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Operator Product Expansion of the Lowest Weight CPOs in $\mathcal{N}=4$ SYM$_4$ at Strong Coupling

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Abstract

We present a detailed analysis of the 4-point functions of the lowest weight chiral primary operators $O^j \sim \text{tr} \phi^{(i} \phi^{j)}$ in $\mathcal{N}=4$ SYM$_4$ at strong coupling and show that their structure is compatible with the predictions of AdS/CFT correspondence. In particular, all power-singular terms in the 4-point functions exactly coincide with the contributions coming from the conformal blocks of the CPOs, the R-symmetry current and the stress tensor. Operators dual to string modes decouple at strong coupling. We compute the anomalous dimensions and the leading $1/N^2$ corrections to the normalization constants of the 2- and 3-point functions of scalar and vector double-trace operators with approximate dimensions 4 and 5 respectively. We also find that the conformal dimensions of certain towers of double-trace operators in the $105$, $84$ and $175$ irreps are non-renormalized. We show that, despite the absence of a non-renormalization theorem for the double-trace operator in the $20$ irrep, its anomalous dimension vanishes. As by-products of our investigation, we derive explicit expressions for the conformal block of the stress tensor, and for the conformal partial wave amplitudes of a conserved current and of a stress tensor in $d$ dimensions.

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1 Introduction

The AdS/CFT correspondence [1–3] is arguably the best currently available way of getting nontrivial dynamical information for the strong coupling behavior of certain conformal field theories. In particular, the $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory in four dimensions (SYM$_4$) at large $N$ and at strong 't Hooft coupling $\lambda = g^2_M N$ is dual to type IIB supergravity on the $AdS_5 \times S^5$ background. The supergravity fields are dual to certain quasi-primary operators in SYM$_4$. According to [2, 3], the generating functional for the connected Green functions of these operators coincides with the on-shell value of type IIB supergravity action which has to be further modified by the addition of definite boundary terms [4]. Thus, computing $n$-point correlation functions in the supergravity approximation is generally divided into two independent problems - finding first the supergravity action up to the $n$-th order and then evaluating its on-shell value. Although a covariant action for type IIB supergravity is unknown, one can use the covariant equations of motion [5–7] and the quadratic action [8] to find cubic actions [9–11] for its physical fields and to compute the corresponding 3-point functions using the technique developed in [12].

Computing 4-point functions [13]-[26] in the supergravity approximation in general requires the derivation of the supergravity action up to fourth order. The part of the action relevant to the massless modes, corresponding to the dilaton and axion fields, was already known as pointed out in [13] where the calculation of the corresponding 4-point functions was initiated. The complete expression for the 4-point functions was obtained in [20] and was further analyzed in [23]. Unfortunately, these modes are dual to the rather complicated operators $\text{tr} F^2 + \ldots$ and $\text{tr} F \tilde{F}$ and the analysis performed in [23] was unavoidably incomplete.

It is known that all operators dual to the type IIB supergravity fields belong to short representations of the conformal superalgebra $SU(2,2|4)$ and are supersymmetric descendants of Chiral Primary Operators (CPOs) of the form $O_k^i = \text{tr}(\phi^{ij} \cdots \phi^{ij})$. CPOs are dual to scalar fields $s^l$ that are mixtures of the five form field strength on $S^5$ and the trace of the graviton on $S^5$. The relevant part of the quartic action of type IIB supergravity for the scalars $s^l$ was found in [27] and was then used in [28] to compute the 4-point functions of the simplest CPOs $O^l = \text{tr}(\phi^{ij} \phi^{i})$. In the present paper we use these 4-point functions to analyze in detail the Operator Product Expansion (OPE) of the lowest weight CPOs at strong coupling.

It is widely believed that the structure of a Conformal Field Theory (CFT) is encoded in the OPE since knowledge of the latter allows, in principle, the calculation of all $n$-point
functions. Thus, in the context of AdS/CFT correspondence one would eventually like to prove that the 4-point functions, (and in general $n$-point functions) of CPOs in the boundary CFT computed in the supergravity approximation admit an OPE interpretation. This is a rather complicated problem because an infinite number of quasi-primary operators may in principle appear in the OPE of two CPOs. Therefore, the best one can presently do is to show that the leading terms in a double OPE expansion of the 4-point functions exactly match the contributions of the conformal blocks of the first few quasi-primary operators with the lowest conformal dimensions. This is the main line of investigation which we follow in the present work in our analysis of the 4-point function of the lowest weight CPOs in $\mathcal{N} = 4$ SYM$_4$.

Our study shows that there are four singular terms in the OPE of two lowest weight CPOs corresponding to the identity operator, the lowest weight CPO itself, the $R$-symmetry vector current and the stress tensor. These three nontrivial operators are dual to the scalars $s^i$, the vector fields $A_\mu$ and the graviton $h_{\mu\nu}$ that appear in the exchange Feynman diagrams of type IIB supergravity. The most singular terms in the 4-point functions computed in the supergravity approximation exactly coincide with the contributions coming from the conformal blocks of the above three operators.

We compare the strong coupling OPE with the free field theory OPE, and explicitly observe, at weak coupling, the splitting of the $R$-symmetry current and of the stress tensor into 2 and 3 terms respectively which belong to different supermultiplets. Only one term in each splitting is dual to a supergravity field and survives at strong coupling while the others acquire large dimensions and decouple. A similar type of splitting also occurs in the case of the double-trace operators transforming in the 84 and 175 irreps.

We also analyze the leading nonsingular terms in the OPE which are due to double-trace operators of the schematic form $:\partial^n O^i \partial^n O^j :$ with free field conformal dimensions $4 + m + n$. A generic property of any correlation function computed in the supergravity approximation is the appearance of logarithmic terms. In an unitary CFT logarithmic terms have a natural interpretation in terms of anomalous dimensions of operators [29] and such an interpretation was used in the past in studies of the $O(N)$ vector model [30, 31]. Since the operators dual to the supergravity fields have protected conformal dimensions, the logarithmic terms in the correlation functions of supergravity can only be attributed to anomalous dimensions of double-trace operators.

We show that among the scalar double-trace operators with free field conformal dimension 4, the only one acquiring an anomalous dimension is the operator $: O^i O^j :$, which transforms in the trivial representation of the $R$-symmetry group $SO(6) \sim SU(4)$.  

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The anomalous dimension of this operator is found to be $-16/N^2$ and coincides with the anomalous dimension of the operator $B$ which was calculated in [23]. This is consistent with the fact that $B$ is a supersymmetric descendent of $:O^4O^4:$ . It is worth noting that among the non-renormalized operators we find a double-trace scalar operator in the 20 irrep of $SO(6)$ whose non-renormalization property does not follow from the shortening condition discussed in [32, 33].

Finally, we compute the anomalous dimensions of the double-trace vector operators with free field conformal dimension 5 transforming in the 15 and 175 irreps respectively. We show that there are several towers of traceless symmetric tensor operators in the 105, 84 and 175 irreps, whose anomalous conformal dimensions vanish. Some of these tensor operators are not subject to any known non-renormalization theorem.

The 4-point functions of CPOs also allow us to find the leading $1/N^2$ corrections to the normalization constants of the 3-point functions involving two CPOs and one double-trace operator with low conformal dimension. In the case when a double trace operator has protected dimension we interpret these corrections as manifestation of the splitting of the free field theory operator in two orthogonal parts carrying different representations of supersymmetry. The first one has protected both the dimension and the normalization constant, the other one acquires infinite anomalous dimension and disappears at strong coupling. To make this interpretation precise one should further show that the linear splitting arising due to the difference between normalization constants in free theory and at strong coupling is consistent with the fact that the split fields transform in different representations of supersymmetry. It would be quite interesting to investigate such a property in more detail.

The plan of the paper is as follows. In section 2 we recall how logarithmic terms are related to anomalous conformal dimensions in an unitary CFT and in the framework of the AdS/CFT correspondence. In section 3 we discuss the structure of the OPE of the lowest weight CPOs in free field theory and at strong coupling. In section 4 we compute anomalous dimensions and first corrections to the 2- and 3-point normalization constants of double-trace operators of approximate dimensions 4 and 5. A discussion of the results obtained and our conclusions are presented in section 5. Several technical issues are considered in five Appendices. In the Appendix A we discuss a decomposition of a biloval operator which is a normal-ordered product of two quasi-primary scalar operators into a sum of conformal blocks of local tensor primary operators. In the Appendix B explicit formulae for conformal partial amplitudes of scalar, conserved vector current and stress tensor are derived. A convenient series representation used throughout the paper is obtained in the Appendix C. In the Appendix D we discuss the projectors which single
out the contributions of irreps occurring in the decomposition $20 \times 20$ of $SO(6)$ from the 4-point function of CPOs. In the Appendix E an explicit formula for the conformal block of the stress tensor is derived.

2 Anomalous dimensions and logarithmic terms in CFT

An arbitrary unitary CFT is completely characterized by a set of quasi-primary operators $O_i$ of conformal dimensions $\Delta_i$ and by their OPE

$$O_i(x)O_j(y) = \sum_k \frac{1}{|x - y|^{\Delta_i + \Delta_j - \Delta_k}} C^k_{ij}(x - y, \partial_y)O_k(y).$$

(2.1)

Here the sum runs over the set of all the quasi-primary operators and $i, j, k$ are multi-indices which in general include the indices of the $R$-symmetry and of the Lorentz groups. The operator algebra structure constants $C^k_{ij}(x - y, \partial_y)$ can be decomposed in a power series in $x - y$ and $\partial_y$. Without loss of generality one can assume that the operators $O_i$ are orthogonal

$$\langle O_i(x)O_j(0) \rangle = C_i \frac{\delta_{ij}}{x^{2\Delta_i}},$$

where $C_i$ is a normalization constant of the 2-point function. Then the operator algebra structure constants are fixed by the conformal dimensions $\Delta_i, \Delta_j, \Delta_k$, and by the ratio $C_{ijk}/C_k$, where the structure constants $C_{ijk}$ appear in the 3-point functions

$$\langle O_i(x)O_j(y)O_k(z) \rangle = \frac{C_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k} |x - z|^{\Delta_i + \Delta_k - \Delta_j} |y - z|^{\Delta_j + \Delta_k - \Delta_i}}.$$  

(2.2)

The conformal dimensions and the structure constants depend on the coupling constants of the CFT. In principle, the OPE (2.1) allows one to compute any correlation function in the CFT. In particular, 4-point functions are given by the following (schematic) double OPE expansion

$$\langle O_i(x)O_j(y)O_k(z)O_l(w) \rangle = \sum_m \frac{1}{|x - y|^{\Delta_i + \Delta_j - \Delta_m} |z - w|^{\Delta_i + \Delta_l - \Delta_m}}$$

$$\times C^m_{ij}(x - y, \partial_y) C^m_{kl}(z - w, \partial_w) C^m_{il} \left( \frac{y - w}{|y - w|^{2\Delta_m}} \right).$$

(2.3)

Thus we see that the short distance expansion of exact CFT correlation functions does not contain logarithmic terms. Suppose, however, that one can only calculate correlation functions up to some order in the coupling constant or another small parameter of the CFT. Then it is clear from (2.3) that logarithmic terms would appear due to the nontrivial
dependence of conformal dimensions on the coupling or on the small parameter. These terms can be easily found representing the conformal dimensions as \( \Delta = \Delta^{(0)} + \Delta^{(1)} \), where \( \Delta^{(0)} \) is the “canonical” part and \( \Delta^{(1)} \) is the “anomalous” coupling constant dependent part. Such a representation leads then to an expansion for the two-point functions of the form \( |x|^\Delta^{(1)} = 1 + \Delta^{(0)} \log |x| + ... \), connecting the logarithmic terms to the anomalous dimensions, that may be used to compute the latter. It is worthwhile to note that at the \( n \)-th order of perturbation theory one encounters terms of the form \((\log |x|)^n\).

The \( \mathcal{N} = 4 \) SYM\(_4\) theory provides an example of such a logarithmic behavior of correlation functions, both in the weak coupling standard perturbation expansion [34]-[37] and also in the supergravity approximation [20,28]. Due to superconformal invariance all quasi-primary operators of SYM\(_4\) belong either to short or long representations of the conformal superalgebra \( SU(2,2|4) \) and in the framework of the AdS/CFT correspondence fall into three classes:

\( i \) Chiral operators dual to the type IIB supergravity fields which belong to short representations and have protected conformal dimensions. The simplest operators in this class are the lowest weight CPOs \( O^I = \text{tr}(\phi^I \phi^I) \).

\( ii \) Operators dual to multi-particle supergravity states which are obtained as “normal-ordered” products of the chiral operators, e.g. the double-trace operators : \( O^I O^J \) :. They may belong either to short or long representations and have conformal dimensions restricted from above.

\( iii \) Operators dual to string states (single- or multi-particle) which belong to long representations and whose conformal dimensions grow as \( \lambda^{1/4} \) in the strong coupling limit. The simplest example of such an operator is the Konishi operator \( \text{tr}(\phi^I \phi^I) \).

In the supergravity approximation to the AdS/CFT correspondence the operators dual to string states decouple from the spectrum and one can calculate the connected \( n \)-point functions of chiral operators dual to the supergravity fields to leading order which is \( 1/N^{n-2} \). Since the expansion parameter is \( 1/N^2 \), an \( n \)-point function contains logarithmic terms of the form \((\log |x|)^{[(n-2)/2]} \). In particular, a 4-point function can have only \( \log |x| \)-dependent terms, and cannot have, say, terms of the form \((\log |x|)^2 \). Moreover, since chiral operators have protected conformal dimensions only the operators dual to multi-particle supergravity states contribute to \( \log \)-dependent terms.

The AdS/CFT correspondence predicts a simple form of the OPE of chiral operators in the strong coupling limit. Let \( O_1 \) and \( O_2 \) be operators dual to the supergravity fields \( \varphi_1 \) and \( \varphi_2 \) respectively and let the supergravity action contain the non-vanishing cubic couplings \( \frac{1}{N} \lambda_{12k} \varphi_1 \varphi_2 \varphi_k \) with some fields \( \varphi_k \). Then, the OPE of \( O_1 \) and \( O_2 \) takes the form
(suppressing the indices of the operators and structure constants)

\[
O_1(x)O_2(y) = \frac{1}{N} \sum_k \frac{1}{|x-y|^{\Delta_1+\Delta_2-\Delta_k}} C_{12}^k(x-y, \partial_y)O_k(y) + [: O_1(x)O_2(y) :], \quad (2.4)
\]

where \( O_k \) is an operator dual to \( \varphi_k \). Here we denote by \([ : O_1(x)O_2(y) : ]\) an infinite sum of tensor quasi-primary operators and their descendents, which are dual to multi-particle supergravity states. In general these operators acquire anomalous dimensions and are responsible for the appearance of logarithms in correlation functions. An important property of the operators dual to multi-particle supergravity states is that their structure constants are of order 1, while the structure constants of the operators dual to supergravity fields are of order \( 1/N \). Due to such a property, the sum of these operators coincides in the limit \( N \to \infty \) with the corresponding free field theory normal-ordered operator \( : O_1^{fr}(x)O_2^{fr}(y) : \). This can be seen as follows. A 4-point function of chiral operators is given by a sum of a disconnected contribution which is of order 1 and a connected Green function which is of order \( 1/N^2 \). Since the structure constants of the operators dual to supergravity fields are of order \( 1/N \), they do not contribute to the disconnected part of the 4-point function. Thus only the “normal-ordered” operators contribute. The disconnected part is given by a sum of products of 2-point functions of chiral operators, hence it does not depend on the coupling constant and \( N \) (we assume that all the chiral operators are orthonormal) and coincides with the free field disconnected part. Therefore, in the limit \( N \to \infty \) the sum \([ : O_1(x)O_2(y) :]\) has to coincide with the free field normal-ordered product \( : O_1^{fr}(x)O_2^{fr}(y) :\), \(^4\) that is decomposed into a sum of local tensor quasi-primary operators. However, at finite \( N \) an infinite number of the tensor operators acquire anomalous dimensions and their structure constants get \( 1/N^2 \) corrections to their free field values. For this reason it seems hardly possible to prove that a 4-point function computed in the supergravity approximation admits an OPE interpretation. This would require the knowledge of the conformal partial wave amplitude of an arbitrary tensor operator. Another reason that complicates the analysis of 4-point functions is that in general one should split the free field theory double-trace operators into a sum of operators with the same free field theory dimensions, each one transforming irreducibly under the superconformal group. In the context of the present work we are able to successfully deal with both the above problems.

