On the Negative Discrete Spectrum of a Periodic Elliptic Operator in a Waveguide–Type Domain, Perturbed by a Decaying Potential

M. Sh. Birman
M. Solomyak

ON THE NEGATIVE DISCRETE SPECTRUM OF A
PERIODIC ELLIPTIC OPERATOR IN A WAVEGUIDE-TYPE
DOMAIN, PERTURBED BY A DECAYING POTENTIAL

BY
M. SII. BIRMAN AND M. SOLOMYAK

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ABSTRACT. Let $\Omega \subset \mathbb{R}^d$ be an unbounded domain, periodic along a chosen direction
(a waveguide-type domain), $P$ be a self-adjoint elliptic second order operator in
$L^2(\Omega)$, periodic along the same direction, and $V$ be a real-valued decaying potential.
We suppose that the bottom of the spectrum of $P$ is $\lambda = 0$, and study the asymptotic
behaviour of the number of negative eigenvalues of the operator $P - \alpha V$ as the
parameter $\alpha$ tends to $+\infty$. We show that typically the Weyl asymptotic law for this
quantity is violated, and find a substitute for this law.

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INTRODUCTION

For a self-adjoint operator $A$ whose negative spectrum is discrete, we denote by $N_-(A)$ the number of its negative eigenvalues. The object of our study in this paper is the function $N_-(P - \alpha V)$ where $P$ is a second order elliptic periodic operator, $V$ is a decaying real-valued potential, and $\alpha > 0$ is the large parameter. More precisely, the unperturbed operator $P$ acts in the space $L_2(\Omega)$ where a domain $\Omega \subset \mathbb{R}^d$ is periodic in one direction (a waveguide-type domain) and the coefficients of $P$ are periodic along the same direction. Moreover, $P$ is an operator in divergence form, with $L_\infty$-coefficients. We suppose that its spectrum, Spec$(P)$, is non-negative, and we are interested in the asymptotic behaviour of $N_-(P - \alpha V)$ as $\alpha \to \infty$. This behaviour heavily depends on whether the number $\min\{\lambda \in \text{Spec}(P)\}$ is positive or is equal to zero. In the first case, under some mild assumptions on $V$, the function $N_-(P - \alpha V)$ behaves according to the classical Weyl’s law. Periodicity of $P$ is of no importance here. This law is in general violated in the second case. Our main goal is to analyze this case and to find a substitute for Weyl’s law. The new asymptotic formula that we establish, looks as follows:

$$N_-(P - \alpha V) \sim [\text{Weyl's term}] + N_-(\beta^2 - \alpha Q), \quad \alpha \to \infty.$$  

The second term here corresponds to an auxiliary Schrödinger operator in $L_2(\mathbb{R})$ with the “effective potential” $Q = Q_{P,V}$. Its description involves some objects coming from the Floquet – Bloch decomposition of the operator $P$. The construction and separation of this auxiliary operator is the main contents of the paper.

This work was preceded by the paper [BLapSus], where the same problem was solved for operators $P$ on $\mathbb{R}^2$, periodic with respect to the lattice $\mathbb{Z}^2$. We basically follow the scheme developed in [BLapSus], though many important technical distinctions appear. In particular, we do not restrict ourselves to the powerlike behaviour of $N_-(P - \alpha V)$, as it was in [BLapSus].

The results of this paper were reported in the Mini-conference on Spectral Theory (Jerusalem, December 1998), on the German – Israeli meeting in Spectral Theory and Scattering (Jerusalem, May 1999) and in the Conference on Analysis and Mathematical Physics in Honour of L.Garding (Lund, August 1999).

Recently another approach to this class of problems was suggested by Ivrii [Iv]. It applies to operators with the smooth coefficients.

Let us describe the structure of the paper. In Section 1 we present a short but rather complete account of the Floquet – Bloch theory as applied to operators of the type we are dealing with. In Section 2 a rigorous setting of the problem is given, and in Theorem 2.2 the main result is formulated for the case of non-negative potentials $V$. Sections 3 and 4 contain auxiliary material from the theory of Hilbert space and from the spectral theory of differential operators on the line. Only then, in Section 5, we are in a position to formulate our result for sign-indfinite $V$ (Theorem 5.1). We also start its proof there. The proof is finished in Section 6, Theorem 2.2 is also proved there. Section 7 contains various complementary remarks, including a short review of relevant results. Some technical material is taken out to three appendices placed in the end of the paper.

The notation we use is rather standard. As usual, $H^1(D)$ stands for the Sobolev space in a domain $D$ (on $\mathbb{R}^d$ or on a smooth manifold) and $H^{1,0}(D)$ stands for its
subspace where the class $C_0^\infty(D)$ is dense. Sometimes we write $u \in C(\overline{D})$ and use the notation $\|u\|_{C(\overline{D})}$ for a function $u$, defined only on $D$. This always means that $u$ extends to $\overline{D}$ as a continuous function, and we identify $u$ with its extension.

A function and the operator of multiplication by this function are denoted by the same symbol, except for the cases when such notation might be ambiguous. If so, we denote by $[\varphi]$ the operator $u \mapsto \varphi u$. The symbol $1_\delta$ stands for the characteristic function of a set $\delta$; the symbol "$\sim"$ denotes asymptotic equality.

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1. Periodic operators

There are many good expositions of the Floquet–Bloch theory for the partial differential operators, see, e.g. [Kuch], [RSim], [Skr]. However, it is rather difficult to suggest a source covering all our needs (operators in divergency form with the $L_\infty$-coefficients, acting in non-cylindrical domains). For this reason, we give here a self-contained exposition of this material.

1.1. We realize $\mathbb{R}^d$ as $\mathbb{R} \times \mathbb{R}^{d-1}$ and denote points $x \in \mathbb{R}^d$ as $x = (t, y)$ where $t \in \mathbb{R}$ and $y \in \mathbb{R}^{d-1}$. Let $\Omega \subset \mathbb{R}^d$ be a domain, periodic along the $t$-axis. More exactly, we always suppose that $\Omega$ obeys the following condition.

Condition ($\Omega$). The domain $\Omega \subset \mathbb{R}^d$ has the Lipschitz boundary and

$$(t, y) \in \Omega \iff (t + 1, y) \in \Omega.$$  

The projection of $\Omega$ on $\mathbb{R}^{d-1}$ along the $t$-axis is bounded.

Identifying the points $x = (t, y) \in \Omega$ which differ by a shift by the period, we get a domain

$$\Omega^\# \subset S^1 \times \mathbb{R}^{d-1}.$$  

Here $S^1$ is understood as the segment $[0, 1]$ whose edges are glued together. In other words, $\Omega^\#$ is the image of $\Omega$ under the canonical covering

$$\pi : \mathbb{R}^d \to S^1 \times \mathbb{R}^{d-1}.$$  

Evidently, $\Omega^\#$ is a bounded domain with the Lipschitz boundary.

Fix a "cell" (fundamental domain) $\Omega_0 \subset \Omega$, that is a bounded domain such that the mapping $\pi \mid \Omega_0$ is injective and $\overline{\pi(\Omega_0)} = \overline{\Omega^\#}$. It is always possible to choose $\Omega_0 \subset \Omega$ having the Lipschitz boundary, and we assume that this property is satisfied for the selected $\Omega_0$. By $\Omega_n$, $n \in \mathbb{Z}$ we denote the domain in $\mathbb{R}^d$, which is $\Omega_0$ shifted by $n$ along the $t$-axis. Denote $\partial_+ \Omega_n = (\partial \Omega_n \cap \partial \Omega_{n+1}) \cap \Omega$ and $\partial_- \Omega_n = \Omega_n \cap \partial \Omega_{n}$, then...
the mapping \( \pi \) is still injective on \( \Omega'_n \) and \( \pi(\Omega'_n) = \Omega^# \) for any \( n \in \mathbb{Z} \). Evidently, \( \Omega = \bigcup_{n \in \mathbb{Z}} \Omega'_n \).

Given a point \( x \in \Omega^# \), let \( t(x) \) stand for the number such that \( x(\pi(x)) = \pi(x) = x \). The function \( t(x) \) is well-defined, as soon as the cell \( \Omega_0 \) is chosen. It is useful to note that

\[
x = (t, y) \in \Omega'_n, \quad x = \pi(x) = t = n.
\]

We always identify a periodic function \( a(x) \) on \( \Omega \) with the function \( a(x) := a(x(x)) \) on \( \Omega^# \).

Consider a mapping (notation \( # \)) which transforms a function on \( \Omega \) into a function on \( \Omega^# \), depending on the parameter \( \xi \in \mathbb{R} \) (\textit{quasi-momentum}). Namely, for an arbitrary function \( u \in C(\mathbb{R}) \) which is zero for large \( |t| \), let

\[
(\#u)(\xi; t, y) = (2\pi)^{-\frac{1}{2}} e^{-i\xi t} \sum_{n \in \mathbb{Z}} e^{-i\xi n} u(t + n, y).
\]

Then \( u^#(\xi; t, y) \) is periodic in \( t \) and thus is well-defined as a function on \( \Omega^# \). Further, \( e^{i\xi t} u^#(\xi; t, y) \) is periodic in \( \xi \) with the period \( 2\pi \); for this reason, we usually consider \( \xi \in [-\pi, \pi] \).

Applying to \( e^{i\xi t} u^#(\xi; t, y) \) Parseval’s identity in \( \xi \), we obtain, for all \( (t, y) \in \Omega \):

\[
\sum_{n \in \mathbb{Z}} |u(t + n, y)|^2 = \int_{-\pi}^{\pi} |u^#(\xi; t, y)|^2 d\xi.
\]

Integration over \( \Omega_0 \) gives

\[
\int_{\Omega} |u(x)|^2 dx = \int_{-\pi}^{\pi} \int_{\Omega^#} |u^#(\xi; x)|^2 d\xi dx.
\]

Here \( dx \) stands for the volume element on \( S^1 \times \mathbb{R}^{d-1} \). The equality (1.3) allows one to extend by the continuity the operator \( # \) to the whole of \( L_2(\Omega) \). The extended operator (we keep for it the same notation \#) is an isometry of \( L_2(\Omega) \) into the Hilbert space

\[
\mathcal{H}^# := L_2((-\pi, \pi); L_2(\Omega^#)).
\]

It is easy to see that actually \( \text{Ran}(\#) = \mathcal{H}^# \). Indeed, let \( \varphi \in \mathcal{H}^# \). Consider the Fourier expansion (in \( \xi \)) of the function \( e^{i\xi t(x)} \varphi(\xi; x) \):

\[
e^{i\xi t(x)} \varphi(\xi; x) = (2\pi)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-i\xi n} \widehat{\varphi_n}(x).
\]

The coefficients \( \widehat{\varphi_n} \) belong to \( L_2(\Omega^#) \), and

\[
\sum_{n \in \mathbb{Z}} \| \widehat{\varphi_n} \|^2_{L_2(\Omega^#)} = \| \varphi \|^2_{\mathcal{H}^#} < \infty.
\]

Take

\[
u(x) = \widehat{\varphi_n}(x(x)) \quad \text{for } x \in \Omega'_n, \quad n \in \mathbb{Z}.
\]
Then \(u \in L_2(\Omega)\) by (1.5), and it is easy to see that \(u^\#(\xi; x) = \varphi(\xi; x)\).

Let now \(a \in L_\infty(\Omega)\) be a periodic function. Multiplying both sides of (1.2) by \(a(t, y)\) and integrating over \(\Omega_n\), we obtain an equality which generalizes (1.3) and holds true for an arbitrary \(u \in L_2(\Omega)\):

\[
\int_\Omega a(x) |u(x)|^2 \, dx = \int_{-\pi}^{\pi} \int_{\Omega^\#} a(x)|u^\#(\xi; x)|^2 \, d\xi dx.
\]

(1.7)

It follows from (1.7) that for any two functions \(u_1, u_2 \in L_2(\Omega)\) one has

\[
\int_\Omega a(x)u_1(x)u_2(x) \, dx = \int_{-\pi}^{\pi} \int_{\Omega^\#} a(x)u_1^\#(\xi; x)u_2^\#(\xi; x) \, d\xi dx.
\]

(1.8)

The function \(e^{i\xi t}\) is not well-defined on \(\Omega^\#\), however the following operator is well-defined:

\[
\nabla^\xi = \nabla_x^\xi : \varphi \mapsto e^{-i\xi t} \nabla(e^{i\xi t} \varphi) := \{\partial_t \varphi + i\xi \varphi, \nabla_y \varphi\}.
\]

(1.9)

Suppose that \(u \in C^1(\overline{\Omega})\) and \(u(t, y) = 0\) for large \(|t|\). A direct calculation shows that

\[
(\nabla u)^\#(\xi; x) = (\nabla^\xi u^\#)(\xi; x).
\]

(1.10)

The equalities (1.10) and (1.3) imply

\[
\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2) \, dx = \int_{-\pi}^{\pi} \int_{\Omega^\#} (|\nabla_x^\xi u^\#(\xi; x)|^2 + |u^\#(\xi; x)|^2) \, d\xi dx
\]

\[
= \int_{-\pi}^{\pi} \int_{\Omega^\#} \left( |\nabla_x u^\#|^2 - 2\text{Im}(u^\# \partial_t u^\#) + (\xi^2 + 1)|u^\#|^2 \right) \, d\xi dx.
\]

(1.11)

For any \(\xi \in \mathbb{R}\) the inner integral in the third term of (1.11) is equivalent to the standard metric form of the space \(H^1(\Omega^\#)\), and the corresponding metrics are mutually equivalent uniformly with respect to \(\xi \in [-\pi, \pi]\). The equality (1.11) shows that the mapping \(^\#\) is an isomorphism of the space \(H^1(\Omega)\) onto a subspace of the Hilbert space \(L_2((-\pi, \pi); H^1(\Omega^\#))\). Actually, the image is the whole of \(L_2((-\pi, \pi); H^1(\Omega^\#))\). To see this, it is enough to prove that the image contains any function of the type

\[
\varphi(\xi; x) = \sqrt{\pi/2} e^{-i\xi k} v(x), \quad v \in C^1(\overline{\Omega^\#}), \quad k \in \mathbb{Z},
\]

since they span the target space. For the function \(\varphi(\xi, \cdot)\) the coefficients in the expansion (1.4) are

\[
\varphi_n(x) = \frac{\sin \pi(t(x) - k + n)}{t(x) - k + n} v(x), \quad n \in \mathbb{Z}.
\]

Construct the function \(u\) by (1.6), then \(\varphi = u^\#\) and \(u \upharpoonright \Omega_n \in C^1(\overline{\Omega_n})\) for each \(n\). Let now \(x_0 \in \partial \Omega_{n-1} \cap \partial \Omega_n\). It follows from (1.1) and from the continuity of \(v\) on \(\Omega^\#\) that the limiting values of the functions \(u \upharpoonright \Omega_{n-1}\) and \(u \upharpoonright \Omega_n\) at the point \(x_0\) are equal. Hence, \(u \in C(\overline{\Omega})\) and therefore \(u \in H^{1, \text{loc}}(\Omega)\). Now an evident estimation of integrals shows that actually \(u \in H^1(\Omega)\) on the whole of \(\Omega\). So we have proved that

\[
\# H^1(\Omega) = L_2((-\pi, \pi); H^1(\Omega^\#)),
\]

(1.12)

It is also easy to see that

\[
\# H^{1, \text{loc}}(\Omega) = L_2((-\pi, \pi); H^{1, \text{loc}}(\Omega^\#)),
\]

(1.13)
1.2. Suppose that on $\Omega$ a quadratic form is given,

$$
P[u] = \int_{\Omega} \left( (g(x)\nabla u, \nabla u) + p(x)|u|^2 \right) dx
$$

$$
= \int_{\Omega} \left( \sum_{j,k=1}^{d} g_{jk}(x) \partial_j u \partial_k u + p(x)|u|^2 \right) dx. \tag{1.14}
$$

Here are our assumptions on $P$.

**Condition (P).** The functions $g_{jk}$ and $p$ are real-valued, belong to $L_\infty(\Omega)$ and are periodic in $t$ with the period 1. The matrix $g(x)$ is symmetric and uniformly positive definite with respect to $x \in \Omega$.

We consider the quadratic form (1.14) either on $\mathcal{D}_N = H^1(\Omega)$, or on $\mathcal{D}_P = H^{1,0}(\Omega)$. The corresponding self-adjoint operators in $L_2(\Omega)$ are denoted by $\mathcal{P}_N$ or $\mathcal{P}_P$, but usually we drop the index and write simply $\mathcal{D}$ for the domain of the quadratic form and $\mathcal{P}$ for the operator.

Along with $P[u]$, consider the family of quadratic forms on $\Omega^\#$:

$$
P_{\xi}^\#[\varphi] = \int_{\Omega^\#} \left( (g^{\neg}(\xi)\varphi, \nabla^{\neg}(\xi)\varphi) + p|\varphi|^2 \right) dx \tag{1.15}
$$

on the domain which does not depend on $\xi$:

$$
\mathcal{D}_N := \text{Dom}(P_{\xi}^\#_{\mathcal{N}}) = H^1(\Omega^\#) \quad \text{(case } \mathcal{N}),
$$

$$
\mathcal{D}_P := \text{Dom}(P_{\xi}^\#_{\mathcal{P}}) = H^{1,0}(\Omega^\#) \quad \text{(case } \mathcal{D}). \tag{1.16}
$$

Like above, the index $\mathcal{N}$ or $\mathcal{D}$ is usually dropped.

It follows from (1.7), (1.8) and (1.10) that

$$
P[u] = \int_{-\pi}^{\pi} P_{\xi}^\#[u^\#(\xi, \cdot)] d\xi. \tag{1.17}
$$

The equalities (1.16) together with (1.12) and (1.13) show that $u \in \mathcal{D}$ if and only if $u^\# \in L_2((-\pi, \pi); d\theta)$.

For each $\xi \in \mathbb{R}$ the quadratic form $P_{\xi}^\#$ generates in $L_2(\Omega^\#)$ a self-adjoint operator, say $\mathcal{P}_{\xi}^\#$. If necessary, we shall use the more detailed notations, like $\mathcal{P}_{\xi}^\#_{\mathcal{N}}$ or $\mathcal{P}_{\xi}^\#_{\mathcal{P}}$. The spectrum of $\mathcal{P}_{\xi}^\#$ is discrete. Let $\{E_k(\xi)\}$ be its eigenvalues in the order of increase,

$$
E(\xi) := E_1(\xi) \leq E_2(\xi) \leq \ldots
$$

We denote by $\{\omega_k(\xi; x)\}$ the corresponding orthonormal system of the eigenfunctions. We assume

$$
E(0) = 0. \tag{1.18}
$$

This can be always achieved by shifting $p(x)$ by an appropriate constant. As we shall see at the end of Subsection 1.3, (1.18) is equivalent to the similar assumption on the original operator $\mathcal{P}$:

$$
\min\{\lambda \in \text{Spec}(\mathcal{P})\} = 0. \tag{1.19}
$$

The following statement collects the properties of the eigenvalues $E_k(\xi)$ and eigenfunctions $\omega_k(\xi; x)$. Their proofs are given in Appendix A1.
Theorem 1.1. Let the conditions (Ω), (P) and (1.18) be satisfied. Then the following assertions hold true.

1° Each function $E_k(\xi)$ is continuous, even and $2\pi$-periodic. It is possible to choose the eigenfunctions in such a way that $\omega_k(\xi, \cdot)$ is measurable as a $H^1(\Omega^\#)$-valued function on $\mathbb{R}$.

2° Let $E_k(\xi_0)$ be a simple eigenvalue of $P_{\xi_0}^\#$. Then in some neighborhood $\Xi \ni \xi_0$ the function $E_k(\xi)$ is real analytic, and there exists a real analytic branch of the corresponding eigenfunction, $\omega_k(\xi, \cdot): \Xi \to H^1(\Omega^\#)$.

3° The eigenvalue $E(0) = 0$ is simple and $E(\xi) > 0$ for $\xi \not\in 2\pi \mathbb{Z}$. Moreover, at $\xi = 0$ one has

$$E(\xi) = \beta \xi^2 + O(\xi^4), \quad \beta > 0. \quad (1.20)$$

4° All the eigenfunctions $\omega_k(\xi, \cdot)$ are continuous on $\Omega^\#$. There exist constants $C(k)$ such that

$$|\omega_k(\xi; x)| \leq C(k), \quad x \in \Omega^\#, \xi \in \mathbb{R}, k \in \mathbb{N}. \quad (1.21)$$

The eigenfunction branch $\omega_k(\xi; \cdot)$ described in 2°, considered as a $C(\Omega^\#)$-valued function on $\Xi$, is real analytic.

5° The eigenfunction $\omega(x) := \omega_1(0; x)$ (principal eigenfunction of the operator $P_0^\#$) can be chosen positive on $\Omega$. In the case $N$ this function is bounded away from zero.

