Einstein–Weyl Geometry, the dKP Equation and Twistor Theory

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Einstein–Weyl geometry, the dKP equation and twistor theory

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Abstract

It is shown that Einstein–Weyl (EW) equations in 2+1 dimensions contain the dispersionless Kadomtsev–Petviashvili (dKP) equation as a special case: If an EW structure admits a constant weighted vector then it is locally given by \( h = dy^2 - 4dzdt - 4udt^2, \) \( V = -4u_x dt, \) where \( u = u(x, y, t) \) satisfies the dKP equation \( (u_t - u_{xx})_x = u_y. \)

Linearised solutions to the dKP equation are shown to give rise to four-dimensional anti-self-dual conformal structures with symmetries. All four-dimensional hyper-Kähler metrics in signature (\(+ + - -\)) for which the self-dual part of the derivative of a Killing vector is null arise by this construction.

Two new classes of examples of EW metrics which depend on one arbitrary function of one variable are given, and characterised.

A Lax representation of the EW condition is found and used to show that all EW spaces arise as symmetry reductions of hyper-Hermitian metrics in four dimensions.

The EW equations are reformulated in terms of a simple and closed two-form on the \( \mathbb{CP}^1 \)-bundle over a Weyl space.

It is proved that complex solutions to the dKP equations, modulo a certain coordinate freedom, are in a one-to-one correspondence with minkwistor spaces (two-dimensional complex manifolds \( Z \) containing a rational curve with normal bundle \( O(2) \)) that admit a section of \( \kappa^{-1/4} \), where \( \kappa \) is the canonical bundle of \( Z \). Real solutions are obtained if the minkwistor space also admits an anti-holomorphic involution with fixed points together with a rational curve and section of \( \kappa^{-1/4} \) that are invariant under the involution.

1 Three-dimensional Einstein–Weyl spaces

The aim of this paper is to study the Einstein–Weyl (EW) equations in relation to integrable systems, and in particular the dispersionless Kadomtsev–Petviashvili equation.

We begin by collecting various definitions and formulae concerning three-dimensional Einstein–Weyl spaces (see [25] for a fuller account). In section 2 we construct and characterise a class of new EW structures in 2+1 dimensions out of solutions to the dKP equation. We then show that the dKP solutions give rise to hyper-Kähler metrics in four dimensions. We abuse terminology and call hyper-Kähler (hyper-complex, hyper-Hermitian) metrics which in signature \((+ + - -)\) should be referred to as pseudo-hyper-Kähler (pseudo-hyper-complex, pseudo-hyper-Hermitian).

A null vector field (with conformal weight) will play a central role in our discussion so most of our constructions only make sense for Einstein-Weyl spaces with Lorentzian signature, or complex holomorphic EW spaces (i.e. the complexification of real analytic EW spaces) and for the most part we work with the latter and restrict to a real slice when reality conditions play a role.

In section 3 we construct some new examples of EW structures. We obtain all solutions of the dKP equation with the property that the associated EW space admits a family of divergence-free, shear-free geodesic congruences. These solutions give rise to new EW metrics depending on one arbitrary function of one variable.

In section 4 a Lax representation of the general EW equations is given, together with a reformulation of the EW equations in terms of a closed and simple two-form on the bundle of spinors. A full twistor characterisation of dKP Einstein–Weyl structures and the corresponding hyper-Kähler metrics will be given in section 5. In section 6 we summarise our present knowledge of conformal reductions of four-dimensional hyper-Kähler metrics in split signature. In the Appendix we show

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how to obtain the dKP equation as a reduction of Plebański’s second heavenly equation [26]. Parts of this work appeared in the DPhil thesis of one of the authors (MD) [5].