\(^4\) One can easily see that the normal-ordered product \( : O_1^{fr}(x)O_2^{fr}(y) :\) is the only term of order 1 in the free field OPE of chiral operators.
3 OPE of the lowest weight CPOs

In this section we study the OPE of the lowest weight CPOs in free field theory and at strong coupling. Recall that the normalized lowest weight CPOs in $\mathcal{N} = 4$ SYM$_4$ are operators of the form

$$O^l(x) = \frac{2^{3/2} \pi^2}{\lambda} C^{ij}_{ij} \text{tr}(: \phi^i(x) \phi^j(x) :),$$

where the symmetric traceless tensors $C^l_{ij}$, $i, j = 1, 2, \ldots, 6$ form a basis of the 20 of SO(6) and satisfy the orthonormality condition

$$C^l_{ij} C^j_{ij} = \delta^l_i.$$

Using for the Wick contractions the following propagator

$$\langle\phi^i(x)\phi^j(x)\rangle = \frac{g_{YM}^2 \delta^i_j}{(2\pi)^2 x^2_{12}}, \quad \text{(3.1)}$$

where $a, b$ are color indices and $x_{ij} = x_i - x_j$, one finds the following expressions for the free field theory 2-, 3- [9] and 4-point functions of $O^l$:

$$\langle O^l(x_1)O^b(x_2) \rangle_{fr} = \frac{\delta^l b}{x^2_{12}} ,$$

$$\langle O^l(x_1)O^b(x_2)O^b(x_3) \rangle_{fr} = \frac{1}{N} 2^{3/2} C^l_{12} C^l_{23} ,$$

$$\langle O^l(x_1)O^b(x_2)O^b(x_3)O^l(x_4) \rangle_{fr} = \left[ \frac{\delta^l b \delta^b b}{x^2_{12} x^2_{34}} + \frac{\delta^l b \delta^b l_4}{x^4_{13} x^4_{24}} + \frac{\delta^l b \delta^l l_4}{x^4_{13} x^4_{23}} \right]$$

$$+ \frac{4}{N^2} \left[ \frac{C^l_{12} C^l_{23} C^l_{34} C^l_{41}}{x^2_{12} x^2_{34} x^2_{24} x^2_{41}} + \frac{C^l_{13} C^l_{24} C^l_{32} C^l_{41}}{x^2_{13} x^2_{24} x^2_{24} x^2_{41}} + \frac{C^l_{14} C^l_{23} C^l_{32} C^l_{41}}{x^2_{14} x^2_{23} x^2_{23} x^2_{41}} \right] , \quad \text{(3.2)}$$

where the first term in the 4-point function represents the contribution of disconnected diagrams. We have also introduced the shorthand notations $C^l_{123} = C^l_{112} C^l_{213} C^l_{314}$ and $C^l_{1234} = C^l_{1112} C^l_{1213} C^l_{1314} C^l_{1415}$ for the trace products of matrices $C^l$.

3.1 Free field theory OPE

The simplest way to derive the OPE in free field theory is to apply Wick’s theorem. Using the propagator (3.1) we find the following formula for the product of two CPOs

$$O^l(x_1)O^b(x_2) = \frac{\delta^l b}{x^2_{12}} + \frac{2^{3/2} \pi^2}{\lambda N x^2_{12}} C^l_{i12} C^b_{j2} : \text{tr}(\phi^i(x_1) \phi^j(x_2)) :$$

$$+ :O^l(x_1)O^b(x_2): , \quad \text{(3.3)}$$
On the r.h.s. of (3.3) we have bi-local operators of the form : \( O^\alpha(x_1)O^\beta(x_2) : \), where \( O^\alpha \) is either \( \phi^i \) or \( O^l \) and \( O^\beta \) is either \( \phi^j \) or \( O^l \). To find the operator content of the r.h.s. of (3.3) one should perform the Taylor expansion of the operator \( O^\alpha \) and rearrange the resulting series as a sum of conformal blocks of local quasi-primary operators. It is clear that in free field theory any bilocal operator : \( O^\alpha(x_1)O^\beta(x_2) : \) may be represented as an infinite sum of conformal blocks of symmetric traceless rank \( l \) tensor operators with dimensions \( \Delta_\alpha + \Delta_\beta + l + 2k \),

\[
: O^\alpha(x)O^\beta(0) : = \sum_{l, k=0}^{\infty} \frac{1}{(l + 2k)!} x^{2k} x_1^{\mu_1} \cdots x_l^{\mu_l} [O^{(k)}_{\mu_1 \cdots \mu_l}(0)],
\]

where the square brackets \([ \ ]\) are used to denote the whole conformal block of a quasi-primary operator. In an interacting theory the tensor quasi-primary operators may acquire anomalous dimensions. Explicit expressions of the tensor operators through \( O^\alpha, O^\beta \) are unknown and the best we can do is to find the first few terms in the series. In particular, as shown in Appendix A, the terms up to two derivatives are given by the following formula

\[
: O^\alpha(x)O^\beta(0) : = : O^\alpha(0)O^\beta(0) : + x^\mu : \partial_\mu O^\alpha(0)O^\beta(0) : + \frac{1}{2} x^\mu x^\nu : \partial_\mu \partial_\nu O^\alpha(0)O^\beta(0) : \\
= [O^{\alpha\beta}(0)] + x^\mu [O^{\alpha\beta}\mu(0)] - \frac{1}{2} x^\mu x^\nu [T^{\alpha\beta}_{\mu\nu}(0)] + \frac{1}{2} x^2 [T^{\alpha\beta}(0)],
\]

Here the quasi-primary operators are given by

\[
O^{\alpha\beta} = : O^\alpha O^\beta :, \\
O^{\alpha\beta}_\mu = \frac{1}{2} : (\partial_\mu O^\alpha O^\beta - O^\alpha \partial_\mu O^\beta) :, \\
T^{\alpha\beta}_{\mu\nu} = \frac{1}{2} : (\partial_\mu O^\alpha \partial_\nu O^\beta + \partial_\nu O^\alpha \partial_\mu O^\beta) : - \frac{\Delta}{2(2\Delta + 1)} \partial_\mu \partial_\nu : O^\alpha O^\beta : \\
+ \frac{\delta_{\mu\nu}}{8} \left( \frac{\Delta + 1}{2\Delta + 1} \partial^2 : O^\alpha O^\beta : + \partial^2 O^\alpha O^\beta : + : O^\alpha \partial^2 O^\beta : \right), \\
T^{\alpha\beta} = \frac{1}{8} \left( \frac{\Delta - 1}{2\Delta - 1} \partial^2 : O^\alpha O^\beta : + \partial^2 O^\alpha O^\beta : + : O^\alpha \partial^2 O^\beta : \right),
\]

where \( \Delta \) is the conformal dimension of the operators \( O^\alpha, O^\beta \) which takes the values 1 and 2 in the cases under consideration.

Obviously the conformal dimensions of the scalar operators \( O^{\alpha\beta} \) and \( T^{\alpha\beta} \) are equal to \( 2\Delta \) and \( 2\Delta + 2 \) respectively, the dimension of the vector operator is \( 2\Delta + 1 \) and the dimension of the traceless symmetric tensor operator is \( 2\Delta + 2 \). Consider first the case when \( \Delta = 1 \). The scalar operator \( \text{tr}(\phi^i \phi^j) \) is decomposed into a sum of the traceless part in the 20 - which is a lowest weight CPO \( O^l \) - and the trace part. The trace part is the normalized Konishi scalar field \( \mathcal{K} = \frac{2}{3 \sqrt{\lambda}} \text{tr}(\phi^2) \). If \( \Delta = 1 \) the vector and tensor operators
are conserved and the operator $T^{ij}$ vanishes because of the on-shell equation $\partial^2 \phi^j = 0$. In fact the conserved current transforms in the 15 irrep of $SO(6)$ and is the $R$-symmetry current of the free field theory of 6 scalars $\phi^j$. Decomposing the tensor operator $T^{ij}_{\mu \nu}$ into irreducible representations of the $R$-symmetry group $SO(6)$, i.e., into the traceless and trace parts with respect to the indices $i, j$, one sees that the trace part $T^{ii}_{\mu \nu}$ coincides with the stress tensor of the free field theory. The Konishi scalar and the traceless part of $T^{ij}_{\mu \nu}$ are dual to string modes and are expected to decouple in the strong coupling limit.

To complete the consideration of the free field theory OPE we have to decompose the remaining operators into irreducible representations of $SO(6) \sim SU(4)$. One has the general decomposition of the $20 \times 20$ of $SU(4)$ as

$$20 \times 20 = [0, 0, 0] + [0, 2, 0] + [0, 4, 0] + [2, 0, 2] + [1, 0, 1] + [1, 2, 1].$$

(3.6)

The representations in the first and the second lines of (3.6) are symmetric and antisymmetric in the indices of the 20's $I_1, I_2$, respectively. The dimensions of the representations are

$$D([0, 0, 0]) = 1, \quad D([0, 2, 0]) = 20, \quad D([0, 4, 0]) = 105, \quad D([2, 0, 2]) = 84, \quad D([1, 0, 1]) = 15, \quad D([1, 2, 1]) = 175.$$ (3.7)

Introducing the orthonormal Clebsch-Gordan coefficients $C^{l_1 l_2}_{J_D}$

$$C^{l_1 l_2}_{J_D} C^{l_1 l_2}_{J_D'} = \delta_{J_D J_D'},$$

where $J_D$ is the index of an irrep of dimension $D$, as well as the operators

$$O^J_D = C^{l_1 l_2}_{J_D} : O^{l_1} O^{l_2} :, \quad O^J_D = C^{l_1 l_2}_{J_D} O^{l_1} O^{l_2},$$

(3.8)

we can write

$$: O^{l_1} O^{l_2} : = \delta^{l_1 l_2} O^l_1 + C^{l_1 l_2}_{J_D} O^J_D + C^{l_1 l_2}_{J_{05}} O^{J_{05}} + C^{l_1 l_2}_{J_{84}} O^{J_{84}},$$

$$O^l_\mu = \frac{1}{2} \left(: \partial_\mu O^l_1 O^{l_2} : - : O^l \partial_\mu O^{l_2} :\right) = C^{l_1 l_2}_{J_{15}} O^{J_{15}} + C^{l_1 l_2}_{J_{175}} O^{J_{175}},$$

and a similar decomposition for $T_{\mu \nu}^{l_1 l_2}$ and $T^{l_1 l_2}$. Note that the operators have the following free field theory 2-point functions$^5$

$$\langle O^{J_1} (x_1) O^{J_2} (x_2) \rangle = \left(2 + O\left(\frac{1}{N^2}\right)\right) \delta^{J_1 J_2},$$

$$\langle O^J_\mu (x_1) O^J_\nu (x_2) \rangle = \left(4 + O\left(\frac{1}{N^2}\right)\right) \frac{I_{\mu \nu} (x_{12})}{x_{12}^{10}} \delta^{J_1 J_2},$$

$^5$The only exception is the operator $O_1 = \frac{1}{\sqrt{N}} : O^{l_1} O^{l_2} :$ in the singlet representation, whose normalization constant is $\frac{1}{N} + O\left(\frac{1}{N^2}\right)$. 

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where \( I_{\mu \nu}(x) = \delta_{\mu \nu} - \frac{2x_\mu x_\nu}{x^2} \). The precise values of the normalization constants will be determined in the next section. Due to the definition of the double-trace operators, the 3-point normalization constants which appear in the following 3-point functions

\[
\langle O_j \beta x_1 \beta O_j \beta x_2 \beta O_j \beta x_3 \beta \rangle = C_{O \beta O \beta} \frac{C_{\beta}^{\mu \nu}}{|x_{12}|^{1-\Delta_\beta} |x_{13}|^{\Delta_\beta} |x_{23}|^{\Delta_\beta}},
\]

\[
\langle O_j \beta x_1 \beta O_j \beta x_2 \beta O_j \beta x_3 \beta \rangle = C_{O \beta O \beta} \frac{C_{\beta}^{\mu \nu}}{|x_{12}|^{1-\Delta_\beta} |x_{13}|^{\Delta_\beta} |x_{23}|^{\Delta_\beta} + 1}
\]

are equal to the 2-point normalization constants \( C_{O \beta} \).

Combining all pieces together we obtain the first few terms in the free field OPE of the CPOs as

\[
O_j \beta x_1 \beta O_j \beta x_2 \beta = \frac{\delta_{j1} l_j}{x_{12}^2} + \frac{2^{3/2}}{N} C_{\beta}^{\mu \nu} O_j \beta x_1 \beta \frac{1}{x_{12}^2} [O_j \beta] + \frac{2}{3^{1/2} N} \delta_{j1} l_j \frac{1}{x_{12}^2} [K]
\]

\[
+ \frac{2^{3/2} \pi^2 x_{12}^{\mu} x_{12}^{\mu}}{\lambda N} C_{\beta}^{\mu \nu} [J^{\beta} \beta J^{\beta} \beta J^{\beta} \beta] - \frac{2^{3/2} \pi^2 x_{12}^{\mu} x_{12}^{\mu}}{3 \lambda N} C_{\beta}^{\mu \nu} x_{12}^{\mu} x_{12}^{\mu} [T^{\mu \nu}] + \frac{4 \pi^2}{\lambda N} x_{12}^{\mu} x_{12}^{\mu} C_{\beta}^{\mu \nu} [T^{\mu \nu}]
\]

\[
+ \delta_{j1} l_j [O_1 \beta] + C_{\beta}^{\mu \nu} [O_\beta \beta] + C_{\beta}^{\mu \nu} [O_\beta \beta] + C_{\beta}^{\mu \nu} [O_\beta \beta] + \ldots
\]  

(3.9)

Here \( T^{\mu \nu} \) is the stress tensor of the free field theory of six scalar fields, while the normalized R-symmetry current \( J^{\mu \nu} \beta \) is defined as follows

\[
J^{\mu \nu} \beta = C_{i j}^{\beta} \frac{1}{2} \text{tr} \left( : \partial_\mu \phi^i \partial_\nu \phi^j : - : \phi^i \partial_\mu \phi^j : \right),
\]

where the antisymmetric tensors \( C_{ij}^{\beta} \) form a basis of the 15 of \( SO(6) \) and satisfy the orthogonality condition \( C_{ij}^{\beta} C_{ij}^{\beta} = \delta_{\beta \beta} \). The R-symmetry current has the following 2-point function

\[
\langle J^{\mu \nu} \beta x_1 \beta J^{\mu \nu} \beta x_2 \beta \rangle = \frac{\lambda^2}{8 \pi^2} \delta^{\mu \nu} \frac{1}{x_{12}^{\mu \nu}}.
\]

We would like to stress that in addition to the above fields the OPE contains infinite towers of both single-trace as well as double-trace operators.

### 3.2 Strong coupling OPE

As was explained in the previous section, the strong coupling OPE of CPOs is easily determined from the cubic terms in the scalars \( \lambda^l \) dual to the lowest weight CPOs in the type IIB supergravity action. There are three different cubic vertices in the action describing the cubic couplings among the three scalars \( \lambda^l \), the interaction of the scalars
with the graviton and the interaction with the $SO(6)$ vector fields. Thus, according to the discussion in the previous section the strong coupling OPE has the form

$$O^1(x_1)O^2(x_2) = \frac{\delta^{l_1 l_2}}{x_{12}^{12}} + \frac{2^{3/2}}{N} C^{l_1 l_2 1}_{12} \frac{1}{x_{12}^2} [O^1] + \frac{2^{7/2} \pi^2}{3 \lambda N} x_{12}^{\mu \nu} C^{l_1 l_2}_{\Delta_5} [R_{\mu \nu}]$$

$$- \frac{2 \pi^2}{15 \lambda N} \delta^{l_1 l_2} x_{12}^{\mu \nu} \left[ T_{\mu \nu} \right] + \delta^{l_1 l_2} x_{12}^{\Delta_1} \left[ O_1 \right]$$

$$+ C^{l_1 l_2 \Delta_2}_{\Delta_2} x_{12}^{\Delta_2} \left[ O_{\Delta_2} \right] + C^{l_1 l_2 \Delta_3}_{\Delta_3} x_{12}^{\Delta_3} \left[ O_{\Delta_3} \right] + C^{l_1 l_2 \Delta_4}_{\Delta_4} x_{12}^{\Delta_4} \left[ O_{\Delta_4} \right]$$

$$+ C^{l_1 l_2 \Delta_5}_{\Delta_5} x_{12}^{\mu \nu} \left[ O_{\Delta_5} \right] + C^{l_1 l_2 \Delta_6}_{\Delta_6} x_{12}^{\mu \nu} \left[ O_{\Delta_6} \right] + \ldots \quad (3.10)$$

Here $R_{\mu \nu}$ is the $R$-symmetry current and $T_{\mu \nu}$ is the stress tensor of $N = 4$ SYM. The structure constants of the operators $O^1$, $R_{\mu \nu}$, $T_{\mu \nu}$ are found by requiring that the above OPE reproduces the known 3-point functions of two CPOs with another CPO, the $R$-symmetry current and the stress tensor respectively, as the latter were computed in the supergravity approximation in [9, 10]. The operator algebra structure constants of the double-trace operators in (3.10) are chosen to be 1, which means that their 2- and 3-point normalization constants are kept equal. The anomalous dimensions $\Delta_1, \Delta_2, \ldots, \Delta_{15}$ of the double-trace operators will be determined in the next section by studying the 4-point functions of the CPOs.

