\gamma(b)(a \ast_N n) \\
&- ((\sigma \otimes \text{id}) \otimes_H \text{id})b \ast_M \gamma(a)(n) + \gamma(a)(b \ast_N n) \\
&= ((a \ast \gamma)(b))(n) - ((\sigma \otimes \text{id}) \otimes_H \text{id})((b \ast \gamma)(a))(n)
\end{align*}
(recall that the action of $L$ on $\text{Chom}(N, M)$ was defined in Remark 10.7). The last equation means that $d\gamma = 0$.
If we choose another splitting of $H$-modules $E = M \oplus' N = \{ m +' n \mid m \in M, n \in N \}'$, then it will differ by an element $\varphi$ of $\text{Hom}_H(N, M)$: $m + n = (m + \varphi(n)) +' n$. Then the corresponding
\[ \gamma'(a)(n) = a \ast_M \varphi(n) - (\text{id}_H \otimes_H \text{id}_H \varphi)(a \ast_N n) + \gamma'(a)(n). \]
Since $\text{Hom}_H(N, M) \cong k \otimes H \text{Chom}(N, M) = C^0(L, \text{Chom}(N, M))$ (see Remark 10.3), we get $\gamma(a) = a \ast \varphi + \gamma'(a)$, i.e., $\gamma = d\varphi + \gamma'$. Conversely, given an element of $H^1(L, \text{Chom}(N, M))$, we can choose a representative $\gamma \in C^1(L, \text{Chom}(N, M))$ and define an action $\ast_E$ of $L$ on $E = M \oplus N$ by (15.6), (15.7), which will depend only on the cohomology class of $\gamma$. This proves (i).

The proof of (ii) is similar. Write $\hat{L} = L \oplus C = \{ a + c \mid a \in L, c \in C \}$ as $H$-modules. Denoting the pseudobracket of $\hat{L}$ by $[a \ast b]$, we have for $a, b \in L$, 

$c, c_1 \in C$:

$[a * c] = a * c,$

$[c * c_1] = 0,$

$[a * b] - [a * b] =: \gamma(a \otimes b) \in H^{\otimes 2} \otimes H C.$

It is clear that $\gamma \in C^2(L, C)$, and the Jacobi identity for $\hat{L}$ implies $d\gamma = 0$.

(iii) A first-order deformation of $L$ is the structure of a Lie $H$-pseudoalgebra on $\hat{L} = L[c]/(c^2) = L \oplus L\epsilon$, where $H$ acts trivially on $\epsilon$, such that the map $\hat{L} \to L$ given by putting $\epsilon = 0$ is a homomorphism of Lie pseudoalgebras. This means that $0 \to L\epsilon \to \hat{L} \to L \to 0$

is an abelian extension of Lie pseudoalgebras, so (iii) follows from (ii).

15.3. Relation to Gelfand–Fuchs cohomology. Let again $L$ be a Lie $H$-pseudoalgebra and $\mathcal{L} = \mathcal{A}(L) := X \oplus H L$ be its annihilation Lie algebra. Recall that (by Proposition 9.4) any $L$-module $M$ has a natural structure of an $\mathcal{L}$-module, given by $(x \otimes m) \cdot a = x \cdot m \ (a \in L, x \in X, m \in M)$, where $a \cdot m$ is the $x$-product defined by (cf. (9.6)):

\[ a \cdot m = \sum_i \langle S(x), g_i \rangle v_i \quad \text{if} \quad a \cdot m = \sum_i (g_i \otimes 1) \otimes H v_i \in H^{\otimes 2} \otimes H M. \]

Similarly, for $\gamma \in C^n(L, M)$ and $x_1, \ldots, x_n \in X$, we define

\[ \gamma_{x_1, \ldots, x_n}(a_1 \otimes \cdots \otimes a_n) = \sum_i \langle S(x_1), g_{i,1} \rangle \cdots \langle S(x_n), g_{i,n} \rangle v_i \]

if

\[ \gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i (g_{i,1} \otimes \cdots \otimes g_{i,n}) \otimes v_i \in H^{\otimes n} \otimes M. \]

The $H$-polylinearity (15.2) of $\gamma$ implies that the map $\mathcal{A}_\gamma : C^{\infty}(\mathcal{L}) \to M$, given by

\[ (\mathcal{A}_\gamma)(x \otimes a_1) \otimes \cdots \otimes (x \otimes a_n) := \gamma_{x_1, \ldots, x_n}(a_1 \otimes \cdots \otimes a_n), \]

is well defined. Moreover, $\mathcal{A}_\gamma$ is skew-symmetric (i.e., it is map from $\wedge^n \mathcal{L}$ to $M$) because of skew-symmetry (15.3) of $\gamma$.

Therefore, we can consider $\mathcal{A}_\gamma$ as an $n$-cochain for the Lie algebra $\mathcal{L}$ with coefficients in $M$. It is not difficult to check that the map $\mathcal{A} : C^{\infty}(L, M) \to C^{\infty}(\mathcal{L}, M)$ commutes with the differentials (this also follows from the results of Section 7.2). The following result is proved in the same way as Proposition 9.4.

Proposition 15.2. The above map $\mathcal{A} : C^{\ast}(L, M) \to C^{\ast}(\mathcal{L}, M)$ is an isomorphism from the complex $C^{\ast}(L, M)$ to the subcomplex $C^{\ast}_{GF}(\mathcal{L}, M)$ of $C^{\ast}(\mathcal{L}, M)$ consisting of local cochains, i.e., cochains $\mathcal{A}_\gamma$ satisfying

\[ (\mathcal{A}_\gamma)(x_1 \otimes a_1) \otimes \cdots \otimes (x_n \otimes a_n) = 0, \quad \text{for} \quad x_1 \in F_k X, \ k \gg 0, \]

for any fixed $x_2, \ldots, x_n$ and $a_1, \ldots, a_n$.

Note that the locality condition (15.8) means that $\mathcal{A}_\gamma$ is continuous when $M$ is endowed with the discrete topology and $\mathcal{L}$ with the topology defined in Section 7.4. Therefore we have:

Corollary 15.3. The basic cohomology $H^{\ast}(L, M)$ of a Lie pseudoalgebra $L$ is isomorphic to the Gelfand–Fuchs cohomology $H^{\ast}_{GF}(\mathcal{L}, M)$ of its annihilation Lie algebra $\mathcal{L}$. 