1.3. In the reasonings below we consider $\omega_k(\xi, \cdot)$ as periodic functions on $\Omega$. For each $k \in \mathbb{N}$ define the integral operator $\Psi_k : L_2(\Omega) \to L_2(-\pi, \pi)$,

$$\Psi_k u(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\Omega} e^{-i\xi t} \omega_k(\xi; t, y) u(t, y) dt dy. \quad (1.22)$$

If $u$ has a bounded support, the integral in (1.22) can be written as

$$\Psi_k u(\xi) = (2\pi)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} \int_{\Omega} e^{-i\xi t} \omega_k(\xi; t, y) u(t + n, y) dt dy.$$

Due to the periodicity in $t$, it follows

$$\Psi_k u(\xi) = (2\pi)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} \int_{\Omega} e^{-i\xi t} \omega_k(\xi; t, y) u(t, y) dt dy$$

$$= \int_{\Omega^\#} u^\#(\xi; x) \omega_k(\xi; x) dx = (u^\#(\xi; \cdot), \omega_k(\xi; \cdot))_{L_2(\Omega^\#)}. \quad (1.23)$$

We see that $(\Psi_k u)(\xi)$ is nothing but the $k$-th Fourier coefficient of the function $u^\#(\xi, \cdot)$ with respect to the complete orthonormal system $\{\omega_k(\xi, \cdot)\}$. By Parseval’s identity,

$$\sum_{k=1}^{\infty} |(\Psi_k u)(\xi)|^2 = \int_{\Omega^\#} |u^\#(\xi; x)|^2 d\mathbf{x}.$$ 

Integrating over $\xi \in (-\pi, \pi)$ and using (1.3), we arrive at

$$\sum_{k=1}^{\infty} \|\Psi_k u\|_{L_2(-\pi, \pi)}^2 = \|u\|_{L_2(\Omega)}^2. \quad (1.24)$$
This shows in particular that all the operators $\Psi_k$ are bounded. By the continuity, (1.24) extends to the whole of $L_2(\Omega)$. Along with (1.24), one has

$$
\sum_{k=1}^{\infty} (\Psi_k u_1, \Psi_k u_2)_{L_2(-\pi, \pi)} = (u_1, u_2)_{L_2(\Omega)}, \quad u_1, u_2 \in L_2(\Omega).
$$

Consider now the family of linear subspaces $F_k \subset L_2(\Omega)$, $k \in \mathbb{N}$:

$$
F_k = \{ u \in L_2(\Omega) : u^\#(\xi; x) = z(\xi) \omega_k(\xi; x); \quad z \in L_2(-\pi, \pi) \}.
$$

Evidently, for $u \in F_k$ one has $\|u\|_{L_2(\Omega)} = \|z\|_{L_2(-\pi, \pi)}$, and from (1.23) it follows that $\Psi_k u = z$ and $\Psi_l u = 0$ for $l \neq k$. In other words, $\Psi_k$ is a partial isometry of $L_2(\Omega)$ onto $L_2(-\pi, \pi)$, with $F_k$ as the isometry domain. This yields that each $F_k$ is a closed subspace, they are mutually orthogonal, and

$$
L_2(\Omega) = \bigoplus_{k=1}^{\infty} F_k.
$$

(1.25)

The operator $\Psi_k^*$ is an isometry of $L_2(-\pi, \pi)$ onto $F_k$, its analytic representation is

$$
(\Psi_k^* z)(t, y) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} e^{i \xi t} \omega_k(\xi; t, y) z(\xi) d\xi.
$$

(1.26)

The operator in $L_2(\Omega)$

$$
\Pi_k = \Psi_k^* \Psi_k
$$

(1.27)

acts as the orthoprojection onto $F_k$. Note that $\sum_k \Pi_k = I$ which is equivalent to (1.25). Taking into account (1.17) and (1.23), we derive from here

$$
P[u] = \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} E_k(\xi) |(\Psi_k u)(\xi)|^2 d\xi, \quad u \in \mathcal{D}.
$$

So, the orthogonal decomposition (1.25) diagonalizes the quadratic form $P[u]$ and therefore, for each $k$ the subspace $F_k$ reduces the operator $P$. The part $P_k$ of $P$ in $F_k$ is unitary equivalent to the operator of multiplication $z(\xi) \mapsto E_k(\xi) z(\xi)$ in $L_2(-\pi, \pi)$. This implies that the spectrum of $P_k$ coincides with the segment

$$
\Delta_k = [\min_{\xi} E_k(\xi), \max_{\xi} E_k(\xi)]
$$

and the spectrum of $P$ is

$$
\text{Spec}(P) = \bigcup_{k \in \mathbb{N}} \Delta_k.
$$

(1.28)

It follows from (1.28) and Theorem 1.1, 3° that the relations (1.18) and (1.19) are equivalent to each other.

The projections (1.27) commute with $P$ and thus, with any function of $P$: for any bounded Borel function $f(\lambda)$ on $\mathbb{R}$ one has

$$
f(P) = \sum_{k=1}^{\infty} \Psi_k^* f(E_k(\xi)) \Psi_k.
$$

(1.29)

1.4. It is often convenient to extend the functions $\Psi_k u$ (see (1.22)) outside the segment $[-\pi, \pi]$ by zero and hence, to treat the operators $\Psi_k$ as acting from $L_2(\Omega)$ to $L_2(\mathbb{R})$. Respectively, the operators $\Psi_k^*$ act from $L_2(\mathbb{R})$ to $L_2(\Omega)$. The analytic expression (1.26) for $\Psi_k^*$ does not change.
2. Setting of the problem. Main result for non-negative perturbations

2.1. Let $\mathcal{P}$ (more precisely, $\mathcal{P}_N$ or $\mathcal{P}_B$) be an operator of the type, described in Section 1. We consider it as the unperturbed operator. The perturbed operator is $\mathcal{P} - \alpha V$, where $\alpha > 0$ is the large parameter and $V$ is a real-valued function on $\Omega$. The restrictions on $V$ depend on the dimension. We always suppose $V \in X(\Omega)$ where

$$X(\Omega) = L_2^d(\Omega) \quad (d > 2); \quad X(\Omega) = l_1(\{\mathcal{O}(\Omega_n)\}), \quad \mathcal{O} = L \log L \quad (d = 2). \quad (2.1)$$

Here the case $d = 2$ needs an explanation. Recall first the definition of the Orlicz space $\mathcal{O} = L \log L$ on an arbitrary set $B$ of finite measure. Namely, $f \in \mathcal{O}(B)$ if and only if

$$\mathcal{O}_B(f) := \int_B |f| \log(\varepsilon + |f|) dx < \infty.$$ 

The above expression is not a norm; one of ways to introduce the norm is

$$\|f\|_{\mathcal{O}(B)} := \inf\{\delta > 0 : \mathcal{O}_B(\delta^{-1} f) \leq 1\}.$$ 

Finally, a function $f \in \mathcal{O}_{oc}(\Omega)$ belongs to the space $l_1(\{\mathcal{O}(\Omega_n)\})$ if and only if

$$\|f\|_{l_1(\{\mathcal{O}(\Omega_n)\})} := \sum_{n \in \mathbb{Z}} \|f\|_{\mathcal{O}(\Omega_n)} < \infty. \quad (2.2)$$

Note that $\|f\|_{L_1(B)} \leq \|f\|_{\mathcal{O}(B)}$ and therefore $X(\Omega) \subset L_2^d(\Omega)$ also for $d = 2$. However, for $d = 2$ the inclusion $V \in L_1(\Omega)$ does not guarantee validity of the estimate (2.5) and of the asymptotics (2.6) described below, in Proposition 2.1. Note also that for any $\sigma > 1$ the inequality is satisfied

$$\|f\|_{\mathcal{O}(B)} \leq C \|f\|_{L_\sigma(B)}, \quad C = C(\sigma, B) < \infty.$$ 

It follows that

$$\|f\|_{X(\Omega)} \leq C(\sigma, \Omega_0) \sum_{n \in \mathbb{Z}} \|f\|_{L_\sigma(\Omega_n)}, \quad \sigma > 1, \quad d = 2. \quad (2.3)$$

Let the conditions (Ω), (P) be satisfied and and let $V \in X(\Omega)$. Then for any $\alpha > 0$ the self-adjoint operator $\mathcal{P} - \alpha V$ in $L_2^d(\Omega)$ is well-defined as the form-sum and its negative spectrum is discrete. Hence,

$$N_-(\mathcal{P} + \gamma^2 I - \alpha V) < \infty, \quad \text{any } \alpha, \gamma \neq 0.$$ 

Below we give the estimate and the asymptotic formula for $N_-(\mathcal{P} + \gamma^2 I - \alpha V)$ considered as a function in $\alpha$, when $\gamma \neq 0$ is fixed. Given a non-negative function $F$ on $\Omega$, we denote

$$\Theta(g, F) := \frac{1}{(2\sqrt{\pi})^d \Gamma(1 + \frac{d}{2})} \int_{\Omega} \frac{F^*}{\sqrt{\det g}} dx. \quad (2.4)$$

This is the well-known coefficient in the Weyl-type asymptotic formula.
Proposition 2.1. Let the conditions \((\Omega), (P)\) and (1.18) be satisfied, \(V \in X(\Omega)\) and \(\gamma \neq 0\). Then the following estimate and asymptotic formula hold true:

\[
N_-(P + \gamma^2 I - \alpha V) \leq C(\gamma, \Omega)\|V\|_{X(\Omega)}^{\frac{\alpha}{\gamma^2}} \quad \text{any } \alpha > 0; \quad (2.5)
\]

\[
N_-(P + \gamma^2 I - \alpha V) \sim \Theta(g, V_+)\alpha^{\frac{\sigma}{\gamma}}, \quad \alpha \to \infty. \quad (2.6)
\]

The facts listed in Proposition 2.1 should be considered as essentially known, actually they are true without the periodicity assumptions. See Appendix A.2 for more detail.

The estimate (2.5) and the inequality (2.3) imply

\[
N_-(P + \gamma^2 I - \alpha V) \leq C(\gamma, \sigma, \Omega) \sum_{n \in \mathbb{Z}} \|V\|_{L^\sigma(\Omega_n)}, \quad \sigma > 1, d = 2. \quad (2.7)
\]

The estimate (2.7) was obtained in [BBor]. It is rougher than (2.5), but allows one to avoid using the Orlicz spaces.

2.2. The asymptotics (2.6) is not uniform with respect to \(\gamma\) and in general it fails for \(\gamma = 0\). Our goal is to investigate the asymptotic behaviour of \(N_-(P - \alpha V)\). The new asymptotic formula which we establish, involves some objects appearing in the Floquet–Bloch decomposition of the operator \(P\).

Namely, let us introduce a family of auxiliary operators acting in \(L^2(\mathbb{R})\):

\[
A_{\alpha Q} = -\partial_t^2 - \alpha Q, \quad \alpha > 0 \quad (2.8)
\]

with the “effective potential”

\[
Q(t) = Q_{\mathcal{P},V}(t) = \beta^{-1} \int_{\Omega(t)} V(t, y)\omega^2(t, y)dy. \quad (2.9)
\]

In (2.9) \(\Omega(t)\) is the cross-section of \(\Omega\), \(\Omega(t) = \{y \in \mathbb{R}^{d-1} : (t, y) \in \Omega\}\), \(\beta\) is the coefficient from (1.20), and \(\omega(x)\) is the principal eigenfunction of the operator \(\mathcal{P}_0\) (see Theorem 1.1, 5°). The function \(\omega(x)\) is considered here as a periodic function on \(\Omega\), rather than a function on \(\Omega^\#\).

Our main results (Theorems 2.2 and 5.1) basically say that under certain assumptions the asymptotic behaviour of \(N_-(P - \alpha V)\) is given by the relation

\[
N_-(P - \alpha V) \sim \Theta(g, V_+)\alpha^{\frac{\sigma}{\gamma}} + N_-(\alpha^2 - \alpha Q_{\mathcal{P},V}), \quad \alpha \to \infty. \quad (2.10)
\]

Comparing (2.6) and (2.10), we see that for \(\gamma = 0\) an additional term appears in the asymptotic formula. This can be interpreted as a “threshold effect”. We discuss it in more detail in Section 7.

The inclusion \(V \in X(\Omega)\) does not determine the behaviour of this additional term in (2.10), actually it even does not guarantee its finiteness for all \(\alpha > 0\). In particular, a necessary condition for this finiteness is \(Q_{\mathcal{P},V} \in L^1(\mathbb{R})\) which is not implied by \(V \in X(\Omega)\), cf. Remark 2 to Theorem 5.1. If the function \(N_-(A_{\alpha Q_{\mathcal{P},V}})\) is finite for all \(\alpha > 0\), its behaviour as \(\alpha \to \infty\) can be quite different. This is illustrated by the examples presented in Subsection 7.4. All this shows that additional
assumptions about $V$ are necessary. It is convenient to state them directly in terms of appropriate regularity conditions of the function $N_-(A_{\alpha Q_p, V})$.

Let $\Phi(\alpha)$ be a positive function on $\mathbb{R}_+$. It is called \textit{regularly varying} if there exists a number $q \in \mathbb{R}$ (\textit{order} of $\Phi$) such that

$$\frac{\Phi(\theta \alpha)}{\Phi(\alpha)} \to \theta^q \text{ as } \alpha \to \infty \quad \text{for any } \theta > 0.$$ (2.11)

We denote by $\mathfrak{S}_q$ the set of all regularly varying functions of order $q$. The theory of regularly varying function is presented in the book [Sen].

Now we are in a position to formulate our main result for the case $V \geq 0$.

\textbf{Theorem 2.2.} Let the conditions (\Omega), (P) and (1.18) be satisfied, and let $V \in X(\Omega)$, $V \geq 0$. Then

1° The negative spectrum of the operator $\mathcal{P} - \alpha V$ in $L_2(\Omega)$ is finite for all $\alpha > 0$ at once if and only if the same is true for the operators $-\partial_t^\alpha - \alpha Q_{\mathcal{P}, V}$ in $L_2(\mathbb{R})$.

2° The asymptotic formula (2.6) holds true also for $\gamma = 0$ if and only if

$$N_-( -\partial_t^\alpha - \alpha Q_{\mathcal{P}, V}) = o(\alpha^{\frac{d}{2}}), \quad \alpha \to \infty.$$ (2.12)

3° Suppose that there is a regularly varying function $\Phi$ of some order $q \geq \frac{d}{2}$, such that

$$N_-( -\partial_t^\alpha - \alpha Q_{\mathcal{P}, V}) \sim \Phi(\alpha), \quad \alpha \to \infty.$$

Then the asymptotic relation (2.10) is satisfied.

The proof is postponed until Subsection 6.4.

\textbf{Remarks.} 1. The assertion 3° remains true also for $q < \frac{d}{2}$. However, the result of 2° is stronger because it does not involve any regularity requirements on the behaviour of $N_-( -\partial_t^\alpha - \alpha Q_{\mathcal{P}, V})$.

2. The relation (2.12) can be reformulated in terms of the effective potential $Q_{\mathcal{P}, V}$ itself, without addressing to the corresponding operator (2.8). Namely, under the assumption $V \in X$ (2.12) is equivalent to the inclusion

$$\left\{ \int_{2^{k-1} < |t| < 2^k} t Q_{\mathcal{P}, V}(t) dt; \ k \in \mathbb{N} \right\} \in \mathfrak{I}_{q, \infty}^0,$$

cf. Lemma 4.1, 2° in Section 4 below. The classes $\mathfrak{I}_{q, \infty}^0$ are defined in Subsection 3.1.

3. We would like to stress that under the assumptions of Theorem 2.2, 3° the asymptotic behaviour of $N_-(\mathcal{P} - \alpha V)$ is not of the Weyl type and not necessarily powerlike. In particular, for $q > \frac{d}{2}$ the first ("Weylian") term on the right-hand side of (2.10) is suppressed by the second term and can be dropped. The most interesting case is, of course, $q = \frac{d}{2}$ when both terms in (2.10) may contribute to the asymptotics.

In Theorem 5.1 the assumption $V \geq 0$ will be withdrawn. We need some preparatory material in order to give this refined formulation.
3. Auxiliary information from the Hilbert space theory

3.1. We start with notations concerning number sequences. The distribution function of a sequence \( h = \{h_j\} \) is defined as

\[
n(\lambda, h) = \# \{ j : |h_j| > \lambda \}, \quad \lambda > 0.
\]

Given \( 0 < q \leq \infty \), we denote \( \|h\|_q := \|h\|_q \), and if \( 0 < q < \infty \), then

\[
\|h\|_{q, \infty} = \sup_{\lambda > 0} \lambda^{n(\lambda, h)}^{1/q}.
\]  

(3.1)

The “weak \( l_q \)-space” \( l_{q, \infty} \) is defined as \( l_{q, \infty} = \{ h : \|h\|_{q, \infty} < \infty \} \). This is a complete linear quasi-normed non-separable space with respect to the quasi-norm (3.1). The subset

\[
l_{q, \infty}^0 := \{ h \in l_{q, \infty} : \Delta_q(h) = 0 \}
\]

is a linear closed separable subspace in \( l_{q, \infty} \). The set of sequences with a finite number of nonzero elements is dense in \( l_{q, \infty}^0 \).

More general linear spaces of number sequences can be introduced with the help of regularly varying functions, see their definition in (2.11). Recall that \( \mathcal{F}_q \) stands for the set of all regularly varying functions, satisfying (2.11) with a given \( q \in \mathbb{R} \). The simplest example of \( \Phi \in \mathcal{F}_q \) is

\[
\Phi_q(\alpha) := \alpha^q.
\]

Regularly varying functions of order 0 are called slowly varying. Any function \( \Phi \in \mathcal{F}_q \) can be represented as

\[
\Phi(\alpha) = \alpha^q \Lambda(\alpha), \quad \Lambda \in \mathcal{F}_0.
\]

It is useful to note that any smooth positive function \( \Lambda(\alpha) \) on \( \mathbb{R}_+ \), such that

\[
\frac{\alpha \Lambda'(\alpha)}{\Lambda(\alpha)} \to 0 \quad \text{as} \ \alpha \to \infty,
\]

belongs to \( \mathcal{F}_0 \) (see [Sen], p.7). Based upon this test, it is easy to check that the functions which for \( \alpha \gg 0 \) are equal \((\log \alpha)^p\), \((\log \alpha)^{p_1}(\log \log \alpha)^{p_2}\) (where \( p_1, p_2 \in \mathbb{R} \)), etc. lie in \( \mathcal{F}_0 \). More exotic but interesting examples are

\[
\Lambda(\alpha) = 2 + \sin(\log \alpha)^p, \quad p < 1
\]

and

\[
\Lambda(\alpha) = 1 + \log \log \alpha \sin^2 \log \log \log \alpha.
\]

The function (3.2) is bounded and bounded away from zero, but has no limit as \( \alpha \to \infty \). The function (3.3) is unbounded but does not tend to infinity as \( \alpha \to \infty \).

The classes \( l_{\Phi, \infty} \) are introduced by analogy with the spaces \( l_{q, \infty} \):

\[
h \in l_{\Phi, \infty} \iff \sup_{\alpha > 0} \frac{n(\alpha^{-1}, h)}{\Phi(\alpha)} < \infty.
\]
Evidently, \( l_{\Phi, \infty} = l_{q, \infty} \). We do not discuss here the topological properties of the general spaces \( l_{\Phi, \infty} \) and of the corresponding spaces of operators introduced in the next subsection. In this respect see [W] where spaces of quite a similar nature were studied in detail.

On \( l_{\Phi, \infty} \) the following functionals are finite:

\[
\Delta_{\Phi}(\hbar) = \lim_{\alpha \to \infty} \sup \frac{n(\alpha^{-1}, \hbar)}{\Phi(\alpha)}, \quad \delta_{\Phi}(\hbar) = \lim_{\alpha \to \infty} \inf \frac{n(\alpha^{-1}, \hbar)}{\Phi(\alpha)}.
\]

We write \( \Delta_q, \delta_q \) instead of \( \Delta_{\Phi(q)} \) and \( \delta_{\Phi(q)} \):

\[
\Delta_q(\hbar) = \lim_{\alpha \to \infty} \sup \alpha^{-q} n\left(\alpha^{-1}, \hbar\right), \quad \delta_q(\hbar) = \lim_{\alpha \to 0} \inf \alpha^{-q} n\left(\alpha^{-1}, \hbar\right).
\]

In each class \( l_{\Phi, \infty} \) we select the subclass

\[ l_{\Phi, \infty}^0 = \{ \hbar \in l_{\Phi, \infty} : \Delta_{\Phi}(\hbar) = 0 \} \]

Clearly, \( l_{\Phi(q), \infty}^0 = l_{q, \infty}^0 \).

3.2. By \( B \) and \( C \) we denote the spaces of all bounded and of all compact linear operators, acting between two (possibly, coinciding) Hilbert spaces. As a rule, in the notations for norms and scalar products we do not reflect the spaces, unless this may lead to an ambiguity.

The sequence of singular numbers of an operator \( T \in C \) (see [GoKr] or [Sim]) is denoted by \( s(T) = \{ s_k(T) \} \). If \( T \) is self-adjoint, then \( \lambda^+(T) = \{ \lambda_k^+(T) \} \) stands for the sequence of its positive eigenvalues and \( \lambda^-(T) := \lambda^+(-T) \). If \( T \geq 0 \), then \( s(T) = \lambda^+(T) \). The corresponding distribution functions are denoted by \( n(\cdot, T) := n(\cdot, s(T)) \) and \( n_{\pm}(\cdot, T) := n(\cdot, \lambda^\pm(T)) \).

