Hidden Symmetries of the Open N=2 String

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Supported by Federal Ministry of Science and Transport, Austria
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Abstract

It is known for ten years that self-dual Yang-Mills theory is the effective field theory of the open $N=2$ string in $2+2$ dimensional spacetime. We uncover an infinite set of abelian rigid string symmetries, corresponding to the symmetries and integrable hierarchy of the self-dual Yang-Mills equations. The twistor description of the latter naturally connects with the BRST approach to string quantization, providing an interpretation of the picture phenomenon in terms of the moduli space of string backgrounds.
1 Introduction

The pioneering work of Ooguri and Vafa [1] revealed an intimate connection between self-dual field theories and (classical) \( N=2 \) string theories, formulated in four spacetime dimensions. In particular, non-abelian gauge fields on Kleinian flat space \( \mathbb{R}^{2,2} \) of ultrahyperbolic signature \((++--)\) and a self-dual field strength arise as exact (to all orders in \( \alpha' \)) classical background configurations for the open \( N=2 \) string. Indeed, the only physical string degree of freedom in this case is a massless \( \text{Lie-algebra-valued scalar field} \) whose (tree-level) dynamics takes the form of Leznov's [2] or Yang's [3] equation, which both describe self-dual Yang-Mills (SDYM), albeit in different gauges. Although the absence of an infinite tower of massive excitations indicates a sort of caricature of a string, this quality makes it amenable to exact solutions, a fact quite rare in string theory. Yet \( N=2 \) strings may not only serve as a testing ground for certain issues in string theory in general but, being consistent quantum theories, they can also help us guiding the quantization of self-dual Yang-Mills theory.

To set the stage for the comparison of string theory with field theory, we review in Sections 2 and 3 the twistor description of the self-dual Yang-Mills equations, their symmetries and hierarchy, on flat Kleinian space \( \mathbb{R}^{2,2} \). Although our treatment mainly follows Refs. [4, 5], we reformulate their results in a language amenable to string theory. In particular, a real form of the integrable SDYM hierarchy corresponding to affine extensions of spacetime translations is characterized. Notice that the importance of the SDYM equations in \( \mathbb{R}^{2,2} \) is also motivated by the conjecture [6] that the SDYM model may be a universal integrable model. Indeed, it has been shown that most (if not all) integrable equations in \( 1 \leq D \leq 3 \) dimensions can be obtained from the SDYM (or their hierarchy) equations by suitable reductions (see [6, 7, 8, 5] and references therein). Therefore open \( N=2 \) strings can also provide a consistent quantization of integrable models in \( 1 \leq D \leq 3 \) dimensions.

If \( N=2 \) string theory "predicts" self-dual Yang-Mills, its wealth of symmetries should be obtainable from the stringy description. More precisely, we expect the SDYM hierarchy related to the non-local abelian symmetries of the SDYM equations to be visible in \( N=2 \) open string quantum mechanics. An analogous connection should exist between the symmetries of the closed \( N=2 \) string and the self-dual \emph{gravity} hierarchy. Indeed, one of the authors (together with Jüennemann and Popov) has recently identified part of these hidden closed string symmetries [9] and has succeeded in relating them to the self-dual gravity hierarchy [10]. Quite surprisingly, the stringy root of such symmetries is technically the somewhat obscure picture phenomenon [11] which is present whenever covariant quantization meets worldsheet supersymmetry. Global symmetries unbroken by the string background under consideration may be classified with the help of BRST cohomology, and the latter unexpectedly displays a picture dependence [12] (see also [13]).

In Sections 4 and 5, we briefly review this issue in the context of the open \( N=2 \) string. Although the field-theoretic description of SDYM is simpler than that of self-dual gravity, its string-theoretic version is more involved because open string mechanics must take into account worldsheet boundaries, cross-caps, and boundary punctures. Nevertheless, the hidden symmetries emerge as in the closed-string case. Furthermore, they are seen to be responsible for the vanishing of almost all (tree-level) open-string scattering amplitudes.

After concluding, two Appendices collect standard material about line bundles over the Riemann sphere and about twistor spaces.
2 Holomorphic bundles and self-dual Yang-Mills fields

In this Section, we use some facts about line bundles over the Riemann sphere and geometry of the twistor spaces which are recalled in two Appendices.

2.1 Vector bundles over $\mathcal{P}$ and the Penrose-Ward correspondence

By a holomorphic rank $r$ vector bundle we mean a collection of five objects $(E, X, p, \mathcal{U}, f)$, where $E$ and $X$ are complex manifolds, $p : E \rightarrow X$ is a holomorphic projection, $\mathcal{U} = \{U_i\}$ is a covering of the manifold $X$ and $f = \{f_{ij}\}$ is a collection of holomorphic transition functions $f_{ij}$ on $U_i \cap U_j$ taking values in complex $r \times r$ matrices. In particular, for the twistor space $\mathcal{P}$, one can always introduce a covering $\mathcal{U} = \{U_+, U_\}$ such that

$$U_+ = H^2_+ \times \mathbb{C}^2, \quad U_- = H^2_- \times \mathbb{C}^2, \quad \mathcal{P}_0 = U_+ \cap U_- \simeq S^1 \times \mathbb{R}^4,$$

where

$$H^2_+ := H^2_+ \cup S^1 = \{\zeta \in \mathbb{C} \cup \{\infty \} : \Im \zeta \geq 0\},
H^2_- := H^2_- \cup S^1 = \{\zeta \in \mathbb{C} \cup \{\infty \} : \Im \zeta \leq 0\},$$

and $S^1 = H^2_+ \cap H^2_-$ = $\{\zeta \in \mathbb{C} \cup \{\infty \} : \Im \zeta = 0\}$. For the covering $\mathcal{U}$ of $\mathcal{P}$, holomorphic bundles $E \rightarrow \mathcal{P}$ are defined by real-analytic transition functions $f_{+-}$ on $\mathcal{P}_0 = U_+ \cap U_- \text{annihilated by the vector fields}$

$$V_\alpha = \frac{\partial}{\partial x^{1,\bar{\alpha}}} - \zeta \frac{\partial}{\partial x^{0,\bar{\alpha}}}$$

with real $\zeta$ and $\bar{\alpha} = 0, 1$. These transition functions extend holomorphically in $\eta^\alpha$ and $\zeta$ to an open neighbourhood $\mathcal{U}$ of $\mathcal{P}_0$ in $\mathcal{P}$.

Let us consider holomorphic rank $r$ vector bundles $E$ over the twistor space $\mathcal{P}$ and suppose that bundles $E$ satisfy the following conditions:

(i) restriction $E|_{\sigma_x}$ to every real holomorphic section $\sigma_x \in \Gamma_{\mathbb{R}}(\mathcal{P})$ is trivial,

(ii) $\det E$ is trivial,

(iii) $E$ has a real structure $\tau^*$. We shall show that such bundles correspond to self-dual gauge fields on $\mathbb{R}^{2,2}$ [4, 5].

In terms of transition functions $f_{+-}$, the condition (i) means that $f_{+-}(\eta^\alpha, \zeta)|_{\sigma_x}$ is the transition function for a trivial bundle over $\mathbb{CP}^1_x$ and therefore it can be factorized:

$$f_{+-}(x^{0,\bar{\alpha}} + \zeta x^{1,\bar{\alpha}}, \zeta) = \psi^+(x, \zeta) \psi^-(x, \zeta),$$

where $x^{0,\bar{\alpha}} + \zeta x^{1,\bar{\alpha}} = \eta^\alpha|_{\sigma_x}$, $x = \{x^{\alpha,\bar{\alpha}}|_{\alpha = 0, 1} \} \in \mathbb{R}^{2,2}$, and the matrices $\psi^+(x, \zeta)$ and $\psi^-(x, \zeta)$ are holomorphic with respect to $\zeta$ in the upper and lower half-planes, respectively. The condition (ii) means that the structure group $GL(r, \mathbb{C})$ of the bundle $E$ is reduced to the structure group $SL(r, \mathbb{C})$. In terms of transition functions $f_{+-}$ this condition has the form

$$\det f_{+-} = 1.$$  

The real structure $\tau^*$ in (iii) is induced from the real structure $\tau$ on $\mathcal{P}$ described in Appendix A.2. Namely, we put

$$\tau^*(f_{+-}(\eta^\alpha, \zeta)) = f_{+-}^\dagger(\bar{\eta}^\alpha, \zeta),$$

where $\dagger$ means the Hermitian conjugation. Then stable “points” of the map $\tau^*$ are transition functions satisfying

$$\tau^*(f_{+-}) = f_{+-} \quad \Leftrightarrow \quad f_{+-}^\dagger(\bar{\eta}^\alpha, \zeta) = f_{+-}(\eta^\alpha, \zeta).$$  

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It is not difficult to see that (2.6) takes place if we impose the following conditions on $\psi_\pm$:

$$\det \psi_+ = \det \psi_- = 1 \ , \quad \psi_+^{-1}(x, \zeta) = \psi_-^1(x, \zeta) .$$ (2.7)

Then $f_\pm|_{\sigma_x} = \psi_\pm^1(x, \zeta) \psi_-^1(x, \zeta)$ is Hermitian on real sections $\sigma_x \in \Gamma_F(\mathcal{P})$. Notice that if the matrix $f_\pm$ satisfies the reality conditions (2.6), then the factorization (2.4) exists for any $x \in \mathbb{R}^{2,2}$, by the results of Gohberg and Krein (see e.g. [5]).

Matrices $f_\pm|_{\sigma_x} = f_\pm(x^{0\dot{\alpha}} + \zeta x^{1\dot{\alpha}}, \zeta) \equiv f_\pm(x, \zeta)$ are annihilated by the vector fields (2.3),

$$V_\alpha f_\pm(x, \zeta) = 0 .$$ (2.8)

Substituting (2.4) into (2.8), we obtain

$$(V_\alpha \psi_+) \psi_\pm^{-1} = (V_\alpha \psi_-) \psi_\mp^{-1} .$$ (2.9)

The left-hand side of (2.9) is holomorphic in finite $\zeta \in H_\pm^2$, the right-hand side is holomorphic in finite $\zeta \in H_\pm^2$ and both sides have the simple pole at $\zeta = \infty$. Hence by an extension to Liouville’s theorem both the sides of (2.9) must be linear in $\zeta$,

$$(V_\alpha \psi_+) \psi_\pm^{-1} = (V_\alpha \psi_-) \psi_\mp^{-1} = -A_{\alpha \beta} + \zeta A_{\alpha \beta} ,$$ (2.10)

where $A_{\alpha \beta}$ are some $r \times r$ matrices which depend only on $x$, $\alpha = 0, 1$, $\beta = 0, 1$. From the conditions (2.7) it follows that $A_{\alpha \beta}$ are trace-free anti-Hermitian $r \times r$ matrices and they can be identified with components of a Yang-Mills gauge potential $A = A_{\alpha \beta} dx^{0\dot{\beta}}$ on $\mathbb{R}^{2,2}$ taking values in the Lie algebra $su(r)$.