Recall that $H$ acts on $\mathcal{L} = X \otimes_H L$ via its left action on $X$: $h(x \otimes_H a) = hx \otimes_H a$ ($h \in H, x \in X, a \in L$). Using the comultiplication $\Delta^{(n-1)}(h) = \sum h_{(1)} \otimes \cdots \otimes h_{(n)}$, we also get an action of $H$ on $\mathcal{L}^{\otimes n}$. It follows from (2.18), (2.25) that for $h \in H, a \in \mathcal{L}^{\otimes n}$, $\gamma \in \bar{C}^n(L, M)$, one has:

\[(A_\gamma)(ha) = (A(\gamma \cdot h))(a),\]

where $\gamma \cdot h \in \bar{C}^n(L, M)$ is defined by:

\[(\gamma \cdot h)(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \Delta^{(n-1)}(h) \otimes v_i \]

if

\[\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes v_i \in H^{\otimes n} \otimes M.\]

Considering $C^n(L, M)$ instead of $\bar{C}^n(L, M)$ amounts to replacing $\otimes$ with $\otimes_H$, i.e., to factoring by the relations

\[(\gamma \cdot h)(a_1 \otimes \cdots \otimes a_n) = (1^{\otimes n} \otimes h) \gamma(a_1 \otimes \cdots \otimes a_n), \quad h \in H.\]

In terms of $A_\gamma$, this corresponds to factoring by

\[h((A_\gamma)(a)) - (A_\gamma)(ha) = (h \circ (A_\gamma) - (A_\gamma) \circ h)(a).\]

This implies the next result.

**Proposition 15.4.** The isomorphism $A: \bar{C}^* (L, M) \cong C^*_{GF}(L, M)$ induces an isomorphism from $C^* (L, M)$ to the quotient complex of $C^*_{GF}(L, M)$ with respect to the subcomplex $\{h \circ c - c \circ h \mid c \in C^*_{GF}(L, M), h \in H\}$.

When $H = U(\mathfrak{d})$, we can define an action of $H$ on $C^*_{GF}(L, M) \equiv C^*_{GF}(L, M)$ by $h \cdot c := h \circ c + c \circ S(h)$. This action commutes with the differential $d$, and $A$ induces an isomorphism from $C^* (L, M)$ to the quotient complex $C^*_{GF}/H \cdot C^*_{GF}$. The Lie algebra $\mathfrak{d}$ acts on $C^*_{GF}$, and clearly $C^*_{GF}/H \cdot C^*_{GF} = C^*_{GF}/\mathfrak{d} \cdot C^*_{GF}$. We have an exact sequence of complexes

\[0 \to \mathfrak{d} \cdot C^*_{GF} \to C^*_{GF} \to C^*_{GF}/\mathfrak{d} \cdot C^*_{GF} \to 0,\]

which gives a long exact sequence for cohomology

\[
\cdots \to H^i(\mathfrak{d} \cdot C^*_{GF}) \to H^i(C^*_{GF}) \to H^i(C^*_{GF}/\mathfrak{d} \cdot C^*_{GF}) \\
\to H^{i+1}(\mathfrak{d} \cdot C^*_{GF}) \to H^{i+1}(C^*_{GF}) \to \cdots. 
\]

**Remark 15.5 ([BK]).** If $\dim \mathfrak{d} = 1$, then $\mathfrak{d}$ acts freely on $C_i^*_{GF}$ for $i > 0$, and we have $H^i(\mathfrak{d} \cdot C^*_{GF}) \simeq H^i(C^*_{GF})$ for $i > 0$.

**Proposition 15.6.** Assume that $\mathfrak{d}$ acts on $\mathcal{L}$ by inner derivations and that the action of $\mathfrak{d}$ on $M$ coincides with that of its image in $\mathcal{L}$. Then for any $i \geq 0$, we have isomorphisms

\[(15.10) \quad H^i(L, M) \simeq H^i_{GF}(L, M) \oplus H^{i+1}(\mathfrak{d} \cdot C^*_{GF}).\]

If, in addition, $\dim \mathfrak{d} = 1$, then we have

\[(15.11) \quad H^i(L, M) \simeq H^i_{GF}(L, M) \oplus H^{i+1}_{GF}(L, M).\]
Proof. Since the adjoint action of $\mathcal{L}$ on $H^*_{\text{GF}}(C, M)$ is trivial, we obtain that $H^i(\mathcal{L} \cdot \mathcal{L}^*)$ maps to zero in the exact sequence (15.9). Therefore we have exact sequences
\[ 0 \to H^i(C^*_{\mathcal{L}}) \to H^i(C^*_{\mathcal{L}}/\mathcal{L} \cdot \mathcal{L}^*) \to H^{i+1}(\mathcal{L} \cdot \mathcal{L}^*) \to 0, \]
which lead to isomorphisms (15.10). Formula (15.11) follows from Remark 15.5.

Note that in general we have:
\[ \dim H^i(L, M) \leq \dim H^i_{\text{GF}}(C, M) + \dim H^{i+1}(\mathcal{L} \cdot \mathcal{L}^*). \]

The above results provide a tool for computing the cohomology of Lie pseudoalgebras, by making use of the known results on Gelfand–Fuchs cohomology of Lie algebras of vector fields [Fu].

15.4. Central extensions of finite simple Lie pseudoalgebras. In this section we determine by a direct computation all nontrivial central extensions of a finite simple Lie pseudoalgebra $L$ with trivial coefficients.

Such a central extension of $L$ is isomorphic as an $H$-module to $\hat{L} = L \oplus k1$, where the action of $H$ on $1$ is given by $h \cdot 1 = \varepsilon(h)1$. The pseudoBracket is then
\[ [a \ast b] = [a \ast b] + \gamma(a, b) \ast_H 1, \quad a, b \in L, \]
where $\gamma(a, b) \in H \ast H$.

Notice that a tensor product $(h_1 \otimes \cdots \otimes h_n) \otimes_H 1 \in H^\otimes n \otimes_H k$ can always be re-expressed as $(h_1 h_{-1} \otimes \cdots \otimes h^{n-1} h_{-(n-1)} \otimes 1) \otimes_H 1$, and this coefficient is unique in $H^\otimes (n-1) \otimes 1$ (see Lemma 2.5).