Comparing (3.10) with (3.9), we see that the structure of the strong coupling OPE is simpler than the corresponding free field theory one. Instead of having an infinite number of single-trace operators as in (3.9), we find in (3.10) only three single-trace operators giving rise to the most singular terms. The coefficients in front of the $R$-symmetry current and the stress tensor are, however, different from the ones in (3.9). The reason is that the free field operators $J_{\mu}^{\Delta_5}$ and $T_{\mu \nu}$ receiving contribution only from bosons may be represented as

$$J_{\mu}^{\Delta_5} = \frac{1}{3} R_{\mu}^{\Delta_5} + \frac{2}{3} K_{\mu}^{\Delta_5}, \quad T_{\mu \nu} = \frac{1}{2} T_{\mu \nu} + \frac{10}{35} \kappa_{\mu \nu} + \frac{18}{35} \Xi_{\mu \nu}, \quad (3.11)$$

where $K_{\mu}$ and $K_{\mu \nu}$ are vector and tensor operators from the Konishi supermultiplet which has as leading component that scalar $K$, while $\Xi_{\mu \nu}$ is the leading component of a new supersymmetry multiplet. The splitting (3.11) is explained by the fact that $T_{\mu \nu}$, $K_{\mu \nu}$ and $\Xi_{\mu \nu}$ have pairwise vanishing two-point functions [38, 39] and belong to different supersymmetry multiplets. The operators in the Konishi supermultiplet as well as $\Xi_{\mu \nu}$ are dual to string modes and therefore decouple in the strong coupling limit.

A splitting analogous to (3.11) may also occur for the free field theory double-trace operators. However, there is an important difference. If we assume that all operators
have free field theory 2-point normalization constants of order 1, then the splitting has the following schematic form

\[ O^{fr} = O^{gr} + \frac{1}{N} O^{str}, \]

where a free field theory double-trace operator \( O^{fr} \) is split into a sum of operators \( O^{gr} \) dual to supergravity multi-particle states, and operators \( O^{str} \) dual to string states. As follows from the discussion in the previous section the coefficient in front of \( O^{str} \) has to be of order \( 1/N \), because otherwise one would not reproduce the disconnected part of the 4-point function. Such a splitting manifests itself in the \( 1/N^2 \) corrections to 2- and 3-point normalization constants of double-trace operators. In what follows we will be mostly interested in double-trace operators with free-field dimensions 4 and 5. We will see that such a splitting does occur for all the operators except the operators in the 20 and 105 irreps.

4 Anomalous dimensions of double-trace operators

In this section we determine the anomalous dimensions of double-trace operators and the leading \( 1/N^2 \) corrections to their 2- and 3-point function normalization constants \( C_D (N) \). To this end, we study the asymptotic behavior of the 4-point functions of CPOs in the direct channel \( x_{12}^2, x_{34}^2 \rightarrow 0 \). Since we know all the 4-point functions, we do not need to consider the crossed channels. It is well-known that a conformally-invariant 4-point function is given as a general analytic function of two variables, which are here conveniently chosen to be the “biharmonic ratios”

\[ u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}. \]

We also use in the following the variable \( Y = 1 - \frac{1}{u} \). The biharmonic ratios above and the variable \( Y \) have the property that \( u, v, Y \rightarrow 0 \) as \( x_{12}^2, x_{34}^2 \rightarrow 0 \).

To perform the computation we need to know the contributions of various quasi-primary operators and their descendents in the 4-point functions of CPOs, i.e. the conformal partial wave amplitudes of quasi-primary operators. We restrict ourselves mainly to the contributions of scalar, vector and second rank symmetric traceless tensor operators. Let the OPE of CPOs be of the form

\[ O^{l_1} (x_1) O^{l_2} (x_2) = C^{l_1 l_2}_{\mathcal{J}} \left( \frac{C_{OS}}{C_S} \frac{1}{x_{12}^{4 - \Delta_S}} [S^{\mathcal{J}}] + \frac{C_{OT}}{C_T} \frac{x_{12}^{\mu} x_{12}^{\nu}}{x_{12}^{6 - \Delta_T}} [T_{\mu \nu}^{\mathcal{J}}] + \frac{C_{OV}}{C_V} \frac{x_{12}^{\mu}}{x_{12}^{6 - \Delta_V}} [V_{\mu}^{\mathcal{J}}] + \cdots \right), \quad (4.1) \]
where $\mathcal{J}$ denotes an index of an irreducible representation of the $R$-symmetry group $SO(6)$, $C_{j}^{i j_2}$ are the Clebsch-Gordon coefficients and $\Delta_S$, $\Delta_T$, $\Delta_V$ are the conformal dimensions of the scalar, tensor and vector operators respectively. For any of the operators, $C_{\mathcal{O}}$ and $C_{\gamma \delta}$ denote the normalization constant in the 2-point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle$ and the coupling constant in the three-point function $\langle \mathcal{O}^j(x_1)\mathcal{O}^j(x_2)\mathcal{O}(x_3) \rangle$, respectively. Then, one can show that the short-distance expansion of the conformal partial amplitudes of the scalar $S$, tensor $T$ and vector $V$ operators can be written as [31]

$$\langle O^l_i (x_1)O^l_j (x_2)O^l_k (x_3)O^l_h (x_4) \rangle = \frac{C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2}}{x_{1 2} x_{3 4}} \times \left[ C_{\mathcal{O}}^{2} v_{4 Y}^{\Delta_S} \left( 1 + \frac{\Delta_S}{4} v + \frac{\Delta_S^2}{16(\Delta_S - 1)(\Delta_S + 1)} v^2 \right) + C_{\mathcal{T}}^{2} v_{4 Y}^{\Delta_T} \left( \frac{1}{4} v Y^2 - \frac{1}{\Delta_T} v - \frac{1}{\Delta_T} v Y \right) \right] + C_{\mathcal{V}}^{2} v_{4 Y}^{\Delta_V} \left( \frac{1}{2} Y + \cdots \right). \quad \text{(4.2)}$$

The formulas for the leading contributions of a rank-2 traceless symmetric tensor and a vector can be generalized to the case of a rank-$l$ traceless symmetric tensor of dimension $\Delta_l$ and one gets a leading term of the form

$$v^{\frac{\Delta_l - 1}{2}} Y^l.$$

For this reason a term of the form $v^{\Delta/2} F(Y)$ in a 4-point function contains, in principle, the contributions not only from a scalar operator, but also from any symmetric tensor operator of rank $l$ and conformal dimension $\Delta + l$. Moreover, (4.2) shows that the anomalous dimensions are related to terms of the type $v^{\Delta_l - 1}/Y$ for scalar operators, $v^{\Delta_l (0)} - Y \log v$ for vector operators and $v^{\Delta_l (0) - 2} - Y^2 \log v$ for rank-2 tensor operators.

The 4-point functions of CPOs were computed in the supergravity approximation in [28] and can be written as follows:

$$\langle O^l_i (x_1)O^l_j (x_2)O^l_k (x_3)O^l_h (x_4) \rangle = \frac{\delta_{j}^{i l_2} \delta_{k}^{l_2 l_4}}{x_{1 2} x_{3 4}} + \frac{\delta_{j}^{i l_2} \delta_{l_4}^{l_2 l_4}}{x_{1 3} x_{2 4}} + \frac{\delta_{j}^{i l_2} \delta_{l_4}^{l_2 l_4}}{x_{1 4} x_{2 3}}$$

$$+ \frac{8}{N^2 \pi^2} \left[ \frac{C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2}}{x_{1 2} x_{3 4}} \left( 2(x_{1 3} x_{2 4} - x_{1 4} x_{2 3}) D_{2222} \right. \right.$$

$$\left. - x_{2 4} x_{2 1 2} + x_{1 4} D_{2121} + x_{1 4} D_{2121} + x_{2 3} D_{1221} \right)$$

$$+ \delta_{j}^{i l_2} \delta_{l_4}^{l_2 l_4} \left( \frac{1}{2 x_{1 2} x_{3 4}} D_{2221} + \frac{(x_{1 3} x_{2 4}^2 + x_{1 4} x_{2 3}^2 - x_{1 4} x_{2 3}^2) x_{3 4} - 3 D_{2222}}{x_{3 4}} \right)$$

$$+ 2 C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2} C_{j}^{i j_2} \left( \frac{1}{x_{1 2} x_{3 4}} D_{2221} + 4 x_{3 4} D_{2223} - 3 D_{2222} \right) + t + u \right], \quad \text{(4.3)}$$

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where \( C_{l_1l_2l_3l_4}^\pm = \frac{1}{2} (C_{l_1l_2l_3l_4} \pm C_{l_2l_1l_3l_4}) \) and \( t \) and \( u \) stand for the contributions of the \( t \)- and \( u \)-channels obtained by the interchange \( 1 \leftrightarrow 4 \) and \( 1 \leftrightarrow 3 \), respectively. The

**D-functions are defined as**

\[
D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \int \frac{d^{d+1}x}{[x_0^2 + (x - x_1)^2]^{\Delta_1} [x_0^2 + (x - x_2)^2]^{\Delta_2} [x_0^2 + (x - x_3)^2]^{\Delta_3} [x_0^2 + (x - x_4)^2]^{\Delta_4}}.
\]

It is convenient to represent \( D \)-functions in the form

\[
D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \frac{\bar{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y)}{(x_1^2)^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}(x_2^2)^{\Delta_1+\Delta_3-\Delta_2-\Delta_4}(x_3^2)^{\Delta_1+\Delta_4-\Delta_2-\Delta_3}(x_4^2)^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}}.
\]

As shown in Appendix C, a \( \bar{D} \)-function is given by a convergent series in \( v \) and \( Y \). In terms of the biharmonic ratios \( u \) and \( v \) the 4-point function acquires the form

\[
\langle O^b(x_1) O^b(x_2) O^b(x_3) O^b(x_4) \rangle = \frac{1}{x_1^4 x_2^4 x_3^4 x_4^4} \left[ \delta^{l_1l_2} \delta^{l_3l_4} + u^2 \delta^{l_1l_4} \delta^{l_3l_2} + v^2 \delta^{l_1l_3} \delta^{l_2l_4} \right] + \frac{8}{\pi^2 N^2} \left( C_{h, l_1l_2l_3l_4} \left[ \bar{D}_{2222} \left( 2v - 2v^2 u - vu - v^2 u - v^3 - u + v^3 \right) \right.ight.
\]
\[
\left. + \bar{D}_{1212} \left( \frac{2v^2}{u} - v^2 - \frac{v^3}{u} \right) \right) + \bar{D}_{2112} \left( -2v - v^2 + v^3 \right) + \bar{D}_{2211} \left( v^2 - v^2 + v^3 \right) + \bar{D}_{3232} \left( -\frac{4v^3}{u} \right) + 4v^2 \bar{D}_{3332} \right]
\]
\[
+ \left( C_{h, l_1l_2l_3l_4} \left[ \bar{D}_{2222} \left( -12v^2 - vu - \frac{v^3}{u} + v^2 u + v^3 \right) \right.ight.
\]
\[
\left. + \bar{D}_{1212} \left( v^2 + \frac{v^3}{u} \right) + \bar{D}_{2112} \left( vu + v^2 \right) \right) + \bar{D}_{2211} \left( 2v - vu - v^2 \right) + 8v^2 \bar{D}_{3232} + \frac{4v^3}{u} \bar{D}_{2323} + 4v^2 \bar{D}_{3323} \right]
\]
\[
+ \left( C_{h, l_1l_2l_3l_4} \left[ \bar{D}_{2222} \left( -6v^2 + vu + \frac{v^3}{u} - v^2 u - v^3 \right) \right.ight.
\]
\[
\left. + \bar{D}_{1212} \left( v^2 - \frac{v^3}{u} \right) + \bar{D}_{2112} \left( -vu + v^2 \right) \right) + \bar{D}_{2211} \left( vu + v^2 \right) + \frac{4v^3}{u} \bar{D}_{2323} + 4v^2 \bar{D}_{3323} \right]
\]
\[
+ \delta^{l_1l_2} \delta^{l_3l_4} \left[ \left( \frac{3}{2} v^2 \bar{D}_{2222} - \frac{1}{2} v \bar{D}_{2211} + \bar{D}_{3223} \left( v + \frac{v^3}{u} - v^2 \right) \right) \right]
\]
\[
+ \delta^{l_1l_3} \delta^{l_2l_4} \left[ \left( \frac{3}{2} v^2 \bar{D}_{2222} - \frac{1}{2} v^2 \bar{D}_{1212} + \bar{D}_{3223} \left( v^2 - \frac{v^3}{u} + v^3 \right) \right) \right]
\]
\[
+ \delta^{l_1l_4} \delta^{l_2l_3} \left[ \left( \frac{3}{2} v^2 \bar{D}_{2222} - \frac{1}{2} v^2 \bar{D}_{2112} + \bar{D}_{3223} \left( -v^2 + \frac{v^3}{u} + v^3 \right) \right) \right].
\]
This 4-point function is given as a sum of contributions from quasi-primary operators transforming in the six irreducible representations (3,6) of SO(6). It is clear that to obtain a contribution of operators belonging to a $D$-dimensional irrep one should multiply the 4-point function by a SO(6) tensor $\mathcal{C}_{fD}^{l_1 l_2 l_3 l_4}$ which is a projector onto the irrep.

In what follows it will be sometimes useful to compare the short-distance expansion of the 4-point function \((4.5)\) with the one of the free field 4-point function \((3.2)\), which in terms of the biharmonic ratios takes the form

$$
\langle O^{l_1} (x_1) O^{l_2} (x_2) O^{l_3} (x_3) O^{l_4} (x_4) \rangle_{f_{r}} = \frac{1}{x_{12}^{4} x_{34}^{4}} \left[ \delta^{l_1 l_2} \delta^{l_3 l_4} + u^{2} \delta^{l_1 l_3} \delta^{l_2 l_4} + v^{2} \delta^{l_1 l_4} \delta^{l_2 l_3} + \frac{4}{N^{2}} \left( (u + v) C_{l_1 l_2 l_3 l_4}^{+} + (v - u) C_{l_1 l_2 l_3 l_4}^{-} + uv C_{l_1 l_2 l_3 l_4} \right) \right]. \tag{4.6}
$$

### 4.1 Projection on the singlet

First we project the 4-point function on the singlet part that amounts to applying to it \(\frac{1}{460} \delta^{l_1 l_2} \delta^{l_3 l_4}\). From the strong coupling OPE (3.10) we expect to find the stress tensor contribution and a contribution of the double-trace scalar operator $O_1$ of approximate dimension 4.

The result for the connected part is

$$
\langle O^{l_1} (x_1) O^{l_2} (x_2) O^{l_3} (x_3) O^{l_4} (x_4) \rangle_{1} = \frac{8}{20 \pi^{2} N^{2} x_{12}^{4} x_{34}^{4}} \left[ \delta^{l_1 l_2} \delta^{l_3 l_4} \right]
$$

\[\begin{align*}
\bar{D}_{2222} & \left( -9 v^{2} - \frac{3 v^{3}}{u} - 3 v u + 3 v^{2} u + 3 v^{3} \right) \\
+ \bar{D}_{1212} \left( \frac{19}{6} v^{2} + 3 v^{3} \right) & + \bar{D}_{2112} \left( 3 v u + \frac{19}{6} v^{2} \right) \\
+ D_{2211} \left( -\frac{10}{3} v - 3 v u - 3 v^{2} \right) & + D_{3222} \left( 20 \frac{v^{2}}{u} + 20 v + \frac{20}{3} v^{2} \right) \\
+ D_{2323} \left( \frac{41}{3} v^{2} + \frac{v^{3}}{u} + v^{3} \right) & + D_{3223} \left( \frac{41}{3} v^{2} + \frac{v^{3}}{u} + v^{3} \right)
\end{align*}\]

Using the formulas for the $\bar{D}$-functions from the Appendix C, we can find that the most singular terms of the $v$-expansion are

$$
\langle O^{l_1} (x_1) O^{l_2} (x_2) O^{l_3} (x_3) O^{l_4} (x_4) \rangle_{1} = \frac{\delta^{l_1 l_2} \delta^{l_3 l_4}}{N^{2} x_{12}^{4} x_{34}^{4}} \left[ v F_{1}(Y) + v^{2} F_{2}(Y) + v^{2} \log v G_{2}(Y) \right]. \tag{4.7}
$$

where

$$
F_{1}(Y) = \frac{4 Y^{2} - 8 Y}{Y^{3}} + \frac{4(-6 + 6 Y - Y^{2}) \log(1 - Y)}{3 Y^{3}},
$$

$$
F_{2}(Y) = \frac{4 Y^{2} - 8 Y}{Y^{3}} + \frac{4(-6 + 6 Y - Y^{2}) \log(1 - Y)}{3 Y^{3}}.
$$

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\[ F_2(Y) = \frac{-1680 + 3360Y - 2108Y^2 + 428Y^3 - 21Y^4}{15(1 - Y)Y^4}, \]
\[ - \frac{4(1140 - 1890Y + 962Y^2 - 151Y^3 + 5Y^4)}{15Y^5} \log(1 - Y) \]
\[ + \frac{16(Y - 2)(6 - 6Y + Y^2)}{Y^5} \text{Li}_2(Y), \]
\[ G_2(Y) = \frac{4(6 - 6Y + Y^2)}{3Y^4} \left( \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6(Y - 2) \log(1 - Y)}{Y} \right). \]

Expanding the functions in powers of \( Y \) we then obtain
\[
\langle O^{l_1}(x_1) O^{l_2}(x_2) O^{l_3}(x_3) O^{l_4}(x_4) \rangle \bigg|_1 = \frac{1}{N^2} \frac{\delta^{l_1 l_2} \delta^{l_3 l_4}}{x_{12} x_{34}^4} \left[ \frac{2}{45} vY^2 + v^2 \left( \frac{47}{225} - \frac{4}{5} \log v \right) - \frac{43}{225} v^2 Y \right].
\] (4.8)

Comparing this asymptotics with (4.2), we see that the contribution from a scalar field of dimension 2 is absent, as it should be, since the Konishi field acquires large anomalous dimension and decouples in the strong coupling limit. We also get the relation:
\[
\frac{C_{\alpha OT}^2}{4C_T} = \frac{2}{45N^2}.
\]
Since for \( C_{\alpha OT} \) one has \( C_{\alpha OT} = \frac{4}{3\pi^2} \lambda^6 \) one finds at strong coupling
\[
C_T = \frac{10\lambda^2}{\pi^4},
\]
which represents the normalization of the complete stress tensor of the \( \mathcal{N} = 4 \) SYMc [43].