Let us rewrite (2.10) in the form

$$D_{\alpha} \psi_+ = 0 ,$$ (2.11)

where

$$D_{\alpha} := D_{1\alpha} - \zeta D_{0\alpha} = \frac{\partial}{\partial x^{1\alpha}} + A_{1\alpha} - \zeta(\frac{\partial}{\partial x^{0\dot{\alpha}}} + A_{0\dot{\alpha}}) ,$$ (2.12)

and $D_{\alpha} := \partial_{\alpha} + A_{\alpha}$ are covariant derivatives, $\partial_{\alpha} := \partial/\partial x^{\alpha\dot{\alpha}}$. Similar equations hold for $\psi_-$. The compatibility condition of the linear system of differential equations (2.11) is

$$[D_{\alpha}, D_{\beta}] = 0 ,$$ (2.13)

and equating the coefficients of $\zeta^0, \zeta^1$ and $\zeta^2$ in (2.13) to zero, we obtain the self-dual Yang-Mills (SDYM) equations which in the coordinates $x^{0\dot{\alpha}}$ have the form

$$[D_{00}, D_{01}] = 0 , \quad [D_{10}, D_{11}] = 0 , \quad [D_{00}, D_{11}] + [D_{10}, D_{01}] = 0 .$$ (2.14)

So, transition functions $f_\pm$ determining holomorphic bundles $E$ over $\mathcal{P}$ and satisfying conditions (2.4)-(2.6) encode all the information about self-dual gauge fields in $\mathbb{R}^{2,2}$.

From formula (2.10) it follows that the gauge potential $A$ does not change its form under the transformations $\psi_+ \mapsto \psi_+ h_+(x^{0\alpha} + \zeta x^{1\dot{\alpha}}, \zeta)$, $\psi_- \mapsto \psi_- h_-(x^{0\dot{\alpha}} + \zeta x^{1\alpha}, \zeta)$, since such $h_\pm$ are annihilated by $V_\alpha$. Under these transformations $f_\pm$ transforms into the transition function $h_\pm^{-1} f_\pm h_-$ of a bundle equivalent to $E$. At the same time, gauge transformations $A \mapsto g^{-1} Ag + g^{-1} dg$ with $g = g(x) \in SU(r)$ correspond to transformations $\psi_\pm \mapsto g^{-1} \psi_\pm$ under which $f_\pm$ does
not change. So, there is a one-to-one correspondence between gauge equivalence classes of $su(r)$-valued solutions to the SDYM equations (2.14) on the Kleinian space $\mathbb{R}^{2,2}$ and equivalence classes of holomorphic rank $r$ vector bundles $E$ over $\mathcal{P}$ satisfying conditions (i)-(iii).

2.2 Gauge fixing

Let us consider Eqs. (2.11) rewritten in the form

\begin{equation}
(D_{1\alpha} - \zeta D_{0\alpha})\psi_+ = 0 \ , \tag{2.15}
\end{equation}

and analogous equations for $\psi_-$. Recall that $\psi_{\pm}$ are smooth functions for $x \in \mathbb{R}^{2,2}$, $\zeta = \cot \frac{\theta}{2} \in \mathbb{R} \cup \{\infty\} = S^1$. Considering $\zeta \to \infty$ in (2.15), one obtains that

\begin{equation}
A_{0\alpha}(x) = \psi_+(x, \zeta) \partial_{0\alpha} \psi_+^{-1}(x, \zeta)|_{\zeta=\infty} \ , \tag{2.16}
\end{equation}

where $\zeta = \infty$ corresponds to $\theta = 0$. Using a gauge transformation $\psi_+ \mapsto g^{-1}\psi_+$ generated by $g(x) = \psi_+(x, \zeta = \infty)$, one may transform $A_{0\alpha}$ to zero, which is equivalent to imposing conditions $\psi_+(x, \zeta = \infty) = \psi_-(x, \zeta = \infty) = 1$. Moreover, we impose the standard asymptotic conditions on $\psi_+$ and $\psi_-$,

\begin{equation}
\psi_+(x, \zeta) = 1 + \zeta^{-1} \Psi(x) + O(\zeta^{-1}) \ , \quad \psi_-(x, \zeta) = 1 + \zeta^{-1} \Psi(x) + O(\zeta^{-1}) \tag{2.17}
\end{equation}

as $\zeta \to \infty$ in their respective domains (cf.[14]). At the same time, when $\zeta \to 0$ we have

\begin{equation}
\psi_+(x, \zeta) = \Phi^{-1}(x) + O(\zeta) \ , \quad \psi_-(x, \zeta) = \Phi^{-1}(x) + O(\zeta) \tag{2.18}
\end{equation}

where $\Phi(x) := \psi_+^{-1}(x, \zeta = 0) = \psi_-^{-1}(x, \zeta = 0)$.

By substituting (2.18) into (2.15), we have

\begin{equation}
A_{0\alpha} = 0 \ , \quad A_{1\alpha} = \Phi^{-1} \partial_{1\alpha} \Phi \ . \tag{2.19}
\end{equation}

This is the Yang gauge for which the SDYM equations (2.14) are replaced by the Yang equation [3],

\begin{equation}
\partial_{\infty} (\Phi^{-1} \partial_{11} \Phi) - \partial_{01} (\Phi^{-1} \partial_{10} \Phi) = 0 \ . \tag{2.20}
\end{equation}

Analogously, by substituting (2.17) into (2.15), we find that

\begin{equation}
A_{0\alpha} = 0 \ , \quad A_{1\alpha} = \partial_{0\alpha} \Psi \ , \tag{2.21}
\end{equation}

and Eqs. (2.14) are reduced to the Leznov equation [2],

\begin{equation}
\partial_{10} \partial_{01} \Psi - \partial_{00} \partial_{11} \Psi + [\partial_{00} \Psi, \partial_{01} \Psi] = 0 \ . \tag{2.22}
\end{equation}

3 Symmetries and hierarchy of the SDYM equations

3.1 Deformations of bundles and symmetries of the SDYM equations

In Section 2 we have introduced the covering $\mathcal{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ of the twistor space $\mathcal{P}$ and holomorphic bundles $E \to \mathcal{P}$ defined by transition functions $f_{\pm}$. The transition functions $f_{\pm}$ are holomorphic on an open neighbourhood $\mathcal{U}$ of $\mathcal{P}_0 = \mathcal{U}_+ \cap \mathcal{U}_-$ and real-analytic on the real twistor space $\mathcal{T}$. It has been shown that transition functions $f_{\pm}$ of holomorphic bundles $E \to \mathcal{P}$ encode
all the information about self-dual gauge potentials $A$ on the Kleinian space $\mathbb{R}^{2,2}$. We have written down the explicit formulae expressing $A$ through matrix-valued functions $\psi_{\pm}(x, \zeta)$ defining a trivialization of the bundle $E$ on real holomorphic sections of the bundle $\mathcal{P} \rightarrow \mathbb{C}P^1$.

Any holomorphic perturbation of $f_{+-}$ on $\mathcal{U}$ preserving conditions (2.5) and (2.6) is allowed since small enough deformations of the bundle $E$ preserve the property (2.4) of its trivializability on $\mathbb{C}P^1_\mathbb{R} \hookrightarrow \mathcal{P}$. Using the Penrose-Ward correspondence $f_{+-} \leftrightarrow A$, to each infinitesimal change $\delta f_{+-}$ of the transition function $f_{+-}$ of the bundle $E$ one can correspond an infinitesimal change $\delta A$ of the self-dual gauge potential $A$. By construction, such $\delta A$ satisfy the linearized SDYM equations and called \textit{infinitesimal symmetries} of the SDYM equations. In this Section we describe all infinitesimal symmetries of the SDYM equations in order to compare them later with hidden symmetries of open $N = 2$ strings.

We consider a bundle $E \rightarrow \mathcal{P}$, the covering $\mathcal{U} = \{\mathcal{U}_+, \mathcal{U}_-\}$ of $\mathcal{P}$, a transition function $f_{+-}$ for $E$ satisfying the conditions (2.4)-(2.6) and the self-dual gauge potential $A$ corresponding to $f_{+-}$. On real holomorphic sections $\sigma_x \in \Gamma_{\mathbb{R}}(\mathcal{P})$ we have $f_{+-}|_{\sigma_x} = \psi_+^{-1}(x, \zeta) \psi_-(x, \zeta), x \in \mathbb{R}^{2,2}$. Let us consider a perturbation $\delta f_{+-}$ of $f_{+-}$ and the factorization of the perturbed transition function,

$$f_{+-} + \delta f_{+-} = (\psi_+ + \delta \psi_+)^{-1}(\psi_+ + \delta \psi_-) , \quad (3.1)$$

supposing that $f_{+-} + \delta f_{+-}$ satisfies the conditions (2.5) and (2.6) up to the first order in $\delta f_{+-}$. Then introduce a Lie-algebra-valued function

$$\varphi_{+-} := \psi_+(\delta f_{+-}) f_{+-}^{-1} \psi_+^{-1} = \psi_+(\delta f_{+-}) \psi_-^{-1} , \quad (3.2)$$

defining an \textit{infinitesimal deformation} of the bundle $E \rightarrow \mathcal{P}$.

For fixed $x \in \mathbb{R}^{2,2}$ we have a function $\varphi_{+-}(x, \zeta)$ on $S^1_x \subset \mathbb{C}P^1_x$ with $\zeta \in S^1 \subset \mathbb{C}P^1$. As usual, $\varphi_{+-}(x, \zeta)$ can be factorized,

$$\varphi_{+-}(x, \zeta) = \varphi_+(x, \zeta) - \varphi_-(x, \zeta) , \quad (3.3)$$

where Lie-algebra-valued functions $\varphi_+$ and $\varphi_-$ can be extended to functions holomorphic in $\zeta$ in upper and lower half-planes, respectively. To find $\varphi_\pm$ means to solve the infinitesimal variant of the Riemann-Hilbert problem and a solution to (3.3) always exists [14, 5]. Suppose $\chi_{\pm}(\eta^\zeta, \zeta)$ are matrix-valued real-analytic functions on the real twistor space $T$ extendible to holomorphic functions on $\mathcal{U}_\pm \rightarrow \mathcal{P}_0$. Then $\varphi_{+-}$ of the form

$$\varphi_{+-} = \psi_+ \chi_+ \psi_+^{-1} - \psi_- \chi_- \psi_-^{-1} , \quad (3.4)$$

defines a trivial perturbation of the transition function $f_{+-}$.

\textbf{Remark.} Functions $\varphi_{+-}$ defined by formula (3.2) are elements of the space of 1-cocycles $Z^1(\mathcal{U}, \operatorname{ad} E)$ of the covering $\mathcal{U}$ with values in the bundle $\operatorname{ad} E$ of endomorphisms. Functions (3.4) form a subspace $B^1(\mathcal{U}, \operatorname{ad} E)$ of 1-cocycles (trivial 1-cocycles). Non-trivial infinitesimal deformations of the bundle $E \rightarrow \mathcal{P}$ are defined by the first cohomology group $H^1(\mathcal{U}, \operatorname{ad} E) = Z^1(\mathcal{U}, \operatorname{ad} E)/B^1(\mathcal{U}, \operatorname{ad} E)$. For cohomological description of symmetries to the SDYM equations see [15, 16].