Therefore, the above bracket is uniquely determined by the unique $\beta(a, b) \in H$ such that
\[ \gamma(a, b) \otimes_H 1 = (\beta(a, b) \otimes 1) \otimes_H 1; \]
we will call this map $\beta: L \otimes L \to H$ the cocycle representing the central extension. Then $H$-bilinearity and skew-symmetry of the pseudoBracket give the following properties of this cocycle:
\[ \beta(ha, b) = h\beta(a, b), \quad \beta(a, hb) = \beta(a, b)S(h), \quad \beta(a, b) = -S(\beta(b, a)), \]
for all $a, b \in L, h \in H$.

We consider two central extensions equivalent if they are isomorphic Lie pseudoalgebras. An isomorphism of $\hat{L}$ is given by an embedding $L \hookrightarrow \hat{L}$ projecting to the identity on $L$. All such embeddings are uniquely determined by $H$-linear maps $\phi: L \to k$. Then the cocycles representing the two equivalent central extensions differ by $\tau_\phi(a, b)$ such that
\[ (\tau_\phi(a, b) \otimes 1) \otimes_H 1 = (\text{id}_H \otimes_H \phi)([a \ast b]). \]
This is called a trivial cocycle.

If $L = He$ is an $H = U(\mathfrak{d})$-module which is free on the generator $e$, such that
\[ [e \ast e] = a \ast_H e, \quad a = r + s \otimes 1 - 1 \otimes s, \quad r \in \mathfrak{d} \wedge \mathfrak{d}, \quad s \in \mathfrak{d}, \]
then a cocycle $\beta(a, b)$ is completely determined by its value $\beta = \beta(e, \epsilon) \in H$. Trivial cocycles are of the form
\[ \tau = \tau(e, \epsilon) = \phi(\epsilon)(2s - x), \quad \phi(\epsilon) \in k, \]
where
\[
   x = \frac{1}{2} \sum_{i,j} r^{ij} [\delta_i, \delta_j] \quad \text{if} \quad r = \sum_{i,j} r^{ij} \delta_i \circ \delta_j.
\]

**Lemma 15.7.** Let \( L = H e \) be a Lie pseudoalgebra as above. Then \( H^2(L,k) \simeq B/k(2s - x) \), where \( B \) is the space of elements \( \beta \in H \) satisfying the following two conditions:

\[
   \beta = -S(\beta),
\]
\[
   a \Delta(\beta) = (\beta \circ 1 + 1 \circ \beta) a + \beta \circ (3s - x) - (3s - x) \circ \beta.
\]

Moreover, when \( r \neq 0 \), then \( \beta \in \mathfrak{d} \), and (15.18) becomes equivalent to the following system of equations:

\[
   [s, \beta] = 0, \quad [r, \Delta(\beta)] = \beta \circ (3s - x) - (3s - x) \circ \beta.
\]

**Proof.** Let \( \hat{L} = H e + k1 \) be a central extension of \( L \) with a pseudobracket
\[
   [\epsilon \ast \epsilon] = a \circ_H e + (\beta \circ 1) \circ_H 1,
\]
where \( h \cdot 1 = \epsilon(h)1 \) for \( h \in H \).

The skew-symmetry of \([\epsilon \ast \epsilon]\) is equivalent to (15.17). The Jacobi identity is equivalent to Jacobi identity for \([\epsilon \ast \epsilon]\) together with the following cocycle condition for \( \gamma = \beta \circ 1 \) (cf. Proposition 4.1):

\[
   (\alpha \circ 1) (\Delta \circ \text{id})(\gamma) \circ_H 1 = (1 \circ \alpha) (\text{id} \circ \Delta)(\gamma) \circ_H 1
\]
\[
   - \langle \sigma \circ \text{id} \rangle (1 \circ \alpha) (\text{id} \circ \Delta)(\gamma) \circ_H 1.
\]

With the usual notation \( r_{12} = r \circ 1 \), \( s_1 = s \circ 1 \circ 1 \), etc., we have:

\[
   (\alpha \circ 1) (\Delta \circ \text{id})(\gamma) \circ_H 1 = (\alpha \Delta(\beta) \circ 1) \circ_H 1,
\]
\[
   (1 \circ \alpha) (\text{id} \circ \Delta)(\gamma) \circ_H 1 = (r_{23} + s_2 - s_3) \beta_1 \circ_H 1 = \beta_1 (r_{23} + s_2 - s_3) \circ_H 1
\]
\[
   = \beta_1 (-r_{21} - x_2 + s_1 + 2s_2) \circ_H 1 = \beta_1 (a_{12} + 3s_2 - x_2) \circ_H 1.
\]

From here it is easy to see that (15.21) is equivalent to (15.18).

Let now \( r \) be nonzero. Rewrite (15.18) in the form
\[
   a (\Delta(\beta) - \beta \circ 1 - 1 \circ \beta) = [\beta \circ 1 + 1 \circ \beta, a] + \beta \circ (3s - x) - (3s - x) \circ \beta.
\]

If \( \beta \notin \mathfrak{d} + k \), then the degree of the left hand side equals \( \deg \beta + 2 \) while that of the right hand side is at most \( \deg \beta + 1 \), giving a contradiction. So \( \beta \in \mathfrak{d} + k \), and (15.17) shows that \( \beta \in \mathfrak{d} \). \( \square \)

**Proposition 15.8.** Let \( \mathfrak{h} \subset \mathfrak{d} \) be finite-dimensional Lie algebras, \( H = U(\mathfrak{d}) \), \( H' = U(\mathfrak{h}) \), and let \( L = \text{Cur}^H_M, W(\mathfrak{h}) \).

(i) If \( \dim \mathfrak{h} = 1 \), then \( H^2(L,k) \text{ is 1-dimensional.} \)

(ii) If \( \mathfrak{d} \text{ is abelian and } \dim \mathfrak{h} > 1 \), then \( H^2(L,k) = 0 \).