As it was discussed above, a term of the form \( vF(Y) \) contains, in general, contributions from all traceless symmetric tensor operators of rank \( l \) and dimension \( 2 + l \). However, comparing \( F_1(Y) \) in (4.7) with the corresponding term in the conformal partial wave amplitude of the stress tensor (7.15) we see that they coincide. Thus, the strong coupling OPE does not contain single-trace rank-\( l \) traceless symmetric tensors with dimension \( 2 + l \) in its singlet part. Nevertheless, it may in principle contain tensors of dimension \( 4 + l \) or higher. However, as it was shown in section 3 a possible single-trace scalar operator of dimension 4 vanishes. Thus the only scalar operator of approximate dimension 4 is the double-trace operator \( O_4.7 \)

6This value of the coupling constant is fixed by a conformal Ward identity [40], the same value was also obtained in the supergravity approximation in [10].

7The free field theory operator \( O_4^{tr} \) probably splits into a linear combination of \( O_4 \) and an operator \( O_4^{tr} \) dual to a string mode. However, the coefficient in front of \( O_4^{tr} \) is of order \( 1/N \), and even if the
The formula (4.8) also allows us to determine the anomalous dimension of $O_1$. Assuming the existence at strong coupling of a scalar field with dimension $\Delta = \Delta^{(0)} + \Delta^{(1)}$, where $\Delta^{(0)} = 4$ and $\Delta^{(1)}$ is the anomalous dimension, we find that

$$v^{\frac{\Delta}{2}} = v^2 + \frac{1}{2} \Delta^{(1)} v^2 \log v + \ldots$$

Since there is only one operator of approximate dimension 4, we do not face the problem of operator mixing and from (4.2) we get

$$\frac{1}{2} C_{O_1}^2 \Delta^{(1)} = \frac{4}{5N^2}.$$ 

Since $\Delta^{(1)}$ is of order $1/N^2$ we use for $\frac{C_{O_1}^2}{c_{O_1}}$ the $O(1)$ result which is $1/10$. In this way we obtain

$$\Delta^{(1)} = -\frac{16}{N^2}, \quad (4.9)$$

for the anomalous dimension of $O_1$. This coincides with the anomalous dimension of the operator $B$ considered in [23], as it should be, since $B$ is a descendent operator of $O_1$.

We can also find the leading $1/N^2$ correction to the 2- and 3-point normalization constant $C_{O_1}$. Writing as

$$C_{O_1} = \frac{1}{10} \left( 1 + \frac{1}{N^2} C_{O_1}^{(1)} \right),$$

and taking into account that $C_{O_1} = C_{O_0 O_1}$, we find from the term of order $v^2$

$$C_{O_1}^{(1)} = \frac{38}{15}.$$ 

Finally, we can make a consistency check of our computation. Namely, since we know corrections to the conformal dimension, $\Delta^{(1)} = -16/N^2$ and to the structure constant we can compute the term of order $v^2 Y$ by using (4.2), in order to compare it with the corresponding value obtained from our 4-point function. Taking into account the contribution of the stress tensor we get from (4.2) and from the expansion of our 4-point function the same number $-\frac{43}{225}$. This also confirms that there is only one operator of approximate dimension 4 in the strong coupling OPE, and that the operator $O_1^{(tr)}$ decouples in the strong coupling limit.

latter operator does not decouple in the strong coupling limit it cannot contribute to log-dependent terms in 4-point functions. In the following, when discussing double-trace operators in other irreps we simply assume that operators such as $O_1^{(tr)}$ above do decouple, making at the same time a consistency check to confirm our assumption.

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We can also compute the 2-point normalization constant in free field theory by using (3.1) and the definition of the operator. A simple calculation gives

\[ C_{O_1} = \frac{1}{10} \left( 1 + \frac{2}{3N^2} \right). \]

Thus, not only the conformal dimension but also the 2- and 3-point normalization constants get \( \frac{1}{N^2} \) corrections in the strong coupling limit.

### 4.2 Projection on 20

According to (4.2), to obtain the contribution of the operators transforming in a D-dimensional irrep, we should multiply the 4-point function by the projector onto the representation

\[
(P_D)_{l_1 l_2 l_3 l_4} = \frac{1}{\nu_D} C_{J_D}^{l_1 l_2} C_{J_D}^{l_3 l_4},
\]

(4.10)

where

\[
\nu_D = \sum_{l_i} C_{J_D}^{l_1 l_2} C_{J_D}^{l_3 l_4} C_{J_D}^{l_1 l_2} C_{J_D}^{l_3 l_4},
\]

is the dimension of the irrep so that \( P_D^2 = \frac{1}{\nu_D} \).

The projector on the 20 can be easily found by taking into account that the Clebsch-Gordon coefficient \( C_{\frac{D}{50}}^{l_1 l_2} \) is proportional to the \( SO(6) \) tensor \( C_{l_1 l_2 l_3 l_4} \). Then, one can show that

\[
(P_{20})_{l_1 l_2 l_3 l_4} = \frac{3}{100} \left( C_{l_1 l_2 l_3 l_4}^+ - \frac{1}{6} \delta_{l_1 l_2} \delta_{l_3 l_4} \right). \]

(4.11)

Using the Table 1 from Appendix D for the contractions of the projector with the \( SO(6) \) tensors appearing in the 4-point function, we find the contribution of the operators in the 20 to the connected part of the 4-point function

\[
\langle O^l (x_1) O^l (x_2) O^l (x_3) O^l (x_4) \rangle_{20} = \frac{8}{\pi^2 N^2} \frac{C_{J_{50}}^{l_1 l_2} C_{J_{50}}^{l_3 l_4}}{x_{12}^4 x_{34}^4} \left[ \right. \]

\[ \bar{D}_{2222} \left( -18v^2 - \frac{3}{2} vu - \frac{3v^3}{2u} + \frac{3}{2} v^2 u + \frac{3}{2} v^3 \right) \]

\[ + \bar{D}_{1212} \left( \frac{4}{3} v^2 + \frac{3v^3}{2u} \right) + D_{2112} \left( \frac{3}{2} vu + \frac{4}{3} v^2 \right) + D_{2211} \left( \frac{10}{3} v - \frac{3}{2} vu - \frac{3}{2} v^2 \right) \]

\[ + \frac{40}{3} v^2 \bar{D}_{3322} + \bar{D}_{2323} \left( v^2 + \frac{19v^3}{3u} + v^3 \right) + \bar{D}_{3323} \left( \frac{19}{3} v^2 + \frac{v^3}{u} + v^3 \right) \].

\]

(4.12)
Expanding the $\tilde{D}$-functions in powers of $v$, we obtain

$$\langle O^h(x_1)O^{I_2}(x_2)O^{I_3}(x_3)O^{I_4}(x_4) \rangle_{20} = \frac{1}{N^2} \frac{C_1^{h}C_2^{I_2}C_3^{I_3}C_4^{I_4}}{x_{12}^{1}\cdot x_{34}^{4}} \left[ \right.$$

$$vF_1(Y) + v^2 F_2(Y) + v^2 \log v \, G_2(Y) \left. \right], \quad (4.13)$$

where

$$F_1(Y) = -\frac{40 \log(1 - Y)}{3Y},$$
$$F_2(Y) = -\frac{8(65 - 65Y + 6Y^2)}{3(1 - Y)Y^2} - \frac{20(74 - 49Y + 2Y^2)}{3Y^3} \log(1 - Y)$$
$$+ \frac{160(Y - 2)}{Y^3} \text{Li}_2(Y),$$
$$G_2(Y) = \frac{40}{3Y^2} \left( \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6(Y - 2) \log(1 - Y)}{Y} \right).$$

Expanding the above functions in powers of $Y$ we finally obtain

$$\langle O^h(x_1)O^{I_2}(x_2)O^{I_3}(x_3)O^{I_4}(x_4) \rangle_{20} = \frac{1}{N^2} \frac{C_1^{h}C_2^{I_2}C_3^{I_3}C_4^{I_4}}{x_{12}^{1}\cdot x_{34}^{4}} \left[ \right.$$

$$\frac{40}{3} v + \frac{26}{9} v^2 (1 + Y)$$
$$- \frac{4}{3} v^2 Y^2 \log v \left. \right]. \quad (4.14)$$

The analysis of the results obtained follows the one in the previous subsection. Firstly, comparing $F_1(Y)$ in (4.13) with the corresponding term of the conformal partial amplitude of a scalar operator of dimension 2 (7.2), we see that they coincide. Therefore, all single-trace rank-$l$ traceless tensors of dimension $2 + l$ transforming in the $20$ are absent in the OPE. Then, the only scalar operator of approximate dimension 4 is the double-trace operator $O_{20}$. Moreover, we see that log $v$-dependent terms appear starting from the term $v^2 Y^2 \log v$. Thus we conclude from (4.2) that the double-trace operator $O_{20}$ has protected conformal dimension. It is worth noting that the non-renormalization of the conformal dimension of this operator is not related to the shortening condition discussed in [32] and is a prediction of the AdS/CFT correspondence. The first operators which acquire anomalous dimensions are scalar and tensor operators of approximate dimension 6.

The first $1/N^2$ correction to the 2- and 3-point normalization constant $C_{O_{20}}$ can also be easily found. Writing the constant as

$$C_{O_{20}} = 2 \left( 1 + \frac{1}{N^2} C_{O_{20}} \right),$$

\footnote{Recall that for the lowest weight CPOs one has $\frac{C_{O_{20}}^2}{C_{O_{20}}} = \frac{48}{3}$.}
and taking into account the contribution of the single-trace operator $O^I$ and that $C_{O_{20}} = C_{O_{20}O_{20}}$, we find from the term of order $v^2$

$$C_{O_{20}}^{(1)} = \frac{1}{3}.$$  

The 2-point normalization constant can be also computed in free field theory by using (3.1) and the definition of the operator (3.8) and appears to coincide with the value obtained in the strong coupling limit

$$C_{O_{20}} = 2 \left(1 + \frac{1}{3N^2}\right).$$

Thus, both the conformal dimension and the 2-point function normalization constant of the double-trace operator in the 20 are non-renormalized in the strong coupling limit. This also shows that in this case there is no splitting, and the free field theory double-trace operator coincides with $O_{20}$.

### 4.3 Projection on 105

The free field theory OPE (3.9) and the strong coupling OPE (3.10) do not contain single-trace operators transforming in the 105 irrep. Thus, only double-trace operators contribute to this part of the 4-point function. The corresponding connected contribution can be easily found using the Table 1 from Appendix D and is given by

$$\langle O^I(x_1)O^{I_2}(x_2)O^{I_3}(x_3)O^{I_4}(x_4)\rangle_{105} = \frac{8}{\pi^2 N^2} \frac{C_{J_{105}J_{105}}^{I_1I_2} C_{J_{105}J_{105}}^{I_3I_4}}{x_{12}^4 x_{34}^4} \left[\right.$$  

\begin{align*}
&\tilde{D}_{2222} \left(-3v^2 + vu + \frac{v^3}{u} - v^2 u - v^3\right) \\
&+ \tilde{D}_{1212} \left(\frac{1}{2}v^2 - \frac{v^3}{u}\right) + \tilde{D}_{2112} \left(-vu + \frac{v^2}{2}\right) + \tilde{D}_{2211} \left(vu + v^2\right) \\
&+ \tilde{D}_{2323} \left(3v^3 + v^3 + v^2\right) + \tilde{D}_{3223} \left(3v^2 + v^3 + \frac{v^3}{3}\right). \quad (4.15)
\end{align*}

Expanding the $\tilde{D}$-functions in powers of $v$, we obtain

$$\langle O^I(x_1)O^{I_2}(x_2)O^{I_3}(x_3)O^{I_4}(x_4)\rangle_{105} = \frac{1}{N^2} \frac{C_{J_{105}J_{105}}^{I_1I_2} C_{J_{105}J_{105}}^{I_3I_4}}{x_{12}^4 x_{34}^4} \left[\right.$$  

\begin{align*}
v^2 F_2(Y) + v^3 F_3(Y) \\
+ v^4 F_4(Y) + v^4 \log v G_4(Y) \right], \quad (4.16)
\end{align*}

where

$$F_2(Y) = \frac{4}{1 - Y},$$

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\[ F_3(Y) = \frac{4(Y - 2)}{(1 - Y)^{Y^2}} - \frac{8}{Y^5} \log(1 - Y), \]

\[ F_4(Y) = \frac{4(28 - 28Y + 3Y^2)}{(1 - Y)^{Y^2}} - \frac{8(38 - 25Y + Y^2)}{Y^5} \log(1 - Y) \]

\[ + \frac{96(Y - 2)}{Y^5} \operatorname{Li}_2(Y), \]

\[ G_4(Y) = \frac{8}{Y^4} \left( \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6(Y - 2) \log(1 - Y)}{Y} \right). \]

Since only double-trace operators contribute, it is useful to compare (4.16) with the corresponding part of the free field theory 4-point function (4.6)

\[ \langle O^{h_1}(x_1)O^{h_2}(x_2)O^{h_3}(x_3)O^{l_4}(x_4) \rangle_{fr} \bigg|_{105} = \frac{1}{N^2} C_{j_{105}}^{l_1} C_{j_{105}}^{l_2} C_{j_{105}}^{l_3} C_{j_{105}}^{l_4} \left[ v^2 \left( 1 + \frac{1}{(1 - Y)^2} \right) + \frac{v^2}{N^2} \frac{4}{Y} \right]. \]

(4.17)

The first term on the r.h.s. of this equation shows the disconnected part of the free field theory 4-point function. Comparing the term of order 1/N^2 in (4.17) with the term v^2F_2(Y) in (4.16), we see that they coincide. This means that the conformal dimensions and the leading corrections in 1/N^2 to 2- and 3-point functions normalization constants of any symmetric traceless rank-2k tensor operator of dimension 4+2k transforming in the 105 coincide with the ones computed in free field theory. Thus, all these operators are non-renormalized in the strong coupling limit. The first correction to the 2- and 3-point functions normalization constant of the double-trace operator O_{105} can be easily found from (4.17) and is given by

\[ C_{O_{105}} = 2 \left( 1 + \frac{2}{N^2} \right). \]

The non-renormalization of the double-trace operator O_{105} follows from the shortening conditions derived in [32,33], and was also checked in perturbation theory at small YM coupling in [32,35,44,37].

The expansion (4.16) also shows that the first log v-term appears at order v^4. Therefore, all symmetric traceless rank-2k tensor operators of dimension 6+2k transforming in the 105 have protected conformal dimensions. Note, however, that the normalization constants of their 2- and 3-point functions certainly receive corrections at strong coupling, which are encoded in the function F_3(Y). The vanishing of anomalous dimensions of these tensor operators does not seem to follow from any known non-renormalization theorem. These results also demonstrate that the free field theory symmetric traceless rank-2k tensor operators of dimension 4+2k do not split, while the ones with dimension 6+2k do.
Since $G_4(Y) = -\frac{1}{10} + \ldots$ the first double-trace operator in the 105 which acquires anomalous dimension is the scalar operator with approximate dimension 8.