Notice that

$$\varphi_{+-} = \psi_+(\delta f_{+-}) \psi_-^{-1} = -(\delta \psi_+) \psi_+^{-1} + (\delta \psi_-) \psi_-^{-1} = \varphi_+ - \varphi_- , \quad (3.5)$$

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and therefore 
\[ \delta \psi_+ = -\varphi_+ \psi_+ , \quad \delta \psi_- = -\varphi_- \psi_- . \] (3.6)

At the same time, from formulae (2.10) it follows that
\[ \delta A_{1\alpha} - \zeta \delta A_{0\alpha} = \delta \psi_+ (\partial_{1\alpha} - \zeta \partial_{0\alpha}) \psi^-_+ + \psi_+ (\partial_{1\alpha} - \zeta \partial_{0\alpha}) \delta \psi^-_+ = \]
\[ = \delta \psi_- (\partial_{1\alpha} - \zeta \partial_{0\alpha}) \psi^-_- + \psi_- (\partial_{1\alpha} - \zeta \partial_{0\alpha}) \delta \psi^-_- . \] (3.7)

Substituting (3.6) into (3.7), we obtain
\[ \delta A_{1\alpha} - \zeta \delta A_{0\alpha} = (D_{1\alpha} - \zeta D_{0\alpha}) \varphi_+ = (D_{1\alpha} - \zeta D_{0\alpha}) \varphi_- . \] (3.8)

From (3.8) it follows that
\[ (D_{1\alpha} - \zeta D_{0\alpha}) \varphi_+ = 0 , \] (3.9)

which can also be obtained from the definition (3.2) and equations (2.10) on \( \psi_\pm \).

It is easy to see that for \( \varphi_\pm = \psi_\pm \chi_\pm \psi^-_\pm \) defining the 1-coboundary (3.4) we have \( \delta A_{\xi} = 0 \) (trivial symmetries). At the same time, for infinitesimal gauge transformations \( \delta A_{\alpha} = D_{\alpha} \varphi \) we have \( \varphi_+ = \varphi_- = \varphi(x) \) (i.e. \( \varphi_\pm \) do not depend on \( \zeta \)) and therefore \( \varphi_+ = \varphi_- = 0 \) that leads to \( \delta f_+ = 0 \). For the case of non-trivial symmetries we have
\[ \delta A_{0\alpha} = D_{0\alpha} \varphi(x, \zeta = \infty) = D_{0\alpha} \varphi(x, \zeta = \infty) , \] (3.10)
\[ \delta A_{1\alpha} = D_{1\alpha} \varphi(x, \zeta = 0) = D_{1\alpha} \varphi(x, \zeta = 0) . \] (3.11)

Thus, to each transformation \( f_+ \rightarrow f_+ + \delta f_+ \) of the transition function \( f_+ \) in the bundle \( E \rightarrow P \), formulae (3.2), (3.3), (3.6), (3.10) and (3.11) correspond a symmetry transformation \( A \rightarrow A + \delta A \) of the self-dual gauge potential \( A \). By construction, such \( \delta A \) satisfy the linearized SDYM equations.

In Subsection 2.2 we considered gauge fixing conditions for \( \psi_\pm \) and \( \{ A_{\alpha} \} \). To preserve these conditions, it is enough to impose the asymptotic conditions
\[ \varphi_+ \rightarrow 0 , \quad \varphi_- \rightarrow 0 \] (3.12)
as \( \zeta \rightarrow \infty \) in their respective domains. Then from (3.10) we obtain \( \delta A_{0\alpha} = 0 \) and therefore gauge fixing conditions \( A_{0\alpha} = 0 \) are preserved. Transformations of Leznov’s prepotentials \( \Psi \) can be obtained from formulae (2.17), (3.6) and have the form
\[ \delta \Psi(x) = - \lim_{\zeta \rightarrow \infty} \zeta \varphi_+(x, \zeta) . \] (3.13)

Analogously, transformations of Yang’s prepotential \( \Phi \) follow from formulae (2.18), (3.6) and have the form
\[ \Phi^{-1} \delta \Phi = \varphi_+(x, \zeta = 0) . \] (3.14)

Thus, the knowledge of \( \varphi_+(x, \zeta) \) permits one to find \( \delta A_{\alpha} \), \( \delta \Psi \) and \( \delta \Phi \).

### 3.2 Hierarchy of the SDYM equations

For the twistor space \( \mathcal{P} = \mathcal{O}(1) \oplus \mathcal{O}(1) \) with the covering \( \mathcal{U} = \{ \mathcal{U}_+, \mathcal{U}_- \} \), on an open set \( \mathcal{U} \supset \mathcal{U}_+ \cap \mathcal{U}_- \) one may define an infinite number of matrix-valued holomorphic functions \( f_{\pm} \) satisfying conditions (2.4)-(2.6). To each such matrix \( f_{\pm} \) there corresponds a holomorphic vector bundle \( E \) over \( \mathcal{P} \) and a gauge equivalence class of self-dual gauge potentials \( A \) on \( \mathbb{R}^{2,2} \). So, we have
infinite-dimensional spaces of matrices $f_{+-}$ and self-dual gauge potentials $A$. The moduli space of holomorphic vector bundles $E$ over $\mathcal{P}$ defined by $f_{+-}$ is bijective to the moduli space $\mathcal{M}$ of self-dual gauge fields on $\mathbb{R}^{2,2}$. Recall that $\mathcal{M} = \mathcal{N}/\mathcal{G}$, where $\mathcal{N}$ is the solution space and $\mathcal{G}$ is the group of gauge transformations. Non-trivial perturbations $\delta f_{+-}$ of transition functions $f_{+-}$ are vector fields on the moduli space of holomorphic bundles $E$, and non-trivial perturbations $\delta A$ of self-dual gauge potentials $A$ are vector fields on the moduli space $\mathcal{M}$. The Penrose-Ward correspondence $\delta f_{+-} \leftrightarrow \delta A$ described in Subsection 3.1 determines an isomorphism between the space $H^1(\mathcal{P}, \text{ad} E)$ of infinitesimal deformations of the bundle $E \to \mathcal{P}$ and the space Vect($\mathcal{M}$) of vector fields on the moduli space $\mathcal{M}$ of self-dual gauge fields.

Symmetries of the SDYM equations were considered in many papers (see e.g. [5, 17, 18] and references therein). In particular, homomorphisms of various Kac-Moody-Virasoro type algebras into the algebra Vect($\mathcal{N}$) of vector fields on the solution space $\mathcal{N}$ of the SDYM equations have been described. In this paper we are mainly interested in affine extensions of spacetime translations [17] and in a hierarchy of the SDYM equations [5] corresponding to them. Later we show that just these symmetries correspond to abelian string symmetries.

The above-mentioned non-local abelian symmetries of the SDYM equations can be defined in the following way. Consider translations in $\mathbb{R}^{2,2}$ generated by a vector field

$$\check{T} = \sum_{\alpha=0}^{1} \left( t^{0\alpha} \frac{\partial}{\partial x^{0\alpha}} + t^{1\alpha} \frac{\partial}{\partial x^{1\alpha}} \right) ,$$

(3.15)

where $t^{\alpha\dot{\alpha}} \in \mathbb{R}$ are constant parameters, $\alpha = 0, 1$, $\dot{\alpha} = 0, 1$. The induced action of $\check{T}$ on $f_{+-}(x^{0\alpha} + \zeta x^{1\dot{\alpha}}, \zeta)$ is

$$\check{T} f_{+-} = \sum_{\alpha=0}^{1} \left( t^{0\alpha} + \zeta t^{1\dot{\alpha}} \right) \frac{\partial}{\partial \eta^{\alpha}} f_{+-} =: \check{T} f_{+-} ,$$

(3.16)

where $\eta^{\dot{\alpha}} = x^{0\dot{\alpha}} + \zeta x^{1\alpha}$ and $\zeta$ are the local coordinates on the real twistor space $\mathcal{T}$ (see Appendix A.2). So, $\check{T}$ is a local vector field on $\mathcal{T} \subset \mathcal{P}$. Now let us consider vector fields $T_{n\dot{\alpha}}$ on $\mathcal{T}$,

$$T_{n\dot{\alpha}} := \zeta^{n} \frac{\partial}{\partial \eta^{\dot{\alpha}}} , \quad n = 0, 1, ..., 2J ,$$

(3.17)

where $2J$ is any positive integer or infinity. When $n = 0, 1$ these vector fields correspond to the translations $\partial/\partial x^{0\alpha}$, $\partial/\partial x^{1\dot{\alpha}}$ in $\mathbb{R}^{2,2}$.

Let us define the transformations

$$f_{+-} \mapsto \delta_{n\alpha} f_{+-} := \zeta^{n} \frac{\partial}{\partial \eta^{\alpha}} f_{+-}$$

(3.18)

of transition functions $f_{+-}$ in the bundle $E$ restricted to $\mathcal{T}$. From (3.18) it is easy to see that $[\delta_{m\dot{\alpha}}, \delta_{n\dot{\beta}}] f_{+-} = 0$ (commutativity). Further, by the algorithm from Subsection 3.1, one may correspond the perturbations $\delta_{n\alpha} A$, $\delta_{n\alpha} \Phi$ and $\delta_{n\alpha} \Psi$ to the perturbations (3.18). Recall that $\delta_{n\alpha} A$, $\delta_{n\alpha} \Phi$ and $\delta_{n\alpha} \Psi$ are components of vector fields on the space of matrices $f_{+-}$ and on the solution spaces to Eqs. (2.14), (2.20) and (2.22), respectively. To all these vector fields one may correspond dynamical systems on the solution spaces and try to solve these differential equations.

Integral trajectories of dynamical systems on the space of transition functions can be described explicitly. Namely, consider the following system of differential equations:

$$\frac{\partial}{\partial t} f_{+-} = \zeta^{n} \frac{\partial}{\partial \eta^{\alpha}} f_{+-} ,$$

(3.19)
where \( t^{\alpha\hat{\alpha}} \in \mathbb{R} \) are real parameters, \( \hat{\alpha} = 0, 1, n = 0, \ldots, 2J \) and \( 2J \) is any positive integer number or \( 2J = \infty \). Equations (3.19) can be easily integrated and we obtain

\[
f_{+-}(x, t, \zeta) = f_{+-}(x^{0\hat{\alpha}} + t^{0\hat{\alpha}} + \zeta(x^{1\hat{\alpha}} + t^{1\hat{\alpha}}) + \sum_{n=2}^{2J} t^{n\hat{\alpha}} \zeta^n, \zeta), \tag{3.20}
\]

where \( t = (t^{0\hat{\alpha}}, t^{1\hat{\alpha}}, \ldots, t^{2J\hat{\alpha}}) \). Any point of the space \( \mathbb{R}^{2,2} \) can be obtained by the shift of the point \( x^{0\hat{\alpha}} = 0 \) and therefore in (3.20) one may put \( x^{0\hat{\alpha}} = 0 \). Then we have

\[
f_{+-}(t, \zeta) = f_{+-} \left( \sum_{n=0}^{2J} t^{n\hat{\alpha}} \zeta^n, \zeta \right), \tag{3.21}
\]

where \( t^{0\hat{\alpha}} \) and \( t^{1\hat{\alpha}} \) are coordinates in \( \mathbb{R}^{2,2} \), and \( t^{n\hat{\alpha}} \) with \( n \geq 2 \) are extra (moduli) parameters.