**Proof.** (i) The Lie pseudoalgebra \( L = H e \) is free of rank one, with \( e = 1 \circ s \), \( s \in \mathfrak{h} \setminus \{0\} \), hence we can use Lemma 15.7. In this case \( \alpha = s \circ 1 - 1 \circ s \), and equation (15.18) becomes
\[
   (s \circ 1 - 1 \circ s) (\Delta(\beta) - \beta \circ 1 - 1 \circ \beta) = 3(\beta \circ s - s \circ \beta).
\]
for $\beta \in H$. We choose a basis $\{\partial_i\}$ of $\mathfrak{d}$ such that $\partial_1 = s$, and express $\beta$ in a Poincaré–Birkhoff–Witt basis as $\beta = \sum I \beta_I \partial^{(I)}$, $\beta_I \in k$ (see Example 2.3). Then the above equation becomes:

$$\sum_{I,J \neq 0} \beta_{I+J}(\partial_I \partial^{(I)} \odot \partial_J) - \partial^{(I)} \odot \partial_I \partial^{(J)} = \sum_I 3\beta_I (\partial^{(I)} \odot \partial_J - \partial_I \odot \partial^{(J)}).$$

Comparing terms of the form $h \odot \partial_J$ ($j \neq 1$) we find that $\beta_J$ is zero unless $I = (i, 0, \ldots, 0)$ for some $i$. Hence $\beta = \sum_i \beta_i s^i$, $\beta_i \in k$. Substituting and comparing coefficients, we obtain that $\beta = \beta_3 s^3$. This obviously satisfies (15.17). The trivial cocycles are multiples of $2s$, hence $s^3$ is the unique extension up to scalar multiples. This is the well known Virasoro central extension.

(ii) Choose a basis of $\mathfrak{d}'$ and let $\beta$ be a cocycle representing a central extension of $L \cong H \otimes \mathfrak{d}'$. Then for each basis element $a$, $\beta$ restricts to a cocycle of $H \otimes a \subset L$, which is a current Lie pseudoalgebra over $W(\mathfrak{ka})$. By part (i) we can then add to $\beta$ a trivial cocycle as to make $\beta_1(1 \otimes a, 1 \otimes a) = c_a a^3$, $c_a \in k$, for every such basis element $a \in \mathfrak{d}'$. Denoting $\beta = \beta(1 \otimes a, 1 \otimes b)$, the Jacobi identity for elements

$$\sum (a \otimes 1 - 1 \otimes a)\Delta(a) = c_a (a^3 \otimes b - b \otimes a^3) + (\beta \otimes 1 - 1 \otimes \beta) \Delta(a).$$

Let $a, b$ be distinct elements in the above basis, which we extend to a basis $\{\partial_i\}$ of $\mathfrak{d}$ with $\partial_1 = a$, $\partial_2 = b$. We substitute the Poincaré–Birkhoff–Witt basis expression $\beta = \sum I \beta_I \partial^{(I)}$ in (15.22), to get:

$$c_a (\partial_1^3 \odot \partial_2 - \partial_2 \odot \partial_1^3) = \sum_{I,J} \beta_{I+J}(\partial_I \partial^{(I)} \odot \partial_J) - \partial^{(I)} \odot \partial_I \partial^{(J)} - \sum_{I,J} \beta_I(\partial_J \partial^{(J)} \odot 1 + \partial^{(J)} \odot \partial_J - \partial_I \odot \partial^{(J)} - 1 \otimes \partial_I \partial^{(J)}).$$

Comparing coefficients of the form $h \odot \partial_J$ for $j \neq 1$, we find that $\beta_J$ can be nonzero only when $I = (2, 1, 0, \ldots, 0)$, in which case $\beta_J = 2c_a$, and when $I = (i, 0, \ldots, 0)$ for some $i$. This means that 

$$\beta = \beta(1 \otimes a, 1 \otimes b) = f(a) + c_a a^3 b$$

for some polynomial $f$. 

We can repeat the same argument after switching the roles of $a$ and $b$, to get:

$$\beta(1 \otimes b, 1 \otimes a) = g(b) + c_b b^3 a.$$ 

Then the skew-symmetry $\beta(1 \otimes a, 1 \otimes b) = -\beta(1 \otimes b, 1 \otimes a)$ implies: $f(a) + c_a a^3 b = -g(b) + c_b b^3 a$. This is possible only when $f = 0$, $c_a = 0$. Therefore $\beta$ is identically zero.

### Proposition 15.9

Let $\mathfrak{d}' \subset \mathfrak{d}$ be abelian finite-dimensional Lie algebras, $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and let $L = \text{Cur}^H_{H'} H(\mathfrak{d}', \chi, \omega)$. Then $H^2(L, k)$ is isomorphic to $\mathfrak{d}$ if $\chi = 0$, and is trivial otherwise.

**Proof.** $L$ is free of rank one and $r \neq 0$, hence (15.18) becomes $3(\beta \otimes s - s \otimes \beta) = 0$, $\beta \in \mathfrak{d}$. This is satisfied only by multiples of $s$ if $s \neq 0$ and by all elements of $\mathfrak{d}$ otherwise. Since $\mathfrak{d}$ is abelian, then $x = 0$ and trivial cocycles are multiples of $s$. \hfill $\Box$

### Proposition 15.10

Let $\mathfrak{d}'$ be the Heisenberg Lie algebra of dimension $N = 2n + 1 \geq 3$, and $\mathfrak{d} = \mathfrak{d}' \oplus \partial_1$ be the direct sum of $\mathfrak{d}'$ and an abelian Lie algebra $\partial_1$. Let $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and $L = \text{Cur}^H_{H'} K(\mathfrak{d}', \theta)$. Then $H^2(L, k) = 0$.

**Proof.** $L$ is free of rank one, and

$$\alpha = \sum_i (a_i \odot b_i - b_i \odot a_i) - c \odot 1 + 1 \odot c,$$
where \( \{a_i, b_i, c\} \) is a basis of \( \mathfrak{g}' \) with the only nonzero commutation relations
\[
[a_i, b_i] = c, \quad 1 \leq i < n \quad \text{ (see Example 8.18)}.
\]

It is immediate to check that \([r, d \odot 1 + 1 \odot d] = c \odot d - d \odot c \) for all \( d \in \mathfrak{g}' \).
Moreover, the element \( x \) from (15.16) equals \( nc \). Then, if \( \beta = \beta' + \beta_0 \) with \( \beta' \in \mathfrak{g}' \),
\( \beta_0 \in \mathfrak{g}_0 \), equation (15.20) becomes:
\[
\beta' \circ c - c \circ \beta' = (n + 3)(\beta \circ c - c \circ \beta).
\]

All solutions \( \beta \) of this equation are multiples of \( c \). Trivial cocycles are multiples of
\[
2s - x = -(n + 2)c, \quad \text{hence all cocycles are trivial.}
\]

**Proposition 15.11.** Let \( \mathfrak{g}' \subset \mathfrak{g} \) be abelian finite-dimensional Lie algebras such
that \( \dim \mathfrak{g}' > 2 \), let \( H = U(\mathfrak{g}), H' = U(\mathfrak{g}') \), and let \( L = \text{C}^{1,H}(S(\mathfrak{g}',0)) \). Then
\( H^2(L,k) = 0 \).