### 4.4 Projection on 84

Just as it was the case for the operators in the 105, only double-trace operators transforming in the 84 irrep can contribute to this part of the 4-point function. The corresponding connected contribution is again found by using the Table 1 from Appendix D:

$$
\left. \langle O_l^i(x_1)O_l^j(x_2)O_l^k(x_3)O_l^l(x_4) \rangle \right|_{84} = \frac{8}{\pi^2 N^2} \frac{C_{j_{84}}^{l_{12}} C_{j_{84}}^{l_{13} l_{4}}}{x_{12}^4 x_{34}^4} \left[ \hat{D}_{2222} \left( 6v^2 - u + \frac{uv}{2} + \frac{v^3}{2} \right) \\
+ \hat{D}_{1212} \left( -v^2 + \frac{v^3}{2u} \right) + \hat{D}_{2112} \left( \frac{uv}{2} - v^2 \right) - \hat{D}_{2211} \left( \frac{uv}{2} + \frac{v^3}{2} \right) \\
+ \hat{D}_{2323} \left( v^2 + v^3 - \frac{3v^3}{u} \right) + \hat{D}_{3223} \left( -3v^2 + v^3 + \frac{v^3}{u} \right) \right]. \quad (4.18)
$$

Expanding the $\hat{D}$-functions in powers of $v$, we obtain

$$
\left. \langle O_l^i(x_1)O_l^j(x_2)O_l^k(x_3)O_l^l(x_4) \rangle \right|_{84} = \frac{1}{N^2} \frac{C_{j_{84}}^{l_{12}} C_{j_{84}}^{l_{13} l_{4}}}{x_{12}^4 x_{34}^4} \left[ v^2 F_2(Y) + v^3 F_3(Y) \\
+ v^3 \log v G_3(Y) \right], \quad (4.19)
$$

where

$$
\begin{align*}
F_2(Y) & = \frac{8 (3 - 3Y + Y^2)}{(1 - Y)Y^2} + \frac{12 (Y - 2)}{Y^3} \log(1 - Y), \\
F_3(Y) & = \frac{8 (Y - 2) (21 - 21Y + 2Y^2)}{(1 - Y)Y^4} + \frac{4 (-228 + 264Y - 80Y^2 + 3Y^3)}{Y^5} \log(1 - Y) \\
& \quad - \frac{144 (Y - 2)^2}{Y^5} \text{Li}_2(Y), \\
G_3(Y) & = -\frac{12 (Y - 2)}{Y^4} \left( \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6 (Y - 2) \log(1 - Y)}{Y} \right).
\end{align*}
$$

Since the first $\log v$-term appears at order $v^3$, all symmetric traceless rank-2k tensor operators of dimension $4 + 2k$ transforming in the 84 have protected conformal dimensions. The first double-trace operator in the 84 which acquires an anomalous dimension is the scalar operator with approximate dimension 6. However, contrary to the case of the 105 irrep, the leading $1/N^2$ corrections to the normalization constants of the 2- and 3-point
functions of these operators differ from their free field theory values. To see this we compare (4.19) with the corresponding part of the free field theory 4-point function (4.6)

$$
\left\langle O^{l_1}(x_1)O^{l_2}(x_2)O^{l_3}(x_3)O^{l_4}(x_4) \right\rangle_{\text{free}} \bigg|_{84} = \frac{1}{N^2} \frac{C_{J_{15}}^{l_1} C_{J_{15}}^{l_2} C_{J_{15}}^{l_3} C_{J_{15}}^{l_4}}{x_{12} x_{34}} \left[ v^2 \left( 1 + \frac{1}{(1 - Y)^2} \right) + \frac{v^2}{N^2} \frac{2}{1 - Y} \right].
$$

(4.20)

Expanding (4.19) and (4.20) in powers of $Y$, we obtain the normalization constants of 2- and 3-point functions of the operator $O_{84}$ at strong coupling and in free field theory correspondingly as

$$
C_{O_{84}}^{\text{str}} = 2 \left( 1 - \frac{3}{N^2} \right),
$$

$$
C_{O_{84}}^{\text{free}} = 2 \left( 1 - \frac{1}{N^2} \right).
$$

The vanishing of the anomalous dimensions of the double-trace operator $O_{84}$ follows from the shortening conditions discussed in [32, 33] and was also shown in perturbation theory at small YM coupling in [44, 37]. The difference between $C_{O_{84}}^{\text{str}}$ and $C_{O_{84}}^{\text{free}}$ again may find a natural explanation in the fact that the corresponding free field theory operator undergoes a linear splitting on $O_{84}$ and $K_{84}$, where $O_{84}$ has protected both its dimension and the normalization constants of the 2- and 3-point functions, while the operator $K_{84}$ belongs to the Konishi multiplet [37] and, therefore, decouples at strong coupling.

4.5 Projection on 15

By using the projector $(P_{15})_{l_1 l_2 l_3 l_4}$ constructed in the Appendix D and the results of Table 1 we find the following contribution of the operators in 15 to the connected part of the 4-point function

$$
\left\langle O^{l_1}(x_1)O^{l_2}(x_2)O^{l_3}(x_3)O^{l_4}(x_4) \right\rangle_{15} = \frac{8}{\pi^2 N^2} \frac{C_{J_{15}}^{l_1} C_{J_{15}}^{l_2} C_{J_{15}}^{l_3} C_{J_{15}}^{l_4}}{x_{12} x_{34}} \left[ D_{2222} \left( 4v + 2uv - \frac{4v^2}{u} - 2uv^2 + 2v^3 - \frac{2v^3}{u} \right) + \tilde{D}_{1212} \left( -3v^2 + 4v^2 + 2v^3 \right) + \tilde{D}_{2112} \left( -4v - 2uv + \frac{3v^2}{2} \right) \right.
$$

$$
\left. + \tilde{D}_{2211} \left( 2vu - 2v^2 \right) + \left( -v^2 - v^3 - \frac{7v^3}{u} \right) \tilde{D}_{2323} + \left( 7v^2 + v^3 + \frac{v^3}{2u} \right) \tilde{D}_{3223} \right].
$$

Expansion of the $D$-functions in powers of $v$ produces now the following expression for leading terms

$$
\left\langle O^{l_1}(x_1)O^{l_2}(x_2)O^{l_3}(x_3)O^{l_4}(x_4) \right\rangle_{15} = \frac{1}{N^2} \frac{C_{J_{15}}^{l_1} C_{J_{15}}^{l_2} C_{J_{15}}^{l_3} C_{J_{15}}^{l_4}}{x_{12} x_{34}} \left[ v F_1(Y) + v^2 F_2(Y) \right]
$$

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Here the functions $F_1, F_2$ and $G_2$ are given by

\[ F_1(Y) = \frac{16}{Y^2} (-2Y + (Y - 2) \log(1 - Y)), \]
\[ F_2(Y) = -\frac{4(Y - 2)(56 - 56Y + 5Y^2)}{(Y - 1)Y^3} + \frac{8(-152 + 176Y - 53Y^2 + 2Y^3)}{Y^4} \log(1 - Y) - \frac{192(Y - 2)^2}{Y^4} L_2(Y), \]
\[ G_2(Y) = -\frac{16(Y - 2)}{Y^3} \left[ \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6(Y - 2) \log(1 - Y)}{Y} \right]. \]

Expansion in powers of $Y$ produces the following leading terms

\[ \langle O^b_1(x_1)O^b_2(x_2)O^b_3(x_3)O^b_4(x_4) \rangle_{15} = \frac{C_{ij5}^b C_{ij5}^b}{N^2 x_1^4 x_2^4} \left[ \frac{8}{3} vY + \frac{12}{25} v^2 Y - \frac{16}{5} v^2 Y \log v \right]. \] (4.22)

The absence of the terms $vY \log v$ shows that the vector operator of the dimension 3, which is the $R$-symmetry current $R^{\alpha}_{\mu}^{\beta_5}$, has protected conformal dimension. According to the discussion above, the function $F_1(Y)$ may receive contributions from single-trace rank $2k + 1$ traceless tensors of dimension $2k + 3$ transforming in 15, which is what indeed happens in the free field theory limit. However, comparing the function $F_1(Y)$ with the relevant part of the conformal partial amplitude of the conserved vector current of dimension 3 (7.10) one concludes that they coincide, therefore, the corresponding tensors are absent in the strong-coupling OPE. Next, comparing (4.22) with eq. (4.2) we read off the value of the ratio

\[ \frac{C_{OOR}}{2C_R} = \frac{8}{3N^2}. \]

Since the value of $C_{OOR}$ is fixed by the conformal Ward identity to be $C_{OOR} = \frac{\lambda^2}{2\pi^2 N}$ one finds

\[ C_R = \frac{3\lambda^2}{8\pi^4} \]

which corresponds to the normalization of the two-point function of the complete $R$-symmetry current of the $\mathcal{N} = 4$ SYM$_4$ [40, 41].

The function $F_2(Y)$ receives contributions both from the $R$-symmetry current and from traceless symmetric rank $2k + 1$ tensors with approximate dimension $2k + 5$. Since $R^{\alpha}_{\mu}^{\beta_5}$ is non-renormalized, the presence of the function $G_2$ shows that operators from the above tensor tower acquire anomalous dimensions. We can now find the anomalous dimension of the lowest current $O_{15}$ in this tower whose free field theory counterpart $O_{\mu}^{\beta_5}$
with conformal dimension $\Delta^{(i)} = 5$ was discussed in section 3. Comparing the coefficient in front of $v^2 Y \log v$ in (4.22) with the asymptotic (4.2) one finds

$$\frac{1}{4} C^5_{O_{O15}} \Delta^{(i)} = -\frac{16}{5N^2}.$$  

The free field result gives $\frac{C^5_{O_{O15}}}{C^5_{O15}} = 4$, therefore we find

$$\Delta^{(i)} = -\frac{16}{5N^2},$$  (4.23)

for the anomalous dimension of $O_{O15}^5$. Since the conformal partial amplitude of the conserved vector current is known one can also find the correction to the normalization constant $C_{O15}$ in the same manner as was done in the previous cases. Indeed, the $v^2 Y$ term in (4.22) is split as

$$\frac{12}{25} v^2 Y = \frac{8}{15} v^2 Y - \frac{4}{75} v^2 Y.$$  

Here the first term is a contribution of the conformal partial amplitude of the $R$-symmetry current, while the second one is related to the correction to the normalization constant $C_{O15}$. One can easily see that in the free field theory $C_{O_{O15}} = C_{O15} = 4$ is an exact result, i.e. it does not receive $1/N^2$ corrections. Thus, if we write

$$C_{O15} = 4 \left( 1 + \frac{1}{N^2} C^{(i)}_{O_{O15}} \right),$$

then from (4.2) it follows that

$$C^{(i)}_{O15} = -\frac{2}{75}.$$  

4.6 Projection on 175

Only double-trace operators transforming in the $175$ appear in the free field theory OPE (3.9) and in the strong coupling OPE (3.10). Applying the projector $(P_{175})^l_{hk} L^l_{1234}$ constructed in Appendix D to the 4-point function we find the following expression for the contribution of the operators in the $175$:

$$\langle O^{(l_1} (x_1) O^{l_2} (x_2) O^{l_3} (x_3) O^{l_4} (x_4)) \rangle_{175} = \frac{8}{\pi^2 N^2} \frac{C^h_{\bar{J}_{175}} C^i_{\bar{J}_{175}}}{x_{12}^4 x_{34}^4} \left[ \frac{v^2}{2} \bar{D}_{1212} + \frac{v^2}{2} \bar{D}_{2112} + \left( v^2 + v^3 - \frac{v^3}{u} \right) \bar{D}_{2323} + \left( v^2 - v^3 - \frac{v^3}{u} \right) \bar{D}_{3232} \right].$$

Expanding $\bar{D}$ functions in $v$ we keep the leading terms $v^2$ and $v^3$

$$\langle O^{(l_1} (x_1) O^{l_2} (x_2) O^{l_3} (x_3) O^{l_4} (x_4)) \rangle_{175} = \frac{1}{N^2} \frac{C^h_{\bar{J}_{175}} C^i_{\bar{J}_{175}}}{x_{12}^4 x_{34}^4} \left[ v^2 F_2(Y) + v^3 F_3(Y) + v^3 \log v G_3(Y) \right]$$

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with
\[
F_2(Y) = \frac{-4(Y(Y - 2) + 2(Y - 1)\log(1 - Y))}{Y^2(Y - 1)},
\]
\[
F_3(Y) = \frac{4(28 - 28Y + 3Y^2)}{Y^3(Y - 1)} - \frac{8(38 - 25Y + Y^2)\log(1 - Y)}{Y^4},
\]
\[
+ \frac{96(Y - 2)}{Y^4} \text{Li}_2(Y),
\]
\[
G_3(Y) = \frac{8}{Y^3} \left( \frac{12 - 12Y + Y^2}{Y - 1} + \frac{6(Y - 2)\log(1 - Y)}{Y} \right).
\]

The function \(F_2\) receives contributions from tensor operators of rank \(2k + 1\) with approximate dimensions \(2k + 5\). Since the term proportional to \(v^2\log v\) is absent, we conclude that these tensor operators have protected conformal dimensions. The lowest current \(O_{\mu}^{J_{175}}\) among them, with dimension 5, was discussed in section 3. Note that these operators also contribute to \(F_3\) together with operators of rank \(2k + 1\) and approximate dimensions \(2k + 7\). For the two terms of the \(Y\)-expansion one finds
\[
\langle O_{175}^{I_1}(x_1)O_{175}^{I_2}(x_2)O_{175}^{I_3}(x_3)O_{175}^{I_4}(x_4) \rangle_{175} = \frac{1}{N^2} \frac{C_{175}^{I_1 I_2} C_{175}^{I_3 I_4}}{x_{12}^2 x_{34}^2} \left[ -4 \frac{v^2 Y - 2v^2 Y^2}{3} \right].
\]

This allows us to determine the \(1/N^2\) correction to the 2- and 3-point normalization constant \(C_{175}\) of the operator \(O_{\mu}^{J_{175}}\). Taking into account that in free field theory \(C_{\mu\nu\mu\nu_{175}}^{fr} = C_{\mu\nu_{175}}^{fr} = 4\) as can be easily seen from the free field theory 4-point function (4.6), we write as
\[
C_{O_{175}} = 4 \left( 1 + \frac{1}{N^2} C_{O_{175}}^{(1)} \right).
\]

Then from the first term of order \(v^2\) in (4.24) one finds
\[
C_{O_{175}}^{(1)} = -\frac{2}{3}.
\]

Apparently, the splitting mechanism is again at work, i.e. the corresponding free field theory operator is split in two orthogonal parts carrying different representation of the supersymmetry; one has protected both its dimension and the normalization constants, while the other one is dual to a string mode and decouples at strong coupling.

5 Conclusions

We studied in detail the 4-point functions of the lowest weight CPOs and we showed that they have a structure compatible with the OPE of CPOs predicted by the AdS/CFT correspondence. We demonstrated that all power-singular terms in the 4-point functions
exactly match the corresponding terms in the conformal partial wave amplitudes of the CPOs, of the $R$-symmetry current and of the stress tensor. As these operators are dual to type IIB supergravity fields, we concluded that the operators dual to string modes, which appear in the free field theory OPE, decouple in the strong coupling limit.

We also computed the anomalous dimensions and the leading $1/N^2$ corrections to the normalization constants of the 2- and 3-point functions of the scalar double-trace operators with approximate dimension 4 and of vector operators with approximate dimension 5. The only scalar double-trace operator that acquires an anomalous dimension appears to be the operator in the singlet of the $R$-symmetry group $SO(6)$. The double-trace operator in the 20 seems to be protected, however as this does not follow from the shortening condition discussed in [32, 33] we do not have a satisfactory explanation for such a non-renormalization property.

The anomalous dimension of the singlet operator is negative, hence this operator is relevant and can be used to study non-conformal deformations of the $\mathcal{N} = 4$ SYM$_4$. All other scalar double-trace operators have protected dimension 4 and are marginal. They can be added to the Lagrangian in order to study conformal deformations. Nevertheless, it is unclear at present how dual deformations of type IIB supergravity (or string theory) can be described.

We have also found several towers of traceless symmetric double-trace operators in the 105, 84 and 175 irreps, whose anomalous conformal dimensions vanish. The rank-2k tensor operators of dimension $6+2k$ satisfy the shortening condition $A'$ of [33]. However, even if they contain the highest weight states of the $SU(2,2|4)$ superalgebra the shortening condition $A'$ does not imply non-renormalization of the corresponding multiplets. On the other hand operators from other towers are certainly not the highest weight states, and at present we are not aware if the lowest weight states of their supermultiplets satisfy the shortening condition responsible for non-renormalization.