Recall that according to the variational principle the eigenvalues of a self-adjoint operator \( T \in C \) are critical values of its Rayleigh functional (or Rayleigh quotient)

\[
R_T(u) = \frac{(Tu, u)}{\|u\|^2}, \quad u \in \mathcal{H} \setminus \{0\}.
\]

Respectively, the squared singular numbers of an arbitrary operator \( T \in C \) are the non-zero critical values of either of the Rayleigh functionals \( R_{T^*T} \) and \( R_{TT^*} \).

Let \( \Phi \) be a regularly varying function of positive order. The corresponding class \( C_{\Phi, \infty} \) is introduced in a natural way:

\[ T \in C_{\Phi, \infty} \iff s(T) \in l_{\Phi, \infty}. \]

We also put

\[
\Delta_{\Phi}(T) := \Delta_{\Phi}(s(T)), \quad \delta_{\Phi}(T) := \delta_{\Phi}(s(T)) \tag{3.4}
\]

and, for \( T = T^* \),

\[
\Delta_{\Phi}^{\pm}(T) := \Delta_{\Phi}(\lambda^\pm(T)), \quad \delta_{\Phi}^{\pm}(T) := \delta_{\Phi}(\lambda^\pm(T)) \tag{3.5}
\]
In each $C_{\Phi,\infty}$ we select the subclass

$$C_{\Phi,\infty}^0 := \{ T \in C_{\Phi,\infty} : \Delta_{\Phi}(T) = 0 \}.$$ 

The classes $C_{\Phi_{(q)},\infty}$ are nothing but the “weak Schatten ideals” $C_{q,\infty}$, and $C_{\Phi_{(q)},\infty}^0$ coincide with their separable subspaces $C_{q,\infty}^0$. We write $\Delta_q$ instead of $\Delta_{\Phi_{(q)}}$ etc., and denote

$$||T||_{q,\infty} := ||\Phi(T)||_{q,\infty}.$$ 

The symbol $|| \cdot ||_q$ stands for the quasi-norm (norm if $q \geq 1$) of an operator in the usual Schatten ideal $C_q$.

Many properties of the classes $C_{\Phi,\infty}$ are parallel to the well-known facts from the theory of the spaces $C_{q,\infty}$.

**Lemma 3.1.** Let $\Phi \in \mathcal{S}_q$ for some $q > 0$. Then for any $T_1, T_2 \in C_{\Phi,\infty}$ the inequality is satisfied

$$|D_{\Phi}(T_1)^\frac{1}{\sigma_\Phi} - D_{\Phi}(T_2)^\frac{1}{\sigma_\Phi}| \leq \Delta_{\Phi}(T_1 - T_2)^\frac{1}{\sigma_\Phi}. \tag{3.6}$$

where $D_{\Phi}$ stands for any of the functionals (3.4), (3.5).

**Proof.** We restrict ourselves with the case $D_{\Phi} = \Delta_{\Phi}$. By Weyl’s inequality,

$$n(\alpha^{-1}, T_1) \leq n(\theta \alpha^{-1}, T_2) + n((1 - \theta)\alpha^{-1}, T_1 - T_2), \quad \text{any } \theta \in (0, 1).$$

Dividing both parts of this inequality by $\Phi(\alpha)$, we find

$$\frac{n(\alpha^{-1}, T_1)}{\Phi(\alpha)} \leq \frac{\Phi(\theta \alpha)}{\Phi(\alpha)} \frac{n(\theta \alpha^{-1}, T_2)}{\Phi(\theta \alpha)} + \frac{\Phi((1 - \theta)\alpha)}{\Phi(\alpha)} \frac{n((1 - \theta)\alpha^{-1}, T_1 - T_2)}{\Phi((1 - \theta)\alpha)}.$$

Passing here to the upper limits as $\alpha \to \infty$ and taking (2.11) into account, we obtain

$$\Delta_{\Phi}(T_1) \leq \theta^{-q} \Delta_{\Phi}(T_2) + (1 - \theta)^{-q} \Delta_{\Phi}(T_1 - T_2).$$

After minimization of the right-hand side over $\theta \in (0, 1)$ this gives

$$\Delta_{\Phi}(T_1) \leq (\Delta_{\Phi}(T_2)^\frac{1}{\sigma_\Phi} + \Delta_{\Phi}(T_1 - T_2)^\frac{1}{\sigma_\Phi})^{q+1}.$$

Exchanging the roles of $T_1$ and $T_2$, we arrive at the inequality (3.6) for the functionals $\Delta_{\Phi}$. $\square$

For $\Phi = \Phi_{(q)}$ Lemma 3.1 coincides with [BSol 3], Cor.XI.6.5; actually, the proof is the same. See also [W].

It follows from Lemma 3.1 that

$$T_1, T_2 \in C_{\Phi,\infty}, \Delta_{\Phi}(T_1 - T_2) = 0 \implies D_{\Phi}(T_1) = D_{\Phi}(T_2). \tag{3.7}$$

For the spaces $C_{q,\infty}$ this fact is usually referred to as Weyl – Ky Fan’s Lemma. We preserve this name also for the general case.

Now we obtain one more useful inequality.
Lemma 3.2. Let \( \Phi \) be a regularly varying function of order \( q > 0 \), \( T_1^*T_1 \in \mathcal{C}_\Phi,\infty \) and \( T_2^*T_2 \in \mathcal{C}_\Phi,\infty \). Then also \( T_1^*T_2 \in \mathcal{C}_\Phi,\infty \) and
\[
\Delta_\Phi^2(T_1^*T_2) \leq \Delta_\Phi(T_1^*T_1)\Delta_\Phi(T_2^*T_2).
\]

Proof. By Horn’s inequality, we have for any \( \theta > 0 \)
\[
n(\alpha^{-1}, T_1^*T_2) \leq n(\theta\alpha^{-\frac{1}{2}}, T_1) + n(\theta^{-1}\alpha^{-\frac{1}{2}}, T_2) = n(\theta^2\alpha^{-1}, T_1^*T_1) + n(\theta^{-2}\alpha^{-1}, T_2^*T_2).
\]
Dividing both parts of this inequality by \( \Phi(\alpha) \) and passing to the upper limits, we obtain
\[
\Delta_\Phi(T_1^*T_2) \leq \theta^{2q}\Delta_\Phi(T_1^*T_1) + \theta^{-2q}\Delta_\Phi(T_2^*T_2), \quad \text{any } \theta > 0.
\]
Minimization over \( \theta > 0 \) gives (3.8). \( \square \)

3.3. Let \( a[u] \) be a positive (that is, \( a[u] > 0 \) for \( u \neq 0 \)) quadratic form in a Hilbert space \( \mathcal{H} \), with the dense domain \( \text{Dom}(a) = d \). The corresponding sesquilinear form defined on \( d \times d \) is denoted by \( a[u, v] \). Suppose that \( a \) is closed. Let \( A \) be the self-adjoint operator in \( \mathcal{H} \), associated with the quadratic form \( a \). Introduce the Hilbert space \( \mathcal{H}_a \), which is the completion of \( d \) with respect to the scalar product
\[
(u, v)_a = a[u, v] = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v).
\]
The corresponding norm is denoted by \( \|u\|_a \).

Let \( b[u] \) be another real quadratic form defined on \( d \). In this paper we always deal with the case when \( b[u] \) corresponds to a self-adjoint operator \( B \) in \( \mathcal{H} \):
\[
b[u] = (Bu, u), \quad u \in \text{Dom}(B).
\]
Suppose that \( \text{Dom}(|B|^{\frac{1}{2}}) \supset d \) and
\[
\|B^{\frac{1}{2}}u\|^2 \leq Ca[u], \quad u \in d.
\]
Then also \( |b[u]| \leq Ca[u] \) on \( d \). It follows that \( b \) extends to \( \mathcal{H}_a \) by the continuity and defines there a self-adjoint bounded operator, say \( T(a, b) \). The restriction of its Rayleigh functional to the original domain \( d \) is
\[
\mathcal{R}_{T(a, b)}(u) = \frac{b[u]}{a[u]} = \frac{(|\text{sign} B|B|^{\frac{1}{2}}u, B|^{\frac{1}{2}}u)}{\|A^{\frac{1}{2}}u\|^2}, \quad u \in d \setminus \{0\}.
\]
Further, let \( G : \mathcal{H}_a \to \mathcal{H} \) be an operator metrically equal to \( A^{\frac{1}{2}} \), that is, \( \|Gu\|_{\mathcal{H}} = \|A^{\frac{1}{2}}u\|_{\mathcal{H}} \) for any \( u \in d \) (and therefore, for any \( u \in \mathcal{H}_a \)). Then \( |B|^{\frac{1}{2}}G^{-1} \) is a well-defined and bounded operator in \( \mathcal{H} \). Consider the operator:
\[
T'(a, b, G) = (|B|^{\frac{1}{2}}G^{-1})^{\ast} \text{ sign } B (|B|^{\frac{1}{2}}G^{-1}).
\]
It follows from (3.10) that the non-zero spectra of the operators \( T(a, b) \) (acting in \( \mathcal{H}_a \)) and \( T'(a, b, G) \) (acting in \( \mathcal{H} \)) coincide.

Consider now one more object associated with the quadratic forms \( a[u] \) and \( b[u] \). Under the assumption (3.9) for any \( \alpha \in \mathbb{R} \) the quadratic form \( a_\alpha[u] = a[u] - \alpha b[u] \)
is well-defined on \( d \). If \( T(a, b) \in \mathcal{C}(\mathcal{H}_a) \), then \( a_\alpha \) is semi-bounded from below and closed in \( \mathcal{H} \). The corresponding self-adjoint operator in \( \mathcal{H} \) is \( A - \alpha B \) (understood as the form-sum). It is well known that under the above assumptions
\[
N_-(A - \alpha B) = n_+((\alpha^{-1}, T(a, b)) = n_+((\alpha^{-1}, T'(a, b, G))),
\]
see [B 1], or a detailed exposition in [BSol 5], §1. The equality (3.11) simplifies if \( B \geq 0 \) and \( A^{\frac{1}{2}} \) is taken as \( G \):
\[
N_-(A - \alpha B) = n((\alpha^{-1}, T(a, b)) = n((\alpha^{-\frac{1}{2}}, B^{\frac{1}{2}}A^{\frac{1}{2}})), \quad \alpha > 0.
\]
4. SPECTRAL PROBLEMS ON THE LINE AND ON THE HALF-LINE

In this Section we discuss the behaviour of the function \( N_\alpha (-\partial_t^2 - \alpha Q) \) depending on the properties of \( Q \). We also analyze certain related problems. Here we consider \( Q \) as an “independent parameter”, not necessarily appearing as the effective potential (2.9).

4.1. Consider at first the eigenvalue problem on the real line:
\[
-\lambda w''(t) = Q(t) w(t) \quad \text{for } t \neq 0; \quad w(0) = 0.
\] (4.1)

Evidently, the problem (4.1) splits into two similar problems on \( \mathbb{R}_\pm \); we shall use this later. In (4.1) \( Q \) is a real-valued function on \( \mathbb{R} \) and we always suppose \( Q \in L_1(\mathbb{R}) \). This assumption is not necessary for our analysis, but simplifies the presentation.

In order to state the problem in a proper way, let us introduce the “homogeneous Sobolev space”
\[
\mathcal{H}^1(\mathbb{R}) = \{ w \in H^1_{\text{loc}}(\mathbb{R}) : w(0) = 0; \| w \|^2_{\mathcal{H}^1(\mathbb{R})} := \int_{\mathbb{R}} |w'(t)|^2 \, dt < \infty \}. \tag{4.2}
\]

Consider also the quadratic form
\[
b_Q[w] = \int_{\mathbb{R}} Q(t)|w|^2 \, dt, \tag{4.3}
\]
it is well-defined at least for \( w \in C_0^\infty(\mathbb{R}) \). If
\[
\sup_{s>0} \left( s \int_{|t|>s} |Q(t)| \, dt \right) < \infty, \tag{4.4}
\]
then the quadratic form \( b_Q \) is bounded in \( \mathcal{H}^1(\mathbb{R}) \) and therefore, generates there a bounded self-adjoint operator
\[
\mathcal{T}_Q = T(\| \cdot \|^2_{\mathcal{H}^1(\mathbb{R})}, b_Q), \tag{4.5}
\]
cf. Subsection 3.3. If
\[
s \int_{|t|>s} |Q(t)| \, dt \to 0 \quad \text{as } s \to \infty, \tag{4.6}
\]
then the operator \( \mathcal{T}_Q \) is compact. For non-negative \( Q \in L_1(\mathbb{R}) \) the above boundedness and compactness conditions are also necessary. The eigenpairs of \( \mathcal{T}_Q \) satisfy (4.1). The corresponding Rayleigh quotient is
\[
\mathcal{R}_{\mathcal{T}_Q}(w) = \frac{\int_{\mathbb{R}} Q(t)|w|^2 \, dt}{\int_{\mathbb{R}} |w|^2 \, dt}, \quad w \in \mathcal{H}^1(\mathbb{R}), \ w \neq 0. \tag{4.7}
\]

Substituting in the definitions (4.2), (4.3), (4.5) and in (4.7) \( \mathbb{R}_\pm \) for \( \mathbb{R} \), we obtain the spaces \( \mathcal{H}^1(\mathbb{R}_\pm) \), the quadratic forms \( b^\pm_Q \), the operators \( \mathcal{T}_Q^\pm \) and their Rayleigh quotients. It is clear that
\[
\mathcal{H}^1(\mathbb{R}) = \mathcal{H}^1(\mathbb{R}_+) \oplus \mathcal{H}^1(\mathbb{R}_-) \quad \text{and} \quad \mathcal{T}_Q = \mathcal{T}_Q^+(\mathbb{R}_+) \oplus \mathcal{T}_Q^-(\mathbb{R}_-).
4.2. When analyzing the eigenvalue behaviour of the above operators, we concentrate on the operator \( T^+_Q \). The passage to the other two is evident.

There are various ways to study the spectrum of \( T^+_Q \). The one we present, was initiated in [BBor], the further steps being made in [BSol 2], [BSol 4], [BLap 2] and [NSol]. A comprehensive survey can be found in [BLapSol]. The results given below, partly extend the material of [BLapSol] to the case of non-powerlike behaviour of \( n_\pm(\lambda, T^+_Q) \). To this end we use the classes \( C_{\Phi, \infty} \) introduced in Subsection 3.2.

With any function \( Q \in L_1_{loc}(\mathbb{R}_+) \) we associate the number sequence \( \eta(Q) = \{\eta_k(Q)\}_{k \in \mathbb{Z}_+} \),

\[
\eta_0(Q) = \int_0^1 |Q(t)|^2 dt; \quad \eta_k(Q) = \int_{2^{k-1}}^{2^k} t|Q(t)|^2 dt, \quad k \in \mathbb{N}. \tag{4.8}
\]

**Lemma 4.1.** Let \( Q \in L_1(\mathbb{R}_+) \). Then
1° If \( \eta(Q) \in L_\infty \), then the operator \( T^+_Q \) is bounded and \( \|T^+_Q\| \leq C \sup_k \eta_k(Q) \).
2° If \( \eta_k(Q) \to 0 \) as \( k \to \infty \), then \( T^+_Q \in C \).
3° For any monotone regularly varying function \( \Phi \) of order \( q > \frac{1}{2} \) the inclusion \( \eta(Q) \in L_{\Phi, \infty} \) implies \( T^+_Q \in C_{\Phi, \infty} \) and, moreover,

\[
\Delta_\Phi(T^+_Q) \leq C(\Phi) \Delta_\Phi(\eta(Q)). \tag{4.9}
\]

In particular,

\[
\eta(Q) \in L^0_{\Phi, \infty} \implies T^+_Q \in C^0_{\Phi, \infty}. \tag{4.10}
\]

For \( Q \geq 0 \) the inequality (4.9) and the implication (4.10) can be reversed.

It is useful to have examples of such functions \( Q \geq 0 \) that the corresponding operator \( T^+_Q \) satisfies

\[
n(\alpha^{-1}, T^+_Q) \sim C(\Phi)(\alpha), \quad \alpha \to \infty \tag{4.11}
\]

with a prescribed regularly varying function \( \Phi \). A rather wide family of such examples is described below.

Let \( \Phi(\alpha) = \alpha^q \Lambda(\alpha) \) be a monotone and smooth function from \( \mathcal{F}_q \), such that \( \Lambda(0+) < \infty \). The inverse function, say \( \Psi \), lies in \( \mathcal{F}_{q^{-1}} \), see [Sen], Section 1.5.

**Lemma 4.2.** Let \( \Phi \) and \( \Psi \) be as above, and let \( 2q > 1 \). Take

\[
Q(t) = t^{-2}(\Psi(\log t))^{-1} (t \geq 1); \quad Q(t) = 0 (t < 1). \tag{4.12}
\]

Then the asymptotic formula (4.11) is satisfied, with

\[
C = C(q) = 4^{q-1} \frac{\Gamma(q-\frac{1}{2})}{\sqrt{\pi} \Gamma(q)}. \tag{4.13}
\]

For \( \Lambda \equiv 1 \) this example and its \( d \)-dimensional analogs were analyzed in [BSol 4] and [BLap 2]. The proofs of both Lemmas 4.1 and 4.2 are given in Appendix A3.
4.3. Here we discuss the negative discrete spectrum of the operator in $L_2(\mathbb{R})$:

$$\mathcal{A}_{\alpha Q} = -\partial_t^2 - \alpha Q,$$

see (2.8). The operator $\mathcal{A}_{\alpha Q}$ is generated by the quadratic form

$$a_{\alpha Q}[u] = \int_{\mathbb{R}} (|w'|^2 - \alpha Q|w|^2) dt, \quad w \in H^1(\mathbb{R}).$$

If $Q \in L_1(\mathbb{R})$ and the condition (4.6) is fulfilled, then this quadratic form is semi-bounded from below and closed in $L_2(\mathbb{R})$, and therefore the operator $\mathcal{A}_{\alpha Q}$ is well-defined. Imposing the additional condition $w(0) = 0$, we obtain another quadratic form, say $a'_{\alpha Q}[u]$. Let $\mathcal{A}'_{\alpha Q}$ be the corresponding operator. By virtue of (4.6), $T_Q \in \mathcal{C}$. Then the negative spectrum of the operator $\mathcal{A}'_{\alpha Q}$ is finite for any $\alpha > 0$, and by (3.11) $N_-(\mathcal{A}'_{\alpha Q}(\alpha)) = n_+(\alpha^{-1}, T_Q)$. It follows

$$0 \leq N_-(\mathcal{A}_{\alpha Q}(\alpha)) - n_+(\alpha^{-1}, T_Q) \leq 1 \quad \text{for any } \alpha > 0. \quad (4.14)$$

Our next goal is to realize, for the case discussed, the scheme presented in Subsection 3.3. Let $a'$ stand for the quadratic form $a'_{\alpha Q}$ with $\alpha = 0$, then $\mathcal{F}_{a'} = H^1(\mathbb{R})$. Let $\mathcal{A}'$ be the self-adjoint operator in $L_2(\mathbb{R})$, generated by the quadratic form $a'$, that is the “free” operator. The quadratic form (4.3) corresponds to the self-adjoint in $L_2(\mathbb{R})$ operator $[Q]$ of multiplication by $Q$, and clearly $|[Q]| = |Q|$. If the functions $Q(\pm t)$ meet the condition of Lemma 4.1, then the inequality (3.9) is satisfied for the operator $|B| = |Q|$ and the quadratic form $a[w] = a'[w]$.

Consider the mapping

$$W: \ z(\xi) \mapsto w(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{i\xi t} - 1}{|\xi|} z(\xi) d\xi. \quad (4.15)$$

If $z \in L_2(\mathbb{R})$, then $w$ is absolute continuous and $w(0) = 0$. By Plancherel’s formula, $\|z\|_{L_2(\mathbb{R})} = \|w'\|_{L_2(\mathbb{R})}$. This shows that $W$ is an isometry of $L_2(\mathbb{R})$ into $H^1(\mathbb{R})$; it is easy to see that actually this is an isometry onto $H^1(\mathbb{R})$. The operator $W^{-1}$ acts from $H^1(\mathbb{R})$ to $L_2(\mathbb{R})$ and is metrically equal to $\sqrt{A'}$.

Given a function $h$ on $\mathbb{R}$, define an integral operator $J_h$ in $L_2(\mathbb{R})$:

$$J_hz(t) = \frac{h(t)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{i\xi t} - 1}{|\xi|} z(\xi) d\xi, \quad (4.16)$$

then

$$|Q|^\frac{1}{2} W = J_{|Q|^{1/2}}. \quad (4.17)$$

Now we derive from (3.11) and (4.17):

$$N_-(\mathcal{A}'_{Q}(\alpha)) = n_+(\alpha^{-1}, T_Q) = n_+(\alpha^{-1}, J_{|Q|^{1/2}} \text{sign} Q J_{|Q|^{1/2}}). \quad (4.18)$$

If $Q \geq 0$, then (3.12) applies, and we obtain

$$N_-(\mathcal{A}'_{Q}(\alpha)) = n(\alpha^{-1}, T_Q) = n(\alpha^{-1}, J_{|Q|^{1/2}}). \quad (4.19)$$
5. Main result (general case) and plan of proof

5.1. Now we are in a position to formulate our result in its full generality. The auxiliary objects involved in the formulation are the operators $\mathcal{T}_{Q_P,V}$ and $\mathcal{T}_{Q_P,|V|1}$, cf. (4.5). Recall that the definition of the “effective potential” $Q_{P,V}$ was given in (2.9).