**Proof.** By Proposition 8.3, \( L \) is spanned over \( H \) by elements
\[
e_{ab} = a \odot b - b \odot a, \quad a, b \in \mathfrak{g}',
\]
satisfying the relations \( e_{ab} = -e_{ba} \) and
\[
\begin{align*}
(15.23) & \quad ae_{bc} + be_{ca} + ce_{ab} = 0.
\end{align*}
\]

The pseudobrackets are (see (8.24)):
\[
[e_{ab} \star e_{cd}] = (a \odot d) \odot_H e_{bc} + (b \odot c) \odot_H e_{ad} - (a \odot c) \odot_H e_{bd} - (b \odot d) \odot_H e_{ac},
\]
and in particular
\[
\begin{align*}
[e_{ab} \star e_{ac}] &= -(a \odot c) \odot_H e_{ab} + (b \odot a) \odot_H e_{ac} - (a \odot a) \odot_H e_{ac}, \\
[e_{ab} \star e_{ab}] &= (b \odot a - a \odot b) \odot_H e_{ab}.
\end{align*}
\]

Trivial cocycles \( \tau_\phi \) are determined by the identity (see (15.15)):
\[
(\tau_\phi(e_{ab}, e_{cd}) \odot 1) \odot_H 1
= (a \odot d) \odot_H \phi_{bc} + (b \odot c) \odot_H \phi_{ad} - (a \odot c) \odot_H \phi_{bd} - (b \odot d) \odot_H \phi_{ac},
\]
where \( \phi_{ab} = \phi(e_{ab}) = -\phi_{ba} \in k \), which gives:
\[
(\tau_\phi(e_{ab}, e_{cd}) \odot 1) \odot_H 1
= -ad\phi_{bc} - bc\phi_{ad} + ac\phi_{bd} + bd\phi_{ac}.
\]

Let \( \beta \) be a cocycle for \( L \) representing a central extension. Write \( \beta_{ab,cd} = \beta(e_{ab}, e_{cd}) \) for short.
Equations (15.14), (15.23) give the identities:
\[
\begin{align*}
(15.25) & \quad \beta_{ab,cd} = -\beta_{ab,dc} = -\beta_{ab,da} = -S(\beta_{d,ab}), \\
(15.26) & \quad a\beta_{bc,cd} + b\beta_{c,a,cd} + c\beta_{ab,c,cd} = 0.
\end{align*}
\]

Using this, Jacobi identity for the elements \( e_{ab}, e_{ab}, e_{ac} \) gives the following equation
for \( \beta \):
\[
(15.27)
\begin{align*}
(b \odot a - a \odot b) \odot (\Delta(\beta_{ab,ac}) - \beta_{ab,ac} \odot 1 - 1 \odot \beta_{ab,ac})
&= ab \odot \beta_{ab,ac} - \beta_{ab,ac} \odot ab + \beta_{ab,ac} \odot a^2 - a^2 \odot \beta_{ab,ac} \\
&\quad + \beta_{ab,ac} \odot ac - ac \odot \beta_{ab,ac}.
\end{align*}
\]

This is a homogeneous equation and can be solved degree by degree. If \( \beta \) is homogeneous of degree one, then the left hand side is zero, and we immediately
see that \( \beta_{ab,ac} = \beta_{ab,ca} = \beta_{ab,ba} = 0 \). Then, by (15.26), \( \beta_{ab,cd} = 0 \).

If \( \beta \) is homogeneous of degree other than one, then \( \beta_{ab,ab} = 0 \), since \( \beta \) restricts
to a cocycle of the free rank one Lie pseudoalgebra \( H e_{ab} \), which has been shown in
Proposition 15.9 to take values in \( \mathfrak{d} \). Then equations (15.25), (15.26) give \( a_{b,b,ac} = b_{c,b,ac} \). Hence if \( a \) and \( b \) are linearly independent, we can find some \( p = p_{bc}^e \in H \) such that
\[
\beta_{ab,ac} = ap_{bc}, \quad \beta_{ab,bc} = bp_{bc}.
\]
We substitute this into (15.27) and after simplification obtain \( \Delta(p) = p \otimes 1 + 1 \otimes p \).
Therefore, \( p \in \mathfrak{d} \), hence the only nonzero solutions to (15.27) occur in degree two.

Now using (15.26) and (15.28), we get:
\[
\beta_{ab,c} = ap_{bd} - bp_{ed}.
\]
The skew-symmetry of \( \beta_{ab,c} \) gives the equations \( p_{bc} = -p_{bc} \) and
\[
\alpha p_{bd} - bp_{ed} = cp_{bd} - dp_{bc}.
\]
From this we obtain that \( p_{bd} \) lies in the linear span of \( a, b, d \). Comparing the coefficients in front of \( ac \) in (15.30), we see that the coefficient of \( a \) in \( p_{bd} \) is equal to the coefficient of \( c \) in \( p_{bd} \). Call this coefficient \( \phi_{bc} \); then \( \phi_{bd} = -\phi_{ab} \). Then comparison of other coefficients in (15.30) shows that
\[
p_{bc} = a\phi_{bc} + b\phi_{ac} + c\phi_{ab}
\]
for all \( a, b, c \in \mathfrak{d} \). Substitute this in (15.29) to obtain that \( \beta = \tau_p \) is trivial (cf. (15.24)).

**Proposition 15.12.** Let \( H = U(\mathfrak{d}) \), and let \( g \) be a simple finite-dimensional Lie algebra. If \( L = \text{Cur}_g \), then \( H^2(L, k) \simeq \mathfrak{d} \).

**Proof.** Let \( \beta \) be a cocycle for \( L \). We will write \( \beta(a, b) = \beta(1 \otimes a, 1 \otimes b) \) for \( a, b \in g \).

Then Jacobi identity leads to the equation
\[
\beta([a, b], c) = 1 - 1 \otimes \beta([b, [a, c]]) = \Delta(\beta([a, b], c)).
\]
This immediately implies: \( \beta([a, b], c) \in \mathfrak{d} + k \). Since \([g, g] = g\) this shows that \( \beta(a, b) \in \mathfrak{d} + k \) for all \( a, b \in g \).