There are two interesting facts related to the structure of the leading log-dependent terms in the 4-point functions. Namely, all the functions $G(Y)$ which appear in (4.7), (4.13) and so on, differ from each other by some simple rational factors. We expect that this is an indication that the anomalous dimensions of all double-trace operators may be related by some relatively simple formula. Then, the leading log $v$-dependent terms appear in the 4-point functions exactly at the same order of $v$ where the dilogarithm $Li_2$ appears for the first time.
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6 Appendix A. Free field OPE and conformal blocks

A quasi-primary field of the CFT appearing in the OPE together with all its derivative
descendents is known as a conformal block. If two fields \(O^\alpha\) and \(O^\beta\) transforming in some
representation of an \(R\)-symmetry group have the one and the same conformal dimension
\(\Delta\) then their OPE has the following structure

\[
O^\alpha(x)O^\beta(0) = \frac{1}{(x^2)^{\Delta - \Delta_0}} C(x, \partial) O^{\alpha\beta}(0)
\]

\[+ \frac{1}{(x^2)^{2(\Delta - \Delta_0 + 1)}} C^\mu(x, \partial) J^{\alpha\beta}_{\mu}(0)
\]

\[+ \frac{1}{(x^2)^{2(\Delta - \Delta_0 + 2)}} C^\mu_{\nu}(x, \partial) T^{\alpha\beta}_{\mu\nu}(0) + \ldots
\]

Here we identify the leading quasi-primary fields with conformal dimensions \(\Delta_0\), \(\Delta_J\) and
\(\Delta_T\) as a scalar \(O^{\alpha\beta}\), a vector current \(J^{\alpha\beta}_{\mu}\) and a symmetric traceless second rank tensor \(T^{\alpha\beta}_{\mu\nu}\)
respectively. The OPE coefficient \(C(x, \partial)\) denotes a power series in derivatives generating
the conformal block \([O^{\alpha\beta}]\) of the scalar \(O^{\alpha\beta}\). Similarly we denote the OPE coefficients for
for the other fields.

The structure of the conformal blocks is uniquely fixed by the conformal symmetry
and it may be found by requiring consistency of the OPE with 2- and 3-point functions
of the fields involved. In particularly, the conformal block of a scalar field with dimension
\(\Delta\) is given by the following differential operator [47, 31]:

\[
C(x, \partial_y) = \frac{1}{B_{(1/2\Delta_0, 1/2\Delta_0)}} \sum_{m=0}^{\infty} \frac{1}{m!(\Delta_0 - \eta + 1)_m}
\times \int_0^1 dt[t(1-t)]^{\Delta_0 - 1} \left(-\frac{1}{4}t(1-t)x^2\Delta_y\right) e^{tx\partial_y},
\]

where the Euclidean space-time dimension \(d\) enters as \(d = 2\eta, x \partial_y = x^\mu \partial_{y,\mu} \), \(\Delta_y = \partial_y^2\) and
we use the Pochhammer symbol \((a)_n = \Gamma(a + n)/\Gamma(a)\). In what follows we need to specify
explicitly the first three terms of \(C(x, \partial_y)\) in the derivative expansion:

\[
C(x, \partial_y) = 1 + \frac{1}{2} (x \partial_y) + \frac{\Delta_0 + 2}{8(\Delta_0 + 1)} (x \partial_y)^2 - \frac{\Delta_0}{16(\Delta_0 + 1)(\Delta_0 + 1 - \eta)} x^2 \Delta_y + \ldots
\]

The conformal blocks of a conserved vector current and a conserved second rank tensor
with canonical dimensions \(2\eta - 1\) and \(\eta\) respectively are also available. For a vector current
one has [30]

\[
C^\mu(x, \partial_y) = \frac{x^\mu}{B(\eta, \eta)} \sum_{m=0}^{\infty} \frac{1}{m!(\eta)_m} \int_0^1 dt[t(1-t)]^{\eta - 1} \left(-\frac{1}{4}t(1-t)x^2\Delta_y\right)^m e^{tx\partial_y} (6.4)
\]

\[= x^\mu + \frac{1}{2} x^\mu (x \partial_y) + \frac{x^\mu}{4(\eta + 1)} \left((\eta + 1)(x \partial_y)^2 - \frac{1}{2} x^2 \Delta_y\right) + \ldots
\]
and for a conserved symmetric traceless tensor one finds (see Appendix E)

\[
C_{\mu\nu}(x, \partial_y) = \frac{x_\mu x_\nu}{B(\eta + 1, \eta + 1)} \sum_{m=0}^{\infty} \frac{1}{m!(\eta + 1)_m} \int_0^1 dt[t(1 - t)]^\eta \left( -\frac{1}{4} t(1 - t)x^2 \Delta_\nu \right)^m e^{tx_\gamma} d\gamma \\
= x_\mu x_\nu + \ldots
\]

Using the above formulae, one can now consider the operator product : \( O^\alpha(x)O^\beta(0) : \) in a free field theory and find explicit expressions for \( J_{\mu}^{\alpha\beta} \) and \( T_{\mu\nu}^{\alpha\beta} \). Indeed, from the Taylor expansion one sees that the leading component is a quasi-primary field \( O^{\alpha\beta} = O^\alpha(x)O^\beta(0) : \) with conformal dimension \( \Delta_\alpha = 2\Delta \), therefore it should appear in the OPE with its whole conformal block. Subtracting from the Taylor expansion the first three terms of the conformal block of the scalar with dimension \( 2\Delta \) we find at the next level another quasi-primary operator \( O_{\mu}^{\alpha\beta} \) that turns out to be a vector current \( J_{\mu}^{\alpha\beta} = \frac{1}{2} : (\partial_\mu O^\alpha O^\beta - O^\alpha \partial_\mu O^\beta) : \) with dimension \( \Delta_J = 2\Delta + 1 \). Now subtracting from what we get the first two terms of the conformal block of the vector current\(^9\) and decomposing the resulting second rank tensor on the traceless and trace parts we are left with two new fields, one is a tensor and another one is a new scalar, which are given by

\[
T_{\mu\nu}^{\alpha\beta} = \frac{1}{2} \left( \partial_\mu O^\alpha \partial_\nu O^\beta + \partial_\nu O^\alpha \partial_\mu O^\beta \right) - \frac{\Delta}{2(2\Delta + 1)} \partial_\mu \partial_\nu : O^\alpha O^\beta : \\
+ \frac{\delta_{\mu\nu}}{4\eta} \left( -\frac{\Delta + 1}{2\Delta + 1} \partial^2 : O^\alpha O^\beta : + : \partial^2 O^\alpha O^\beta : + : O^\alpha \partial^2 O^\beta : \right),
\]

\[
T^{\alpha\beta} = \frac{1}{4\eta} \left( -\frac{\Delta - \eta + 1}{2\Delta + 1 - \eta} \partial^2 : O^\alpha O^\beta : + : \partial^2 O^\alpha O^\beta : + : O^\alpha \partial^2 O^\beta : \right).
\]

The transformation properties of these fields under the conformal group show that they are both quasi-primary. Thus, for \( \eta = 2 \) we get the desired result (3.5). Note that \( T_{\mu\nu}^{\alpha\beta} \) is conserved while \( T^{\alpha\beta} \) vanishes on-shell as soon as \( \eta = \Delta + 1 \). Clearly with the knowledge of the conformal blocks of the higher rank tensor operators the procedure of identifying the quasi-primary operators on the r.h.s of (3.4) may be extended to any desired order.

7 Appendix B. Conformal partial wave amplitudes of a scalar, a conserved vector current and the stress tensor

The full contribution of the conformal block of an operator carrying and irreducible representation of the conformal group into the 4-point function is known as the conformal partial wave amplitude (CPWA). The scalar CPWA was computed in [47] by evaluating\(^9\)We do not assume here that \( J_{\mu}^{\alpha\beta} \) is conserved, however the first two terms in the conformal blocks of the conserved and non-conserved vector currents are the same.
the corresponding scalar exchange diagram. If we consider operators with the same conformal dimension, then the CPWA of a scalar operator with dimension $\Delta_S$ contributes to its 4-point function as [31]:

$$\mathcal{H}_S(v, Y) = v^{\Delta_S} \sum_{n=0}^{\infty} \frac{v^n}{n! (\Delta_S + 1 - \eta)^n} \frac{1}{(\Delta_S + n)(\Delta_S + n + 1 - \eta)} \Delta_S F_1 \left( \frac{1}{2} \Delta_S + n; \frac{1}{2} \Delta_S + n + 1 - \eta; \frac{1}{2} \Delta_S + 2n; Y \right),$$

(7.1)

where we have represented the result as the convergent series in conformal variables $v$ and $Y$. The first few terms of the $v, Y$ expansion of $\mathcal{H}_S(v, Y)$ are given in (4.2). In particular, for $\Delta_S = 2$ the first term of $v$-expansion reads as

$$\mathcal{H}_S(v, Y) = \frac{3}{40} v F_1(Y) + \cdots,$$

(7.2)

where $F_1(Y)$ is defined in section 4.2.

The CPWA of traceless symmetric tensors of dimension $\Delta$ and rank $l$, corresponding to irreducible representations of dimension $\Delta$ and spin $l$ of $SO(d, 2)$, can be also calculated in CFT as the relevant graphs reduce to sums of scalar exchanges. Using the following normalization prescriptions [45, 46] for the 2- and 3-point functions of the exchanged tensor fields

$$\langle M_{\mu_1, \ldots, \mu_l}(x_1)M_{\nu_1, \ldots, \nu_l}(x_2) \rangle = C_{\Delta, l} \frac{\mathcal{N}((\Delta + l)(x_{12})^\Delta)}{(2\pi)^{2d} \Gamma(\frac{d}{2} - \frac{l}{2}) \Gamma(\Delta + l + 1 - \frac{d}{2})} \left\{ I_{\mu_1, \nu_1}(x_{12}) \cdots I_{\mu_l, \nu_l}(x_{12}) \right\}_{\text{sym} - \text{traces}},$$

$$\langle O(x_1)O(x_2)M_{\mu_1, \mu_2, \ldots, \mu_l}(x_3) \rangle = \langle O(x_1)O(x_2)M_{\mu_1, \mu_2, \ldots, \mu_l}(x_3) \rangle = \frac{g^{2\Delta} \mathcal{N}((\Delta + \Delta)(x_{13}x_{23})^{\Delta})}{(2\pi)^{2d} \Gamma(\Delta + \Delta + l)(d - \Delta - l)} \left\{ I_{\mu_1, \nu_1}(x_{13}) \cdots I_{\mu_l, \nu_l}(x_{13}) \right\}_{\text{sym} - \text{traces}},$$

where the normalization constants are taken to be

$$\mathcal{N}(\Delta, l) = \frac{2^\Delta \Gamma(\Delta + l)(d - \Delta - 1)}{(2\pi)^{2d} \Gamma(\frac{d}{2} - l)(d + l - 1)}$$

$$\mathcal{N}((\Delta + \Delta), l) = \frac{2^{2\Delta} \Gamma(\Delta + \Delta + l)(d - \Delta - l)}{(2\pi)^{2d} \Gamma(\Delta + \Delta + l + 1)(d - \Delta - l)} \left\{ I_{\mu_1, \nu_1}(x_{13}) \cdots I_{\mu_l, \nu_l}(x_{13}) \right\}_{\text{sym} - \text{traces}},$$

and

$$\xi_{\mu}(1, 2; 3) = \frac{(x_{13})_{\mu}}{x_{13}^2} - \frac{(x_{23})_{\mu}}{x_{23}^2}, \quad \xi^2(1, 2; 3) = \frac{x_{12}^2}{x_{13}x_{23}},$$

the contribution of the tensor field to the 4-point function of a scalar operator with dimension $\Delta$ takes the form

$$\beta_{\Delta}(x_1, x_3; x_2, x_4; \Delta, l) = \beta_{\Delta; \Delta, l} \frac{(x_{13}^2)^{\Delta-\Delta}(x_{24}^2)^{\Delta-\Delta}}{(x_{13}^2)^{\Delta-\Delta}(x_{24}^2)^{\Delta-\Delta}} \times \int d^4 x^5 \left\{ e_{\mu_1} \cdots e_{\mu_l} - \text{traces} \right\} \left\{ e'_{\mu_1} \cdots e'_{\mu_l} - \text{traces} \right\} \left( x_{13}^2 x_{24}^2 \right)^{\Delta-\Delta} \left( x_{13}^2 x_{24}^2 \right)^{\Delta-\Delta} \cdots \left( x_{13}^2 x_{24}^2 \right)^{\Delta-\Delta},$$

(7.3)
The constant $\beta_{\tilde{\Delta}; \Delta, l}$ is then given by

$$
\beta_{\tilde{\Delta}; \Delta, l} = \frac{g_{\tilde{\Delta}\Delta, l}^2 2^{2\tilde{\Delta}+\frac{1}{2}d+\frac{l}{2}} \Gamma(\tilde{\Delta} - \frac{1}{2}d + \frac{l}{2}) \Gamma(\tilde{\Delta} + \frac{1}{2}d + \frac{l}{2} - \frac{d}{2})}{C_{\Delta, l} (2\pi)^{d/2} \Gamma(\frac{1}{2}d - \Delta + \frac{1}{2}d + \frac{l}{2}) \Gamma(d - \Delta - \frac{1}{2}d + \frac{l}{2})}.
$$

where we have introduced the concise notation

$$
e_{\mu} = \frac{\xi_{\mu}(1, 3; 5)}{[\xi^2(1, 3; 5)]^{1/2}}, \quad e'_{\mu} = \frac{\xi_{\mu}(2, 4; 5)}{[\xi^2(2, 4; 5)]^{1/2}}, \quad e \cdot e' = e'_{\mu} e_{\mu} = 1.
$$

One can show that for the general tensor exchange (7.3) is reduced to a finite sum of four-star integrals $S(a_1, a_2; a_3, a_4)$:

$$
S(a_1, a_2; a_3, a_4) = \int \frac{d^4 x_5}{x_{15}^{2a_1} x_{25}^{2a_2} x_{35}^{2a_3} x_{45}^{2a_4}} \int d^4 x_6 \frac{e \cdot e'}{(x_{12}^{2a_2} x_{16}^{2a_3} x_{26}^{2a_4} x_{36}^{2a_5})^{1/2}}.
$$

which can be directly evaluated. The final result is obtained after dropping the “shadow series” of the four-star integral, as the latter corresponds to the exchange of the “shadow tensor” field with dimension $d - \Delta$.

Here, we apply the general formula (7.3) to the two cases we are interested in the paper; the case of the conserved vector current with $\Delta = d - 1$ and $l = 1$ and the stress tensor with $\Delta = d$ and $l = 2$. Choosing to work directly in $d = 4$, the contribution of a conserved vector field in the scalar four-point function is given by

$$
\beta_2(x_1, x_2; x_3, x_4; 3, 1) = \beta_{2; 3, 1} \frac{1}{(x_{12}^{2a_2} x_{16}^{2a_3} x_{26}^{2a_4} x_{36}^{2a_5})^{1/2}} \int \frac{d^4 x_6}{x_{15}^{2a_1} x_{25}^{2a_2} x_{35}^{2a_3} x_{45}^{2a_4} x_{56}^{2a_5}} e \cdot e'.
$$

The inner product $e \cdot e'$ can be written as

$$
e \cdot e' = \frac{1}{2} \left( \frac{x_{12}^{2a_2} x_{16}^{2a_3} x_{26}^{2a_4} x_{36}^{2a_5}}{x_{12}^{2a_2} x_{16}^{2a_3} x_{26}^{2a_4} x_{36}^{2a_5}} \right)^{1/2} \left[ x_{14}^{2a_2} - x_{14}^{2a_2} + x_{15}^{2a_2} - x_{15}^{2a_2} \right].
$$

Substituting (7.6) into (7.5) we obtain four 4-star functions as

$$
\beta_2(x_1, x_2; x_3, x_4; 3, 1) = \frac{1}{2} \beta_{2; 3, 1} \frac{1}{(x_{12}^{2a_2} x_{16}^{2a_3} x_{26}^{2a_4} x_{36}^{2a_5})^{1/2}} \times

\left[ x_{24}^{2a_2} S (1 + \epsilon, 2 + \epsilon; -\epsilon, 1 - \epsilon) - x_{14}^{2a_2} S (2 + \epsilon, 1 + \epsilon; -\epsilon, 1 - \epsilon) + x_{13}^{2a_2} S (2 + \epsilon, 1 + \epsilon; -\epsilon, 1 - \epsilon) - x_{23}^{2a_2} S (1 + \epsilon, 2 + \epsilon; -\epsilon, 1 - \epsilon) \right].
$$

Note that we have also regularized the dimension of the vector field as $\Delta = 3 + 2\epsilon$ to deal with the singularities contained in the four-star functions involved into (7.7). The singularities are avoided by keeping the regulating parameter $\epsilon$ non-zero in the intermediate stages of the calculation. The analyticity of the exchange graph then ensures that taking
the limit $\epsilon \to 0$ at the end of the calculation one recovers the correct result. Using the expression for the four-star function derived in [26] we then obtain (here we present the formula for general $d$ and $\Delta$ to ensure a wider applicability of our result)

$$
\beta_\Delta(x_1, x_2; x_3, x_4; \Delta, 1) = -\frac{2}{3}\bar{\beta}_\Delta \int \frac{dx_3 x_4}{(x_{12} x_{13} x_{14} x_{15} x_{16})^{\Delta}}
$$

where the function $H_V(v, Y)$ represents the CPWA of the vector current

$$
H_V(v, Y) = -\frac{3}{4}v^{\frac{d-4}{2}} \sum_{n,m=0}^\infty \frac{v^n Y^m}{n!m!} (1 - \eta + \Delta)^n (\Delta + 1)^m
$$

(7.8)

For $\Delta = d - 1 = 3$ the CPWA of the vector current simplifies to give

$$
H_V(v, Y) = -\frac{3}{4}v^{\frac{d-4}{2}} \sum_{n,m=0}^\infty \frac{v^n Y^m}{n!m!} (1 - \eta + \Delta)^n (\Delta + 1)^m
$$

(7.9)

and it is normalized to start as $H_V(v, Y) = \frac{1}{2}Y + \cdots$ (cf. (4.2)). To make a comparison with the supergravity results in section 4.5 we need to single out in eq. (7.9) the leading-$v$ contribution. Putting in the previous formula $n = 0$ and performing the summation in $m$ we obtain

$$
H_V(v, Y) = \frac{3}{16}v F_1(Y) + \cdots
$$

(7.10)

where $F_1(Y)$ is defined in section 4.5.