Notice that for finite \( 2J \) the polynomials

\[
\eta^\hat{\alpha}(\zeta) = \sum_{n=0}^{2J} t^{n\hat{\alpha}} \zeta^n \tag{3.22}
\]

define real sections of the bundle \( \mathcal{O}(2J) \oplus \mathcal{O}(2J) \rightarrow \mathbb{C}P^1 \), and matrices (3.21) are transition functions of the bundle \( E \rightarrow \mathcal{O}(2J) \oplus \mathcal{O}(2J) \) restricted to real sections \( S_1 \rightarrow \mathcal{O}(2J) \oplus \mathcal{O}(2J) \). In other words, local vector fields (3.17) generate the change of topology of the twistor space \( \mathcal{O}(1) \oplus \mathcal{O}(1) \) to \( \mathcal{O}(2J) \oplus \mathcal{O}(2J) \). Using the homogeneous coordinates on \( S^1 \) from Appendix A.1, one can rewrite Eqs. (3.19) for finite \( J \) in the form

\[
\frac{\partial}{\partial t^{n\hat{\alpha}}} f_{+-} = g^J \binom{2J}{n} t^{n\hat{\alpha}} \sin^{2J-n} \frac{\theta}{2} \cos^n \frac{\theta}{2} f_{+-}, \tag{3.23}
\]

where parameters \( t^{n\hat{\alpha}} \in \mathbb{R} \) differ from those in (3.19) by the multipliers, but we shall not introduce new notations for them. The general solution of Eqs. (3.23) has the form

\[
f_{+-}(t, \theta) = f_{+-} \left( g^J \sum_{n=0}^{2J} t^{n\hat{\alpha}} \sin^{2J-n} \frac{\theta}{2} \cos^n \frac{\theta}{2}, \theta \right) =
\]

\[
= f_{+-} \left( g^J \sum_{M=-J}^{J} \binom{2J}{J+M} t^{J+M\hat{\alpha}} \sin^{2J-M} \frac{\theta}{2} \cos^{J+M} \frac{\theta}{2}, \theta \right), \tag{3.24}
\]

where \( 0 \leq \theta \leq 2\pi \).

Now one should use the Penrose-Ward transformation starting from the factorization

\[
f_{+-}(t, \zeta) = \psi_{+}^{-1}(t, \zeta) \psi_{-}(t, \zeta), \tag{3.25}
\]

where now \( \psi_{\pm} \) depend on \( t = (t^{n\hat{\alpha}}) \) and \( \zeta, \hat{\alpha} = 0, 1, n = 0, \ldots, 2J \). As in Subsection 2.2, we use the gauge in which

\[
\psi_{\pm}(t, \zeta) = \Phi^{-1}(t) + O(\zeta) \quad \text{for} \quad \zeta \rightarrow 0, \tag{3.26}
\]

\[
\psi_{\pm}(t, \zeta) = 1 + \zeta^{-1} \Psi(t) + O(\zeta^{-1}) \quad \text{for} \quad \zeta \rightarrow \infty. \tag{3.27}
\]
The group-valued function $\Phi(t)$ and the algebra-valued function $\Psi(t)$ give (implicit) solutions of the differential equations
\begin{align}
\partial_{n_\alpha} \cdot \Phi &= \delta_{n_\alpha} \cdot \Phi, \\
\partial_{n_\alpha} \cdot \Psi &= \delta_{n_\alpha} \cdot \Psi,
\end{align}
and describe commuting flows on the space of solutions to Eqs. (2.20) and (2.22), respectively. These flows are integral curves for the dynamical systems (3.28) and (3.29), where $\partial_{n_\alpha} := \partial/\partial t^{n_\alpha}$.

Notice that the transition functions (3.21) (and (3.24)) are annihilated by the vector fields
\begin{equation}
V_{n_\alpha} := \frac{\partial}{\partial t^{n_\alpha + 1}} - \zeta \frac{\partial}{\partial t^{n_\alpha}},
\end{equation}
i.e. we have
\begin{equation}
V_{n_\alpha} f_{n_\alpha}(t, \zeta) = 0,
\end{equation}
where $n = 0, \ldots, 2J - 1$, $\alpha = 0, 1$. Substituting (3.25) into Eqs. (3.31) and applying the standard arguments (see Subsection 2.1), we obtain
\begin{equation}
\psi_+ (t, \zeta) V_{n_\alpha} \psi^{-1}_+ (t, \zeta) = \psi_+ (t, \zeta) V_{n_\alpha} \psi^{-1}_+ (t, \zeta) = A_{n_\alpha} (t) - \zeta \tilde{A}_{n_\alpha} (t),
\end{equation}
where $A_{n_\alpha} (t)$ and $\tilde{A}_{n_\alpha} (t)$ are some $sa(r)$-valued functions of $t = (t^{n_\alpha})$. We remark that \{\tilde{A}_{n_\alpha} (t), A_{n_\alpha} (t)\} coincide with components \{\tilde{A}_{n_\alpha} \} of gauge potential on $\mathbb{R}^{2, 2}$. The compatibility conditions of Eqs. (3.32) have the form
\begin{equation}
[D_{m+1_\alpha} - \zeta \tilde{D}_{m_\alpha}, D_{n+1_\beta} - \zeta \tilde{D}_{n_\beta}] = 0 \iff \begin{cases} \end{equation}
\begin{align}
[D_{m+1_\alpha}, D_{n+1_\beta}] &= 0, \quad [\tilde{D}_{m_\alpha}, \tilde{D}_{n_\beta}] = 0, \quad [D_{m+1_\alpha}, \tilde{D}_{n_\beta}] + [\tilde{D}_{m_\alpha}, D_{n+1_\beta}] = 0,
\end{align}
where $D_{m+1_\alpha} := \partial_{m+1_\alpha} + A_{m+1_\alpha}$, $\tilde{D}_{m_\alpha} := \partial_{m_\alpha} + \tilde{A}_{m_\alpha}$, $\partial_{n_\alpha} := \partial/\partial t^{n_\alpha}$. Equations (3.33) are equations of the truncated SDYM hierarchy. The SDYM hierarchy equations are obtained when $J \to \infty$ [5].

It is not difficult to verify that for the gauge fixing conditions (3.26), (3.27) we have
\begin{equation}
\tilde{A}_{m_\alpha} = 0, \quad A_{m+1_\alpha} (t) = \Phi^{-1} (t) \partial_{m+1_\alpha} \Phi (t) = \partial_{m_\alpha} \Psi (t).
\end{equation}
When we represent $A_{m+1_\alpha}$ by $\Phi$, Eqs. (3.33) reduce to
\begin{equation}
\partial_{m_\alpha} \left( \Phi^{-1} \partial_{n+1_\beta} \Phi \right) - \partial_{n_\beta} \left( \Phi^{-1} \partial_{m+1_\alpha} \Phi \right) = 0,
\end{equation}
and when we represent $A_{m+1_\alpha}$ by $\Psi$, Eqs. (3.33) reduce to
\begin{equation}
\partial_{m+1_\alpha} \partial_{n_\beta} \Psi - \partial_{n+1_\beta} \partial_{m_\alpha} \Psi + [\partial_{m_\alpha} \Psi, \partial_{n_\beta} \Psi] = 0
\end{equation}
where $m, n = 0, \ldots, 2J - 1$. When $m = n = 0$, Eqs. (3.36) coincide with the SDYM equations in the Leznov form (2.22). When $n \geq 1$, $m = 0$, Eqs. (3.36) are equations on symmetries $\delta_{n_\alpha} \Psi = \partial_{n_\alpha} \Psi$,
\begin{equation}
\partial_{\delta_{l_\beta} m_\alpha} \Psi - \partial_{\delta_{l_\beta} n_\alpha} \Psi + [\delta_{n_\beta} \Psi, \partial_{0_\beta} \Psi] = 0,
\end{equation}
where $\dot{\alpha}, \dot{\beta} = 0, 1$. 

9
4 Review of the open N=2 string

From the worldsheet point of view, critical open N=2 strings in flat Kleinian space $\mathbb{R}^{2,2}$ are a
theory of N=2 supergravity $(\hat{h}, \chi, A)$ on a (pseudo) Riemann surface with boundaries, coupled to
two chiral N=2 massless matter multiplets $(y, \psi)$. The latter’s components are complex scalars
(the four string coordinates) and SO(1, 1) Dirac spinors (their four NSR partners). The N=2 string
Lagrangian, as first written down by Brink and Schwarz [19], reads

$$\mathcal{L} = \sqrt{h} \left\{ \frac{1}{2} h_{pq} \partial_{y^p} y^\pi \partial_{y^q} y^\sigma + i \frac{1}{2} \psi^+ \psi^+ \gamma^\rho \partial_{y^\rho} \psi^+ + A_q \psi^+ \gamma^\rho \gamma^\sigma \psi^+ \gamma^\omega \psi^+ \right\} \eta_{\pi\rho\omega} \quad (4.1)$$

where $h_{pq}$ and $A_q$, with $p, q = 0, 1$, are the (real) worldsheet metric and $U(1)$ gauge connection,
respectively. The worldsheet gravitino $\chi_\sigma$ is as well as the matter fields $y^A$ and $\psi^+ \gamma^\sigma \psi$ are complex
valued, so that the spacetime index $A, A = 1, 2$ runs over two values only. Complex conjugation reads

$$(y^A)^* = \chi_\sigma^a \quad \text{but} \quad (\psi^+)^* = \psi^- \quad \text{and} \quad (\chi^+_\sigma)^* = \chi^-_{\sigma} \quad (4.2)$$

and $\eta_{\pi\rho\omega} = \text{diag}(+ - )$ is the flat metric in $\mathbb{C}^{1,1}$. As usual, $\{ \gamma^\rho \}$ are a set of SO(1, 1) worldsheet
gamma matrices, $\psi = \psi^+ \gamma^\sigma \psi$, and $D_\rho$ denotes the worldsheet gravitationally covariant derivative.

Since open-string world-sheets have boundaries (and possibly cross-caps), boundary conditions are to be specified. As usual, the auxiliary supergravity fields remain free, while the string coordinates $y$ are subject to

$$\partial_{\text{normal}} y^A \big|_{\text{boundary}} = 0 \quad \text{or} \quad y^A \big|_{\text{boundary}} = \bar{y}_0 = \text{constant} \quad (4.3)$$

parallel resp. orthogonal to whatever D-branes are present. The two components of the spinors $\psi$
are related at each boundary segment $\Gamma_s$ by multiplication with a phase $e^{i \rho_s}$. A boundary puncture
(vertex) separating segments $\Gamma_s$ and $\Gamma_i$ thus carries a “twist” $\rho_{st} = \rho_s - \rho_i$. The latter is a property of the
corresponding asymptotic string state and interpolates between the traditional Neveu-Schwarz
($\rho_{st} = 1$) and Ramond ($\rho_{st} = -1$) sectors. The abelian $\text{R}$ gauge symmetry of the N=2 worldsheet
supergravity allows one to rotate all these phases to unity; in the superconformal gauge, its remnant
is known as the spectral flow of the N=2 superconformal constraint algebra. Hence, we may restrict
ourselves to the Neveu-Schwarz sector.