We can now solve the homogeneous equation (15.32) degree by degree. Solutions of degree zero are cocycles of the Lie algebra \( g \), hence they are all trivial. Solutions of degree one satisfy \( \beta([a, b], c) = \beta([a, b], c) \), and skew-symmetry implies \( \beta(a, b) = -\beta(b, a) \). Therefore every such \( \beta \) is an invariant symmetric bilinear form on \( g \) with values in \( \mathfrak{d} \). Any such bilinear form can be written as \( \beta(a, b) = (a|b)d \) where \((\cdot|\cdot)\) is the Killing form on \( g \) and \( d \) is some element of \( \mathfrak{d} \). Such cocycles \( \beta \) are nontrivial, hence inequivalent central extensions are in one-to-one correspondence with elements of \( \mathfrak{d} \).

**Theorem 15.13.** Let \( H = U(\mathfrak{d}) \) and \( L \) be a simple Lie \( H \)-pseudomodule. Then \( L \) may have a nontrivial central extension (given by (15.12), (15.13)) only if:

(i) \( L = \text{Cur}_g \), in which case \( H^2(L, k) \simeq \mathfrak{d} \) and cocycles \( \beta \) are given by \( \beta \cdot (1 \otimes a, 1 \otimes b) = (a|b)d \) for \( a, b \in g \), where \( d \in \mathfrak{d} \) and \((\cdot|\cdot)\) is the Killing form.

(ii) \( L = \text{Cur}^H_{g'} \), \( W(g') \) with \( k = \mathfrak{d} \subset \mathfrak{d} \), \( \dim g' = 1 \), in which case \( H^2(L, k) \) is of dimension one, generated by the Virasoro cocycle \( \beta(1 \otimes s, 1 \otimes s) = s^2 \).

(iii) \( L = \text{Cur}^H_{g''} \), \( H(g', \omega', \omega') \) with \( g' \subset \mathfrak{d} \), in which case \( H^2(L, k) \) is isomorphic to the quotient of the space of all solutions \( \beta \in \mathfrak{d} \) to equations (15.19), (15.20) by the subspace \( k(2s - x) \), where \( v \in g' \wedge g' \) is dual to \( \omega' \), \( s \in g' \) is such that \( \omega' = i_s \omega' \), and \( x \) is given by (15.16).
Proof. (i) and (ii) follow from Propositions 15.12 and 15.8(i), and (iii) from a direct application of Lemma 15.7.

For any other simple pseudoalgebra \( L \), the strategy is to construct a continuous family of pseudoalgebras \( L_t \), indexed by \( t \in \mathbb{k} \) endowed with the Zariski topology, that are all isomorphic to \( L \) when \( t \neq 0 \), and whose fiber at \( t = 0 \) is one of the pseudoalgebras already considered in Propositions 15.8(ii), 15.10 and 15.11. Then, since \( H^3(L_t, \mathbb{k}) = 0 \) for \( t = 0 \), it will follow that \( H^3(L_t, \mathbb{k}) = 0 \) whenever \( t \) lies in a neighborhood of 0, hence for all \( t \in \mathbb{k} \).

In the case of a current pseudoalgebra over a \( W \) or \( S \) type Lie pseudoalgebra, choose a basis \( \{ \partial_i \} \) of \( \mathfrak{d} \) that contains a basis of \( \mathfrak{d}' \), and construct the family \( \mathfrak{d}'_t \subset \mathfrak{d}_t \) of Lie algebras indexed by \( t \in \mathbb{k} \) generated by elements \( \{ \partial_i^t \} \) with Lie bracket 
\[
[\partial_i^t, \partial_j^t] = t[\partial_i, \partial_j].
\]
Then for \( t \neq 0 \) we have an isomorphism \( \mathfrak{d}_t \cong \mathfrak{d}'_t \rightarrow t \partial_i \in \mathfrak{d}_t \), whereas \( \mathfrak{d}_0 \) is an abelian Lie algebra. Then \( \text{Cur}^H_{H_t} W(\mathfrak{d}_t') \), \( H_t = U(\mathfrak{d}_t) \), \( H_t' = U(\mathfrak{d}_t') \), is a family of pseudoalgebras all isomorphic to \( \text{Cur}^H_{H_t} W(\mathfrak{d}') \) for \( t \neq 0 \). The fiber of this family at \( t = 0 \) has been shown in Proposition 15.8(ii) to have no nontrivial central extensions. In the same way, if we set \( \chi_t(\partial_i^t) = t \chi(\partial_i) \), then \( \{ \text{Cur}^H_{H_t} S(\mathfrak{d}_t', \chi_t) \} \) is a family of pseudoalgebras all isomorphic to \( \text{Cur}^H_{H_t} S(\mathfrak{d}', \chi) \) for \( t \neq 0 \), and the fiber at \( t = 0 \) is \( \text{Cur}^H_{H_0} S(\mathfrak{d}', \chi_0) \), \( \mathfrak{d}_t' \subset \mathfrak{d}_t \) are isomorphic to \( \mathfrak{d}' \subset \mathfrak{d} \) as vector spaces but have trivial bracket.

If \( L \) is a current pseudoalgebra over \( K(\mathfrak{d}', \mathfrak{b}) \), for finite-dimensional Lie algebras \( \mathfrak{d}' \subset \mathfrak{d} \), choose a basis \( \{ a_1, b_1, s \} \) of \( \mathfrak{d}' \) as in Lemma 8.8, and complete it with \( \{ d_1, \ldots, d_j \} \) to a basis of \( \mathfrak{d}_t \). Then a continuous family \( \{ \partial_i \} \) of Lie algebras can be constructed for \( t \neq 0 \) as \( \mathfrak{d}_t \simeq \mathfrak{d} \) spanned by \( a_1^t = ta_i, b_1^t = tb_i, s^t = t^3 s, d_i^t = t^2 d_i \), and by setting \( a_i^0, b_i^0, c^0 = -s^0 \) to span a Heisenberg algebra, and all brackets involving \( d_i^0 \) to be zero. Define \( \theta_t(\partial_i^t) \) by \( \theta_t(a_i^t) = \theta_t(b_i^t) = 0, \theta_t(s^t) = -1 \). Then \( \text{Cur}^H_{H_0} K(\mathfrak{d}', \mathfrak{b}) \) is the limit of the Lie pseudoalgebras \( \{ \text{Cur}^H_{H_t} K(\mathfrak{d}_t', \mathfrak{b}_t) \} \), which are all isomorphic to \( \text{Cur}^H_{H_t} K(\mathfrak{d}', \mathfrak{b}) \), and the former Lie pseudoalgebra is of the type treated in Proposition 15.10.