Let \( W \) be a 3-dimensional complex manifold (one can also define Weyl spaces in arbitrary dimension) with a torsion-free connection \( D \) and a conformal metric \([h]\). We shall call \( W \) a Weyl space if the null geodesics of \([h]\) are also geodesics for \( D \). This condition is equivalent to

\[
D_i h_{jk} = \nu_i h_{jk}
\]

for some one form \( \nu \). Here \( h_{jk} \) is a representative metric in the conformal class. The indices \( i, j, k,... \) go from 1 to 3. If we change this representative by \( h \rightarrow \phi^2 h \), then \( \nu \rightarrow \nu + 2d\ln \phi \). The one-form \( \nu \) ‘measures’ the difference between \( D \) and the Levi-Civita connection \( \nabla \) of \( h \):

\[
D_i V^j = \nabla_i V^j - \frac{1}{2} \left( \delta^j_i \nu_k + \delta^i_j \nu_k - h_{ik} \nu^j \right) V^k.
\]

The Ricci tensor \( W_{ij} \) and scalar \( W \) of \( D \) are related to the Ricci tensor \( R_{ij} \) and scalar \( R \) of \( \nabla \) by

\[
W_{ij} = R_{ij} + \nabla_i \nu_j - \frac{1}{2} \nabla_j \nu_i + \frac{1}{4} \nu_i \nu_j + h_{ij} \left( -\frac{1}{4} \nu_k \nu^k + \frac{1}{2} \nabla_k \nu^k \right),
\]

\[
W = h^{ij} W_{ij} = R + 2 \nabla^k \nu_k - \frac{1}{2} \nu^k \nu_k.
\]

A tensor object \( T \) which transforms as \( T \rightarrow \phi^m T \) when \( h_{ij} \rightarrow \phi^2 h_{ij} \) is said to be conformally invariant of weight \( m \). The Ricci scalar \( W \), and the Ricci tensor \( W_{ij} \) have weights \(-2\) and \(0\) respectively.

Let \( \beta \) be a \( p \)-form of weight \( m \). The covariant exterior derivative

\[
\overline{D} \beta := d \beta - \frac{m}{2} \nu \wedge \beta
\]

is a well-defined \( p+1 \)-form of weight \( m \). The formula for a covariant weighted derivative of a vector of weight \( m \) is

\[
\overline{D}_i V^j = \nabla_i V^j - \frac{1}{2} \delta^j_i \nu_k V^k - \frac{m+1}{2} \nu_i V^j + \frac{1}{2} \nu^j V_i.
\]

We say that a vector \( K \) is a symmetry of a Weyl structure if it preserves the conformal structure \([h]\), the Weyl connection, and the compatibility (1.1) between those two. These conditions imply

\[
\mathcal{L}_K h = \psi h, \quad \mathcal{L}_K \nu = d\psi,
\]

where \((h, \nu)\) is a Weyl structure, and \( \mathcal{L}_K \) is the Lie derivative along \( K \).

The conformally invariant Einstein–Weyl (EW) condition on \((W, h, \nu)\) is

\[
W_{(ij)} = \frac{1}{3} W h_{ij}.
\]

If the above equation is satisfied and \( \nu \) is a gradient, then \( h \) is conformal to a metric with constant curvature.

In terms of the Riemannian data the Einstein–Weyl equations are

\[
\chi_{ij} := R_{ij} + \nabla_i \nu_j - \frac{1}{2} \nabla_j \nu_i + \frac{1}{4} \nu_i \nu_j - \frac{1}{3} \left( R + \frac{1}{2} \nabla^k \nu_k + \frac{1}{4} \nu^k \nu_k \right) h_{ij} = 0.
\]

Here \( \chi_{ij} \) is a conformally invariant tensor (the trace-free part of the Ricci tensor of the Weyl connection). Weyl spaces which satisfy (1.5) will be called Einstein–Weyl (or EW) spaces.