Analogously, the contribution of the stress tensor is given by

$$
\beta_2(x_1, x_2; x_3, x_4; 4, 1) = \beta_{2;3,1} \int \frac{dx_3 x_4}{(x_{12} x_{13} x_{14} x_{15} x_{16})^{\Delta}}
$$

(7.11)

Using then (7.6) and regularizing the tensor dimension as $\Delta = 4 + 2\epsilon$ we obtain

$$
\beta_2(x_1, x_2; x_3, x_4; 4, 1) = \beta_{2;3,1} \left[ \frac{1}{x_{34}^2} \right] \left[ \frac{1}{x_{12}^2 x_{13}^2} \right] (x_{24}^2 S(1 + \epsilon, 3 + \epsilon; -1 - \epsilon, 1 - \epsilon)
$$
\[ + x_{14}^4 S(3 + \epsilon, 3 + \epsilon; -1 - \epsilon, 1 - \epsilon) + x_{13}^2 S(3 + \epsilon, 1 + \epsilon; 1 - \epsilon, -1 - \epsilon) + x_{23}^2 S(1 + \epsilon, 3 + \epsilon; 1 - \epsilon, -1 - \epsilon) - 2 x_{24}^2 x_{14}^2 S(2 + \epsilon, 2 + \epsilon; -1 - \epsilon, 1 - \epsilon) - 2 x_{24}^2 x_{13}^2 S(2 + \epsilon, 2 + \epsilon; 1 - \epsilon, -1 - \epsilon) + 2 x_{24}^2 x_{13}^2 S(2 + \epsilon, 2 + \epsilon; -1 - \epsilon, -1 - \epsilon) - 2 x_{24}^2 x_{13}^2 S(2 + \epsilon, 2 + \epsilon; -1 - \epsilon, -1 - \epsilon) \right] \frac{1}{4} S(2 + \epsilon, 2 + \epsilon; -1 - \epsilon, -1 - \epsilon). \] (7.12)

One then observes that (7.12) contains a number of four-star functions which are \( O(\epsilon) \) and therefore vanish in the \( \epsilon \to 0 \) limit. These are all the four-star functions with \( -\epsilon \) in the last two positions. Then, by virtue of

\[ \frac{\Gamma(-2 - 2\epsilon)}{\Gamma(-1 - \epsilon)} = \frac{1}{2} + O(\epsilon), \] (7.13)

the remaining four-star functions give a finite result which reads

\[ \beta_2(x_1, x_2; x_3, x_4; 4, 1) = \frac{\pi^2}{12} \beta_{2,4,4} \frac{1}{x_{12}^4 x_{34}^4} (\mathcal{H}_T(v, Y) + \text{shadow part}), \]

where \( \mathcal{H}_T(v, Y) \) represents the CPWA of the stress tensor:

\[ \mathcal{H}_T(v, Y) = \frac{5}{4} v \sum_{\alpha} \frac{v^n Y^m}{n! m!} \frac{1}{(3)_n(4)_{2n+m}} \times \left[ (3)_n^2 (1)_{n+m}^2 + 2 (3)_n (3)_n (3)_{n+m} (1)_{n+m} + (1 - Y)^2 (1)^2 (3)_{n+m}^2 \right. \]

\[ -2 (3)_n (2)_{n+m} (2)_{n+m} - 2 (1 - Y) (1)_{n+m} (3)_{n+m} (2)_{n+m} \]. \] (7.14)

The normalization of \( \mathcal{H}_T(v, Y) \) is fixed such that its \( v, Y \) expansion reproduces the corresponding terms in (4.2). Again to establish a link with supergravity results in section 4.1 we single out the \( v \) term in eq. (7.14) and, performing the summation in \( m \), get

\[ \mathcal{H}_T(v, Y) = \frac{45}{8} v F_1(Y) + \ldots, \] (7.15)

where \( F_1(Y) \) is the function defined in section 4.1. This completes the construction of the CPWA for conserved vector and tensor currents.

8 Appendix C. Series representation for \( \tilde{D} \)-functions

Here we derive a representation for the \( \tilde{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \)-functions in a form of a convergent series in \( v \) and \( Y \) variables by using a technique similar to [18].
We start with the definition (4.4). Standard Feynman parameter manipulations based on the formula
\[ \frac{1}{z^\lambda} = \int_0^\infty dt t^{\lambda-1} e^{-tz}, \]
and two integrals
\[ \int e^{-\sum t_i x_i^2} x^{\frac{d-1}{2} + \sum \Delta_i} dx = \frac{1}{2} (S_t)^{\frac{d-1}{2}} \frac{\Gamma\left(\frac{\sum \Delta_i}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}, \]
\[ \int d^d x e^{-t_i x_i^2 - t_j x_j^2} = \frac{\pi^{d/2}}{S_t^{d/2}} e^{-\frac{1}{S_t} \sum_{i<j} t_{ij} x_i x_j}, \]
lead to
\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = K \int_0^\infty dt_1 \ldots dt_4 t_1^{\Delta_1-1} \ldots t_4^{\Delta_4-1} (S_t)\frac{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4}{2} \exp\left[-\frac{1}{S_t} (t_1 t_2 x_{12}^2 + \ldots + t_4 t_4 x_{44}^2)\right], \]
where the short-hand notations
\[ S_t = t_1 + t_2 + t_3 + t_4, \quad K = \frac{\pi^{d/2} \Gamma\left(\frac{\sum \Delta_i}{2}\right)}{2 \Gamma(\Delta_1) \ldots \Gamma(\Delta_4)}, \]
were introduced. Performing the change of variables
\[ t_i = S_t^{1/2} t'_i = (\sum_i t'_i) t'_i \equiv u t'_i, \quad \det \left(\frac{\partial t_i}{\partial t'_j}\right) = 2u^4, \]
one obtains
\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = 2K \int_0^\infty dt_1 \ldots dt_4 t_1^{\Delta_1-1} \ldots t_4^{\Delta_4-1} \exp\left[-\sum_{i<j} t_{ij} x_{ij}^2\right]. \]
Now we rescale the variables \( t_i \): \( t_i \to \lambda_i t_i \), where the constant parameters \( \lambda_i \) are chosen to induce the following scale transformations
\[ t_1 t_2 \to \frac{1}{x_{12}^2} t_1 t_2, \quad t_1 t_3 \to \frac{1}{x_{13}^2} t_1 t_3, \quad t_1 t_4 \to \frac{1}{x_{14}^2} t_1 t_4, \quad t_2 t_3 \to \frac{1}{x_{23}^2} t_2 t_3, \]
and as the consequence
\[ t_2 t_4 = \frac{t_2 t_3 \cdot t_1 t_4}{t_1 t_3} \to \frac{x_{12}^2}{x_{13}^2 x_{23}^2} t_2 t_4, \quad t_3 t_4 = \frac{t_2 t_3 \cdot t_1 t_4}{t_1 t_2} \to \frac{x_{12}^2}{x_{14}^2 x_{23}^2} t_3 t_4. \]
Under this rescaling the integral transforms into
\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \frac{\bar{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y)}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4}{2}} (x_{13}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4}{2}} (x_{14}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4}{2}} (x_{23}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4}{2}} (x_{14}^2)^{\Delta_4}}, \]
where
\[
\tilde{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y) =
2K \int dt_1 \cdots dt_4 t_1^{\Delta_1 - 1} t_2^{\Delta_2 - 1} t_3^{\Delta_3 - 1} t_4^{\Delta_4 - 1} \exp \left[ -t_1 t_2 - t_1 t_3 - t_1 t_4 - t_2 t_3 - \frac{v}{u} t_2 t_4 - v t_3 t_4 \right],
\]
and the integral is understood as a function of the conformal variables \(v\) and \(Y\).

Next, using the Mellin-Barnes integral representation
\[
\exp[-z] = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} ds \Gamma(-s) z^s, \quad r < 0, \quad |\arg z| < \frac{1}{2}\pi,
\]
for the two exponentials in the last formula which involve \(\frac{v}{u}\) and \(v\) the integral reduces to
\[
\tilde{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y) = 2K \int \frac{d\lambda ds}{(2\pi i)^2} \Gamma(-s) \Gamma(-\lambda) v^\lambda \left( \frac{v}{u} \right)^s \\
\times \int dt_1 \ldots dt_4 t_1^{\Delta_1 - 1} t_2^{\Delta_2 + s - 1} t_3^{\Delta_3 + \lambda - 1} t_4^{\Delta_4 + s + \lambda - 1} \exp \left[ -t_1 t_2 - t_1 t_3 - t_1 t_4 - t_2 t_3 \right].
\]
The following change of variables:
\[
t_1 t_2 = u_1, \quad t_1 t_3 = u_2, \quad t_1 t_4 = u_3, \quad t_2 t_3 = u_4, \quad \det \left( \frac{\partial t_i}{\partial u_j} \right) = \frac{1}{2u_1 u_2},
\]
allows one to perform the \(t\)-integration with the result
\[
\tilde{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y) =
K \int \frac{d\lambda ds}{(2\pi i)^2} \left[ \Gamma(-s) \Gamma(-\lambda) \Gamma\left( \frac{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4}{2} - s \right) \Gamma\left( \frac{\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4}{2} - s \right) \right. \\
\times \left. \Gamma\left( \frac{\Delta_2 + \Delta_3 + \Delta_4 - \Delta_1}{2} + s + \lambda \right) \Gamma\left( \Delta_4 + s + \lambda \right) v^\lambda \left( \frac{v}{u} \right)^s \right].
\]
The \(s\)-integration is then performed by using the integral and series representations for the hypergeometric function \(F(a, b, c; 1 - z)\):
\[
F(a, b, c; 1 - z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(c - a) \Gamma(c - b)} \\
\times \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} ds z^s \Gamma(-s) \Gamma(c - a - b - s) \Gamma(a + s) \Gamma(b + s),
\]
and
\[
F(a, b, c; 1 - z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(a + m) \Gamma(b + m)}{\Gamma(c + m) m!} (1 - z)^m,
\]
where one needs to substitute
\[
a = \frac{\Delta_2 + \Delta_3 + \Delta_4 - \Delta_1}{2} + \lambda, \quad b = \Delta_4 + \lambda, \quad c = \Delta_3 + \Delta_4 + 2\lambda.
\]

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Thus one arrives at the convergent hypergeometric series in the variable $Y$:

$$
\tilde{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(v, Y) = K \sum_{m=0}^{\infty} \frac{Y^m}{m!} \left( \int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \left[ \Gamma(-\lambda) \frac{\Gamma(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)}{\Gamma(\Delta_3 + \Delta_4 + 2\lambda + m)} \times \left( -\frac{1}{n + 1} + \psi(4 + 2n + m) - \psi(n + m + 2) - \frac{1}{2} \ln v \right) \right] \right),
$$

Since for any $\tilde{D}$-function occurring in the 4-point function of CPOs the quantity $\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4$ is an integer, the final Mellin-Barnes integral receives a contribution from double poles and, therefore, the integration can be done by using the general formula

$$
\int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \Gamma^2(-s) g(s) \, v^s = \sum_{n=0}^{\infty} \frac{v^n}{(n!)^2} \left[ 2\psi(n + 1) g(n) - g(n) \ln v - \frac{d}{d\xi} [g(\xi)]_{\xi = n} \right],
$$

valid for any function $g(s)$ regular at $s = 0$. In this way we arrive at the representation for $\tilde{D}$-functions in terms of double convergent series in $v$ and $Y$ variables.

Below we list explicitly the series representations for $\tilde{D}$-functions we used in the paper

$$
\tilde{D}_{2222}(v, Y) = \pi^2 \sum_{m=0}^{\infty} \frac{Y^m}{m! (n!)^2} \frac{\Gamma(n + 2 \, \Gamma(2 + n + m)\right)^2}{\Gamma(4 + 2n + m)} \times \left( -\frac{1}{n + 1} + \psi(4 + 2n + m) - \psi(n + m + 2) - \frac{1}{2} \ln v \right),
$$

$$
\tilde{D}_{2112}(v, Y) = \frac{\pi^2}{2} \sum_{m=0}^{\infty} \frac{Y^m}{m! (n!)^2} \frac{\Gamma(n + 2) \Gamma(n + 1) \Gamma(n + m + 1) \Gamma(n + m + 2)}{\Gamma(3 + 2n + m)} \times \left( -\frac{1}{n + 1} + 2\psi(3 + 2n + m) - \psi(n + m + 1) - \psi(n + m + 2) - \ln v \right),
$$

$$
\tilde{D}_{1212}(v, Y) = \pi^2 \sum_{m=0}^{\infty} \frac{Y^m}{m! (n!)^2} \frac{\Gamma(n + 1)^2 \Gamma(n + m + 2)^2}{\Gamma(3 + 2n + m)} \times \left( \psi(3 + 2n + m) - \psi(n + m + 2) - \frac{1}{2} \ln v \right),
$$

$$
\tilde{D}_{2211}(v, Y) = \frac{-\pi^2}{2} \sum_{m=0}^{\infty} \frac{Y^m}{m! (n!)^2} \frac{n \Gamma(n + 1)^2 \Gamma(n + m + 1)^2}{\Gamma(2 + 2n + m)} \times \left( -\frac{1}{n} - 2\psi(n + m + 1) + 2\psi(2 + 2n + m) - \ln v \right),
$$

$$
\tilde{D}_{3322}(v, Y) = \frac{-\pi^2}{4} \sum_{m=0}^{\infty} \frac{Y^m}{m! (n!)^2} \frac{n \Gamma(n + 2) \Gamma(2 + n + m)^2}{\Gamma(4 + 2n + m)} \times \left( -\frac{3n + 1}{n(n + 1)} + 2\psi(4 + 2n + m) - 2\psi(2 + n + m) - \ln v \right),
$$

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\[
\bar{D}_{2323}(v, Y) = \frac{\pi^2}{2} \sum_{m=0}^{\infty} \frac{Y^m}{m!} \frac{n^n}{(n+1)^2} \frac{\Gamma(n+2)\Gamma(3+n+m)^2}{\Gamma(5+2n+m)} \\
\times \left( \frac{1}{n+1} + \psi(5+2n+m) - \psi(3+n+m) - \frac{1}{2} \ln v \right),
\]

\[
\bar{D}_{3223}(v, Y) = \frac{\pi^2}{4} \sum_{m=0}^{\infty} \frac{Y^m}{m!} \frac{n^n}{(n+1)^2} \frac{\Gamma(n+2)\Gamma(n+3)\Gamma(2+n+m)\Gamma(3+n+m)}{\Gamma(5+2n+m)} \\
\times \left( -\frac{3n+5}{(n+1)(n+2)} + 2\psi(5+2n+m) \\
- \psi(2+n+m) - \psi(3+n+m) - \ln v \right). 
\]

(8.4)

9 Appendix D. Projectors

Here we give an explicit construction of the projectors that single out the contributions of irreps occurring in the decomposition $20 \times 20$ of $SO(6)$ from the 4-point function of the lowest weight CPOs.