16. Application to the Classification of Poisson Brackets in Calculus of Variations

In calculus of variations the phase space consists of \( C^\infty \) vector functions \( u = (u_1(x), \ldots, u_r(x)) \) where \( u_i(x) \) are, for example, functions with compact support on a closed \( N \)-dimensional manifold. We will consider linear local Poisson brackets:

\[
\{u_i(x), u_j(y)\} = \sum_\alpha B_{\alpha ij}(y) \partial^\alpha_y \delta(x - y)
\]

where the sum runs over a finite set of multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N \), the \( B_{\alpha ij} \) are linear combinations of the \( u_k \) and of their derivatives \( u_k^{(\alpha)} := \partial^\alpha u_k \), where \( \partial^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_N} \right)^{\alpha_N} \), and \( \delta(x - y) \) is the delta-function (defined by \( \int f(x) \delta(x - y) dx = f(y) \)). By Leibniz rule and bilinearity, the Poisson bracket (16.1) extends to arbitrary polynomials \( P \) and \( Q \) in the \( u_i \) and their derivatives. Explicitly:

\[
\{P(x), Q(y)\} = \sum_{\alpha, \beta, i, j} \frac{\partial P(x)}{\partial u_i^{(\alpha)}} \frac{\partial Q(y)}{\partial u_j^{(\beta)}} \partial_x^\alpha \partial_y^\beta \{u_i(x), u_j(y)\}.
\]
This bracket, apart from bilinearity and Leibniz rule, should satisfy skew-commutativity and the Jacobi identity.

The basic quantities in calculus of variations are local functionals (Hamiltonians) \( I_P = \int P(x) dx \). Using bilinearity and integration by parts (\( \int \frac{\partial P}{\partial x} \delta Q dx = -\int P \frac{\partial \delta Q}{\partial x} dx \)), we get from (16.2) the following well-known formula:

\[
(16.3) \quad \{ I_P, I_Q \} = \sum_{i,j} \iiint \frac{\delta P(x)}{\delta u_i} \frac{\delta Q(y)}{\delta u_j} \{ u_i(x), u_j(y) \} dx dy,
\]

where

\[
(16.4) \quad \frac{\delta P(x)}{\delta u_i} = \sum_{\alpha} (-\partial x) \frac{\partial P(x)}{\partial u_i^{(\alpha)}}
\]

is the variational derivative.

More generally, one usually considers a class of Poisson brackets of the form:

\[
(16.5) \quad \{ u_i(x), u_j(y) \} = B_{ij}(y, \delta_y^{(\alpha)} \delta(x - y),
\]

where \( B_{ij} \) are differential operators in \( \delta_y^{(\alpha)} \) whose coefficients are polynomials in \( u_k^{(\gamma)}(y) \). Then the \( r \times r \) matrix \( B = (B_{ij}) \) is called a Hamiltonian operator, and, integrating by parts, formula (16.3) can be rewritten in its most familiar form:

\[
(16.6) \quad \{ I_P, I_Q \} = \int \left( \frac{\partial P(x)}{\partial u} \right) \frac{\partial Q(x)}{\partial u} dx.
\]

Given a Hamiltonian \( h = \int P(x) dx \), we have the corresponding Hamiltonian system of evolutionary partial differential equations:

\[
(16.7) \quad \dot{u}_i = \{ h, u_i \} = \sum_j B_{ij} \frac{\partial P}{\partial u_j},
\]

so that if another Hamiltonian \( h_1 \) is in involution with \( h \), i.e., \( \{ h, h_1 \} = 0 \), then \( h_1 \) is an integral of motion of (16.7), i.e., \( \dot{h}_1 = 0 \).

It is shown in [DN1] and [M] that for \( r \geq 2 \), any Poisson bracket of hydrodynamic type (i.e., linear in the derivatives) under certain nondegeneracy conditions can be transformed into a linear Poisson bracket of hydrodynamic type by a change of the field variables. The latter Poisson brackets have been studied rather extensively (see [Do], [DN2] and references there, [GD], [M], [Z]).

Let \( H = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}] \) be the universal enveloping algebra of the \( N \)-dimensional abelian Lie algebra \( \delta \). Let \( F = \bigoplus_{i=1}^r Hu_i \) be the free \( H \)-module of rank \( r \) on generators \( u_i \). Consider a linear Poisson bracket (16.1). For any three subspaces \( A, B, C \) of \( F \), we will use the notation \( \{ A, B \} \subset C \) if for any \( a \in A, b \in B \) all coefficients in front of \( \delta_y^{(\alpha)} \delta(x - y) \) in \( \{ a(x), b(y) \} \) belong to \( C \). We call a linear Poisson algebra any \( H \)-submodule \( L \) of \( F \) which is closed under the Poisson bracket, i.e., such that \( \{ L, L \} \subset L \). By an isomorphism of two such algebras we mean a \( \mathbb{C} \)-linear isomorphism preserving Poisson brackets.

If \( L \) is a linear Poisson algebra, we define the \( \lambda \)-bracket \( (\lambda = (\lambda_1, \ldots, \lambda_N)) \) on \( L \) as the Fourier transform of the linear Poisson bracket (16.1):

\[
(16.8) \quad [u_i, \lambda u_j] = \sum_{\alpha} \lambda^{(\alpha)} B_{\alpha ij}.
\]
Then we get a Lie conformal algebra in $N$ (commuting) indeterminates (defined in the same way as for the $N=1$ case in the introduction). Thus, the classification of linear Poisson algebras follows from the classification of Lie $U(\mathfrak{d})$-conformal algebras, where $\mathfrak{d}$ is the $N$-dimensional abelian Lie algebra.