The Brink-Schwarz formulation (4.1) entails the choice of a complex structure on the Kleinian
target space. A given complex structure breaks the global “Lorentz” invariance of $\mathbb{R}^{2,2}$,

$$\text{Spin}(2, 2) = SU(1, 1) \times SU(1, 1)^t \rightarrow U(1) \times SU(1, 1)^t \simeq U(1, 1) \quad (4.4)$$

The moduli space of complex structures is the two-sheeted hyperboloid $H^2 = H^2_+ \cup H^2_-$ with
$H^2_\pm \simeq SU(1, 1)/U(1)$. It can be completed to $CP^1$ by sewing the two sheets together along a circle,

$$CP^1 = H^2_+ \cup S^1 \cup H^2_- \quad (4.5)$$

Instead of using complex coordinates adapted to $SU(1, 1)^t$, one may alternatively choose a basis
appropriate for $SL(2, \mathbb{R})$ and employ a real notation for the string coordinates,

$$y^1 = x^1 + ix^2 \quad , \quad y^2 = x^3 + ix^4 \quad (4.6)$$
by expressing the real coordinates $x^\mu, \mu, \nu, \ldots = 1, 2, 3, 4$, in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})'$ spinor notation,

$$x^{\alpha \dot{\alpha}} = \sigma^\alpha_{\mu} x^\mu = \left(\begin{array}{c} x^4 + x^2 \\
 x^1 - x^3 \\n x^1 + x^3 \\
 x^4 - x^2 \end{array}\right), \quad \alpha \in \{0, 1\}, \quad \dot{\alpha} \in \{0, \dot{1}\}, \quad (4.7)$$

with the help of chiral gamma matrices $\sigma^\mu_\alpha$ appropriate for the spacetime metric $\eta_{\mu \nu} = \text{diag}(++, --)$.

In the real formulation, the tangent space at any point of $\mathbb{R}^{2,2}$ can be split to $\mathbb{R}^2 \oplus \mathbb{R}^2$ which defines a real polarization or cotangent structure $\mathbb{R}^{2,2} = T^* \mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$. Such a polarization is characterized by a pair of null planes $\mathbb{R}^2$, and the latter are determined by a real null two-form modulo scale or, equivalently, by a real $SL(2, \mathbb{R})$ spinor $v$ modulo scale. Indeed, each null vector $(u_{\alpha \dot{\alpha}})$ factorizes into two real spinors, $u_{\alpha \dot{\alpha}} = v_{\alpha} w^\alpha_{\dot{\alpha}}$. Choosing coordinates such that $(\xi_{\alpha \dot{\alpha}}) = (\xi_{0 \d})$, it becomes clear that a given null plane is stable under the action of

$$B_+ \times SL(2, \mathbb{R})', \quad \text{with} \quad B_+ := \left\{ \begin{pmatrix} a & b \\
 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^*, b \in \mathbb{R} \right\}, \quad (4.8)$$

where $B_+$ acts on $v$ and $SL(2, \mathbb{R})'$ on $w$. The moduli space of cotangent structures thus becomes

$$\text{Spin}(2, 2)/[B_+ \times SL(2, \mathbb{R})'] \simeq SL(2, \mathbb{R})/B_+ \simeq S^1 \quad (4.9)$$

which in fact is just the $S^1$ in (4.5). However, it turns out [20] that the real spinor $v$ also encodes the two string couplings,

$$\left(\begin{array}{c} v_0 \\
 v_1 \end{array}\right) = \sqrt{g} \left(\begin{array}{c} \cos \frac{\theta}{2} \\
 \sin \frac{\theta}{2} \end{array}\right), \quad (4.10)$$

with $g \in \mathbb{R}^+$ being the gauge coupling and $\theta \in S^1$ the instanton angle. Since $v$ (including scale) is inert only under the parabolic subgroup of $B_+$ obtained by putting $a=1$, the space of string couplings is that of nonzero real $SL(2, \mathbb{R})$ spinors,

$$\mathbb{R}^+ \times S^1 \simeq \mathbb{R}^2 - \{0\} \simeq \mathbb{C} - \{0\} \ni \sqrt{g} e^{i\theta/2}. \quad (4.11)$$

Consequently, fixing the values of the string couplings amounts to breaking the global “Lorentz” invariance of $\mathbb{R}^{2,2}$ in a way different from (4.4),

$$\text{Spin}(2, 2) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})' \longrightarrow \mathbb{R} \times SL(2, \mathbb{R})' \quad , \quad (4.12)$$

where $\mathbb{R} \simeq B_+(a=1)$ from Eq. (4.8).

The $N=2$ supergravity multiplet defines a gravitini and a Maxwell bundle over the worldsheet Riemann surface in the presence of boundaries, cross-caps, and boundary punctures. The topology of the total space is labeled by the Euler number $\chi$ of this Riemann surface and the first Chern number (instanton number) $M$ of its Maxwell bundle. It is notationally convenient to replace the Euler number by the “spin”

$$J := -2\chi = n - 4 + 2(\# \text{ boundaries}) + 2(\# \text{ cross-caps}) + 4(\# \text{ handles}) \in \mathbb{Z}. \quad (4.13)$$

The Lagrangian is to be integrated over the string worldsheet of a given topology. The first-quantized string path integral for the $n$-point function $A^{(n)}$ includes a sum over worldsheet topologies ($J, M$), weighted with appropriate powers in the (dimensionless) string couplings $(g, e^{i\theta})$:

$$A^{(n)}(g, \theta) = \sum_{J=n-2}^{\infty} \sum_{M=-J}^{J} e^{iM\theta} A^{(n)}_{J, M} = \sum_{J=n-2}^{\infty} \sum_{M=-J}^{J} e^{iM\theta} \sin^{J-M} \frac{\theta}{2} \cos^{J+M} \frac{\theta}{2} A^{(n)}_{J, M} \quad , \quad (4.14)$$
where the instanton sum has a finite range because bundles with \(|M|>J\) do not contribute. The presence of Maxwell instantons breaks the explicit \(U(1)\) factor in (4.4) but the \(SU(1, 1)\) factor (and thus the whole \(\text{Spin}(2, 2)\)) is fully restored if we let \(\sqrt{g}(e^{i/2}, e^{-i/2})\) transform as an \(SU(1, 1)\) spinor. The partial amplitudes \(\tilde{A}_{JM}^{(a)}\) (complex) and \(\tilde{A}_{JM}^{(a)}\) (real) are integrals over the metric, gravitini, and Maxwell moduli spaces. The integrands may be obtained as correlation functions of boundary vertex operators in the \(N=2\) superconformal field theory on the worldsheet surface of fixed shape (moduli) and topology.

The vertex operators generate from the (first-quantized) vacuum state the asymptotic string states in the scattering amplitude under consideration. They uniquely correspond to the physical states of the \(N=2\) open string and carry their quantum numbers. The physical subspace of the \(N=2\) string Fock space in a covariant quantization scheme turns out to be surprisingly small [21]: Only the ground state \(|k, a\rangle\) remains, a scalar on the massless level, i.e., for center-of-mass momentum \(k^4\) with \(k \cdot k := \eta_{IA} k^I k^A = 0\). In the presence of coincident D-branes, open strings stretch between the various branes. The open-string states encodes this information by carrying a Chan-Paton label \(a\) which transforms in the adjoint representation of the gauge algebra. Thus, the dynamics of this string “excitation” is described by a Lie-algebra-valued massless scalar field,

\[
\Xi(y) = \int d^4k \ e^{-i(k \cdot y + k \cdot \delta)} \tilde{Z}^a(k) T^a \ ,
\]

with \(T^a\) denoting a set of Lie algebra generators. The self-interactions of this field are determined on-shell from the (amputated tree-level) string scattering amplitudes,

\[
\langle \tilde{Z}^a_1(k_1) \tilde{Z}^a_2(k_2) \cdots \tilde{Z}^a_n(k_n) \rangle_{\text{amp}}^{\text{tree}} = : A_{n-2}^{(a)}(\{k_i\}; \{a_j\}; \theta) = : \delta_{k_1 + \cdots + k_n} A_{n-2}^{(a)}(\{k_i\}; \{a_j\}; \theta) \ .
\]

Finally, it has been shown [1, 22] that all tree-level \(n\)-point functions vanish on-shell, except for the two- and three-point amplitudes,

\[
\tilde{A}_{0}^{(a)}(k_1, k_2; a, b; \theta) = k_{ab} \ ,
\]

\[
\tilde{A}_{1}^{(a)}(k_1, k_2, k_3; a, b, c; \theta) = \frac{i}{2} f^{abc} \left( \epsilon_{AB} k_1^A k_2^B e^{i\theta} - \eta_{AB} (k_1^A \tilde{k}_2^B - k_2^A \tilde{k}_1^B) - \epsilon_{AB} k_1^A \tilde{k}_2^B e^{-i\theta} \right) = f^{abc} \epsilon_{\alpha\beta} \left( k_1^{\alpha} \cos \frac{\theta}{2} + k_2^{\alpha} \sin \frac{\theta}{2} ight) \left( k_1^{\beta} \cos \frac{\theta}{2} + k_2^{\beta} \sin \frac{\theta}{2} \right) \ .
\]

Here, the Chan-Paton labels appear on the Killing form \(k^{ab}\) and the (totally antisymmetric) structure constants \(f^{abc}\), and the momenta obey \(k_1 \cdot k_j + k_j \cdot k_i = 0\) due to \(\sum_n k_i = 0\). Note that \(\tilde{A}_{1}^{(a)}\) is totally symmetric in the external state quantum numbers \((k_i, a_i)\).

Since we argue that the string couplings \((g, \epsilon^{i\theta})\) can be changed at will by global “Lorentz” transformations, it is admissible to make a convenient choice of Lorentz frame. First, we may scale \(g \to 1\) (i.e., put the constant dilaton to zero). Second, the instanton angle \(\theta\) is at our disposal. In the real notation, one sees that taking \(\theta = 0\) reduces the amplitude (4.18) to a single term [23],

\[
\tilde{A}_{1}^{(3)}(k_1, k_2, k_3; a, b, c; \theta=0) = f^{abc} \epsilon_{\alpha\beta} k_1^{\alpha} k_2^{\beta} \ ,
\]

which, renaming \(\Xi \to \Psi\), translates to a cubic interaction\(^1\)

\[
\mathcal{L}_{\text{int}} = \frac{1}{6} \epsilon^{\alpha\beta} \text{tr} \Psi \left[ \partial_\alpha \Psi \partial_\beta \Psi \right] \ .
\]

\(^1\)The \(SO(2, 2)\) transformation properties of this interaction become manifest when this term is rewritten as \(\frac{1}{2} T^{(+)}_{\alpha\beta} \epsilon^{\alpha\beta} \Psi \partial_\alpha \Psi \partial_\beta \Psi\), with a self-dual projector \(T^{(+)}\) having nonzero components \(T^{(+)}_{\alpha\beta} = 1\) only.
It is remarkable that (at least at tree-level) no quartic or higher field-theory vertices are needed to reproduce the vanishing string amplitudes, $A_{n=2}^{[n>4]} = 0$, because the Feynman graphs based on (4.20) alone happen to cancel in 2+2 dimensions. In this sense, the cubic Lagrangian (4.20) is tree-level exact. Its resulting equation of motion reads

$$-\Box \Psi + \frac{1}{2} e^{\tilde{\alpha} \tilde{\beta}} \left[ \partial_{\tilde{\alpha} \tilde{\beta}} \Psi, \partial_{\tilde{\alpha} \tilde{\beta}} \Psi \right] = 0 \quad (4.21)$$

which we recognize as Leznev’s equation (2.22) [2]. It describes the dynamics of the single-helicity ($h=+1$) gluon in 2+2 self-dual Yang-Mills theory. More precisely, the self-dual field strength $F_{\alpha \beta}$ is entirely expressed (in light-cone gauge) through Leznev’s prepotential $\Psi$,

$$F_{\alpha \beta} = \partial_{\alpha} \partial_{\beta} \Psi \quad , \quad (4.22)$$

which is subject to the second-order equation (4.21).