In three dimensions the general solution of (1.1)–(1.5) depends on four arbitrary functions of two variables [4]. The equations of the Weyl geodesics are

\[
\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial \mathcal{L}}{\partial x^i} = F_i(x^j, \dot{x}^j)
\]

where \( \mathcal{L} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j \) and \( F_i = \dot{x}_i (\dot{x}^j \nu_j) - (1/2) \nu_i (\dot{x}^j \dot{x}_j) \). Here \( \dot{\cdot} = d/ds \) stands for the derivative with respect to a parameter \( s \). It is evident that for null \( \dot{x}^i \) the geodesics coincide with the null geodesics for \([h]\).
2 Einstein–Weyl structures from the dKP equation

In this section we shall construct Einstein–Weyl structures out of solutions to the dKP equation. In subsection 2.1 we shall find a class of hyper-Kähler metrics in four dimensions which reduce to dKP EW metrics.

The full Kadomtsev–Petviashvili equation for \( U := U(X^i), X^i = (X, Y, T) \)

\[
(U_T - U U_X - (1/12) U_{XXX}) X = U_Y
\]  

(2.6)

arises as a compatibility condition for the linear system \( L_0 \Psi = L_1 \Psi = 0 \), where \( \Psi = \Psi(X, Y, T) \) and

\[
L_0 = \partial_Y - (1/2) \partial_X^2 - U, \quad L_1 = \partial_T - (1/3) \partial_X^3 - U \partial_X - W,
\]

for some \( W = W(X, Y, T) \). To take a dispersionless limit of (2.6) [11] introduce the slow coordinates \( x^i := \epsilon X^i \) (note that our notation for ‘slow’ and ‘fast’ coordinates is different from the usual one), and define \( u(x^i) := U(X^i), \tilde{u}(x^i) := W(X^i) \). The linear system is replaced by

\[
S_y = (1/2) S_x^2 + u, \quad S_t = (1/3) S_x^3 + u S_x + w.
\]  

(2.7)

Here \( S := S(x^i) \) is the action defined by \( \Psi(X^i) = \exp\{\epsilon^{-1} S(x^i)\} \), and higher order terms in \( \epsilon \) have been neglected. Formulae (2.7) can be treated as a pair of Hamilton-Jacobi equations \( S_{\lambda_x} + H_A(S_x, x, t_A) = 0 \), with \( t_A = (y, t) \) and \( H_A = (H_2, H_3) \) where

\[
H_2 := \frac{\lambda^2}{2} + u, \quad H_3 := \frac{\lambda^3}{3} + \tilde{\lambda} u + w
\]

for \( u = u(x, y, t) \) and \( w = u(x, y, t) \).

Now \( x^i \) and \( \partial S/\partial x^i \) form a set of canonically conjugate variables on an ‘extended phase-space’, with the symplectic form

\[
\Pi = dx^i \wedge d \frac{\partial S}{\partial x^i} = dx \wedge d\lambda + dy \wedge dH_2 + dt \wedge dH_3.
\]  

(2.8)

This two-form is closed by definition. It is also simple iff \( u \) and \( w \) satisfy

\[
w_x = u_y, \quad u_t - uu_x = w_y.
\]

Eliminating \( w \) yields the dKP equation

\[
(u_t - uu_x)_x = u_{yy}.
\]  

(2.9)

The simplicity of \( \Pi \) implies \([\partial_y + X H_z, \partial_x + X H_x] = 0 \) where \( X_H := H_x \partial_\lambda - H_\lambda \partial_x \) denotes the Hamiltonian vector field with respect to \( d\lambda \wedge dx \), holding \( t \) and \( y \) constant. This gives a Lax pair for the dKP equation in terms of Hamiltonian vector fields. To obtain a Lax pair which is linear in the spectral parameter put

\[
L_0 := \partial_t + X H_z - \tilde{\lambda} (\partial_y + X H_y) = \partial_t - u \partial_x - \tilde{\lambda} \partial_y + u_y \partial_\lambda, \quad L_1 := \partial_y + X H_z = \partial_y - \tilde{\lambda} \partial_x + u_x \partial_\lambda.
\]  

(10.1)

The dKP equation is equivalent to

\[
[L_0^\dagger, L_1] = -u_x L_1^\dagger.
\]

Define a triad of vectors

\[
\nabla_{11} := \partial_x, \quad \nabla_{y1} := \partial_y, \quad \nabla_{01} := \partial_t - u \partial_x
\]

so \( L_A^\dagger = \pi^A \nabla A^B + f_A^B \partial_\lambda, \) where \( \pi^A^\dagger = (1, -\tilde{\lambda}) \) and \( f_A^B = (u_y, u_x) \).