Recall that the structure of a Lie conformal algebra is equivalent to the structure of a Lie pseudoalgebra (see Section 9). The relationship between the linear Poisson bracket (16.1) and the pseudobracket can be described explicitly as follows:

\[ [u_i \ast u_j] = \sum_k P_i^k (\partial \otimes 1, 1 \otimes \partial) \otimes_H u_k, \]

if

\[ \{ u_i(x), u_j(y) \} = \sum_k P_i^k (\partial_x, \partial_y) (u_k(y) \delta(x - y)) \]

for some polynomials $P_i^k$. Note that equation (16.1) can always be written in the form (16.10). Indeed, if

\[ \{ u_i(x), u_j(y) \} = \sum_k Q_i^k (\partial_y, \partial_x) (u_k(t) \delta(x - y)) \big|_{t = y} \]

for some polynomials $Q_i^k$, then we have (16.10) with $P_i^k(z, w) = Q_i^k(-z, z + w)$.

In this case, the $\lambda$-bracket (16.8) is given by:

\[ \{ u_i(x), u_j(y) \} = \sum_k Q_i^k (\lambda, \partial) u_k. \]

Remark 16.1. The constant terms of $B_{ai,j}$ in (16.1) give a central extension of the linear Poisson algebra corresponding to the $B_{ai,j}$ with constant terms removed. In terms of the associated Lie pseudoalgebras this corresponds to a central extension by $\mathbb{C}$ with a trivial action of $H$. By Theorem 15.1, these central extensions are parameterized by $H^2(L, \mathbb{C})$.

**Examples 16.2** (cf. [M], [DN2], [K4], [BKVV]).

1. General Poisson algebra $W_{r,N}$, where $1 \leq r \leq N$ ($1 \leq i, j \leq r$):

\[ \{ u_i(x), u_j(y) \} = \frac{\partial u_j(y)}{\partial y_i} \delta(x - y) + u_j(y) \frac{\partial}{\partial y_i} \delta(x - y) + u_i(y) \frac{\partial}{\partial y_j} \delta(x - y). \]

2. Special Poisson algebra $S_{r,N,\chi}$, where $2 \leq r \leq N$ and $\chi = (\chi_1, \ldots, \chi_r) \in \mathbb{C}^r$,

is the following subalgebra of $W_{r,N}$:

\[ \left\{ \sum_{i=1}^r P_i (\partial_x) u_i(x) \left| \sum_{i=1}^r \left( \frac{\partial}{\partial x_i} + \chi_i \right) P_i (\partial_x) = 0 \right. \right\}. \]

It is generated over $H = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}]$ by elements

\[ u_i(x) = \left( \frac{\partial}{\partial x_i} + \chi_i \right) u_i(x) - \left( \frac{\partial}{\partial x_j} + \chi_j \right) u_j(x). \]

3. Hamiltonian Poisson algebra $H_{2s,N}$, $2 \leq r = 2s \leq N$:

\[ \{ u(x), u(y) \} = \sum_{i=1}^s \left( \frac{\partial u_i(y)}{\partial y_i} \frac{\partial}{\partial y_i} \delta(x - y) - \frac{\partial u_i(y)}{\partial y_{i+s}} \frac{\partial}{\partial y_{i+s}} \delta(x - y) \right). \]

We have an inclusion $H_{2s,N} \subset W_{2s,N}$ by letting

\[ u(x) = \sum_{i=1}^s \left( \frac{\partial u_{i+s}(x)}{\partial x_i} - \frac{\partial u_i(x)}{\partial x_{i+s}} \right). \]
4. Current Poisson algebra $\text{Cur}_N \mathfrak{g}$ associated to a simple $r$-dimensional Lie algebra $\mathfrak{g}$ with structure constants $c^k_{ij}$ ($1 \leq i, j, k \leq r$):

$$\{v_i(x), v_j(y)\} = \sum_{k=1}^r c^k_{ij} v_k(y) \delta(x - y).$$

5. Semidirect sum of $W_{r,N}$ or one of its subalgebras $S_{r,N,N}, H_{2s,N}$ with $\text{Cur}_N \mathfrak{g}$ defined by ($1 \leq i \leq r$, $v(x) \in \text{Cur}_N \mathfrak{g}$):

$$\{u_i(x), v(y)\} = \frac{\partial v(y)}{\partial y_i} \delta(x - y) + v(y) \frac{\partial}{\partial y_i} \delta(x - y).$$

A subspace $I$ of a Poisson algebra $L$ is called an ideal if it is invariant under taking Poisson brackets with elements of $L$, i.e., if $\{L, I\} \subset I$. A Poisson algebra $L$ is called simple (respectively semisimple) if the Poisson bracket is not identically zero and $L$ contains no nonzero $H$-invariant ideals $I$ such that $I \neq L$ (respectively $\{I, I\} \neq 0$). Note that the Poisson algebras that we consider here are finite, i.e., finitely generated as $H$-modules.

Then Theorems 13.10 and 13.15 and Corollary 13.26 imply:

**Theorem 16.3.** (i) Any simple linear Poisson algebra is isomorphic to one of the Poisson algebras $W_{r,N}, S_{r,N,N}, H_{2s,N}, \text{Cur}_N \mathfrak{g}$.

(ii) Any semisimple linear Poisson algebra is a direct sum of simple ones and of the semidirect sums described in Example 16.2(5).

**Remark 16.4.** It follows from Remark 16.1 and the results of Section 15.4 that all nontrivial central extensions of simple Poisson algebras are described by the following 2-cocycles.

For $\alpha \in \mathbb{C}^r$ let

$$\psi_\alpha(x, y) = \sum_{i=1}^r \alpha_i \frac{\partial}{\partial y_i} \delta(x - y).$$

Then all nontrivial 2-cocycles for $H_{r,N}$ are:

$$\gamma_\alpha(u(x), u(y)) = \psi_\alpha(x, y).$$

All nontrivial 2-cocycles for $\text{Cur}_N \mathfrak{g}$ are:

$$\gamma_\alpha(v_i(x), v_j(y)) = b_{ij} \psi_\alpha(x, y),$$

where $b_{ij} = \{v_i, v_j\}$ is the invariant scalar product.

The Poisson algebra $W_{r,N}$ has a nontrivial central extension iff $r = 1$, and in the latter case it is given by the well known Virasoro cocycle:

$$\gamma(u(x), u(y)) = \left( \frac{\partial}{\partial y} \right)^3 \delta(x - y).$$

The Poisson algebra $S_{r,N,N}$ has no nontrivial central extensions if $r > 2$ or $\chi \neq 0$, and $S_{2,N,0} \simeq H_{2s,N}$ has nontrivial central extensions described above.

**References**


[R2] Differential groups and formal Lie theory for an infinite number of parameters, Ann. of Math. (2) 52 (1950), 708–726.