In the complex notation, the $U(1)$ factor in (4.4) can be restored by averaging over all cotangent structures. In this manner, $\tilde{A}_1^{[3]}$ simplifies to

$$\int \frac{d\theta}{2\pi} \tilde{A}_1^{[3]} (k_1, k_2, k_3; a, b, c; \theta) = \frac{1}{2} \left[ \tilde{A}_1^{[3]} (\theta=0) + \tilde{A}_1^{[3]} (\theta=\pi) \right]$$

$$= -\frac{i}{2} f^{abc} \eta_{AB} (k_1^A k_2^B - k_1^B k_2^A)$$

$$= \frac{1}{2} f^{abc} \epsilon_{\alpha \beta} (k_1^\alpha k_2^\beta + k_1^\beta k_2^\alpha) \quad (4.23)$$

which, renaming $\Xi \rightarrow \phi$, leads to a cubic vertex

$$\mathcal{L}_{int}^{[3]} = \frac{i}{2} \eta^{AB} \text{tr} \phi \left[ \partial_A \phi, \partial_B \phi \right] \quad . \quad (4.24)$$

The Feynman rules based on this vertex yield non-vanishing $n$-point functions for all $n \geq 4$ which, however, may be cancelled recursively (at tree-level) by supplementing an infinite set of judiciously chosen higher vertices,

$$\mathcal{L}_{int} = \mathcal{L}_{int}^{[3]} + \mathcal{L}_{int}^{[4]} + \ldots \quad , \quad (4.25)$$

resulting in a non-polynomial but tree-level exact Lagrangian. Surprisingly, the corresponding equation of motion,

$$-\Box \phi + \frac{i}{2} \eta^{AB} \left[ \partial_A \phi, \partial_B \phi \right] + O(\phi^3) = 0 \quad , \quad (4.26)$$

can be written in closed form after reassembling the Lie-algebra-valued field $\phi$ to the group-valued field $\Phi = e^{i\phi}$,

$$\eta^{AB} \partial_A (e^{-i\phi} \partial_B e^{i\phi}) = 0 \quad . \quad (4.27)$$

The latter is nothing but Yang’s equation (2.20) [3]. Like (4.21), it describes 2+2 self-dual Yang-Mills but in a different parametrization. We conclude that, at tree-level, the $N=2$ open string is indeed identical to self-dual Yang-Mills.
5 Non-local symmetries of the open N=2 string

In Section 3, an infinite number of non-local symmetries of the self-dual Yang-Mills equations have been described. The latter’s intimate connection with the open N=2 string gives rise to the question where all these symmetries hide in the string description. To answer this, the BRST approach offers a systematic procedure for the construction of all conserved charges in a given quantum field theory [24, 25]. One should realize that this issue is a classical one from the spacetime point of view (no string loops) but involves a quantum description of the underlying (super)conformal field theory.

For closed N=2 strings, the BRST quantization has been treated exhaustively in [12] and reviewed in the context of global symmetries in [9, 10]. Here, we shall only collect the facts pertinent to the open string case, especially where the treatment differs from that of the closed string. Relevant for the physics of the open N=2 string is the so-called relative chiral BRST cohomology

\[ H_{rel} = \frac{\ker Q}{\text{im } Q} \quad \text{on } \ker b_0 \cap \ker b'_0 \]  

(5.1)

which is graded by\(^2\)

- ghost number \( g \in \mathbb{Z} \)
- picture numbers \( (\pi_+, \pi_-) \in \mathbb{Z}^2 \) (no loss of generality due to spectral flow)
- spacetime momentum \( (k^A, \bar{k}^A) \) or \( k^{a\dot{a}} \in \mathbb{R}^{2,2} \) as well as Chan-Paton label \( a \)

so that one may restrict the analysis to Fock states of a given ghost number, built on the momentum-dressed picture vacua \( (\pi_+, \pi_-; k, a) \) of ghost number zero (by definition). It is important to distinguish the exceptional case of \( k = 0 \) from the generic case \( (k \neq 0) \) which contains the propagating modes.

It has been shown [12] that the generic BRST cohomology is non-empty only for

- \( g = 1 \)
- any value for \( (\pi_+, \pi_-) \)
- any lightlike momentum, \( k \cdot k = 0 \)

and is one-dimensional in each such case. Moreover, for \( k \neq 0 \) one may construct picture-raising and picture-lowering operators which commute with \( Q \) and do not carry ghost number or momentum [13]. Together with spectral flow, these operators may therefore be used to define an equivalence relation among all pictures. This projection of the BRST cohomology leaves us with a single, massless, physical mode taking value in the Lie algebra of the Chan-Paton group, supporting our assertion of the previous Section.

It is perhaps less well known that the exceptional (zero-momentum) BRST cohomology at ghost number one harbors all unbroken global symmetry charges of the theory [24]. A conserved charge \( \mathcal{A} = \int_{-\ell}^{\ell} \Omega^{(1)} \), with the integration contour connecting the two boundaries of the free open string worldsheet, originates from a (current) one-form \( \Omega^{(1)} \) which is closed up to a BRST commutator. BRST invariance of the charge requires

\[ [Q, \Omega^{(1)}] = d\Omega^{(0)} \]  

(5.2)

\(^2\)We use the notation of Ref. [10].
for some zero-form (function) $\Omega^{(0)}$. Consistency then implies that

$$[Q, \Omega^{(0)}] = 0 \quad \text{and} \quad \Omega^{(0)} \simeq \Omega^{(0)} + [Q, \text{any}] \quad (5.3)$$

which are precisely the defining relations for the BRST cohomology (on zero-form operators instead of Fock states). Taking as $\Omega^{(0)}$ some cohomology class of ghost number one, we may solve the descent equation (5.2) and construct a current $\Omega^{(1)}$ of ghost number zero which yields a conserved charge that can map physical states to physical states. In this way, the classification of global symmetries has been reduced to the computation of the $g=1$ exceptional relative BRST cohomology $H_{rel}^{g=1}(k=0)$. We shall see that the picture dependence of the latter plays a crucial role.

As a simple example, consider for a moment the open bosonic string. Its ghost number one exceptional relative BRST cohomology is spanned by the operators $\Omega^{(0)} = c \partial x^\mu$. It is easy to see that Eq. (5.2) is solved by $\Omega^{(1)} = \partial x^\mu d z$. Obviously, this leads to the charge $A = p^\mu = \int \frac{dx^\mu}{2\pi} \partial x^\mu$; which is nothing but the center-of-mass momentum generating spacetime translations.

A key role in the computation of $H_{rel}^{g=1}(k=0)$ is played by $H_{rel}^{g=0}(k=0)$, the so-called ground ring, because it is contained in any $H_{rel}^{g}(k=0)$ in the sense that

$$\omega^{g} \cdot H_{rel}^{g=0}(k=0) \subseteq H_{rel}^{g}(k=0) \quad \text{for any} \quad \omega^{g} \in H_{rel}^{g}(k=0) \quad (5.4)$$

where we denoted the natural product in the BRST cohomology ring [26] by a dot. On representatives, this multiplication is nothing but the normal ordered product. In our bosonic string example, the only ghost number zero cohomology class is the unit operator, rendering the ground ring trivial. For the $N=2$ string, however, the ground ring was found to be infinite-dimensional, with $\pi+1$ generators in each picture $\pi \geq 1$ [12]. Let us briefly review this result before describing the set of conserved charges and the transformations they generate.

For convenience we change the picture labels from $(\pi_+, \pi_-)$ to “spin” labels $(j, m)$ via\(^\text{3}\)

$$\pi_+ = j + m \quad \text{and} \quad \pi_- = j - m \quad \text{on operators}$$
$$\pi_+ + 1 = j + m \quad \text{and} \quad \pi_- + 1 = j - m \quad \text{on states} \quad . \quad (5.5)$$

One finds [12, 9] that the ground ring is spanned by basis elements

$$O_{j,m}^\ell \quad \text{with} \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad m = -j, -j+1, \ldots, +j \quad \ell = 0, 1, \ldots, 2j \quad . \quad (5.6)$$

which are built from the picture-raising operators $X_\pm$ and the spectral-flow operator $S$ [27]. Under the cohomology product these operators form an infinite abelian algebra on which the picture-number operators $\Pi_\pm$ act as derivations. This algebra and its derivations can be written more concisely in terms of polynomials in two variables $(x, y)$ and vector fields on the $xy$ plane, with the translations $\partial_x$ and $\partial_y$ missing.

Starting from the obvious representatives of $H_{rel}^{g=1}(k=0)$, namely the “translations”\(^\text{4}\)

$$P_{a\alpha} = \epsilon_{a\beta} \epsilon_{\alpha\bar{\beta}} P_{\bar{\beta}\beta} \quad , \quad P_{\beta\bar{\beta}} := i c \partial x^\beta \bar{\beta} - 2i \gamma_{\bar{\beta}} \gamma^\beta \quad , \quad (5.7)$$

one immediately sees that

$$O_{j,m;\alpha}^{(0)} := P_{a\alpha} \cdot O_{j,m}^\ell \quad (5.8)$$

\(^{3}\)The picture offset of the canonical ground state $(-1, -1; 0, a)$ is responsible for the distinction.

\(^{4}\)Worksheet reparametrization and supersymmetry ghosts are denoted by $c$ and $\gamma$, respectively. Due to $N=2$ supersymmetry, the latter has two real components which have been collected in a matrix $\gamma_\alpha^\beta$ subject to $\gamma_0^0 = \gamma_1$ and $\gamma_1^0 = -\gamma_0$. 
comprise a set of $2(2j+1)$ independent operators in the $(j, m)$ sector with $g=1$.

In order to find the symmetry charges $A$, we have to insert our $g=1$ zero-forms into the descent equation (5.2), work out the corresponding one-forms, and integrate those across the worldsheet. The result of this computation reads

$$A^\ell_{j,m;\,a} = \int \frac{dz}{2\pi i} \left[ \int \frac{dw}{2\pi i} b(w) P_{\alpha \beta} (z) \cdot O^\ell_{j,m} (z) \right]. \quad (5.9)$$

Together with the derivations

$$B^\pm_{j,m} = O^\ell_{j,m} \cdot (\Pi^\pm + 1) \quad (5.10)$$

these charges form an enormous non-abelian algebra.

According to Noether’s theorem the conserved charges $A$ must generate global symmetries of the open $N=2$ string. The symmetry transformations of the physical state

$$|k, a\rangle \in \{ |\pi_+, \pi_-, k, a\rangle \} / \text{picture-changing} \quad (5.11)$$

(pictures are identified for $k \neq 0$) are found by evaluating the action of $A$’s on $-1, -1; k, a\rangle$,

$$A^\ell_{j,m;\,a} |k, a\rangle = O^\ell_{j,m} \cdot P_{\alpha \beta} |k, a\rangle = h(k)^{-2j-m+\ell} k_{\alpha \beta} |k, a\rangle, \quad (5.12)$$

with the important phase $[28, 29]$

$$h(k) := \frac{k_{0,0}}{k_{10}} = \frac{k_{0,1}}{k_{11}}. \quad (5.13)$$

The action of the derivations $B^\pm_{j,m}$ obtains by replacing $k_{\alpha \beta} \rightarrow \pi_+ \pm 1$ on the right-hand side of (5.12). The transformations (5.12) constitute an infinity of global symmetries which are unbroken in the flat Kleinian background. Their Ward identities constrain the tree-level scattering amplitudes so severely [9] that all but the three-point function must vanish, consistent with the direct computations alluded to earlier.