**Theorem 5.1.** Let the conditions $(\Omega)$, $(P)$ and (1.18) be satisfied, and let $V \in X(\Omega)$. Then

1° If

$$\mathcal{T}_{Q_P,|V|1} \in \mathcal{C},$$

then for any $\alpha > 0$ the negative spectrum of the operator $P - \alpha V$ is finite. Besides:

2° If

$$n(\alpha^{-1}, \mathcal{T}_{Q_P,|V|1}) = o(\alpha^{\frac{d}{2}}), \quad \alpha \to \infty,$$

then the asymptotic formula (2.6) is satisfied also for $\gamma = 0$.

3° Suppose that there exists a function $\Phi \in \mathfrak{F}_q$, $2q \geq d$, such that

$$\eta(Q_{P,|V|}) \in l_{\Phi, \infty}$$

where $\eta(Q)$ is the sequence (4.8). Suppose also that

$$n_+(\alpha^{-1}, \mathcal{T}_{Q_P,V}) \sim \Phi(\alpha), \quad \alpha \to \infty.$$

Then the asymptotic relation (2.10) is satisfied.

**Remarks.** 1. In view of (4.14) and Lemma 4.1, 3°, all the conditions could be stated directly in terms of the behaviour of the functions $N_+(-\partial_t^2 - \alpha Q_{P,|V|})$ and $N_+(-\partial_t^2 - \alpha Q_{P,V})$.

2. The assumptions of Theorem 5.1 imply in particular that

$$Q_{P,|V|} \in L_1(\mathbb{R}).$$

Indeed, the integral of $Q_{P,|V|}$ over $(-1, 1)$ is finite since $V \in X(\Omega)$, and the integral over the rest of $\mathbb{R}$ is finite because this is necessary for (5.1), cf. (4.6).

3. It follows from (5.3) that $\mathcal{T}_{Q_P,V} \in C_{\Phi, \infty}$. The additional restriction (5.4) means in particular that the sequences (4.8) for the effective potentials $Q_{P,V}$ and $Q_{P,|V|}$ are comparable in size. This forbids “too large” cancellations in the integral (2.9) for sign-indeterminate $V$.

We postpone other remarks and comments until Section 7. In particular, we formulate there Theorem 7.1, result of which for $2q > d$ is slightly more general than the corresponding result of 3°.

5.2. Here we introduce the basic objects appearing in the course of the proof of Theorem 5.1. Fix $\gamma \neq 0$. Let us realize the equality (3.11) for the operators $A = P + \gamma^2 I$, $B = [V]$, and $G = A \frac{1}{2}$. Denote $W = |V|^{\frac{1}{2}}$, then

$$G(\gamma) := W(P + \gamma^2 I)^{-\frac{1}{2}} = |B|^\frac{1}{2}G^{-1}, \quad \gamma \neq 0$$
and (3.11) turns into
\[ N_-(\mathcal{P} + \gamma^2 I - \alpha V) = n_-(\alpha^{-1}, G(\gamma)^n \operatorname{sign} V G(\gamma)), \quad \gamma \neq 0. \quad (5.7) \]

If \( V \geq 0 \), then a simpler relation is implied by (3.12):
\[ N_-(\mathcal{P} + \gamma^2 I - \alpha V) = n_-(\alpha^{-\frac{1}{2}}, G(\gamma)), \quad V \geq 0, \quad \gamma \neq 0. \quad (5.8) \]

Applying the estimate (2.5) and the equality (5.8) with \( V \) replaced by \(|V|\), we obtain
\[ n(\alpha^{-1}, G(\gamma)) \leq C(\gamma, \Omega)\|V\|_X^{\frac{d}{d}}\alpha^d; \quad \gamma \neq 0. \quad (5.9) \]

This yields
\[ G(\gamma) \in C_{d,\infty}, \quad \Delta_d(G(\gamma)) \leq C(\gamma, \Omega)\|V\|_X^{\frac{d}{d}}; \quad \gamma \neq 0. \]

We need a careful analysis of the behaviour of the operators \( G(\gamma) \) as \( \gamma \to 0 \). Using the formula (1.29) with \( f(\lambda) = (\lambda + \gamma^2)^{-\frac{1}{2}} \), we find
\[ G(\gamma) = W \sum_{k=1}^{\infty} \frac{\Psi_k}{\sqrt{E_k(\xi) + \gamma^2}}. \]

For the technical reasons, it is convenient here to interpret \( \Psi_k \) as operators acting from \( L_2(\Omega) \) into \( L_2(\mathbb{R}) \), cf. Subsection 1.4.

In order to understand what happens as \( \gamma \to 0 \), we represent
\[ G(\gamma) = G_1(\gamma) + G_2(\gamma) := G(\gamma)(I - \mathcal{E}_\mu) + G(\gamma)\mathcal{E}_\mu \quad (5.10) \]

where \( \mathcal{E}_\mu \) is the spectral projection of the operator \( \mathcal{P} \), corresponding to the interval \((0, \mu)\). We choose \( \mu > 0 \) so small that \( \min E_2(\xi) > \mu \) and on the interval \((0, \pi)\) there is only one point \( \varepsilon \) such that \( E_1(\varepsilon) = \mu \). The existence of such \( \mu \) is guaranteed by the assertions 2° and 3° of Theorem 1.1. The projection \( \mathcal{E}_\mu \) can be written as
\[ \mathcal{E}_\mu = \Psi_1^*1_{(-\varepsilon, \varepsilon)}(\xi)\Psi_1. \]

Recall that we denote \( E(\xi) := E_1(\xi) \) and \( \omega(t, y) := \omega(0; t, y) \) (see Subsection 1.2). Besides, we shall write \( \omega(\xi; t, y) := \omega_1(\xi; t, y) \) and \( \Psi := \Psi_1 \).

Write the expressions for \( G_1(\gamma) \) and \( G_2(\gamma) \) in a more detailed form:
\[ G_1(\gamma) = W(\mathcal{P} + \gamma^2 I)^{-\frac{1}{2}}(I - \mathcal{E}_\mu), \quad (5.11) \]
\[ G_2(\gamma) = W\Psi^*\frac{1_{(-\varepsilon, \varepsilon)}(\xi)}{\sqrt{E(\xi) + \gamma^2}}\Psi. \quad (5.12) \]

We shall investigate the behaviour of both operators as \( \gamma \to 0 \). The operator \( G_1(\gamma) \) is well-defined by the equality (5.11) for all \( \gamma \geq 0 \).
Lemma 5.2. Let the conditions (Ω), (P) and (1.18) be satisfied, and let \( V \in X(\Omega) \). Then \( G_1(0) \in C_{d, \infty} \) and
\[
\|G_1(\gamma) - G_1(0)\|_{d, \infty} \to 0 \quad \text{as} \quad \gamma \to 0. \tag{5.13}
\]

For the operator
\[
\mathcal{K} := G_1(0)^* \, \text{sign} \, V \, G_1(0) \tag{5.14}
\]
the asymptotic formula is valid, cf. (2.4) and (2.6):
\[
n_{\pm} (\alpha^{-1}, \mathcal{K}) \sim \Theta(g, V_\pm) \alpha^{\frac{\gamma}{2}}, \quad \alpha \to \infty. \tag{5.15}
\]

The proof of Lemma 5.2, as well as proofs of Lemmas 5.3 and 5.4 given below, is postponed until Section 6.

The operators \( G_2(\gamma) \) do not have a limit as \( \gamma \to 0 \); it is easy to see that \( \|G_2(\gamma)\| \to \infty \). However, the situation improves if we subtract from \( G_2(\gamma) \) an appropriate rank one operator. Write \( G_2(\gamma) = G'_2(\gamma) \Psi \) where, according to (5.12) and (1.26), \( G'_2(\gamma) : L_2(\mathbb{R}) \to L_2(\Omega) \) is an integral operator:
\[
(G'_2(\gamma) z)(t, y) = \left(2\pi\right)^{-\frac{1}{2}} W(t, y) \int_{-\varepsilon}^{\varepsilon} e^{i \xi t} \frac{\omega(t, y)}{\sqrt{E(\xi) + \gamma^2}} \xi(\xi) d\xi. \tag{5.16}
\]

Along with \( G'_2(\gamma) \), consider the operators \( Y'(\gamma) : L_2(\mathbb{R}) \to L_2(\Omega) \),
\[
(Y'(\gamma) z)(t, y) = \left(2\pi\right)^{-\frac{1}{2}} W(t, y) \omega(t, y) \int_{-\varepsilon}^{\varepsilon} \frac{z(\xi)}{\sqrt{E(\xi) + \gamma^2}} d\xi. \tag{5.17}
\]

The inclusion (5.5) and the continuity of \( \omega \) imply \( W \omega \in L_2(\Omega) \), therefore \( Y'(\gamma) \) are bounded operators of rank one.

Consider also an operator \( \mathcal{G}' : L_2(\mathbb{R}) \to L_2(\Omega) \) which does not involve the parameter \( \gamma \):
\[
(G' z)(t, y) = \frac{W(t, y) \omega(t, y)}{\sqrt{2\pi \beta}} \int_{\mathbb{R}} e^{i \xi t} \frac{1}{|\xi|} \xi(\xi) d\xi = \frac{W(t, y) \omega(t, y)}{\sqrt{\beta}} (Wz)(t). \tag{5.18}
\]

Recall that \( \beta \) is the coefficient from (1.20) and the operator \( W \) is defined in (4.15). Further, put
\[
Y(\gamma) = Y'(\gamma) \Psi, \quad \mathcal{G} = \mathcal{G}' \Psi,
\]
and
\[
\mathcal{N} = \mathcal{G}^* \, \text{sign} \, V \, \mathcal{G}.
\]

Lemma 5.3. Let the conditions (Ω), (P) and (1.18) be satisfied. Suppose also that \( V \in X(\Omega) \) and (5.1) is fulfilled. Then there exists an operator \( Z \in C_d \) such that
\[
(\text{u})- \lim_{\gamma \to 0} (G_2(\gamma) - Y(\gamma)) = \mathcal{G} + Z \tag{5.19}
\]
(convergence in the operator norm). Further,
\[
n_+(\alpha^{-1}, \mathcal{N}) = n_+(\alpha^{-1}, \mathcal{T}_{\Omega, \nu}) \tag{5.20}
\]
and
\[ n(\alpha^{-1}, \mathcal{G}^* \mathcal{G}) = n(\alpha^{-1}, \mathcal{T}_{Q,1V_1}). \] (5.21)

It follows from Lemmas 5.2 and 5.3 that
\[ (u)^\gamma \lim_{\gamma \to 0} (G(\gamma) - Y(\gamma)) = G_1(0) + \mathcal{G} + Z. \]

Consider the operator
\[ S = (G_1(0) + \mathcal{G} + Z)^* \text{sign} \, V \, (G_1(0) + \mathcal{G} + Z). \] (5.22)

Since \( \text{rank} \, Y(\gamma) = 1 \), the operator
\[ G(\gamma)^* \text{sign} \, V \, G(\gamma) - (G(\gamma) - Y(\gamma))^* \text{sign} \, V \, (G(\gamma) - Y(\gamma)) \]
has rank no greater than two, therefore (5.7) implies
\[ |N_-(\mathcal{P} + \gamma^2 I - \alpha V) - n_+ (\alpha^{-1}, (G(\gamma) - Y(\gamma))^* \text{sign} \, V \, (G(\gamma) - Y(\gamma)))| \leq 2, \quad \gamma \neq 0. \]

Letting here \( \gamma \to 0 \), we find
\[ |N_-(\mathcal{P} - \alpha V) - n_+ (\alpha^{-1}, S)| \leq 2 \] (5.23)
and hence
\[ N_-(\mathcal{P} - \alpha V) \sim n_+ (\alpha^{-1}, S), \quad \alpha \to 0. \] (5.24)

Theorem 5.1 follows from (5.23) and (5.24) almost immediately, except for the assertion 3* for \( q = \frac{d}{2} \). Indeed, in view of (5.21) the assumption (5.1) implies \( \mathcal{G} \in \mathcal{C} \) and therefore, also \( \mathcal{S} \in \mathcal{C} \). Now the finiteness of \( N_-(\mathcal{P} - \alpha V) \) for any \( \alpha > 0 \) follows from (5.23). Thus, 1* is established.

The assumption (5.2) implies \( \mathcal{G} \in \mathcal{C}_{d,\infty}^0 \), hence also \( \mathcal{G} + Z \in \mathcal{C}_{d,\infty}^0 \) and \( \mathcal{S} - \mathcal{K} \in \mathcal{C}_{\Phi,\infty}^0 \); recall that \( \mathcal{K} \) is the operator (5.14). Using (5.15), (5.24) and (3.7) for \( \Phi(\alpha) = \alpha^{\frac{d}{2}} \), we justify the asymptotic formula (2.6) for \( \gamma = 0 \).

Let now (5.3) be satisfied with \( 2q > d \). Then from \( G_1(0) + Z \in \mathcal{C}_{d,\infty} \) we derive
\[ (G_1(0) + Z)^* \text{sign} \, V \, (G_1(0) + Z) \in \mathcal{C}_{\Phi,\infty}^0, \quad (G_1(0) + Z)^* (G_1(0) + Z) \in \mathcal{C}_{\Phi,\infty}^0. \]

By (3.8), this yields also \( (G_1(0) + Z)^* \text{sign} \, V \, \mathcal{G} \in \mathcal{C}_{\Phi,\infty}^0 \). Therefore, \( \mathcal{S} - \mathcal{N} \in \mathcal{C}_{\Phi,\infty}^0 \), and (2.10) follows from (5.20), (5.24) and (3.7).

The rest of the proof of Theorem 5.1 concerns only the case \( \Phi \in \mathfrak{F}_{\Phi,\infty}^0 \). Here we need a somewhat more complicated argument. Let us write the expression (5.22) for the operator \( \mathcal{S} \) in a more detailed form: put
\[ \widehat{\mathcal{N}} := (\mathcal{G} + Z)^* \text{sign} \, V (\mathcal{G} + Z), \]
then
\[ \mathcal{S} = \mathcal{K} + \widehat{\mathcal{N}} + 2\Re((\mathcal{G} + Z)^* \text{sign} \, V G_1(0)). \] (5.25)
The operator $\mathcal{K} + \hat{\mathcal{N}}$ is the $u$-limit of the family
\[
G_1(\gamma)^* \text{sign} V G_1(\gamma) + (G_2(\gamma) - Y(\gamma))^{*} \text{sign} V (G_2(\gamma) - Y(\gamma))
\]
as $\gamma \to 0$. Withdrawing from here $Y(\gamma)$, we obtain an operator which is the
orthogonal sum of its two terms. This follows directly from the definition (5.10).
Therefore, the quantity $n_+(\alpha^{-1}, \cdot)$ for this operator splits into the sum of similar
quantities for these terms. Acting in exactly the same way as when deriving (5.24),
we find from here:
\[
n_+(\alpha^{-1}, \mathcal{K} + \hat{\mathcal{N}}) \sim n_+(\alpha^{-1}, \mathcal{K}) + n_+(\alpha^{-1}, \hat{\mathcal{N}}), \quad \alpha \to \infty. \tag{5.26}
\]
Actually, both sides of (5.26) differ by a number no greater than two, cf. (5.23).

For the rest of the proof we need the following Lemma, which also will be used
when proving Theorem 2.2.

**Lemma 5.4.** Under the assumptions of Lemma 5.3, suppose also that $T_{Q, \gamma, \lambda} \in
\mathcal{C}_{\Phi, \infty}$ where $\Phi \in \mathfrak{F}'$. Then
\[
n(\alpha^{-1}, \mathcal{G}^{*} \text{sign} V G_1(0)) = o(\Phi_1(\alpha)), \quad \alpha \to \infty;
\Phi_1(\alpha) := \Theta(g, V_+)^{\frac{d}{2}} + \Phi(\alpha). \tag{5.27}
\]

Note that the function $\Phi_1$ is regularly varying (of class $\mathfrak{F}'$), as the sum of two
regularly varying functions, see [Sen], Sec.1.5, 3°.

One more fact which we need is
\[
\hat{\mathcal{N}} - \mathcal{N} = Z^{*} \text{sign} V Z + 2 \Re(G^{*} \text{sign} V Z) \in \mathcal{C}_{\Phi_1, \infty}^{0}.
\tag{5.28}
\]
Indeed, the first term belongs to $\mathcal{C}_{\frac{d}{2}} \subset \mathcal{C}_{\Phi_1, \infty}^{0}$, since $Z \in \mathcal{C}_d$. For the second term
we derive from Lemma 3.2:
\[
\Delta_{\Phi_1}^{\alpha} (G^{*} \text{sign} V Z) \leq \Delta_{\Phi_1} (G^{*} G) \Delta_{\Phi_1} (Z^{*} (\text{sign} V)^{2} Z) \leq \Delta_{\Phi} (G^{*} G) \Delta_{\Phi} (Z^{*} Z) = 0,
\]
since the first factor in the last product is finite by (5.3) and (5.20) and the second
is zero.

Having Lemma 5.4 and asymptotic equality (5.28) in our disposal, we easily
complete the proof of Theorem 5.1. Only since now we need the assumption (5.4).
Let us show that
\[
n_+ (\alpha^{-1}, \mathcal{K}) + n_+ (\alpha^{-1}, \hat{\mathcal{N}}) \sim \Phi_1(\alpha), \quad \alpha \to \infty, \tag{5.29}
\]
cf. (2.10). To this end, consider the function
\[
\mu(\alpha) = \frac{n_+ (\alpha^{-1}, \mathcal{K}) + n_+ (\alpha^{-1}, \hat{\mathcal{N}})}{\Phi_1(\alpha)} - 1
\]
\[
= \frac{n_+ (\alpha^{-1}, \mathcal{K}) - \Theta(g, V_+)^{\frac{d}{2}} + n_+ (\alpha^{-1}, \hat{\mathcal{N}}) - \Phi(\alpha)}{\Phi_1(\alpha)}.
\]
By Lemma 5.2 the first term of the last sum tends to 0 as \( \alpha \to \infty \). Consider the second term which we denote \( \mu_1(\alpha) \):

\[
\mu_1(\alpha) \leq \frac{n_+(\theta \alpha^{-1}, \mathcal{N}) - \Phi(\alpha)}{\Phi(\alpha)} \Phi(\alpha) + \frac{n_+((1 - \theta) \alpha^{-1}, \hat{\mathcal{N}} - \mathcal{N})}{\Phi(\alpha)}.
\]

According to (4.14), (5.20) and (5.4),

\[
n_+(\alpha^{-1}, \mathcal{N}) \sim N_+(-\delta^2 - \alpha Q_{r,V}) \sim \Phi(\alpha), \quad \alpha \to \infty.
\]

Using (5.28) and the definition of a regularly varying function, we find from here \( \limsup_{\alpha \to \infty} \mu_1(\alpha) \leq \theta^{\frac{p}{q}} - 1 \). Since \( \theta < 1 \) is arbitrary, we get \( \limsup_{\alpha \to \infty} \mu_1(\alpha) \leq 0 \) which implies also

\[
\limsup_{\alpha \to \infty} \mu(\alpha) \leq 0.
\]

In order to derive the converse inequality, let us write

\[
\mu_1(\alpha) \geq \frac{n_+(\theta^{-1} \alpha^{-1}, \mathcal{N}) - \Phi(\alpha)}{\Phi(\alpha)} \Phi(\alpha) - \frac{n_+((\theta^{-1} - 1) \alpha^{-1}, \hat{\mathcal{N}} - \mathcal{N})}{\Phi(\alpha)}.
\]

Again using (5.28), we derive

\[
\liminf_{\alpha \to \infty} \mu_1(\alpha) \geq (\theta^{\frac{p}{q}} - 1) \limsup_{\alpha \to \infty} \frac{\Phi(\alpha)}{\Phi_1(\alpha)}.
\]

It follows

\[
\liminf_{\alpha \to \infty} \mu(\alpha) = \liminf_{\alpha \to \infty} \mu_1(\alpha) \geq 0
\]

which together with the above inequality for the upper limit shows that \( \mu(\alpha) \to 0 \). This is equivalent to (5.29).

Now we show that

\[
n_+(\alpha^{-1}, S) \sim n_+(\alpha^{-1}, \mathcal{K} + \hat{\mathcal{N}}), \quad \alpha \to \infty.
\]

Indeed, by (5.25)

\[
S - (\mathcal{K} + \hat{\mathcal{N}}) = 2\Re(\mathcal{G}^* \text{sign} V G_1(0)) + 2\Re(Z^* \text{sign} V G_1(0)).
\]

The first term on the right-hand side lies in \( C^{0}_{q_1, \infty} \) by Lemma 5.4. The second lies in \( C^{0}_{q, \infty} \subset C^{0}_{q_1, \infty} \), since \( Z \in C_d \) and \( G_1(0) \in C_{d, \infty} \). Hence, \( S - (\mathcal{K} + \hat{\mathcal{N}}) \in C^{0}_{q_1, \infty} \) and by (3.7) the asymptotic relation (2.6) follows from (5.29).