Matrices $C_{ij}^l$ and $C_{ij}^{J_{15}}$ introduced in section 3 obey to the following summation formulae [28]:

\[
\sum_l C_{ij}^l C_{kl}^l = \frac{1}{2} \delta_{ik} \delta_{jl} + \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{6} \delta_{ij} \delta_{kl}, \quad \sum_{J_{15}} C_{ij}^{J_{15}} C_{kl}^{J_{15}} = \frac{1}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}).
\]

It is then easy to check that the orthonormal Clebsch-Gordon coefficients $C_{J_{20}}^{l_1 l_2}$ and $C_{J_{15}}^{l_1 l_2}$ are given by

\[
C_{J_{20}}^{l_1 l_2} = \frac{1}{2^{1/2}} C_{ij}^{l_1} C_{ik}^{l_2}, \quad C_{J_{15}}^{l_1 l_2} = \frac{1}{2^{1/2}} C_{ij}^{l_1} C_{jk}^{l_2} C_{ik}^{J_{15}}.
\]

(9.1)

The other coefficients are constructed in a similar manner. Irreps 84, 105 and 175 are described by traceless rank 4 tensors $C_{ijkl}^{J_{15}}$ with the normalization condition

\[
C_{ijkl}^{J_{15}} C_{ijkl}^{J_{15}} = \delta_{J_{15} J_{15}}.
\]

Tensor $C_{ijkl}^{J_{15}}$ is antisymmetric in $i, k$ and in $j, l$ and symmetric under permutation of the pairs $ij$ and $kl$. It is also required to obey the condition $\varepsilon_{ijklmn} C_{ijkl}^{J_{15}} = 0$. Then $C_{ijkl}^{J_{15}}$ is a totally symmetric and, finally, $C_{ijkl}^{J_{15}}$ is symmetric in $i, k$ and in $j, l$ and antisymmetric under permutation of the pairs $ij$ and $kl$.

A projector on the contribution of irrep $\mathbf{D}$ into the 4-point function is defined by (4.10) with $\nu_D$ being the dimension of the representation. The sums $C_{J_{20}}^{l_1 l_2} C_{J_{20}}^{l_1 l_2}$ and $C_{J_{15}}^{l_1 l_2} C_{J_{15}}^{l_1 l_2}$ are
computed straightforwardly by using eqs. (9.1). To find the other projectors we introduced the following three tensors $Q_{D}^{h l_2}$ being elements of the corresponding representations:

\begin{align*}
(Q_{84}^{h l_2})_{ij kl} &= C_{ij}^{l_2} C_{kl}^{l_2} - C_{ik}^{l_2} C_{jl}^{l_2} + C_{il}^{l_2} C_{jk}^{l_2} - C_{il}^{l_2} C_{jk}^{l_2} + \frac{1}{4} (C_{im}^{l_2} C_{mj}^{l_2} \delta_{il} - C_{km}^{l_2} C_{mj}^{l_2} \delta_{il} + C_{km}^{l_2} C_{mi}^{l_2} \delta_{kj} - C_{im}^{l_2} C_{mk}^{l_2} \delta_{kj}) \\
&- \frac{1}{5} \delta_{ij}(C_{km}^{l_2} C_{ml}^{l_2} + C_{im}^{l_2} C_{ml}^{l_2}) - \frac{1}{5} \delta_{kl}(C_{im}^{l_2} C_{mj}^{l_2} + C_{jm}^{l_2} C_{mi}^{l_2}) \\
&- \frac{1}{5} \delta_{ik}(C_{jm}^{l_2} C_{ml}^{l_2} + C_{im}^{l_2} C_{mj}^{l_2}) - \frac{1}{5} \delta_{il}(C_{jm}^{l_2} C_{mk}^{l_2} - C_{jm}^{l_2} C_{mk}^{l_2}) \\
&+ \frac{1}{20} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \delta_{l_2} l_2,
\end{align*}

\begin{align*}
(Q_{105}^{h l_2})_{ij kl} &= C_{ik}^{l_2} C_{jl}^{l_2} - C_{ik}^{l_2} C_{jl}^{l_2} - \frac{1}{8} \delta_{ij}(C_{km}^{l_2} C_{ml}^{l_2} - C_{im}^{l_2} C_{mk}^{l_2}) - \frac{1}{8} \delta_{kl}(C_{im}^{l_2} C_{mj}^{l_2} - C_{im}^{l_2} C_{mj}^{l_2}) \\
&- \frac{1}{8} \delta_{il}(C_{km}^{l_2} C_{mj}^{l_2} - C_{jm}^{l_2} C_{mk}^{l_2}) - \frac{1}{8} \delta_{ik}(C_{jm}^{l_2} C_{ml}^{l_2} - C_{jm}^{l_2} C_{ml}^{l_2}).
\end{align*}

Clearly one may write

\[
C_{ij}^{l_2} C_{i j}^{l_2} = \gamma_D (Q_{D}^{h l_2})_{ij kl},
\]

where $\gamma_D$ is a normalization constant. Then one finds

\[
C_{i j}^{h} C_{i j}^{h} = \gamma_D (Q_{D}^{h l_2})_{ij kl} (Q_{D}^{l_2 l_2})_{ij kl} = \gamma_D^2 (Q_{D}^{h l_2})_{ij kl} (Q_{D}^{l_2 l_2})_{ij kl}
\]

with the normalization constant $\gamma_D$ following from

\[
\nu_D = \gamma_D^2 (Q_{D}^{h l_2})_{ij kl} (Q_{D}^{l_2 l_2})_{ij kl}
\]

and, therefore,

\[
(P_{D})_{i_1 l_2 l_3 l_4} = \frac{(Q_{D}^{l_2 l_4})_{ij kl} (Q_{D}^{l_4 l_4})_{ij kl}}{(Q_{D}^{l_2 l_4})(Q_{D}^{l_4 l_4})_{mnsp}}.
\]

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In this way one obtains the following explicit expressions for projectors singling out the contributions of the irreps:

\[
(P_{15})_{l_1 l_2 l_3 l_4} = -\frac{1}{30} C_{l_1 l_2 l_3 l_4}^-, \\
(P_{20})_{l_1 l_2 l_3 l_4} = \frac{3}{100} \left( C_{l_1 l_2 l_3 l_4}^+ - \frac{1}{6} \delta_{l_1 l_2} \delta_{l_3 l_4} \right), \\
(P_{84})_{l_1 l_2 l_3 l_4} = \frac{1}{504} \left( 2 \delta_{l_1 l_5} \delta_{l_2 l_4} + 2 \delta_{l_1 l_4} \delta_{l_2 l_5} + \frac{1}{5} \delta_{l_1 l_5} \delta_{l_3 l_4} - 4 C_{l_1 l_5 l_2 l_4} - 2 C_{l_1 l_5 l_3 l_4} \right), \\
(P_{105})_{l_1 l_2 l_3 l_4} = \frac{1}{1260} \left( 2 \delta_{l_1 l_5} \delta_{l_2 l_4} + 2 \delta_{l_1 l_4} \delta_{l_2 l_5} + \frac{1}{5} \delta_{l_1 l_5} \delta_{l_3 l_4} + 8 C_{l_1 l_5 l_2 l_4} - \frac{16}{5} C_{l_1 l_5 l_3 l_4} \right), \\
(P_{175})_{l_1 l_2 l_3 l_4} = \frac{1}{350} \left( \delta_{l_1 l_5} \delta_{l_2 l_4} - \delta_{l_1 l_4} \delta_{l_2 l_5} + C_{l_1 l_5 l_3 l_4}^- \right).
\]

One may check that together with \(\frac{1}{400} \delta_{l_1 l_2} \delta_{l_3 l_4}\) these projectors provide the orthogonal decomposition of the unity \(\delta_{l_1 l_2} \delta_{l_3 l_4}\).

The following formulae

\[
C_{l_1 l_2 l_3 l_4} C_{l_1 l_2 l_3 l_4} = \frac{380}{3}, \quad C_{l_1 l_2 l_3 l_4} C_{l_2 l_1 l_3 l_4} = \frac{20}{3}, \quad C_{l_1 l_2 l_3 l_4} C_{l_1 l_3 l_2 l_4} = 0, \\
C_{l_1 l_2 l_3 l_4}^+ C_{l_1 l_2 l_3 l_4}^+ = \frac{200}{3}, \quad C_{l_1 l_2 l_3 l_4}^+ C_{l_1 l_2 l_3 l_4}^- = 0, \quad C_{l_1 l_2 l_3 l_4}^+ C_{l_1 l_2 l_3 l_4}^+ = \frac{20}{3},
\]

are helpful to find the contractions of the projectors with tensors describing the 4-point function. The results for contractions are summarized in the Table 1.
<table>
<thead>
<tr>
<th>Tensor</th>
<th>$C^+_{l_1 l_2 l_3 l_4}$</th>
<th>$C^-_{l_1 l_2 l_3 l_4}$</th>
<th>$C_{l_1 l_2 l_3 l_4}$</th>
<th>$\delta_{l_1 l_2} \delta_{l_3 l_4}$</th>
<th>$\delta_{l_1 l_4} \delta_{l_2 l_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{400} \delta_{l_1 l_2} \delta_{l_3 l_4}$</td>
<td>$\frac{1}{6}$</td>
<td>$0$</td>
<td>$\frac{1}{60}$</td>
<td>$\frac{1}{20}$</td>
<td>$\frac{1}{20}$</td>
</tr>
<tr>
<td>$(P_{15})_{l_1 l_2 l_3 l_4}$</td>
<td>$0$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$(P_{20})_{l_1 l_2 l_3 l_4}$</td>
<td>$\frac{5}{3}$</td>
<td>$0$</td>
<td>$\frac{1}{6}$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(P_{84})_{l_1 l_2 l_3 l_4}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-\frac{1}{7}$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(P_{105})_{l_1 l_2 l_3 l_4}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(P_{175})_{l_1 l_2 l_3 l_4}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Table 1. The values of contractions of the projectors with tensors describing the structure of the 4-point function of the lowest weight CPOs.

10 Appendix E. Conformal block of the conserved 2nd rank tensor

Here we sketch the derivation of the conformal block of the conserved second rank tensor. We do not use this result in the paper, however, we feel that it might be useful for subsequent studies of the OPE.

We start with (6.1) and suppress the unessential indices $\alpha$ and $\beta$. The 3-point function is given by the following expression

$$\langle O(x) O(0) T_{\mu \nu} (y) \rangle = \frac{1}{(x^2)^{\frac{1}{2} (\Delta - \Delta + 2)} (y^2)^{\frac{1}{2} (\Delta + 2)} ((y - x)^2)^{\frac{1}{2} (\Delta + 2)}} \left[ \frac{\delta_{\mu \nu} x^2 y^2 (y - x)^2}{d} - y^4 (y - x)_\mu (y - x)_\nu - (x - y)^4 y_{\mu} y_{\nu} + y^2 (y - x)^2 ((y - x)_\mu y_{\nu} + y_{\mu} (y - x)_\nu) \right],$$

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where for simplicity we choose the constant $C_{O\Omega T}$ to be equal to unity. Compatibility of
the 3-point function with the conservation law requires the dimension of the tensor to be canonical, i.e. $\Delta_T = d$, where $d = 2\eta$ is a space-time dimension. However, in what follows we meet certain divergences and that is why we keep in some places $\varepsilon = d - \Delta_T$ as a regularization parameter. Substituting eq.(6.1) into the 3-point function, we get an equation defining the conformal block

$$\frac{1}{\Delta_T (\Delta_T - 2)} \left( \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \partial_\mu \partial_\nu \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} \partial_\mu \partial_\nu \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \right) + \frac{1}{(\Delta_T - 2)^2} \left( \partial_\mu \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \partial_\nu \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} \partial_\nu \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \partial_\mu \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} \right) - \frac{d - \Delta_T}{d\Delta_T} \delta_{\mu \nu} \left( \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} \partial_\lambda \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \partial_\lambda \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} \right) + \frac{2\delta_{\mu \nu}}{d(\Delta_T - 2)^2} \partial_\lambda \frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}} e^{\sigma y} \partial_\lambda \frac{1}{(y^2)^{\frac{1}{2}(\Delta_T - 2)}} = \sum_{k=2}^{\infty} \frac{1}{k!} \Delta^k_{\rho\lambda}(x, \partial_y) \langle T_{\rho\lambda}(0) T_{\mu\nu}(y) \rangle,$$

where we have introduced the following representation

$$C_{\mu\nu}(x, \partial_y) = \sum_{k=2}^{\infty} \frac{1}{k!} \Delta^k_{\rho\lambda}(x, \partial_y).$$

Using the series representation for the exponentials on the l.h.s. of the equation defining the conformal block, one then obtains an equation for $\Delta^k_{\rho\lambda}(x, \partial_y)$.

The 2-point function of the second rank tensor can be written in the form [40,42]

$$\langle T_{\rho\lambda}(0) T_{\mu\nu}(y) \rangle = \frac{1}{(\Delta_T - 3)(\Delta_T - 2)\Delta_T (\Delta_T + 1)} \mathcal{E}_{\mu\nu\rho\lambda\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \frac{1}{(y^2)^{\Delta_T - 2}},$$

where again for simplicity we choose the constant $C_T$ to be the unity. Here $\mathcal{E}_{\mu\nu\rho\lambda\alpha\beta\gamma\delta}$ is a tensor with the following structure

$$\mathcal{E}_{\mu\nu\rho\lambda\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta = \frac{\Delta_T - 3}{4(\Delta_T - 2)} \left( \frac{1}{2} \Box^2 (\delta_{\mu\rho} \delta_{\nu\lambda} + \delta_{\mu\rho} \delta_{\nu\lambda}) + \delta_{\rho\lambda}(\cdots) + \text{longitudinal} \right).$$

If we suppose that the conformal block acting on the 2-point function is symmetric traceless and transversal, then only the first term here is of importance and one gets

$$\Delta^k_{\rho\lambda}(x, \partial_y) \langle T_{\rho\lambda}(0) T_{\mu\nu}(y) \rangle = \frac{4(\Delta_T - 1)(\Delta_T - 1 - \eta)(\Delta_T - \eta)}{(\Delta_T - 2)\Delta_T (\Delta_T + 1)} \Delta^k_{\mu\nu}(x, \partial_y) \frac{1}{(y^2)^{\Delta_T}}.$$

Now we substitute every function $\frac{1}{(y^2)^{\frac{3}{2}(\Delta_T - 2)}}$ appearing in the equation defining the conformal block for its Fourier transform

$$\frac{1}{y^2} = 2^{2(\nu-a)}\pi^\eta \frac{\Gamma(\eta - a)}{\Gamma(a)} \frac{1}{(2\pi)^{2\eta}} \int dp \frac{e^{-ipy}}{(p^2)^{\eta - a}}.$$
and find\footnote{We keep \( \varepsilon \) only there where it is actually needed to compute the limit \( \varepsilon \to 0 \).}

\[
\Delta_{\mu\nu}^k (x, -ip) = \frac{2^4 \Gamma(2\eta)(2\eta+1)}{\pi^\eta \Gamma^2(\eta-1)\Gamma(2-\eta)} \int \frac{dq}{(p^2)^{\eta}} \frac{(p_\mu - q_\mu)(p_\nu - q_\nu)}{((p - q)^2 q^2)^{1+\varepsilon/2}} (-ixq)^k.
\]

Since the conformal block is applied to the traceless transversal operator (2-point function) in the last expression we have omitted all trace and longitudinal terms proportional to \( \delta_{\mu\nu} \) and to \( p_\mu \) respectively. The equation can be then brought to the form

\[
\Delta_{\mu\nu}^k (x, -ip) = -\frac{4}{\pi^\eta \Gamma^2(\eta-1)\Gamma(2-\eta)} \frac{(p^2)^{-\eta}}{\frac{k}{2}(2-\eta)} \partial_\mu \partial_\nu I_k \left( \frac{\varepsilon}{2} - 1; \frac{\varepsilon}{2} + 1 \right),
\]

again modulo unessential trace and longitudinal terms. Here we introduced the following integral

\[
I_k(\alpha_1; \alpha_2) = \int dq \frac{(-ixq)^k}{((p - q)^2)^{\alpha_1}(q^2)^{\alpha_2}}
\]

that is explicitly evaluated to give

\[
I_k(\alpha_1; \alpha_2) = \frac{\pi^\eta}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{m=0}^{[k/2]} \left( \begin{array}{c} k \cr 2m \end{array} \right) \frac{(2m)!}{m!} \left( -ixp \right)^{k-2m} \left( \frac{1}{4x^2p^2} \right)^m (p^2)^{\eta-\alpha_1-\alpha_2} \frac{\Gamma(k + m + \eta - \alpha_2)\Gamma(m + \eta - \alpha_1)}{\Gamma(k + 2\eta - \alpha_1 - \alpha_2)} I_k \left( \alpha_1 + \alpha_2 - m - \eta \right).
\]

Again neglecting the trace and longitudinal contributions, we evaluate the limit \( \varepsilon \to 0 \) and normalize the resulting expression such that the first nontrivial term \( \Delta_{\mu\nu}^2 \) starts as

\[
\Delta_{\mu\nu}^2 (x, -ip) = 2x_\mu x_\nu + \ldots
\]

In this way we find the following expression

\[
\Delta_{\mu\nu}^k (x, \partial_y) = x_\mu x_\nu \frac{\Gamma(2\eta + 2)}{\Gamma(\eta + 1)} \frac{\Gamma(k - m + \eta - 1)}{\Gamma(k + 2\eta)} (x\partial_y)^{k-2m-2} \left( \frac{1}{4x^2} \Delta_y \right)^m.
\]

Finally, performing the summation in \( k \) we recover the expression \( C_{\mu\nu}(x, \partial_y) \) for the conformal block of the conserved second rank tensor given in Appendix A.
References