To make contact with the non-local abelian symmetries of the SDYM equations (see Section 3), it suffices to consider the subalgebra of symmetry charges

$$P_{\alpha \beta} := A^0_{j,-n;\,a} = P_{\alpha \beta} \cdot O^0_{j,-n} \quad (5.14)$$

i.e. putting $\ell=0$. It has the important property that only non-positive powers of the phase $h(k)$ appear when acting by these charges on physical states,

$$\delta_{n\alpha} |k, a\rangle := P_{n\alpha} |k, a\rangle = h(k)^{-n} k_{\alpha \beta} |k, a\rangle \quad (5.15)$$

Note that for $j=0$ we find the translations $\delta_{0\alpha} |k, a\rangle = k_{\alpha \beta} |k, a\rangle$ as it should be. Moreover, for $j=1/2$ we have $\delta_{1\alpha} |k, a\rangle = P_{1\alpha} |k, a\rangle = k_{\alpha \beta} |k, a\rangle$. Although $P_{n\alpha}$ shifts the picture $(\pi_+, \pi_-) \rightarrow (\pi_+ + 2j-n, \pi_- + n)$ of the state representative, by virtue of the picture-changing equivalence (5.11) physical states $|k, a\rangle$ are eigenstates not only of the momentum $P_{\alpha \beta}$ but of all our hidden symmetry generators $P_{n\alpha}$. To employ their eigenvalues $k_{\alpha \beta}$ as further labels of the physical state $|k, a\rangle$ would be customary but superfluous, since (5.15) tells us that these eigenvalues are not independent but completely given by the first two, $k_{\alpha \beta} = h(k)^{-n} k_{\alpha \beta}$.

Recall that a single massless $N=2$ string physical state with zero instanton angle $\theta$ corresponds to a massless spacetime field $\Psi$, and therefore the SDYM symmetries $\delta_{n\alpha} \Psi$ will correspond to the
string symmetries (5.15). To compare symmetries $\delta_{n+1} \Psi$ with the string symmetries (5.15), one should turn to the momentum representation $\Psi \rightarrow \Psi$ and single out terms linear in $\Psi$, since we cannot expect to see non-linear in $\Psi$ transformations in the first-quantized string theory. For this consider Eqs. (3.37) in the momentum representation,

$$
k_{\alpha} \delta_{n+1} \Psi - k_{\alpha} \delta_{n} \Psi + O(\Psi) = 0 \Leftrightarrow \delta_{n+1} \Psi - h(k)^{-1} \delta_{n} \Psi + O(\Psi) = 0
d_k_{\alpha} \Psi = k_{\alpha} \Psi.
$$

where $h(k)$ is given in (5.13). From Eqs. (5.16) we obtain

$$
\delta_n \Psi = h(k)^{-1} k_{\alpha} \Psi + O(\Psi)
$$

since $\delta_{\alpha} \Psi = k_{\alpha} \Psi$. The identity of (5.15) to (5.17) is evident. Thus, the transformations (5.15) of the string ground state $|k, a\rangle$ corresponding to the field $\Psi(k)$ precisely reproduce the linear part of the symmetries (3.29) of the self-dual Yang-Mills equations. Recall that these symmetries generate flows on the moduli space of self-dual gauge fields.

6 Conclusion

This work lifts to a new level the identification of open $N=2$ strings (at tree-level) with self-dual Yang-Mills (SDYM) on 4D manifolds of signature $(++--)$. It provides further evidence that $N=2$ strings inherit integrability from self-dual Yang-Mills theory. After recapitulating the SDYM equations and their symmetries in the twistor framework, as well as reviewing the open $N=2$ string and its rigid symmetries in the first-quantized BRST approach, we have demonstrated complete agreement on the linearized level.

Interestingly, the stringy source of those symmetries is a non-trivial ground ring of ghost number zero operators in the chiral BRST cohomology. Such a phenomenon is familiar from the non-critical 2D string, where the ground ring was exploited to investigate the global symmetries of the theory, with the result that there are more discrete states and associated symmetries in 2D string theory than had been recognized previously [26, 24]. The authors of Ref. [24] have wondered if their findings “could be relevant in a realistic string theory with a macroscopic four-dimensional target space”. The outcomes of [10] and of this paper answer their question in the affirmative by providing the first four-dimensional if not yet realistic string theory (open as well as closed) with a rich symmetry structure based on an infinite ground ring.

More concretely, the symmetry charges are constructed from zero-momentum operators of picture-raising $X_\pm$, picture charge $\Pi_\pm$, spectral flow $S$, and momentum operators $P_{\alpha, \alpha}$. The abelian subalgebra generated by $X_\pm, S$, and $P_{\alpha, \alpha}$ was found to coincide with the algebra of non-local abelian symmetries of the SDYM equations produced by the operators $\partial_{\alpha, \alpha}$ and the recursion operator $R : \delta_{n+1} \Psi \rightarrow \delta_{n+1} \Psi$ defined by Eqs. (3.37).

Our results ascertain that the non-trivial picture structure of the BRST cohomology is not just an irrelevant technical detail of the BRST approach but indispensable for a deeper understanding of the theory. It is, of course, not a simple task to discover the full symmetry group of a string model. Doing so would roughly correspond to having found a useful non-perturbative definition of the theory. In this paper we have worked in the standard first-quantized formalism which is background-dependent and limits our access to unbroken linear global symmetries.

There remain a number of interesting unresolved issues which should be addressed. Prominent among them is the detection of the non-abelian symmetries of the SDYM equations in the $N=2$ string context. For this goal, a string field theoretic setup [30, 31] should be more effective. Another
point concerns the quantum extension of our hidden non-local symmetries. On the string theory side, it appears that their Ward identities forbid any scattering (beyond three-point) not only at tree- but also at the loop-level [32, 33]. On the field theory side, such a feature would seem to select a particular quantum version of SDYM, different from the one yielding the celebrated MHV amplitudes [34]. Finally, it would be interesting to find further examples in which the picture structure yields non-trivial information about a theory, like it happens for the relative zero-momentum cohomology of the Ramond sector of the $N=1$ string in flat 9+1 dimensional spacetime [13]. We hope to return to these problems soon.

Acknowledgements

The work of T.A.I. was partially supported by the Heisenberg-Landau Program and the grant RFBR-99-01-01076. T.A.I. thanks for hospitality the Institut für Theoretische Physik der Universität Hannover, where part of this work was done. O.I. is grateful to the Erwin Schrödinger International Institute for Mathematical Physics in Vienna, where this work was completed.

A Appendices

A.1 Line bundles over the Riemann sphere

The Riemann sphere $\mathbb{C}P^1 \simeq S^2$ is the complex manifold obtained by patching together two coordinate patches $\Omega_+$ and $\Omega_-$, with $\Omega_+$, the neighbourhood of $\zeta = 0$, and $\Omega_-$, the neighbourhood of $\zeta = \infty$. For example, if

$$\Omega_+ = \{ \zeta \in \mathbb{C} : |\zeta| < \infty \}, \quad \Omega_- = \{ \zeta \in \mathbb{C} \cup \{ \infty \} : |\zeta| > 0 \}, \quad (A.1)$$

then we can use $\zeta$ as the coordinate on $\Omega_+$ and $\tilde{\zeta} = \zeta^{-1}$ as the coordinate on $\Omega_-$. 

Consider the holomorphic line bundle $\mathcal{O}(n)$ over $\mathbb{C}P^1$ with the transition function $\zeta^n$, and the first Chern class $c_1(\mathcal{O}(n)) = n$. The space $\mathcal{O}(n)$ is a two-dimensional complex manifold. It can be covered by two coordinate patches, $\mathcal{O}(n) = U_+ \cup U_-$ with the coordinates $(\gamma_+, \zeta)$ on $U_+$ and $(\gamma_-, \tilde{\zeta})$ on $U_-$. The projection

$$\mathcal{O}(n) \to \mathbb{C}P^1 \quad (A.2)$$

is given by $(\gamma_+, \zeta) \to \zeta, (\gamma_-, \tilde{\zeta}) \to \tilde{\zeta}$ in these coordinates. On the overlap region $U_+ \cap U_-$ the coordinates are related by

$$(\gamma_-, \tilde{\zeta}) = (\zeta^{-n} \gamma_+, \zeta^{-1}) \quad (A.3)$$

The space $\Gamma(\mathcal{O}(n))$ of global holomorphic sections of the bundle $\mathcal{O}(n)$ coincides with the space of polynomials of degree $n$ in $\zeta$ with complex coefficients, $\Gamma(\mathcal{O}(n)) = \mathbb{C}^{n+1}$. Points $\sigma_i \in \Gamma(\mathcal{O}(n))$ are complex projective lines $\mathbb{C}P^1_i \hookrightarrow \mathcal{O}(n)$,

$$\mathbb{C}P^1_i = \sigma_i(\zeta) = \left\{ \begin{array}{l} \gamma_+ = \sum_{i=0}^n t_i \zeta^i, \quad \zeta \in \Omega_+ \\ \gamma_- = \sum_{i=0}^n t_i \zeta^{-n}, \quad \zeta \in \Omega_- \end{array} \right., \quad (A.4)$$

parametrized by points $t = \{ t_i \} \in \mathbb{C}^{n+1}$.

On the complex space $\mathcal{O}(n)$ one can introduce a map $\tau : \mathcal{O}(n) \to \mathcal{O}(n)$ called the real structure. It is an antiholomorphic involution, defined by the formula

$$\tau(\gamma, \zeta) = (\gamma, \zeta) \quad (A.5)$$
where \((\gamma, \zeta)\) are local coordinates on \(O(n)\) and the bar denotes the complex conjugate. There are fixed points of the action \(\tau\) on \(O(n)\) and they form a two-dimensional real manifold \(O_\mathbb{R}(n) \cong S^1 \times \mathbb{R}\) fibred over \(S^1\),
\[
O_\mathbb{R}(n) \rightarrow S^1,
\]
and parametrized by the real coordinates \((\gamma_+, \zeta) \in \mathbb{R}^2, (\gamma_-, \zeta^{-1}) \in \mathbb{R}^2\). Here real \(\zeta\) and \(\zeta^{-1}\) parametrize the equator \(\mathbb{R}P^1 = S^1 \cong \mathbb{R} \cup \{\infty\}\) on the sphere \(\mathbb{C}P^1\).

Recall that the sphere \(\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}\) can be decomposed into the disjoint union,
\[
\mathbb{C}P^1 \cong H^+ \cup S^1 \cup H^-,
\]
where \(H^+\) and \(H^-\) are upper and lower half-planes of the extended complex plane \(\mathbb{C} \cup \{\infty\}\). Moreover, (A.7) is exactly the decomposition of \(\mathbb{C}P^1\) into three orbits \(H^+ \cong SL(2, \mathbb{R})/SO(2), H^- \cong SL(2, \mathbb{R})/SO(2)\) and \(S^1 \cong SL(2, \mathbb{R})/B_+\) of the group \(SL(2, \mathbb{R})\) acting on \(\mathbb{C}P^1\). Here
\[
B_+ := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+, b \in \mathbb{R} \right\}
\]
is a two-dimensional subgroup of \(SL(2, \mathbb{R})\) and \(S^1\) is the equator on \(\mathbb{C}P^1\) (the real axis \(\text{Im} \, \zeta = 0\) on \(\mathbb{C} \cup \{\infty\}\)).