The next proposition shows that we can find a one form \( \nu \) such that \( \nabla A^B \) is a null triad for an EW metric:

**Proposition 2.1** Let \( u := u(x, y, t) \) be a solution of the dKP equation (2.9). Then the metric and the one-form

\[
h = dy^2 - 4dx dt - 4u dt^2, \quad \nu = -4u_x dt
\]  

(2.11)

give an EW structure.
\textbf{Proof.} Let \( x^1 := t, \ x^2 := y, \ x^3 := x \). Five (out of six) EW equations \( \chi_{ij} = 0 \) are satisfied identically by ansatz (2.11). The equation \( \chi_{11} = 0 \) is equivalent to (2.9). We also find \( W = -3u_{xx} \).

\textbf{Example:} Solutions which yield EW structures conformal to Einstein metrics (i.e. those for which \( \nu \) is exact) are of the form

\[
u(x, y, t) = x f_1(t) + \frac{1}{2} \left( \frac{df_1(t)}{dt} - f_1(t)^2 \right) y^2 + f_2(t) y + f_3(t), \tag{2.12}\]

where \( f_1(t), f_2(t), f_3(t) \) are arbitrary functions of one variable.

One can verify that the vector \( \partial_x \) in the EW space (2.11) is a covariantly constant null vector in the Weyl connection with weight \(-1/2\). Now we shall prove the converse, and show that solutions (2.11) are characterised by the existence of a constant weighted vector.

\textbf{Proposition 2.2} If a three dimensional EW space has a constant weighted vector field \( l \) then coordinates can be chosen to put the EW metric and 1-form in the form (2.11).

We shall need following lemma:

\textbf{Lemma 2.3} Let \( l \) be a constant weighted vector on a three-dimensional EW space. Then either the EW space is flat or \( l \) is null (so on a real slice the signature is \((+ - -)\) and has weight \(-1/2\).

\textbf{Proof.} Assume that \((h, \nu)\) is a complex EW structure (we shall specify the reality conditions later in the proof). Commuting the Weyl derivatives yields

\[
[D_i, D_j] l^k = \frac{m}{2} (D_i \nu_j - D_j \nu_i) l^k = W^k_{\ \ mi} l^m,
\]

where \( W^k_{\ \ mi} \) is the curvature of the Weyl connection, and \( m \) is the weight of \( l^k \). It can be decomposed as

\[
W^k_{\ \ mi} = -\varepsilon_i \varepsilon_j \varepsilon_m \varepsilon_n S_{pq} - \delta_m^{k_i} F_{ij}, \tag{2.13}\]

where \( F_{ij} = \nabla_k \nu_{pj} \) and \( S_{ij} \) is a conformally invariant tensor of weight 0. If the EW equations are satisfied \( S_{ij} \) is given by

\[
S_{ij} = \frac{1}{2} F_{ij} + \frac{W}{6} h_{ij}. \tag{2.14}\]

Equations (2.13) and (2.14) imply

\[
(m + 1) F_{ij} l^k = -\frac{1}{2} \varepsilon_i \varepsilon_j \varepsilon_m \varepsilon_n F_{pq} l^m l^q + \frac{W}{6} (\delta_i^{k_j} l^j - \delta_j^{k_i} l^i).
\]

In three dimensions any non-zero two-form \( F_{ij} \) has a non-trivial kernel, i.e. there exists a non-zero vector \( L^j \) with \( F_{ij} L^j = 0 \), which implies

\[
F_{ij} = F \varepsilon_{ijk} L^k \tag{2.16}\]

for some non-zero \( F \). We have to consider three cases:

- **Suppose first that** \( L^k \) is a null vector and contract (2.15) with \( L^j \) to find

\[
0 = -\frac{1}{2} \varepsilon_i \varepsilon_j \varepsilon_m \varepsilon_n F_{pq} L^m L^q + \frac{W}{6} (\delta_i^{k_j} L^j - \delta_j^{k_i} L^i). \tag{2.17}\]

Contracting this with \( L_k \) yields \( W L_1 L^j = 0 \). If \( W = 0 \) then (2.17) implies that \( L^i \) and \( L^j \) are proportional, so \( L^i \) is null. If \( W \neq 0 \), so that \( L_j L^j = 0 \) then (2.17) reduces to

\[
0 = \frac{1}{2} F L^m L_i \varepsilon^k_{mq} - \frac{W}{6} L^k L^k
\]

from which again \( L^i \) is null. Therefore \( L^i \) and \( L^j \) are both null and orthogonal and so (as we work in three dimensions) they have to be proportional. Now (2.17) forces \( W = 0 \). Equation (2.15) is now satisfied only if \( m = -1/2 \).

- **If** \( L^i \) is not null, we can choose an orthogonal frame with \( F_{23} = F 
eq 0 \), and \( F_{12} = F_{13} = 0 \), and use (2.15) to examine components of \( F_{ij} l^k \) in this frame. This yields

\[
W L_1 = 0, \quad F L^1 = 0, \quad \frac{1}{2} F L^2 + \frac{1}{6} W L_2 = 0, \quad \frac{1}{2} F V^2 - \frac{1}{6} W L_3 = 0, \quad (m + 1) F L^1 = 0, \quad (m + 1) F L^2 = \frac{1}{6} W L_3 = \frac{1}{2} F L^3, \quad (m + 1) F L^3 = -\frac{1}{6} W L_2 = \frac{1}{2} F L^3. \tag{2.18}\]

Therefore \( L^i = 0 \), and (2.18) imply \( (m + 1/2) F L^2 = 0, (m + 1/2) F L^3 = 0 \). But \( L^i \neq 0 \), so \( m = -1/2 \). Equations (2.18) also imply that \( L^i \) is null.
• If $F = 0 = d\nu = 0$ (Einstein case) choose a conformal gauge in which $\nu = 0$. Now $D_i l^j = \nabla_i l^j = 0$ implies $R = 0$. Therefore the metric $h$ is flat and $l^i$ is a constant vector.

\[\square\]

**Proof of Proposition (2.2).** Lemma 2.3 and the formula (1.3) with $m = -1/2$ imply

\[\hat{D}_i l^j = D_i l^j + \frac{1}{4} \nu l^j = 0.\]  \hspace{1cm} (2.19)

Therefore $D_i l^j = (3/4) \nu l^j$, so $dl = (3/4) \nu \Lambda l$ (here $\Lambda$ is the one form dual to $l$).

This implies that we can rescale the metric and hence $l$ so that $l = -2 dt$ for some function $t$. We must then have $\nu = b dt$ for some function $b$. Choose coordinates $x$ and $y$ so that $l(x) = 1$ and $(x, y, t)$ is a coordinate system. At this point we have

\[h = F dy^2 + G d\nu dt - 4 dx dt - 4 u dt^2, \quad \nu = b dt,
\]

where $F, G, b$ and $u$ are functions of $x, y, t$. The formulae (1.2) and (2.19) imply $\nabla_i l^j = (1/4) \nu l^j - (1/2) l^j$. Symmetrising this expression yields $\nabla_i l^j = - (1/4) \nu l^j$, which implies that $F_x = G_x = 0$, and $4 u_x = - b$. We are still free to change $x \rightarrow x + P(y, t)$, which gives

\[h = F dy^2 + G d\nu dt - 4 dx + P_y dy + P_t dt | dt = 4 u dt^2 \quad \nu = - 4 u_x dt.
\]