In order to complete the proof of Theorem 5.1, it remains to prove Lemmas 5.2 - 5.4. This we do in the next Section.
6. Main results: end of proof

We start with the proof of Lemma 5.3, since one of its statements will be used when proving Lemma 5.2.

6.1. Proof of Lemma 5.3. Consider the operators \( G'_2(\gamma) \) and \( Y'(\gamma) \) defined by (5.16) and (5.17). Our starting point is the equality

\[
G'_2(\gamma) - Y'(\gamma) = Z'(\gamma) + \tilde{G}'_2(\gamma), \quad \gamma \neq 0
\]

where

\[
(Z'(\gamma)z)(t, y) = (2\pi)^{-\frac{d}{2}} W(t, y) \int_{-\varepsilon}^{\varepsilon} e^{i\xi t} \frac{\omega(\xi; t, y) - \omega(t, y)}{\sqrt{E(\xi) + \gamma^2}} z(\xi) d\xi \tag{6.2}
\]

and

\[
(\tilde{G}'_2(\gamma)z)(t, y) = (2\pi)^{-\frac{d}{2}} W(t, y) \omega(t, y) \int_{-\varepsilon}^{\varepsilon} \frac{e^{i\xi t} - 1}{\sqrt{E(\xi) + \gamma^2}} z(\xi) d\xi. \tag{6.3}
\]

The integral operators (6.3) and some other we encounter below, have a specific structure: the variable \( y \) does not appear in the integrand. This allows one to reduce the study of their singular numbers to the similar problem for appropriate operators acting in \( L_2(\mathbb{R}) \). Here we explain the way to do it.

Let \( \mathcal{F} : L_2(\mathbb{R}) \to L_2(\Omega) \) be an integral operator,

\[
(\mathcal{F}z)(t, y) = F_0(t, y) \int_{\mathbb{R}} F_1(t, \xi) z(\xi) d\xi. \tag{6.4}
\]

Along with \( \mathcal{F} \), let us consider the operator \( \tilde{\mathcal{F}} : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \),

\[
(\tilde{\mathcal{F}}z)(t) = \tilde{F}_0(t) \int_{\mathbb{R}} F_1(t, \xi) z(\xi) d\xi \tag{6.5}
\]

where

\[
\tilde{F}_0(t) = \left( \int_{\Omega(t)} |F_0(t, y)|^2 dy \right)^{\frac{1}{2}}. \tag{6.6}
\]

Recall that \( \Omega(t) = \{ y \in \mathbb{R}^{d-1} : (t, y) \in \Omega \} \). It is clear that \( \| \tilde{\mathcal{F}} z \|_{L_2(\mathbb{R})} = \| \mathcal{F} z \|_{L_2(\Omega)} \) for any \( z \in L_2(\mathbb{R}) \). It follows from this property that

\[
\tilde{\mathcal{F}} \in \mathcal{C} \iff \mathcal{F} \in \mathcal{C} \quad \text{and} \quad s(\tilde{\mathcal{F}}) = s(\mathcal{F}).
\]

Below we use this reduction systematically. The symbol "\( \sim \)" always denotes the passage from an operator of the type (6.4) to its counterpart (6.5) - (6.6). Note that the mapping \( \mathcal{F} \mapsto \tilde{\mathcal{F}} \) is linear with respect to the function \( F_1 \): if \( \mathcal{F}' \) and \( \mathcal{F}'' \) are two operators with the kernels \( F_0(t, y)F_1'(t, \xi) \) and \( F_0(t, y)F_1''(t, \xi) \) respectively, then \( \tilde{\mathcal{F}'} + \tilde{\mathcal{F}''} = \tilde{\mathcal{F}'} + \tilde{\mathcal{F}''} \), and \( e\tilde{\mathcal{F}'} = e\tilde{\mathcal{F}'} \).
Lemma 6.1. Let $W \in L_d(\Omega)$. Then for all $\gamma \geq 0$ the operator $Z'(\gamma)$ is well-defined by the equality (6.2) and belongs to $\mathcal{C}_d$. Besides,

$$\|Z'(\gamma) - Z'(0)\|_d \to 0, \quad \gamma \to 0.$$ 

Below we give a unified argument which proves the result for all dimensions $d$ at once. For $d = 2$ a much more elementary reasoning is possible, as it always happens when dealing with the Hilbert–Schmidt class.

Proof of Lemma 6.1. Consider firstly integral operators $R(\gamma)$ which are simpler than $Z'(\gamma)$:

$$(R(\gamma)z)(t, y) = W(t, y) \int_{-\varepsilon}^{\varepsilon} e^{i\xi t} \frac{\xi}{\sqrt{E(\xi) + \gamma^2}} z(\xi) d\xi.$$ 

Each $R(\gamma)$ is an operator of the type (6.4). The corresponding operator $\widehat{R}(\gamma)$ has the kernel

$$\widehat{R}(\gamma; \xi, t) = \overline{W}(t)e^{i\xi t}m(\gamma; \xi), \quad \overline{W}(t) = \left(\int_{\Omega(t)} |V(t, y)| dy\right)^{\frac{1}{2}}.$$ 

where

$$m(\gamma; \xi) = \frac{\xi}{\sqrt{E(\xi) + \gamma^2}}1(-\varepsilon, \varepsilon)(\xi).$$

By Hölder’s inequality,

$$\int_{\Omega(t)} |V(t, y)| dy \leq (\text{meas}_{d-1} \Omega(t))^{1-\frac{1}{p}} \left(\int_{\Omega(t)} |V(t, y)|^p dy\right)^{\frac{1}{p}}.$$ 

The condition $(\Omega)$ and the inclusion $V \in X(\Omega)$ yield $\overline{W}(t) \in L_d(\mathbb{R})$. The function $m(\gamma; \xi)$ is bounded and compactly supported and hence, $m(\gamma; \cdot) \in L_d(\mathbb{R})$. Now we apply the interpolational estimate of $\mathcal{C}_p$-norms of integral operators $E$ in $L_2(\mathbb{R})$ with the kernels $E(t, \xi) = b(t)e^{i\xi t}a(\xi)$, see e.g. [Sim], Theorem 4.1:

$$\|E\|_p \leq (2\pi)^{\frac{1}{2}-\frac{1}{p}} \|a\|_{L_p(\mathbb{R})} \|b\|_{L_p(\mathbb{R})}, \quad p \geq 2.$$ 

Using this estimate with $p = d$, we find

$$\|\widehat{R}(\gamma) - R(0)\|_d = \|\widehat{R}(\gamma) - \widehat{R}(0)\|_d \leq (2\pi)^{\frac{1}{2}-\frac{1}{d}} \|\overline{W}\|_{L_d(\mathbb{R})} \|m(\gamma; \cdot) - m(0; \cdot)\|_{L_d(\mathbb{R})}. \quad (6.7)$$

Further,

$$|m(\gamma, \xi)| \leq C; \quad m(\gamma, \xi) \to m(0; \xi) \text{ as } \gamma \to 0; \quad \xi \neq 0. \quad (6.8)$$

By Lebesgue’s Theorem on the dominated convergence, $\|m(\gamma; \cdot) - m(0; \cdot)\|_{L_d(\mathbb{R})} \to 0$ and therefore (6.7) implies $\|\widehat{R}(\gamma) - R(0)\|_d \to 0$.

The integral operator $Z'(\gamma)$ is obtained from $R(\gamma)$ by multiplication of its kernel by the function $(\sqrt{2\pi \xi})^{-1}(\omega(\xi; t, y) - \omega(t, y))$ which does not involve the parameter $\gamma$. This function is real analytic in $\xi$ on the interval $(-\varepsilon, \varepsilon)$ and all its derivatives
in $\xi$ belong to $C(\bar{\Omega})$ and are bounded (Theorem 1.1, 3° and 4°). Therefore, it is a multiplier in the class $C_d$ (see [BSol 1], §8), which implies
\[ \|Z'(\gamma) - Z'(0)\|_d \leq C \|R(\gamma) - R(0)\|_d \to 0. \]

In order to complete the proof of Lemma 5.3, it remains to consider the operators $G'_2(\gamma)$, introduced in (6.3). Note that the operator $G'_2(0)$ is also well-defined by this relation, at least on the dense in $L_2(\mathbb{R})$ set of functions $z(\xi)$ vanishing at the point $\xi = 0$. The operators $G'_2(\gamma)$ are of the type (6.4), and the corresponding operators (6.5) can be written as
\[ G'_2(\gamma) = \frac{\beta_1}{\sqrt{\beta}} J_{h_{P, \gamma}}[m(\gamma; \xi) \text{ sign } \xi] \]
where
\[ h_{P, \gamma} = q_{P, \gamma}^{1/2}. \]

Recall that the operators $J_{h}$ were defined in (4.16). By (4.19), the assumption (5.1) is equivalent to $J_{h_{P, \gamma}} \in C$, and (6.9) shows that also $G'_2(\gamma) \in C$ for all $\gamma \in \mathbb{R}$. In particular, the operator $G'_2(0)$ extends to the whole of $L_2(\mathbb{R})$ and is compact.

The multiplicity by sign $\xi$ is a unitary operator in $L_2(\mathbb{R})$, and in view of (6.8) the operators $[m(\gamma; \cdot)]$ converge pointwise to $[m(0; \cdot)]$. Since $J_{h_{P, \gamma}} \in C$, (6.8) yields
\[ \|G'_2(\gamma) - G'_2(0)\| = \|G'_2(\gamma) - G'_2(0)\| \to 0 \quad \text{as } \gamma \to 0. \]

Return to the operator $G'$ given by (5.18). It is clear that $\tilde{G}' = J_{h_{P, \gamma}}$ and that the kernel of the integral operator $(2\pi)^{1/2}(\tilde{G}' - \tilde{G}'_2(0))$ is
\[ -h_{P, \gamma}(t)e^{i\xi t} - \frac{1}{\sqrt{E(\xi)}} \left( \frac{\sqrt{\beta}}{\sqrt{E(\xi)}} - \frac{1}{\xi} \right) 1_{(-\varepsilon, \varepsilon)}(\xi) + h_{P, \gamma}(t) \frac{e^{i\xi t} - 1}{|\xi|} (1 - 1_{(-\varepsilon, \varepsilon)}(\xi)). \]

A direct calculation (which uses (1.20) and (5.5)) shows that both terms belong to $L_\infty(\mathbb{R}^2)$ and therefore, $\tilde{G}' - \tilde{G}'_2(0) \in C_2 \subset C_d$. The same inclusion holds true for $G' - G'_2(0)$. It follows that (5.19) is satisfied, with $Z = (Z'(0) + G'_2(0) - G')\Psi$.

The equality (5.21) immediately follows from (4.19). The equality (5.20) reduces to (4.18), however this reduction is not as easy as the previous one, and we present it with more detail.

We start with the equality
\[ n_+(\alpha^{-1}, \mathcal{N}) = n_+(\alpha^{-1}, \mathcal{G}^* \text{ sign } V \mathcal{G}) = n_+(\alpha^{-1}, (\mathcal{G}')^* \text{ sign } V \mathcal{G}'). \]

The quadratic form of the operator $(\mathcal{G}')^* \text{ sign } V \mathcal{G}'$ is equal (cf. (5.18))
\[ (\text{sign } V \mathcal{G}' z, \mathcal{G}' z)_{L_2(\mathbb{R})} = \beta^{-1} \int_{\Omega} \text{sign } V(t, y) W^2(t, y) \omega^2(t, y) |(Wz)(t)|^2 dt dy \]
\[ = \int_{\mathbb{R}} Q_{P, V}(t) |(Wz)(t)|^2 dt = \int_{Q_{P, V}} \frac{1}{\pi} \text{sign } Q_{P, V} \mathcal{G}' \mathcal{G}' z |(z, z)|_{L_2(\mathbb{R})}. \]

Now (5.20) follows from (4.18). \qed
6.2. Proof of Lemma 5.2. Write
\[ G_1(\gamma) - G_1(0) = W(x)((P + \gamma^2 I)^{-\frac{1}{2}} - P^{-\frac{1}{2}})(I - \mathcal{E}_\mu) \]
\[ = G_1(P + \gamma^2 I)^{-\frac{1}{2}}((P + \gamma^2 I)^{-\frac{1}{2}} - P^{-\frac{1}{2}})(I - \mathcal{E}_\mu) = G_1(f(\gamma; P) \right) \]
where
\[ f(\gamma; \lambda) = (\lambda + 1)^{-\frac{1}{2}}((\lambda + \gamma^2)^{-\frac{1}{2}} - \lambda^{-\frac{1}{2}})1_{(\mu, \infty)}(\lambda). \]
The function \( f(\gamma; \lambda) \) tends to zero as \( \gamma \to 0 \), uniformly in \( \lambda \in \mathbb{R}_+ \). This implies \( \|f(\gamma; P)\| \to 0 \). By (5.9), \( G_1(\gamma) \in \mathcal{C}_{d,\infty} \). Finally,
\[ \|G_1(\gamma) - G_1(0)\|_{d,\infty} \leq \|G_1\|_{d,\infty} \|f(\gamma; P)\| \to 0. \]
This proves (5.13).

It follows from (2.6) and (5.7) that the asymptotic formula (5.15) is valid for the operator \( G_1(1)^* \text{ sign } V G_1(1) \) substituted for \( \mathcal{K} \). Now we prove that
\[ G_1(0) - G_1(1) \in \mathcal{C}_{d,\infty}^0. \tag{6.10} \]
This will imply that the operator
\[ \mathcal{K} - G_1(1)^* \text{ sign } V G_1(1) = (G_1(0) - G_1(1))^* \text{ sign } V G_1(0) + G_1(1)^* \text{ sign } V (G_1(0) - G_1(1)) \]
lies in \( \mathcal{C}_{d,\infty}^0 \), and by (3.7) the operators \( \mathcal{K} \) and \( G_1(1)^* \text{ sign } V G_1(1) \) have the same principal term of spectral asymptotics. Thus, to justify the asymptotic formula (5.15) for \( \mathcal{K} \), it is sufficient to prove (6.10). To this end, we use the equality
\[ G_1(0) - G_1(1) = W(P^{-\frac{1}{2}} - (P + I)^{-\frac{1}{2}})(I - \mathcal{E}_\mu) - W(P + I)^{-\frac{1}{2}} \mathcal{E}_\mu =: \mathcal{M} - G_2(1). \]
The operator \( \mathcal{M} \) can be represented as
\[ \mathcal{M} = W(P + I)^{-1} f(P) \]
where \( f(\lambda) = (\lambda + 1)(\lambda^{-\frac{1}{2}} - (\lambda + 1)^{-\frac{1}{2}})1_{(\mu, \infty)}(\lambda) \) is a bounded function. By Lemma A2.2, \( W(P + I)^{-1} \in \mathcal{C}_{d,\infty}^0 \), and therefore the same is true for \( \mathcal{M} \).

Instead of \( G_2(1) \), we analyze the operator \( G_2'(1) \). This is enough, since both operators have the same singular numbers. According to (6.1),
\[ G_2'(1) = Y'(1) + Z'(1) + G_2'(1). \]
Here \( \text{rank } Y'(1) = 1 \) and by Lemma 6.1 \( Z'(1) \in \mathcal{C}_d \). The operator \( \widehat{G_2'(1)} \) (cf. (6.3)) is an integral operator with the kernel
\[ \sqrt{\beta / 2\pi} Q_{P_1}^{1/2} -\frac{e^{i\xi t_1} - 1}{\sqrt{E(\xi) + 1}} 1(1), \xi(1, \xi). \]
In view of (5.5), this kernel belongs to \( L_2(\mathbb{R}^2) \), so that \( \widehat{G_2'(1)} \in \mathcal{C}_2 \). The same is true for the operator \( G_2'(1) \). We conclude that \( G_2'(1) \in \mathcal{C}_{d,\infty}^0 \), hence also \( G_2(1) \in \mathcal{C}_{d,\infty}^0 \).
The inclusion (6.10) is justified, so Lemma is proved. \( \square \)

Let us derive one more useful inequality which we need in the next subsection. Using the representation
\[ G_1(0) = W P^{-\frac{1}{2}} (I - \mathcal{E}_\mu) = G_1(P + I)^{\frac{1}{2}} P^{-\frac{1}{2}} (I - \mathcal{E}_\mu) \]
and the elementary fact that \( (\lambda + 1)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \leq (1 + \mu^{-1})^{-\frac{1}{2}} \) for \( \lambda \geq \mu \), we find
\[ \|G_1(0)\|_{d,\infty} \leq (1 + \mu^{-1})^{-\frac{1}{2}} \|G_1(1)\|_{d,\infty}. \]
Together with (5.9), this implies
\[ \|W P^{-\frac{1}{2}} (I - \mathcal{E}_\mu)\|_{d,\infty} \leq C \|V\|_{\mathcal{X}(\Omega)}^{\frac{1}{2}}, \quad C = C(\mu, \Omega). \tag{6.11} \]
6.3. Proof of Lemma 5.4. Here we slightly change our notations, in order to reflect the dependence of the operators involved on the function $W = |V|^\frac{1}{2}$. On the other hand, we drop other indices. So, we now denote

$$G_{1,W} = WP^{-\frac{1}{2}}(I - \mathcal{E}_\mu)$$

and write $\mathcal{G}'_W$, $\mathcal{G}_W$ instead of $\mathcal{G}'$, $\mathcal{G}$:

$$(\mathcal{G}'_W z)(t, y) = \frac{W(t, y)\omega(t, y)}{\sqrt{\beta}} \int \frac{e^{i\xi t} - 1}{|\xi|} z(\xi)d\xi, \quad \mathcal{G}_W = \mathcal{G}'_W \Psi.$$  \hspace{1cm} (6.13)

Given a number $r > 0$ (it will be chosen later), we let

$$V_r = V \ (|t| < r), \quad V_r = 0 \ otherwise; \quad V^r = V - V_r,$$

and also $W_r = |V_r|^\frac{1}{2}$ and $W^r = |V^r|^\frac{1}{2}$. The key observation is

$$\mathcal{G}'_W \Psi \ sign V \ G_{1,W_r} = \mathcal{G}^*_W \ sign V \ G_{1,W^r} = 0;$$  \hspace{1cm} (6.14)

and these equalities follow directly from (6.12) and (6.13). It follows from (6.14) that

$$\mathcal{G}'_W \ sign V \ G_{1,W} = \mathcal{G}^*_W \ sign V \ G_{1,W_r} + \mathcal{G}^*_W \ sign V \ G_{1,W^r}.$$  \hspace{1cm} (6.15)

The second term on the right-hand side of (6.15) is an operator of the class $C^0_{\frac{3}{2}, \infty}$. Indeed, the effective potential $Q_{\mathcal{T}, W_r}$ lies in $L_1(\mathbb{R})$ and is compactly supported, therefore the problem (4.1) with $Q = Q_{\mathcal{T}, W_r}$ behaves according to the Weyl law, that is $n(\alpha^{-1}, Q_{\mathcal{T}, W_r}) = O(\alpha^{\frac{1}{2}})$. Using the equalities $\mathcal{G}'_W = \mathcal{J}_{\mathcal{T}, W_r}$ and (4.19), we see that $\mathcal{G}'_W \subset C_{1, \infty} \subset C^0_{d, \infty}$. Further, by Lemma 5.2 $G_{1,W} \in C_{d, \infty}$ and the reference to Horn’s inequality completes the argument.

Move on to the first term on the right-hand side of (6.15). We apply to it the inequality (3.8) with the function $\Phi_1(\alpha) = \Theta(g, V_r)\alpha^\frac{d}{2} + \Phi(\alpha)$, cf. (5.27). According to (5.21), (5.4) and (6.11) (with $W^r$ substituted for $W$)

$$\Delta_\Phi_1(\mathcal{G}'_W \mathcal{G}_W) \leq \Delta_\Phi(\mathcal{G}'_W \mathcal{G}_W) = \Delta_\Phi(\mathcal{J}_{Q_{\mathcal{T}, W_r}}) = 1;$$

$$\Delta_\Phi_1(\mathcal{G}^*_W \ sign V \ G_{1,W^r}) \leq C \Delta_d(G_{1,W^r}) \leq C_1 ||V^r||^\frac{3}{2}_{X(\Omega)}.$$  \hspace{1cm} (6.16)

By Lemma 3.2

$$\Delta^2_\Phi_1(\mathcal{G}'_W \ sign V \ G_{1,W^r}) \leq \Delta^2_\Phi_1(\mathcal{G}'_W \mathcal{G}_W) \Delta_\Phi_1(\mathcal{G}^*_W \ (sign V)^2 G_{1,W^r}) \leq C_1 ||V^r||^\frac{3}{2}_{X(\Omega)}.$$  \hspace{1cm} (6.16)

It follows from (6.15), (6.16) and the inclusion $\mathcal{G}'_W \ sign V \ G_{1,W} \in C^0_{\frac{3}{2}, \infty}$ that

$$\Delta^2_\Phi_1(\mathcal{G}'_W \ sign V \ G_{1,W}) = \Delta^2_\Phi_1(\mathcal{G}^*_W \ sign V \ G_{1,W^r}) \leq C_1 ||V^r||^\frac{3}{2}_{X(\Omega)}.$$  \hspace{1cm} (6.16)

The norm $||V^r||_X(\Omega)$ is arbitrarily small provided $r$ is large enough. Therefore $\Delta^2_\Phi_1(\mathcal{G}'_W \ sign V \ G_{1,W}) = 0$, which is equivalent to (5.27). \hspace{1cm} \Box

The proof of Theorem 5.1 is complete.
6.4. Proof of Theorem 2.2. In Theorem 5.1 conditions are imposed on both effective potentials $Q_{r,v}$ and $Q_{r,v}$. For $V \geq 0$ these conditions become one, and in view of Lemma 4.1, $3^\circ$ the inclusion (5.3) is implied by (5.4). So, it remains to prove the part “only if” of the assertions $1^\circ$ and $2^\circ$.