Those sections \(\sigma_i \in \Gamma(O(n))\) which are preserved under the conjugation (A.5) are called the real sections of the bundle (A.2), and form a subset \(\Gamma(O_\mathbb{R}(n)) \subseteq \Gamma(O(n))\). Points \(\sigma_i \in \Gamma(O_\mathbb{R}(n))\) are real projective lines \(\mathbb{R}P^1_i = S^1_i \hookrightarrow O_\mathbb{R}(n) \subset O(n)\),
\[
S^1_i = \sigma_i(\zeta) = \left\{ \begin{array}{ll}
\gamma_+ = \sum_{i=0}^n t_i \zeta^i, & \zeta \in \mathbb{R}^+ \\
\gamma_- = \sum_{i=0}^n t_i \zeta^{-i}, & \zeta \in \mathbb{R} \cup \{\infty\} - \{0\}
\end{array} \right.
\]
parametrized by points \(t = \{t_i\} \in \mathbb{R}^{n+1}\). We also consider real holomorphic sections of the bundle (A.2), which are defined by formulae (A.4) with complex \(\zeta \in \mathbb{C}P^1\), real \(t = \{t_i\} \in \mathbb{R}^{n+1}\) and satisfy the reality condition
\[
\overline{\gamma_+(t, \zeta)} = \gamma_+(t, \overline{\zeta}) \Leftrightarrow \left( \sum_{i=0}^n t_i \overline{\zeta^i} \right) = \sum_{i=0}^n t_i \zeta^i
\]
and analogously for \((\gamma_-(t, \zeta), \overline{\zeta})\). We denote the space of such sections by \(\Gamma_\mathbb{R}(O(n))\). It is easy to see that \(\Gamma_\mathbb{R}(O(n)) \cong \Gamma(O_\mathbb{R}(n))\).

Notice that the linear-fractional transformation
\[
\zeta \mapsto \lambda = \frac{\zeta - i}{\zeta + i}
\]
carries the upper half-plane \(\text{Im} \, \zeta > 0\) to the unit disk \(|\lambda| < 1\), the lower half-plane \(\text{Im} \, \zeta < 0\) to the domain \(|\lambda| > 1\) and the real axis \(\text{Im} \, \zeta = 0\) to the circle \(|\lambda| = 1\). Then the conjugation operation \(\zeta \mapsto \overline{\zeta}\) is replaced by \(\lambda \mapsto \lambda^{-1}\) and the reality condition has a different form. If we parametrize the circle \(|\lambda| = 1\) by \(\lambda = e^{-i\theta}, 0 \leq \theta \leq 2\pi\), then from (A.10) we obtain
\[
\zeta = \cot \frac{\theta}{2},
\]
In some formulae it is convenient to parametrize \(\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}\) by homogeneous coordinates
\[
v_0 = g^{1/2} \sin \frac{\theta}{2}, \quad v_1 = g^{1/2} \cos \frac{\theta}{2} \Leftrightarrow \zeta = \frac{v_1}{v_0} = \cot \frac{\theta}{2},
\]
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where $g$ is a positive constant. In these coordinates, real sections of the bundle \((A.2)\) have the form
\[
\sigma_i(\theta) = g^{n/2} \sum_{i=0}^{n} t_i \cos^{\frac{n}{2}} \sin^{n-i} \frac{\theta}{2} = g^{J} \sum_{M=J}^{\infty} t_{J+M} \cos^{J+M} \frac{\theta}{2} \sin^{J-M} \frac{\theta}{2} ,
\]
where $J = n/2$.

A.2 Twistor spaces

Let us consider the rank 2 holomorphic vector bundle
\[
P = \mathcal{O}(1) \otimes \mathbb{C}^2 = \mathcal{O}(1) \oplus \mathcal{O}(1)
\]
over the Riemann sphere $\mathbb{C}P^1$ with a holomorphic projection
\[
\pi : P \to \mathbb{C}P^1 .
\]
Each fibre of the bundle \((A.15)\) is a copy of $\mathbb{C}^2$. The space $P$ can be covered by two coordinate patches, $P = P_+ \cup P_-$, with the coordinates $(\eta^\alpha_+, \eta^\beta_+, \zeta)$ on $P_+$ and $(\eta^\alpha_-, \eta^\beta_-, \zeta)$ on $P_-$ related by
\[
(\hat{\eta}^\alpha_-, \eta^\beta_-, \zeta) = (\zeta^{-1} \hat{\eta}^\alpha_+, \zeta^{-1} \eta^\beta_+, \zeta^{-1})
\]
on the overlap $P_+ \cap P_-$. The space $P$ will be called the twistor space. It is an open subset of $\mathbb{C}P^3$, $P \simeq \mathbb{C}P^3 - \mathbb{C}P^1 \simeq S^2 \times \mathbb{R}^4$ [4].

Global holomorphic sections $\sigma_x$ of the bundle \((A.15)\) are complex projective lines $\mathbb{C}P^1_x \rightarrow P$, \[
\mathbb{C}P^1_x = \sigma_x(\zeta) = \left\{ \begin{array}{l}
\hat{\eta}^\alpha_+ = x^{00} + \zeta x^{10}, \quad \eta^\beta_+ = x^{01} + \zeta x^{11}, \quad \zeta \in \Omega_+ , \\
\hat{\eta}^\alpha_- = \zeta^{-1} x^{00} + x^{10}, \quad \eta^\beta_- = \zeta^{-1} x^{01} + x^{11}, \quad \zeta \in \Omega_- ,
\end{array} \right.
\]
parametrized by $x = \{x^{\alpha\hat{\alpha}}\} \in \mathbb{C}^4$, $\alpha = 0, 1$, $\hat{\alpha} = 0, 1$.

We introduce a real structure $\tau$ on $P$ by the formula
\[
\tau(\eta^\beta_+, \zeta) = (\overline{\eta^\beta_+}, \zeta) .
\]
There are fixed points of the action $\tau$ on $P$ and they form a 3-dimensional real manifold \[
T = \mathbb{R} P^3 - \mathbb{R} P^1 \simeq S^1 \times \mathbb{R}^2
\]
with a projection \[
T \rightarrow S^1 .
\]
The space $T \subset P$ is called the real twistor space.

Real sections of the bundle \((A.15)\) over $P_+$ are defined by the formulae
\[
\sigma_x(\zeta) = (\eta^\alpha_+(\zeta), \eta^\beta_+(\zeta)) = (x^{00} + \zeta x^{10}, x^{01} + \zeta x^{11}), \quad \zeta \in \mathbb{R} ,
\]
and over $P_-$ by the formulae
\[
\sigma_x(\zeta) = (\eta^\alpha_-(\zeta), \eta^\beta_-(\zeta)) = (\zeta x^{00} + x^{10}, \zeta x^{01} + x^{11}), \quad \zeta = \zeta^{-1} \in \mathbb{R} ,
\]
with \( x = \{ x^\alpha \} \in \mathbb{R}^4 \). Real holomorphic sections of the bundle (A.15) are defined by formulae (A.17) with real \( x = \{ x^\alpha \} \in \mathbb{R}^4 \) and complex \( \zeta \in \mathbb{CP}^1 \).

On the space \( \mathbb{R}^4 \) of real sections of the bundle (A.15) isomorphic to the space of real holomorphic sections, one may introduce the metric \( \eta = \text{diag}(+1, +1, -1, -1) \), and other signatures are not compatible with the real structure (A.18) on the twistor space \( \mathcal{P} \). Namely, we have

\[
ds^2 = -\det(d\dot{x}^\alpha) = \eta_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu ,
\]

where

\[
x^1 := \frac{1}{2}(x^0i + x^{10}) , \quad x^2 := \frac{1}{2}(x^{0i} - x^{11}) , \quad x^3 := \frac{1}{2}(x^{10} - x^{0i}) , \quad x^4 := \frac{1}{2}(x^{00} + x^{11}) .
\]

So, the space of real sections of the bundle (A.15) is the Kleinian space \( \mathbb{R}^{2,2} = (\mathbb{R}^4, \eta) \).

Note that real sections \( \sigma_x \in \Gamma_{\mathbb{R}}(\mathcal{P}) \) of the bundle (A.15) define real projective lines \( S^1_x \hookrightarrow \mathcal{T} \subset \mathcal{P} \),

\[
S^1_x = \sigma_x(\zeta) = \left\{ \begin{array}{ll}
  x^{0i} + \zeta x^{10} , & \zeta \in \mathbb{R} , \\
  \zeta^{-1} x^{00} + x^{10} , & \zeta \in \mathbb{R} \cup \{ \infty \} - \{ 0 \} ,
\end{array} \right.
\]

parametrized by \( x = \{ x^\alpha \} \in \mathbb{R}^{2,2} \). Fibres \( \mathbb{R}^2 \) of the bundle \( \mathcal{T} \to S^1 \) are real null 2-planes (real \( \alpha \)-planes [5]) in the space \( \mathbb{R}^{2,2} \).

The polynomials \( \eta^\alpha_\beta \) in (A.17) are annihilated by the following differential operators:

\[
V_\alpha = \frac{\partial}{\partial x^{1\beta}} - \zeta \frac{\partial}{\partial x^{0\beta}} .
\]

Let us introduce the subspace \( \mathcal{P}_0 = S^1 \times \mathbb{R}^4 \) in \( \mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_- \simeq S^2 \times \mathbb{R}^4 \). Then the vector fields (A.26) together with \( \partial / \partial \zeta \) form a basis of (0,1) vector fields on \( \mathcal{P}_+ - \mathcal{P}_0 \), and the vector fields \( \zeta^{-1} V_\alpha , \partial / \partial \zeta \) form a basis of (0,1) vector fields on \( \mathcal{P}_- - \mathcal{P}_0 \). The space \( \mathcal{P}_0 \) is fibred over \( \mathcal{T} \) by real null 2-planes (real \( \beta \)-planes [5]), and their basis is formed by the vector fields (A.26) with \( \zeta \in \mathbb{RP}^1 \), \( x^{\alpha \hat{\alpha}} \in \mathbb{R} \), \( \alpha = 0,1 \), \( \hat{\alpha} = 0,1 \) (real vector fields). Such vector fields annihilate the real sections (A.21), (A.22) of the bundle (A.15).

Let us introduce the basis \( d\eta^\alpha_\beta = dx^{0\alpha} + \zeta dx^{1\hat{\alpha}} \) of 1-forms on fibres of the bundle (A.15) and consider real \( x = \{ x^\alpha \} \in \mathbb{R}^{2,2} \). Then the 2-forms

\[
d\eta^\alpha_\beta \wedge d\eta^\hat{\beta}_\beta = (dx^{0\alpha} + \zeta dx^{1\hat{\alpha}}) \wedge (dx^{0\hat{\beta}} + \zeta dx^{1\beta}) =
\]

\[
= dx^{0\alpha} \wedge dx^{0\hat{\beta}} + \zeta (dx^{1\hat{\alpha}} \wedge dx^{0\hat{\beta}} + dx^{0\alpha} \wedge dx^{1\beta} + \zeta^2 dx^{1\hat{\alpha}} \wedge dx^{1\beta})
\]

are complex null 2-forms on \( \mathbb{R}^{2,2} \) labelled by \( \zeta \in \mathbb{CP}^1 \). If \( \text{Im} \zeta = 0 \), then (A.27) are real null 2-forms on \( \mathbb{R}^{2,2} \) parametrized by \( S^1 \ni \zeta = \cot \theta \).
References