Making the standard substitution $u = \omega v$, where $v$ is a compactly supported function from $C^1(\overline{\Omega})$, we obtain

$$P[u] - \alpha \int_{\Omega} V |u|^2 dx = \int_{\Omega} \omega^2(x) \left( |g(x) \nabla v, \nabla v| - \alpha V(x) |v|^2 \right) dx, \quad (6.17)$$


Consider the (non-closed) subspace $\mathcal{F} \subset H^1(\Omega)$ whose elements are the functions

$$u(t, y) = \omega(t, y) w(t), \quad w \in C^1(\mathbb{R}), \supp(w) \text{ is compact.}$$

On $\mathcal{F}$ the equality (6.17) reduces to

$$P[u] - \alpha \int_{\Omega} V |u|^2 dx = \int_{\Omega} \omega^2(t, y) (g_{11}(t, y) |w'_t|^2 - \alpha V(t, y) |w|^2) dt dy$$

$$= \int_{\Omega} \omega^2(t, y) g_{11}(t, y) |w'_t|^2 dt dy - \alpha \beta \int_{\mathbb{R}} Q_{r,v}(t) |w|^2 dt.$$

Let

$$m_1 = \max_{x \in \Omega^\#} \omega(x); \quad m_2 = \sup_{t \in [0, 1]} \int_{\Omega(t)} g_{11}(t, y) dy,$$

then

$$P[u] - \alpha \int_{\Omega} V |u|^2 dx \leq m_1 m_2 \int_{\mathbb{R}} |w'_t|^2 dt - \alpha \beta \int_{\mathbb{R}} Q_{r,v}(t) |w|^2 dt$$

$$= m_1 m_2 a_{Ca} Q_{r,v} |u|, \quad c = \beta(m_1 m_2)^{-1}, \ u \in \mathcal{F}.$$

The subspace $\mathcal{F}$ is dense in $H^1(\mathbb{R})$, and by the variational principle

$$N_-(\mathcal{P} - \alpha V) \geq N_-(\alpha \beta^2 - c Q_{r,v}), \quad (6.18)$$

The part “only if” of the assertion $1^\circ$ follows immediately.

It also follows from (6.18) that

$$N_-(\mathcal{P} - \alpha V) = O(\alpha^{\frac{4}{3}}) \Rightarrow N_-(\alpha \beta^2 - c Q_{r,v}) = O(\alpha^{\frac{4}{3}}), \quad \alpha \to \infty.$$

The latter is equivalent to $\mathcal{T}_{Q_{r,v}} \in C_{r, \infty}$, so that the conditions of Lemma 5.4 are satisfied with $\Phi(\alpha) = a^{\frac{4}{3}}$. This implies $\hat{N} - N' \in C_{r, \infty}^0$ which can be proved in exactly the same way as the inclusion (5.28). Now, by (5.26) and (5.15),

$$n_+(\alpha^{-1}, S) \sim \Theta(g, V) a^{\frac{4}{3}} + n_+(\alpha^{-1}, \hat{N}).$$

It follows from here and from (5.24) that if (2.6) is satisfied for $\gamma = 0$, then $n_+(\alpha^{-1}, \hat{N}) = o(\alpha^{\frac{4}{3}})$. This is equivalent to (2.12), so $2^\circ$ is proved. $\square$
7. Discussion

7.1. The negative eigenvalues $\lambda_n(\mathcal{P} - sV)$ are decreasing functions of the real parameter $s$. The number $N_-(\mathcal{P} + \gamma^2 I - \alpha V)$ measures the quantity of these eigenvalues, passing the “observation point” $-\gamma^2$ when $s$ increases from 0 to $\alpha$. Comparing (2.6) and (2.10), we see that for $\gamma = 0$ an additional term appears in the asymptotic formula for $N_-(\mathcal{P} + \gamma^2 I - \alpha V)$. This can be interpreted as a “threshold effect”: when the bottom of $\text{Spec}(\mathcal{P})$ is chosen as the observation point, an additional channel opens whose strength is described by the second term on the right-hand side of (2.10). For slowly decaying potentials $V$ (still, lying in $X(\Omega)$) the contribution from this additional channel can be of the same strength, or even stronger than the contribution from the “basic” channel which is expressed by the Weyl-type term $\Theta(y, V_+) \alpha^{\frac{d}{2}}$.

The formula (2.10) shows that the contributions from these two channels are independent. This is not connected with any orthogonal decomposition of the operator $\mathcal{P} - \alpha V$, and for this reason is rather unexpected. We call this effect “asymptotic orthogonality”.

By now, several other problems are known where this effect exhibits in such a way that some auxiliary differential operator, acting in a “complementary” Hilbert space, appears and contributes to the asymptotics of the original problem. Probably, [JMSim] was the first publication where the effect was revealed. The authors considered the spectral asymptotics for the Neumann Laplacian in unbounded domains, rapidly narrowing down at infinity. It was shown that this asymptotics is given by the classical Weyl term, plus an additional term coming from an auxiliary Schrödinger operator acting in $L_2(\mathbb{R}^+)$. The present paper belongs to a series of articles where the effect of asymptotic orthogonality is investigated for various operators of Mathematical Physics. The starting paper of this series is [BLap 1], where the discrete spectrum of the perturbed Dirac operator $D - \alpha V 1$, lying in the gap $(-1, 1)$, was studied. The auxiliary operator is an appropriate Schrödinger operator in $L_2(\mathbb{R}^3)$.

In [BLap 2] the operator $-\Delta - \alpha V$ in $L_2(\mathbb{R}^2)$ was analyzed. This problem is of a more delicate nature than the one studied in [JMSim], because the complementary space which gives rise to this auxiliary operator, appears in somewhat "hidden" form. The effect exhibits only in dimension two. Technical tools used in [BLap 1] and [BLap 2] are quite different. All the subsequent papers of this series use the general scheme outlined in [BLap 2].

In [Sol 2] the asymptotic behaviour of $N_-(\mathcal{P} + \gamma^2 I - \alpha V)$ in domains $\Omega \subset \mathbb{R}^d$ of the type $|y| < f(t)$, $t > 0$ was considered. The character of the result was similar to those of [JMSim] and [BLap 2]. The basic assumption on $f$ in [Sol 2] was $f'(t) \to 0$ as $t \to \infty$. Prof. S. Agmon, who attended a talk where the results of this paper were reported, stated the question: what happens if the latter assumption is violated, for example if $f$ is periodic? The present paper is a result of our attempt to answer this question.

Probably, the most difficult paper of the series is [BLapSus]: it is necessary to overcome difficulties coming from both sources, namely from the periodicity and from the two-dimensionality. As it was already mentioned in the Introduction, in the present paper we basically follow the scheme developed in [BLapSus]. For our problem two-dimensionality as a source of difficulties disappears, but another one
arises instead: the operators $\mathcal{P}_\xi^\#$ now act on functions in the domain $\Omega^\#$ with the Lipschitz boundary, whereas in [BLapSus] one is dealing with the operators on the torus $\mathbb{T}^2$. As a result, we need somewhat more advanced techniques, see Appendix A1.

Other problems, where one may encounter "competition between two channels", include study of the eigenvalues of $\mathcal{P} - \alpha V$, lying in the finite gaps of the unperturbed operator $\mathcal{P}$ ($\mathbb{R}_-$ is the semi-infinite gap). The approach developed in the present paper, applies to this problem as well. The results will be presented elsewhere. For the operators, studied in [BLapSus], the eigenvalues in gaps are investigated in [BSus].

The case of a domain $\Omega$, periodic with respect to the lattice $\mathbb{Z}^2$, can also be treated in a similar way. One should combine the techniques of the present paper and of [BLapSus]. For domains, periodic with respect to the lattice $\mathbb{Z}^k$ for $k > 2$, the effect does not exhibit.

7.2. Here we present an alternative version of the main result. The changes concern only the assertion 3° of Theorem 5.1.

**Theorem 7.1.** Suppose that the conditions of Theorem 5.1, 3° are satisfied, except for the assumption (5.4). Then

a) if $2q > d$, then

$$\limsup_{\alpha \to \infty} \frac{N_- (\mathcal{P} - \alpha V)}{\Phi(\alpha)} = \Delta_+^+ (\mathcal{T}_Q, \nu) ; \quad \liminf_{\alpha \to \infty} \frac{N_- (\mathcal{P} - \alpha V)}{\Phi(\alpha)} = \delta_+ (\mathcal{T}_Q, \nu) ;$$

b) if $2q = d$ and $\Phi(\alpha) = \alpha^{\frac{d}{2}}$, then

$$\limsup_{\alpha \to \infty} \alpha^{-\frac{d}{2}} N_- (\mathcal{P} - \alpha V) = \Theta(g, V_+) + \Delta_+^+ (\mathcal{T}_Q, \nu) ;$$

$$\liminf_{\alpha \to \infty} \alpha^{-\frac{d}{2}} N_- (\mathcal{P} - \alpha V) = \Theta(g, V_+) + \delta_+^+ (\mathcal{T}_Q, \nu) .$$

For the proof, only some minor changes are necessary in the reasonings presented in Sections 5 and 6. We leave them to the reader.

Clearly, the result of the assertion a) is more general than the one given by Theorem 5.1 for $2q > d$. It is especially important that the function $n_+ (\alpha^{-1}, \mathcal{T}_Q, \nu)$ must not be regularly varying. For non-negative $V$, this automatically improves also the corresponding result of Theorem 2.2. On the other hand, in the case $2q = d$ Theorem 7.1 ignores the possibility of the non-powerlike behaviour of $N_- (\mathcal{P} - \alpha V)$. Still, the result of b) is not covered by Theorem 5.1.

Till now, the authors could not find a way to formulate a result which would cover both Theorems 5.1 and 7.1.

The way of presentation, used in Theorem 7.1, originates from [BLap 1]. The papers [BLap 2] and [BLapSus] also follow this line. The way of presentation, used in Theorems 2.2 and 5.1, stems from [JMSim]. The regularity conditions on the behaviour of the counting function for the auxiliary operator in [JMSim] are slightly weaker than those in the present paper. This is due to the fact that geometry of the domains, considered in [JMSim], is simpler than the geometry of the periodic domain $\Omega$.
7.3. In order to construct the effective potential (2.9), one has to know the “effective mass” $\beta^{-1}$ and the principal eigenfunction $\omega(x)$ of the operator $\mathcal{P}_0^#$. Especially, the determination of $\omega(x)$ is usually a difficult problem. In this connection we point out a case where this problem becomes trivial. Namely, let in (1.14) $p(x) \equiv 0$ and let $\mathcal{P} = \mathcal{P}_\mathcal{V}$. Then evidently (1.18) is satisfied and $\omega(x) = (\text{meas } \Omega^#)^{-\frac{1}{2}} = \text{const.}$

Note also that the choice of the effective potential is, in principle, not unique: there can be other ways to construct a potential, say $Q_{\mathcal{P},\mathcal{V}}(t)$, such that

$$N_-(\mathcal{P} - \alpha \mathcal{P}_\mathcal{V}) \sim N_-(\mathcal{P} - \alpha \mathcal{P}_{\mathcal{V}}), \quad \alpha \to \infty. \quad \text{(7.1)}$$

It may happen that the expression for $Q_{\mathcal{P},\mathcal{V}}$ does not involve $\omega$. A search for such type of the effective potential is important for applications. A rather mild regularity conditions on $\mathcal{V}$, allowing one to remove $\omega$ from the expression for the effective potential, was found in [BLapSus] for the problem considered there. For our problem the situation is different which is illustrated by the following example.

For the potential

$$V(t, y) = \eta(t)\zeta(y) \quad \text{(7.2)}$$

the equality (2.9) gives

$$\beta \mathcal{Q}_{\mathcal{P},\mathcal{V}}(t) = \eta(t)\varphi(t), \quad \varphi(t) = \int_{\Omega(t)} \zeta(y)\omega^2(t, y)dy.$$ 

The function $\varphi$ is periodic. The argument from [BLapSus] applies to it and shows that generically (7.1) is satisfied for the potential (7.2) if we take

$$\beta Q_{\mathcal{P},\mathcal{V}} = M(\zeta)\eta(t), \quad M(\zeta) = \int_{\Omega^#} \zeta(y)\omega^2(t, y)\,dy\,dy;$$

we do not specify the assumptions on $\zeta(y)$, justifying this substitution. Since $\|\omega\|_{L_2(\Omega^#)} = 1$, we see that $M(\zeta) = 1$ for $\zeta \equiv 1$, so in this case $\omega$ can be removed from the expression for $Q_{\mathcal{P},\mathcal{V}}(t)$. However, we also see that in general such a removal is impossible even for the very simple potentials (7.2).

7.4. Numerous concrete examples of operators for which the function $N_-(\mathcal{P} - \alpha \mathcal{V})$ behaves in a non-standard way, can be constructed based upon Lemma 4.2. Here we consider only the simplest case when $\Omega = \mathbb{R} \times \Omega(0)$ is a cylinder, in (1.14) $p(x) \equiv 0$, and $\mathcal{P} = \mathcal{P}_\mathcal{V}$. Then $\omega(x) \equiv (\text{meas}_{d-1} \Omega(0))^{-\frac{1}{2}}$ (cf. Subsection 7.2) and

$$Q_{\mathcal{P},\mathcal{V}}(t) = \beta^{-1}(\text{meas}_{d-1} \Omega(0))^{-1} \int_{\Omega(0)} V(t, y)dy.$$

If $V \geq 0$ does not depend on $y$, then $Q_{\mathcal{P},\mathcal{V}} = \beta^{-1}V$. Choose now any $\Phi \in \mathcal{S}_q$, obeying the conditions of Lemma 4.2. Let $V = V(t)$ be the potential $Q(t)$ from (4.12), extended to $t < 0$ by zero. Then $V \in X(\Omega)$ (for $d = 2$ the easiest way to check this is to use (2.3)). According to Theorem 2.2 and Lemma 4.2, $N_-(\mathcal{P} - \alpha \mathcal{V})$ has the Weyl-type asymptotics (2.6) if $2q < d$. Further,

$$N_-(\mathcal{P} - \alpha \mathcal{V}) \sim \Theta(g, V)\alpha^{\frac{d}{2}} + \beta^{-1}C(q)\Phi(\alpha), \quad \alpha \to \infty \quad (2q = d) \quad \text{(7.3)}$$
and
$$N_\pm (\mathcal{P} - \alpha V) \sim \beta^{-1} C(q) \Phi (\alpha), \quad \alpha \to \infty \quad (2q > d). \quad (7.4)$$

In (7.3) and (7.4) $C(q)$ is the constant (4.13). Let in particular $\Lambda (\alpha)$ be the function (3.2). The product $\alpha^q \Lambda (\alpha)$ is a monotone function for $\alpha$ large enough. Extend it to the whole of $\mathbb{R}_+$ in such a way that the resulting function $\Phi (\alpha)$ is non-decreasing and positive. The specific choice of this extension does not affect the result. Now the formulas (7.3) and (7.4) show that the function $\alpha^{-q} N_\pm (\mathcal{P} - \alpha V)$ asymptotically oscillates, in a prescribed way. Theorem 7.1 also applies but for $2q = d$ gives a result which is much less complete. For $\Lambda$ given by (3.3) Theorem 7.1 gives only a very rough result, while Theorem 2.2 describes the asymptotic behaviour in a precise way.

A1. Proof of Theorem 1.1

A1.1. Recall that Theorem describes properties of the eigenvalues $E_k (\xi)$ and the eigenfunctions $\omega_k (\xi; x)$ of the operators $\mathcal{P}_\xi ^\#$ in $L_2 (\Omega ^\#)$, associated with the quadratic forms $P_\xi ^\#$ given by (1.15); the operator $\nabla^\xi$ appearing in (1.15) was defined in (1.9). According to (1.16), the domain $\mathfrak{d} = \text{Dom} (P_\xi ^\#)$ is independent of $\xi$. For each non-zero element $\varphi \in \mathfrak{d}$ and real $\xi$ the expression $P_\xi ^\#[\varphi]$ is a quadratic polynomial in $\xi$, with the zero order term $P_0 ^\#[\varphi]$. The coefficients in front of $\xi$ and $\xi^2$ are quadratic forms in $\varphi$ which are compact with respect to $P_0 ^\#[\varphi] + M \| \varphi \|_{L_2 (\Omega ^\#)}^2$ where $M > 0$ is large enough. (It follows from the equality (A1.12) in Subsection A1.4 that actually this quadratic form is positive definite in $L_2 (\Omega ^\#)$ for any $M > 0$.) This allows one to extend the family $P_\xi ^\#$ to any $\xi \in \mathbb{C}$, the extended family being holomorphic of type (a), see [K], Section VII.4.2. Correspondingly, the operators $\mathcal{P}_\xi ^\#$ constitute a holomorphic family of type (B). The mapping $\varphi \mapsto e^{2\pi i t} \varphi$ is unitary in $L_2 (\Omega ^\#)$, preserves $\mathfrak{d}$, and $P_\xi ^\# [e^{2\pi i t} \varphi] = P_{\xi + 2\pi} ^\# [\varphi]$ for each $\xi \in \mathbb{R}$. Also, $P_{\xi + 2\pi} ^\# [\varphi] = P_\xi ^\# [\varphi]$, since the coefficients $g_{jk} (x)$ and $p (x)$ are real.

It follows that the spectra of the operators $\mathcal{P}_\xi ^\#, \mathcal{P}_{-\xi} ^\#$ and $\mathcal{P}_{\xi + 2\pi} ^\#$ coincide. Therefore, each function $E_k (\xi)$ is even and periodic. Other properties of $E_k (\xi)$ and of $\omega_k (\xi; \cdot)$, listed in assertions $1^\circ$ and $2^\circ$ of Theorem 1.1, follow from the general theory of holomorphic families of operators.

A1.2. Basically, assertion $4^\circ$ follows from the De Giorgi theory on the smoothness properties of the solutions of elliptic equations in the divergency form with real measurable coefficients, see [GilTr], Theorem 8.24 and 8.29, or [LadUr], Sections III.13 and III.14. For the reader’s convenience, we present here formulations of the results we need in the form, convenient for our purposes. Then, in the next Subsection we prove assertion $4^\circ$.

Consider a differential equation in divergency form in a bounded domain $D \subset \mathbb{R}^d$:
$$Lu := \nabla^* (a(x) \nabla u + b(x) u) + \langle \nabla u , c (x) \rangle + r(x) u = f(x) \quad (A1.1)$$
where the function $r(x)$ and the entries $a_{jk}(x)$, $b_j(x)$, $c_j(x)$ of the uniformly positive definite symmetric matrix $a(x)$ and of the vectors $b(x)$, $c(x)$ are measurable, real
and bounded. As usual, the equation (A.1.1) is understood in the weak sense. This means that \( u \in H^1_{\text{loc}}(D) \) and the equality

\[
\int_D \left( (a(x) \nabla u, \nabla v) + u \langle b(x), \nabla v \rangle + \langle \nabla u, c(x) \rangle + r(x) u \overline{v} \right) dx = \int_D f(x) \overline{v} dx
\]

is satisfied for any function \( v \in C_0^\infty(D) \).

Let now \( S \) be an open subset of \( \partial D \). Suppose that \( u \in H^1(D) \) and the above equality is satisfied for any \( v \in H^1(D) \), such that \( v = 0 \) in a vicinity of the set \( \partial D \setminus S \). Then one says that on \( S \) the solution \( u \) meets the Neumann boundary condition. The Dirichlet boundary condition on \( S \) is understood in the standard sense.

We say that the numbers

\[
\lambda_0 = \text{ess inf}_{x \in D} \min_{\|\xi\| = 1} \langle a(x) \xi, \xi \rangle \quad \text{(ellipticity constant)}
\]

and

\[
\|r\|_{L^\infty(D)} + \sum_{j,k=1}^d \|a_{jk}\|_{L^\infty(D)} + \sum_{j=1}^d (\|b_j\|_{L^\infty(D)} + \|c_j\|_{L^\infty(D)})
\]

are generic constants of the differential expression \( \mathcal{L} \).

**Proposition A.1.1.** Let a function \( u \in H^1(D) \) satisfy the equation (A.1.1) where \( f \in L_q(D) \) for some \( q > d/2 \). Let \( D' \subset D \) be a strictly interior subdomain. Then \( u \in C(\overline{D'}) \) and

\[
\max_{x \in \overline{D'}} |u| \leq C \left( \|a\|_{L_2(D)} + \|f\|_{L_q(D)} \right),
\]

where \( C \) depends on \( q \), the generic constants of \( \mathcal{L} \), and on \( \text{dist}(D', \partial D) \).

Similar results are valid for solutions of the boundary value problems. Recall that the boundary is called Lipschitz at the point \( x \in \partial D \), if \( x \) has a neighbourhood \( U \subset \mathbb{R}^d \), such that in some Cartesian coordinate system the set \( D \cap U \) can be described by the inequality \( x_d < h(x_1, \ldots, x_{d-1}) \) where the function \( h \) is Lipschitz. Let \( L(x) \) stand for the minimal possible Lipschitz constant of \( h \) in such description.

We say that an open subset \( S \subset \partial D \) is Lipschitz, if the above property is satisfied at any point \( x \in S \) and \( L(S) := \sup_{x \in S} L(x) < \infty \).

**Proposition A.1.2.** Suppose that an open subset \( S \subset \partial D \) is Lipschitz and a subdomain \( D' \subset D \) is such that \( \partial D' \cap \partial D \subset S \). Let a function \( u \in H^1(D) \) satisfy the equation (A.1.1) where \( f \in L_q(D) \) for some \( q > d/2 \), and the Dirichlet or the Neumann boundary condition on \( S \). Then \( u \in C(\overline{D'}) \) and the estimate (A.1.3) is fulfilled. The constant \( C \) depends on \( q \), the generic constants of \( \mathcal{L} \), \( \text{dist}(D', \partial D \setminus S) \) and \( L(S) \).

The books quoted give this result only for the Dirichlet boundary condition. The result for the Neumann boundary condition can be easily derived from Proposition A.1.1 by using “straightening the boundary” and then, the standard reflection procedure. If \( S \) is smooth enough (say, of class \( C^2 \)), then the result is contained in [LadUr], Section X.2, were it is proved for elliptic *quasilinear* equations.

We need also an analog of (the global version of) Proposition A.1.2 for the equations, in which the lower order coefficients in (A.1.1) are allowed to be complex.
Proposition A1.3. Let in the equation (A1.1) the matrix $a(x)$ be as above, and the functions $b(x)$, $c(x)$ and $r(x)$ be complex-valued. Suppose that $D$ has the Lipschitz boundary. Let a function $u \in H^1(D)$ satisfy the equation (A1.1) and the Dirichlet or the Neumann boundary condition. Then $u \in C(\overline{D})$ and the estimate (A1.3) is fulfilled. The constant $C$ depends on $D$ and on the generic constants of the equation.

For the Dirichlet boundary condition the result is a particular case of [LadUr], Theorem VII.2.1. For the Neumann boundary condition, one should verify beforehand that

$$\int_D (|u|^4 + |u|^2 \nabla u|^2)\,dx < \infty.$$ 

This can be proved in exactly the same way as the similar fact for the Dirichlet condition, see [LadUr], Section VII.2.

The result applies also to equations on a Riemannian manifold; the constant $C$ then depends also on the metric.

Note also that actually under the assumptions of all Propositions A1.1 - A1.3 the solution $u$ is Hölder continuous in $D'$. We do not use this stronger result.

We are grateful to N.N. Uraltseva for the exhaustive consultation on the subject.

A1.3. Here we prove assertion 4°. As soon as we prove the continuity of the eigenfunctions, 5° becomes a well known property of the principal eigenfunction of elliptic operators with real-valued coefficients, and therefore needs no separate proof.

The reasoning below is a refinement of an argument from [B3], Section 4.9.

The eigenfunction $\omega_m(\xi; x)$ satisfies in $\Omega^\#$ the equation

$$\nabla(\xi)^* (g(x)\nabla(\xi)\omega) + (p(x) - E_m(\xi))\omega = 0. \quad (A1.4)$$

Denote by $g_1(x)$ the first row of the matrix $g(x)$. The equation (A1.4) is a particular case of (A1.1), with $a(x) = g(x)$, $b(x) = c(x) = i\xi g_1(x)$, $r(x) = p(x) + \xi^2 g_{11}(x) - E_m(\xi)$ and $f = 0$. The ellipticity constant does not depend on $\xi$ and the constants (A1.2) are bounded uniformly with respect to $\xi \in [-\pi, \pi]$. The boundary condition $N$ or $D$ is satisfied on $\partial \Omega^\#$. Proposition A1.3 applies, and according to the estimate (A1.3) we obtain

$$\max_{x \in \Omega^\#} |\omega_m(\xi; x)| \leq C_m \|\omega_m(\xi; \cdot)\|_{L^2(D)} = C_m. \quad (A1.5)$$

So, the continuity of the eigenfunctions, and also the estimate (1.21) are proven.

Let now $E_m(\xi_0)$ be a simple eigenvalue of the operator $P_\xi^\#$ and let $\omega_m(\xi; \cdot)$ be a $H^1(\Omega^\#)$-valued real analytic function in a neighbourhood of $\xi_0$, whose values are (normalized in $L^2(\Omega^\#)$) eigenfunctions, corresponding to the eigenvalue $E_m(\xi)$. Our next goal is to show that $\omega_m(\xi; \cdot)$ is real analytic also as a $C(\overline{\Omega^\#})$-valued function. The proof will consist of two steps. First, we prove that $\omega_m(\xi; \cdot)$ is real analytic as a $L^\infty(\Omega^\#)$-valued function. Second, we show that all the derivatives $\omega_m^{(n)}(\xi_0, \cdot)$ (derivatives in $\xi$, taken in the sense of $H^1$-valued functions) are continuous on $\overline{\Omega^\#}$.

Being taken together, these two facts give the desired result.
Below we drop the index $m$ from our notations. The functions $E(\xi)$ and $\omega(\xi; \cdot)$ can be analytically extended to a small complex neighbourhood $U \ni \xi_0$ in such a way that the equation \((A1.4)\) and the corresponding boundary condition are satisfied. For $\xi \not\in \mathbb{R}$ the eigenfunctions $\omega(\xi; \cdot)$ are no longer normalized in $L_2(\Omega^\#)$, however their norms are uniformly bounded in $U$. The above reasoning, based upon Proposition \(A1.3\), applies and shows that the function $\omega(\xi, \cdot)$ is continuous on $\Omega^\#$ and satisfies the estimate \((A1.5)\), though possibly with a bigger constant factor $C$.

Let $C$ be the circle $|z - \xi_0| = \varepsilon$, with a sufficiently small $\varepsilon > 0$. By Cauchy’s formula, the equality

$$\omega^{(n)}(\xi, \cdot) = \frac{n!}{2\pi i} \int_C \frac{\omega(z, \cdot)}{(z - \xi)^{n+1}} dz$$

is satisfied for any $\xi$ inside this circle. For each $z \in C$ the integrand is an element of $C(\Omega^\#)$, but the integral is well-defined only in the sense of $H^1$-valued functions. For this reason we can not immediately claim that the integral also belongs to $C(\Omega^\#)$. However, it follows from \((A1.5)\) that $\omega^{(n)}(\xi, \cdot) \in L_\infty(\Omega^\#)$, and

$$\|\omega^{(n)}(\xi, \cdot)\|_{L_\infty(\partial)} \leq 2C n!(2\varepsilon^{-1})^n, \quad |\xi - \xi_0| \leq \frac{\varepsilon}{2}, \quad n \in \mathbb{N}. \quad (A1.6)$$

The estimate \((A1.6)\) guarantees the convergence in $L_\infty(\Omega^\#)$ of the Taylor series for $\omega(\xi; \cdot)$, which proves its $L_\infty(\Omega^\#)$-analyticity at the point $\xi_0$.

Move on to the second step. It is easy to see that each derivative satisfies in $\Omega^\#$ a certain elliptic equation. Unfortunately, Proposition \(A1.3\), and also more general results of \cite{LadUr}, Chapter VII, do not apply to this equation. The approach we use below, allows us to derive the continuity of the derivatives from Proposition \(A1.2\), however only for real $\xi$ (we shall take $\xi = \xi_0$).

Let us consider \((A1.4)\) as an equation on $\Omega$, rather than on $\Omega^\#$. The function

$$\psi(\xi; x) = e^{i\xi \cdot x} \omega(\xi; x) \quad (A1.7)$$

is a solution of an equation with real coefficients

$$\nabla^*(q(x)\nabla \psi) + (p(x) - E_m(\xi)) \psi = 0. \quad (A1.8)$$

The $n$-th derivative $\psi^{(n)}(\xi_0; x)$ of the function \((A1.7)\) at the point $\xi_0 \in \mathbb{R}$ satisfies the nonhomogeneous elliptic equation

$$\mathcal{P} \psi^{(n)}(\xi_0; \cdot) - E(\xi_0)\psi^{(n)}(\xi_0; \cdot) = \sum_{j=0}^{n-1} \binom{n}{j} E^{(n-j)}(\xi_0) \psi^{(j)}(\xi_0; \cdot) \quad (A1.9)$$

which can be easily derived from \((A1.8)\) written in the weak form. Since all the coefficients in \((A1.9)\) are real, the equation splits into the pair of similar equations for the real and the imaginary parts of $\psi^{(n)}$.

Take $D = \Omega_{-1} \cup \Omega_0' \cup \Omega_1'$ (see Subsection 1.1), then $D$ has Lipschitz boundary. The boundary condition $D$ or $N'$ on $\partial \Omega \cap \partial D$ is satisfied for $\psi^{(n)}(\xi_0; \cdot)$, and therefore for $\Re \psi^{(n)}(\xi; x)$ and $\Im \psi^{(n)}(\xi; x)$. Choose a subdomain $D' \subset D$ such that $\Omega_0 \subset D'$, $\partial D' \cap \partial D \subset S$, and $\text{dist}(D \setminus D', \Omega_0) > 0$. Now, we use the standard inductive procedure. Suppose that $\omega^{(j)}(\xi_0; \cdot) \in C(\Omega^\#)$ for all $j < n$. We already know that this is true for $n = 1$. Applying Proposition \(A1.2\) to $\Re \psi^{(n)}(\xi; x)$ and $\Im \psi^{(n)}(\xi; x)$, we find that $\psi^{(n)}(\xi_0; \cdot) \in C(\Omega^\#)$. Therefore, $\omega^{(n)}(\xi_0; \cdot) \in C(\Omega^\#)$.

The proof of $\mathbf{4}^\#$ is complete.

Note that for $d \leq 5$ this argument can be sharpened, and the real analyticity of $\omega(\xi; \cdot)$ as a $C(\Omega^\#)$-valued function can be proved without using Proposition \(A1.3\).
A1.4. In this Subsection we prove assertion 3°. The fact that the principal eigenvalue \( E(0) \) is non-degenerate, is a well known property of elliptic operators with real-valued coefficients. Proofs of other statements require certain preliminary work.

Since \( E(0) = 0 \), the equation for the eigenfunction \( \omega \) turns into

\[
\int_{\Omega^\#} \left( (g \nabla \omega, \nabla \varphi) + p \omega \varphi \right) d\mathbf{x} = 0, \quad \varphi \in \mathfrak{a}.
\] (A1.10)

For any \( v \in C^1(\overline{\Omega^\#}) \) the product \( \omega |v|^2 \) belongs to \( \mathfrak{a} \). Substituting this function for \( \varphi \) in (A1.10), we get

\[
\int_{\Omega^\#} \left( |v|^2 (g \nabla \omega, \nabla \omega) + \omega (g \nabla \omega, \nabla |v|^2) + p \omega^2 |v|^2 \right) d\mathbf{x} = 0, \quad v \in C^1(\overline{\Omega^\#}).
\] (A1.11)

The function \( \varphi = \omega v \) also lies in \( \mathfrak{a} \), both in the case \( \mathcal{N} \) and in the case \( \mathcal{D} \). Calculating \( P_{\xi}^\# [\omega v] \) and taking (A1.11) into account, we obtain

\[
P_{\xi}^\# [\omega v] = \int_{\Omega^\#} \omega^2 (g \nabla (\xi) v, \nabla (\xi) v) d\mathbf{x}, \quad v \in C^1(\overline{\Omega^\#}).
\] (A1.12)

Along with (A1.12), one has also

\[
P_{\xi}^\# [\varphi] + \int_{\Omega^\#} |\varphi|^2 d\mathbf{x} = \int_{\Omega^\#} \omega^2 ((g \nabla (\xi) v, \nabla (\xi) v) + |v|^2) d\mathbf{x},
\] \( \varphi = \omega v, \quad v \in C^1(\overline{\Omega^\#}), \quad \xi \in \mathbb{R}. \) (A1.13)

The quadratic form on the left-hand side of (A1.13) can be taken as the metric form on \( \mathfrak{a} \). Based upon (A1.13), we now show that the mapping \([\omega]: v \mapsto \omega v\) extends by continuity, to become an isomorphism between certain appropriate Hilbert spaces. Here we consider the cases \( \mathcal{N} \) and \( \mathcal{D} \) separately.

In the case \( \mathcal{N} \) the function \( \omega \) is bounded away from zero (by assertion 5°) and therefore the expression on the right-hand side of (A1.13) also generates the topology of \( H^1(\Omega^\#) \). It follows that the mapping \([\omega]\) extends to the whole of \( H^1(\Omega^\#) \) and is an automorphism of this space.

In the case \( \mathcal{D} \) consider the Hilbert space \( H^1_\omega(\Omega^\#) \), whose elements are functions \( v \) on \( \Omega^\# \) for which the expression on the right-hand side of (A1.13) is finite. We take this expression as the metric form in \( H^1_\omega(\Omega^\#) \). Denote by \( V_\omega \) the closure in \( H^1_\omega(\Omega^\#) \) of \( C^1(\Omega^\#) \). The equality (A1.13) shows that the operator \([\omega]\) maps \( V_\omega \) into \( \mathfrak{a}_\mathcal{D} = H^{1,0}(\Omega^\#) \) and

\[
\|\omega v\|_{H^1(\Omega^\#)} \geq c \|v\|_{H^1_\omega(\Omega^\#)}, \quad c > 0.
\]

Hence, \( \text{Ran}(\omega) \) is closed in \( H^{1,0}(\Omega^\#) \). If \( \varphi \in C^1(\overline{\Omega^\#}) \) and \( \varphi = 0 \) in a neighbourhood of \( \partial \Omega^\# \), then \( \omega^{-1} \varphi \in C^1(\Omega^\#) \), hence \( \text{Ran}(\omega) \) is dense in the space \( H^{1,0}(\Omega^\#) \) and therefore, coincides with it.

We have proved that in the case \( \mathcal{D} \) the operator \([\omega]\) maps \( V_\omega \) onto \( H^{1,0}(\Omega^\#) \). In both cases, it follows that \( P_{\xi}[\varphi] \geq 0 \) on \( \mathfrak{a} \). This implies \( E(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \).
It remains to prove (1.20). Since $E(\xi)$ is even and real analytic at $\xi = 0$, it suffices to show that

$$E(\xi) \geq c\xi^2, \quad c > 0, \xi \in [-\pi, \pi]. \quad (A1.14)$$

The basic observation is the inequality

$$\int_{S^1} |(e^{i\xi t} z(t))'|^2 dt \geq \xi^2 \int_{S^1} |z(t)|^2 dt, \quad z \in H^1(S^1). \quad (A1.15)$$

For the proof, one expands $z(t)$ in the Fourier series and then uses Parseval’s identity.

Let now $X_B = S^1 \times B$ where $B$ stands for the unit ball in $\mathbb{R}^{d-1}$. It immediately follows from (A1.15) that

$$\int_{X_B} |\nabla(x)|^2 \|w\|^2 \geq \xi^2 \int_{X_B} |w|^2 \|x\|^2, \quad w \in H^1(X_B). \quad (A1.16)$$

The inequality (A1.14) will follow from (A1.16), however certain preliminary technical considerations are necessary.

Take any point $x_0 = (t_0, y_0) \in \Omega$, then also $x_1 = (t_0 + 1, y_0) \in \Omega$. Since $\Omega$ is connected, there is a smooth path in $\Omega$ with the initial point $x_0$ and the endpoint $x_1$. Its image in $\Omega^\#$ is a loop. Without loss of generality, we may assume that this loop is smooth and has no self-intersections.

By the theorem on tubular neighbourhood (see e.g. [H], Section 4.5), there exists a smooth embedding $\Psi : S^1 \times B \to \Omega^\#$ such that the image $X := \Psi(X_B)$ is a strictly interior subdomain of $\Omega^\#$. Below $x' = (t', y')$ denotes a point in $X_B$ and $J(x')$ stands for the Jacobian. Let $v \in H^1(X)$ and $w := v \circ \Psi$. Then

$$\int_X |\nabla(x)|^2 \|v\|^2 dx = \int_{X_B} |\nabla(x') (e^{i\xi t'} w)|^2 |J(x')| dx'$$

$$= \int_{X_B} |\nabla(x') (e^{i\xi t'} (e^{i\xi ((t', y')-x') w(x')))|^2 |J(x')| dx'.$$

The function $e^{i\xi ((t', y')-x')}$ is periodic in $t'$, and hence is smooth on $X_B$. Therefore, the inequality (A1.16) applies to the product $w_1(x') = e^{i\xi ((t', y')-x') w(x')}$ and we obtain

$$\int_X |\nabla(x)|^2 \|v\|^2 dx \geq C \xi^2 \int_{X_B} |w_1(x')|^2 dx' \geq C' \xi^2 \int_X |v(x)|^2 dx \quad (A1.17)$$

where $C = \min |J(x')|$ and $C' = C(\max |J(x')|)^{-1}$.

To finalize the proof, we need the following auxiliary statement.

**Lemma A1.4.** Let $X \subset \Omega^\#$ be an arbitrary subset of a positive Lebesgue measure. Then there exists a number $C_X > 0$ such that

$$\int_{\Omega^\#} \|\varphi\|^2 dx \leq C_X (P^\#_\xi \varphi + \int_X \|\varphi\|^2 dx), \quad \varphi \in \mathcal{D}, \text{ any } \xi \in [-\pi, \pi]. \quad (A1.18)$$
Proof. Suppose that (A.18) is violated. Then a number sequence \( \{ \xi_n \} \in [-\pi, \pi] \) and a sequence of functions \( \{ \varphi_n \} \in H^1(\Omega^\#) \) can be found, such that
\[
\int_{\Omega^\#} |\varphi_n|^2 d\mathbf{x} = 1, \quad P_{\xi_n}^\# [\varphi_n] \to 0, \quad \text{and} \quad \int_X |\varphi_n|^2 d\mathbf{x} \to 0. \tag{A.19}
\]
Each quadratic form \( P_{\xi}^\# [\varphi] + \int_{\Omega^\#} |\varphi|^2 d\mathbf{x} \) generates on \( \mathfrak{d} \) the topology of the space \( H^1(\Omega^\#) \), and moreover, in the inequality
\[
c_1 \int_{\Omega^\#} (|\nabla \varphi|^2 + |\varphi|^2) d\mathbf{x} \leq P_{\xi}^\# [\varphi] + \int_{\Omega^\#} |\varphi|^2 d\mathbf{x} \leq c_2 \int_{\Omega^\#} (|\nabla \varphi|^2 + |\varphi|^2) d\mathbf{x}, \varphi \in \mathfrak{d}
\]
the constants \( c_1 > 0 \) and \( c_2 \) do not depend on \( \xi \in [-\pi, \pi] \). Therefore, (A.19) implies the boundedness of \( \{ \varphi_n \} \) in \( H^1(\Omega^\#) \). A subsequence (let it be \( \{ \varphi_n \} \) itself) converges weakly in \( H^1(\Omega^\#) \), and consequently strongly in \( L^2(\Omega^\#) \) and in \( L^2(X) \), to some element \( \varphi_0 \). We can also assume that \( \xi_n \to \xi_0 \). It follows from these properties that \( P_{\xi_0}^\# [\varphi_n] \to 0 \) as \( n \to \infty \) and, further,
\[
P_{\xi_0}^\# [\varphi_0] + \int_{\Omega^\#} |\varphi_0|^2 d\mathbf{x} \leq \liminf_{n \to \infty} (P_{\xi_n}^\# [\varphi_n] + \int_{\Omega^\#} |\varphi_n|^2 d\mathbf{x}) = 1.
\]
Therefore, \( P_{\xi_0}^\# [\varphi_0] = 0 \) which implies \( \varphi_0(t, y) = C e^{-i \xi_0 t}, C = \text{const.} \). Now, we have
\[
\int_{\Omega^\#} |\varphi_0|^2 d\mathbf{x} = |C|^2 \text{ meas } \Omega^\# = 1, \quad \int_X |\varphi_0|^2 d\mathbf{x} = |C|^2 \text{ meas } X = 0,
\]
which is a contradiction. \( \Box \)

Proof of (A.14). Let \( \varphi \in \mathfrak{d} \) and \( v = \omega^{-1} \varphi \). Let \( X \subset \Omega^\# \) be a strictly interior sub-domain on which the inequality (A.17) is satisfied. Let
\[
m_0 = \min_{x \in X} \omega(x), \quad m_1 = \max_{x \in \Omega^\#} \omega(x), \tag{A.20}
\]
then \( m_0 > 0 \). Applying (A.12) and (A.17), we obtain:
\[
P_{\xi}^\# [\varphi] \geq m_0^2 \lambda_0 \int_X |\nabla (\xi) v|^2 d\mathbf{x} \geq C' m_0^2 \lambda_0 \xi^2 \int_X |v|^2 d\mathbf{x} \geq C'' \xi^2 \int_X |\varphi|^2 d\mathbf{x} \tag{A.21}
\]
where \( C'' = C' m_0^2 m_1^{-2} \lambda_0 \). By (A.18) and (A.21),
\[
\int_{\Omega^\#} |\varphi|^2 d\mathbf{x} \leq C_X (1 + (C'' \xi^2)^{-1}) P_{\xi}^\# [\varphi],
\]
which implies (A.14) with \( c = C_X^{-1} C'' (C'' \pi^2 + 1)^{-1} \).

The proof of Theorem 1.1 is complete. \( \Box \)
A2. On Proposition 2.1

A2.1. The conditions of Proposition 2.1 are rather excessive: actually, its statements remain valid for non-periodic operators. Here we present the complete formulation and give necessary references and additional explanations.

For non-periodic domains $\Omega \subset \mathbb{R}^2$ the definition (2.1) of the space $X(\Omega)$ is unsatisfactory. In order to define the appropriate function space, we split $\mathbb{R}^2$ into the union of unit squares $Q_n = (n_1, n_1 + 1) \times (n_2, n_2 + 1)$ where $n = (n_1, n_2) \in \mathbb{Z}^2$. We identify functions on $\Omega$ with their extensions by zero to the whole of $\mathbb{R}^2$. We say that $f \in X(\Omega)$ if and only if

$$\|f\|_{X(\Omega)} = \sum_{n \in \mathbb{Z}^2} \|f|_{\partial Q_n}\| < \infty, \quad \Omega \subset \mathbb{R}^2. \tag{A2.1}$$

It is easy to see that for the domains $\Omega \subset \mathbb{R}^2$ that meet the condition (\Omega) the norms given by (2.2) and by (A2.1) are equivalent.

Proposition A2.1. Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be an arbitrary domain. Let $P[u]$ be the quadratic form (1.14) for which the condition (P) is fulfilled, except for the periodicity assumptions, and let $V \in X(\Omega)$. Suppose also that for the operator $P_{\gamma}$ (1.19) is satisfied and $\gamma \neq 0$. Then the estimate (2.5) and the asymptotic formula (2.6) are valid for $P = P_{\gamma}$.

Suppose in addition that there exists a linear bounded extension operator $\Pi : H^1(\Omega) \to H^1(\mathbb{R}^d)$. Then the above result remains true for $P = P_{\gamma'}$.

Outline of proof. 1. Estimates. Introduce a family of Hilbert spaces $\mathfrak{H}(\gamma)$, $\gamma \neq 0$. Each $\mathfrak{H}(\gamma)$ is $\mathfrak{D} = \text{Dom}(P)$, endowed with the metric form $P[u] + \gamma^2 \|u\|_{L_2(\Omega)}^2$. If $V \in X(\Omega)$, then the quadratic form $\int_{\Omega} V|u|^2$ is bounded in $\mathfrak{H}(\gamma)$. Denote by $T(V; \gamma)$ the corresponding self-adjoint operator. Define also the family of operators acting in $L_2(\Omega)$,

$$G(V; \gamma) := W(P + \gamma^2 I)^{-\frac{1}{2}}, \quad W = |V|^\frac{1}{2}, \tag{A2.2}$$

(cf. (5.6)) and denote

$$T'(V; \gamma) = G(V; \gamma)^* \text{ sign } V G(V; \gamma). \tag{A2.3}$$

According to (3.12),

$$N_{\gamma}(P + \gamma^2 I - \alpha V) = n_+(\alpha^{-1}, T(V; \gamma)) = n_+(\alpha^{-1}, T'(V; \gamma)), \quad \gamma \neq 0. \tag{A2.4}$$

If $\Omega = \mathbb{R}^d$ and $|\gamma|$ is large enough, so that $\|\rho\|_{L_\infty(\Omega)} < \gamma^2$, then (2.5) follows directly from the Rozenblum–Cwikel–Lieb estimate ($d > 2$) and from [Sol 1] ($d = 2$). The passage to an arbitrary $\gamma \neq 0$ is straightforward, since the norms in $\mathfrak{H}(\gamma)$ for all $\gamma \neq 0$ are mutually equivalent. Now the estimate (2.5) for $\Omega \neq \mathbb{R}^d$ follows by the variational argument in the spirit of [BSol 2], Ch.4, see especially §2, 7.

2. Asymptotics. At first step it is convenient to deal with the operators $T(V; \gamma)$. Choose $\gamma_0 \neq 0$ in such a way that $\gamma_0^2 + p(x) \geq 1$ on $\Omega$. Then the proof of (2.6) for the operator $T(V; \gamma_0)$ is elementary: one splits $\Omega$ into the union of bounded sub-domains and uses the Dirichlet–Neumann bracketing, applying in
each sub-domain the Weyl-type asymptotic formula for bounded domains; see [BSol 2], Chapter 5 for the proof of this formula for the operators with $L_\infty$-coefficients.

Let now $\gamma \neq 0$ be arbitrary. Here it is more convenient to switch to the operators $T'(V; \gamma)$. We will show that

$$V \in X(\Omega) \implies T'(V; \gamma) - T'(V; \gamma_0) \in C^0_{d,\infty}, \quad (A2.5)$$

then by (3.7) the asymptotics (2.6) for $T'(V; \gamma)$ follows from the same asymptotics for $T'(V; \gamma_0)$. When proving (A2.5), we shall rely on the following Lemma whose proof is given in Subsection A2.2.

**Lemma A2.2.** Let $V \in X(\Omega)$, $W = |V|^\frac{\beta}{2}$, and $\gamma \neq 0$. Then $W(\mathcal{P} + \gamma^2 I)^{-1} \in C^0_{d,\infty}$.

To simplify our notations, denote $Z_0 = \mathcal{P} + \gamma_0^2 I$ and $Z = \mathcal{P} + \gamma^2 I$. From (A2.2) and (A2.3) we derive

$$T'_V(\gamma) - T'_V(\gamma_0)$$

$$= (Z - \frac{\gamma}{Z_0} - Z_0^{-\frac{\beta}{2}})W \text{sign} \ V \ G(V; \gamma) + G(V; \gamma_0)^* \text{sign} \ V \ W \ (Z - \frac{\gamma}{Z_0} - Z_0^{-\frac{\beta}{2}}) \quad (A2.6)$$

$$= f(Z_0)(WZ_0^{-1})^* \text{sign} \ V \ G(V; \gamma) + G(V; \gamma_0)^* \text{sign} \ V \ WZ_0^{-1}f(Z_0)$$

where $f(\lambda) = \lambda((\lambda + (\gamma^2 - \gamma_0^2))^{-\frac{\beta}{2}} - \lambda^{-\frac{\beta}{2}})$ is a bounded function. In the last sum in (A2.6) each term contains the factor $G(V; \cdot)$ lying in $C^0_{d,\infty}$ (by (2.5) and (A2.4)). The operators $WZ_0^{-1}$ and $(WZ_0^{-1})^*$ belong to $C^0_{d,\infty}$ by Lemma A2.2 and the rest factors are bounded operators. The implication (A2.5), and thus the asymptotics (2.6) follows from here by Horn’s inequality.

**A2.2. Proof of Lemma A2.2.** The estimate (2.5) shows that the linear mapping $V \mapsto T(V; \gamma)$ is continuous as an operator acting from $X(\Omega)$ to $C^0_{d,\infty}$. Applying Lemma 3.1, we find that it suffices to prove Lemma A2.2 for only such $V$ that $W \in C^0(\Omega)$, since these $V$ constitute a dense subset of $X(\Omega)$.

Thus, let $W \in C^0(\Omega)$. Any $W \in C^0(\Omega)$ is a multiplier in Dom$(Z^\frac{\beta}{2})$. Therefore, taking arbitrary $f, h \in L_2(\Omega)$ and denoting $u = Z^{-1}f$, $v = Z^{-1}h$ we get (all the scalar products are in $L_2(\Omega)$)

$$(WZ^{-1}f, h) - (Z^{-1}Wf, h) = (Wu, Zv) - (WZu, v)$$

$$= \int_\Omega (g \nabla (Wu), \nabla v) - (g \nabla u, \nabla (Wv)) dx.$$

After calculations and cancellations, this leads to

$$[W]Z^{-1} - Z^{-1}[W] = \sum_{j=1}^d (\partial_j Z^{-1})^* \alpha_j Z^{-1} - \sum_{j=1}^d Z^{-1}(\alpha_j \partial_j Z^{-1}); \quad (A2.7)$$

recall that $[W]$ stands for the operator of multiplication by a function $W$. The coefficients $\alpha_j$ are bounded and have a compact support in $\Omega$.

Let now $\chi$ be a compactly supported function from $L_\infty(\Omega)$. We have

$$([W]Z^{-1} - Z^{-1}[W])[\chi]$$

$$= \sum_{j=1}^d ((\partial_j Z^{-1})^* (\alpha_j Z^{-\frac{\beta}{2}})(Z^{-\frac{\beta}{2}}[\chi]) - Z^{-\frac{\beta}{2}}(Z^{-\frac{\beta}{2}} \alpha_j \partial_j Z^{-\frac{\beta}{2}})(Z^{-\frac{\beta}{2}}[\chi])).$$
Each term of the last sum contains two factors from $C_{d,\infty}$ (by the estimate (2.5)), other are bounded. This implies that the whole sum belongs to $C_{\frac{d}{2},\infty}$. Further,

$$[W]Z^{-1}[\chi] = [W]Z^{-\frac{1}{2}}Z^{-\frac{1}{2}}[\chi] \in C_{\frac{d}{2},\infty}.$$  

Finally, suppose that the function $\chi$ is chosen in such a way that $W \chi = W$. Then we obtain

$$Z^{-1}[W] = Z^{-1}[W][\chi] = [W]Z^{-1}[\chi] - ([W]Z^{-1} - Z^{-1}[W])[\chi] \in C_{\frac{d}{2},\infty} \subset C_{d,\infty}^{0}$$

which is equivalent to the statement of Lemma. □

For periodic operators somewhat different proof of this statement was given in [B 2].

A3. Proof of Lemmas 4.1 and 4.2

A3.1. Here we give the formulation of a particular case of Theorem 2.7 from [Sen]. It will be used in the proofs of both Lemmas 4.1 and 4.2.

Proposition A3.1. Let a function $\Lambda$ be slowly varying on $\mathbb{R}_+$, such that

$$\Lambda \text{ is bounded on each finite interval } (0, a). \quad (A3.1)$$

Suppose that

$$\int_0^1 \sigma^{-\delta} |f(\sigma)| d\sigma < \infty \quad (A3.2)$$

for a given measurable function $f$ and for some $\delta > 0$. Then

$$\int_0^1 f(\sigma) \Lambda(\alpha \sigma) d\tau \sim \Lambda(\alpha) \int_0^1 f(\sigma) d\sigma, \quad \alpha \to \infty. \quad (A3.3)$$

Note that the assumption (A3.1) is not restrictive for our purposes. Indeed, for any function $\Lambda \in \mathfrak{F}_0$ there exists a number $A_0 > 0$, such that $\Lambda$ is bounded on any finite segment $[A, B]$ with $A \geq A_0$; this immediately follows from [Sen], Theorem 1.2. Replacing $\Lambda(\alpha)$ by an arbitrary constant $c > 0$ for $\alpha < A_0$, we obtain another function, say $\Lambda'$, for which (A3.1) is satisfied.

Let now $q > 0$, $\Phi(\alpha) = \alpha^q \Lambda(\alpha)$ and $\Phi'(\alpha) = \alpha^q \Lambda'(\alpha)$. Evidently, $l\phi,\infty = l\phi,\infty'$ and $\Delta\phi(\eta) = \Delta\phi(\eta)$ for any number sequence $\eta$. The similar is true for the corresponding spaces of compact operators. It follows that in all our reasonings the original function $\Lambda$ can be replaced by $\Lambda'$. For this reason, we always assume below that (A3.1) is fulfilled.

A3.2. Proof of Lemma 4.1. The assertions 1° and 2° are just a reformulation of the classical boundedness and compactness tests (4.4) and (4.6). For $\Phi = \Phi(q)$, a proof of 3°, based on interpolation, is given in [BLapSol]. Below we give a non-interpolational argument which covers the general case.

The starting point is the estimate (cf. [BSol 2], inequality (4.38), or [BLapSol], inequality (2.32)):

$$n(\alpha^{-1}, \mathcal{T}_Q^+) \leq C \alpha \frac{1}{2} \sum_{k=0}^{\infty} \eta_k(Q)^{\frac{1}{2}}. \quad (A3.4)$$
Let us explain the nature of this estimate. Due to the Hardy inequality the estimate for \( n(\alpha^{-1}, T^+_Q) \) is equivalent to the similar estimate for the operator whose Rayleigh quotient is
\[
\frac{\int_{\mathbb{R}^+} Q(t)|w|^2 dt}{\int_{\mathbb{R}^+} (|w'|^2 + t^{-2}|w|^2) dt}, \quad w \in \mathcal{H}^1(\mathbb{R}^+), \ w \neq 0.
\]

To obtain (A3.4), one splits \( \mathbb{R}^+ \) into the sum of intervals \( (0, 1) \) and \((2^{k-1}, 2^k), k \in \mathbb{N}\) and applies the well-known eigenvalue estimate for finite intervals, cf. [BSol 2], Section 4.8. Each term in (A3.4) evaluates the quantity \( n(\alpha^{-1}, \cdot) \) for the Neumann problem on the corresponding interval. The latter quantity is zero if it is smaller than one. Therefore, the estimate (A3.4) can be replaced by
\[
n(\alpha^{-1}, T^+_Q) \leq C \alpha^{1/2} \sum_{k: \eta_k(\eta) > (C^2 \alpha)^{-1}} \frac{\eta_k(\eta)}{\alpha^{1/2}}.
\]

For any \( \varepsilon > 0 \) a number \( \alpha_\varepsilon > 0 \) can be found, such that
\[
n(\alpha^{-1}, \eta(Q)) \leq (\Delta_\Phi + \varepsilon)\Phi(\alpha), \quad \alpha > \alpha_\varepsilon; \quad \Delta_\Phi := \Delta_\Phi(\eta(Q)).
\]

Now we make use of the equality
\[
2 \sum_{k: \eta_k > \rho} \eta_k^{1/2} = \int_\rho^\infty \tau^{-1/2} n(\tau, \eta)d\tau + 2\rho^{1/2} n(\rho, \eta) = \int_0^{\rho^{-1}} \sigma^{-1/2} n(\sigma^{-1}, \eta)d\sigma + 2\rho^{1/2} n(\rho, \eta).
\]

Using the standard representation \( \Phi(\sigma) = \sigma^\gamma \Lambda(\sigma) \), we find for sufficiently large \( \alpha \):
\[
2(C \alpha^{1/2})^{-1} n(\alpha^{-1}, T^+_Q) \leq \int_0^{C^2 \alpha} \sigma^{-\frac{\gamma}{2}} n(\sigma^{-1}, \eta(Q))d\sigma + 2(C \alpha^{1/2})^{-1} n((C^2 \alpha)^{-1}, \eta(Q))
\leq \int_0^{C^2 \alpha} \left( (\sigma^{-\frac{\gamma}{2}} n(\sigma^{-1}, \eta(Q)) - (\Delta_\Phi + \varepsilon)\sigma^{q-\frac{\gamma}{2}} \Lambda(\sigma))d\sigma
\right.
\leq \int_0^{C^2 \alpha} \left( (\sigma^{q-\frac{\gamma}{2}} \Lambda(\sigma) + 2(\Delta_\Phi + \varepsilon)(C^2 \alpha)^{q-\frac{\gamma}{2}} \Lambda(C^2 \alpha) + O(1) + (C^2 \alpha)^{q-\frac{\gamma}{2}} (\Delta_\Phi + \varepsilon) \left( \int_0^{C^2 \alpha} \sigma^{q-\frac{\gamma}{2}} \Lambda(C^2 \alpha)d\sigma + 2\Lambda(C^2 \alpha) \right) \right)
\]

For the function \( f(\sigma) = \sigma^{-\frac{\gamma}{2}} \) the condition (A3.2) is satisfied with any \( \delta \in (0, q - \frac{1}{2}) \), so Proposition A3.1 applies. By (A3.3),
\[
\int_0^{C^2 \alpha} \sigma^{q-\frac{\gamma}{2}} \Lambda(h\sigma)d\sigma \sim (q - \frac{1}{2})^{-1} \Lambda(h), \quad h \to \infty. \quad (A3.5)
\]

Hence,
\[
\limsup_{\alpha \to \infty} \frac{n(\alpha^{-1}, T^+_Q)}{\Phi(\alpha)} \leq (\Delta_\Phi + \varepsilon) \frac{q}{2q - 1} C^{2q}.
\]

Since \( \varepsilon > 0 \) is arbitrary, this implies (4.9).
In order to obtain for $\Delta_\phi(T_Q^+)$ the lower estimate (provided $Q \geq 0$), we make use of the inequality (2.34) in [BLapSol] which leads to

$$n(\mu \alpha^{-1}, \eta) \leq 2n(\alpha^{-1}, T_Q^+) + 1.$$ 

Here $\mu$ is some positive number; the term 1 on the right-hand side appears because the size of $\eta_2(Q)$ is not controlled by $n(\alpha^{-1}, T_Q^+)$. Dividing both sides by $\Phi(\alpha)$ and passing to the upper limits, we find

$$\Delta_\phi(\eta(Q)) \leq 2 \mu^{-\eta} \Delta_\phi(T_Q^+).$$

The implication converse to (4.10) (for $Q \geq 0$) also follows from here. □.

**A3.3. Proof of Lemma 4.2.** The Rayleigh quotient of the operator $T_Q^+$ for $Q$ given by (4.12) is

$$\int_1^\infty (t^2 \Psi(\log t))^{-1} |w|^2 dt - \int_0^\infty |w'|^2 dt, \quad w \in H^1(\mathbb{R}^+).$$

(A3.6)

Substituting $w(t) = \sqrt{t} v(t)$ and $t = e^s$, we reduce (A3.6) to

$$\frac{\int_{E_+} (\Psi(s))^{-1} |v|^2 ds}{\int_E (|v|^2 + 2^{-1}(|v|^2)' + 4^{-1} |v|^2)ds}, \quad v \in H^1(\mathbb{R}).$$

We have $\int (|v|^2)' ds = 0$; imposing the additional condition $v(0) = 0$ does not affect the eigenvalue asymptotics. So, we come to the Rayleigh quotient which corresponds to the eigenvalue problem

$$\lambda(-v'' + 4^{-1} v) = (\Psi(s))^{-1} v, \quad v \in H^{1,0}(\mathbb{R}^+).$$

The semi-classical asymptotic formula applies and gives

$$n(\alpha^{-1}, T_Q^+) \sim \pi^{-1} I_\Psi(\alpha) := \pi^{-1} \int_{E_+} (\alpha (\Psi(s))^{-1} - 4^{-1})^{1/2} ds, \quad \alpha \to \infty. \quad (A3.7)$$

Recall that $\Psi$ is the function inverse to $\Phi(\alpha) = \alpha^q \Lambda(\alpha)$. Changing the variables $\Psi(s) = \sigma$ and integrating by parts, we obtain

$$I_\Psi(\alpha) = \alpha \int_0^{\alpha^q} (\alpha \sigma^{-1} - 4^{-1})^{1/2} \Phi'(\sigma) d\sigma$$

$$= \frac{\alpha}{2} \int_0^{\alpha^{q-1/2}} (\alpha \sigma^{-1} - 4^{-1})^{-1/2} \sigma^{q-2} \Lambda(\sigma) d\sigma = 4^{q-1} \alpha^{q-\frac{1}{2}} \int_0^{(1 - \tau)^{1/2}} \tau^{q-\frac{q}{2}} \Lambda(4\alpha \tau) d\tau.$$

For the function $f(\tau) = (1 - \tau)^{-\frac{1}{2}} \tau^{\frac{q}{2} - \frac{q}{2}}$, the condition (A3.1) is satisfied with any $\delta \in (0, q - \frac{1}{2})$. By (A3.2),

$$I_\Psi(\alpha) \sim 4^{q-1} \alpha^{q-\frac{1}{2}} \Lambda(4\alpha) \int_0^1 (1 - \tau)\tau^{\frac{q}{2} - \frac{q}{2}} d\tau, \quad \alpha \to \infty.$$

Since $\Lambda(4\alpha) \sim \Lambda(\alpha)$, the last relation is equivalent to (4.11), with the constant $C = C(q)$ given by (4.13). □
References


\textbf{St.-Petersburg University, Dept. of Physics,} \\
\textbf{Ulyanov str. 1, 198904 St.-Petersburg, Russia}

\textbf{Department of Theoretical Mathematics,} \\
\textbf{The Weizmann Institute of Science,} \\
\textbf{Rehovot 76100, Israel}

\textit{E-mail address:} sclm@wisdom.weizmann.ac.il