We can find $K$ such that $d_j := \sqrt{F} dy + K dt$ is exact, and eliminate the $d_j dt$ term in the metric by choosing $4 P = - 2 K + G / \sqrt{F}$. This (after redefining $u$ by adding to it a function of $(y, t)$ so that $\nu$ remains unchanged) yields the EW structure (2.11).

\[\square\]

**Remark:** The above coordinate conditions fix the coordinates and $u$ only up to the freedom $(x, y, t) \rightarrow (\tilde{x}, \tilde{y}, \tilde{t}), u(x, y, t) \rightarrow \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ where

\[(x, y, t) = (\tilde{x} - f', \tilde{y} - g', \tilde{t} - f, \tilde{t}'), \quad \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(\tilde{x} - f', \tilde{y} - g', \tilde{t} - f, \tilde{t}').
\]

(2.20)

where $f$ and $g$ are arbitrary functions of $t$ and $\tilde{t}$ denotes the derivative with respect to $t$.

Furthermore the conformal scale is only fixed up to arbitrary functions of $t$, $h \rightarrow \tilde{h} = \Omega^2 h$. Such a rescaling leads to a redefinition of $t$, $t \rightarrow \tilde{t}$ given by $t \rightarrow \Omega(t)$ where $\Omega = \frac{1}{2} \Omega(t)$ now and in the following $\tilde{t}$ denotes the derivative wrt $\tilde{t}$. This leads to the redefinitions $(x, y, t) \rightarrow (\tilde{x}, \tilde{y}, \tilde{t}), u(x, y, t) \rightarrow \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ given by

\[(x, y, t) = (c^{1/3} \tilde{x} + \frac{c''}{6c^2} \tilde{y}^2, c^{1/3} \tilde{y} + c^{1/3} \tilde{t}, c^{1/3} \tilde{t}), \quad \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = \tilde{u}(\tilde{x} - f', \tilde{y} - g', \tilde{t} - f, \tilde{t}').
\]

(2.21)

From the point of view of the Einstein-Weyl spaces, the transformations above are equivalents, however from the point of view of the dKP equations, they map one solution of the dKP equations to another allowing one to deduce solutions depending on 3 functions of one variable from a given solution.

**Corollary 2.4** Let $u(x, y, t)$ be a solution to the dKP equation, then $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ is another solution where $\tilde{u}$ is given in terms of either of the formulae (2.21) or (2.20).

### 2.1 Hyper-Kähler structures from the dKP equation

In this subsection we shall show that EW structures given by (2.11) give rise to four-dimensional hyper-Kähler structures with symmetry. We shall start by summarising some results about anti-self-dual (ASD) four manifolds with Killing vectors, and the Lax representation of hyper-Hermitian four manifolds.

All three-dimensional EW spaces can be obtained as spaces of trajectories of conformal Killing vectors in four-dimensional manifolds with ASD conformal curvature.

**Proposition 2.5 ([16])** Let $(\mathcal{M}, \hat{g})$ be an ASD four-manifold with a conformal Killing vector $\hat{K}$. The EW structure on the space $\mathcal{W}$ of trajectories of $\hat{K}$ (which is assumed to be non-pathological) is defined by

\[h := |K|^{-2} \hat{g} - |K|^{-4} \hat{K} \circ K, \quad \nu := s^* \left(2 |K|^{-2} * \hat{g} (K \wedge dK)\right),\]  \hspace{1cm} (2.22)

\[\square\]
where \(|K|^2 := g_{ab}K^a K^b\), \(K\) is the one form dual to \(K\) and \(s_g\) is taken with respect to \(g\) and \(s : \mathcal{W} \to \mathcal{M}\) is an arbitrary section of the fibration \(\mathcal{M} \to \mathcal{W}\). All EW structures arise in this way.

Conversely, let \((h, \nu)\) be a three-dimensional EW structure on \(\mathcal{W}\), and let \((V, \alpha)\) be a pair consisting of a function of weight \(-1\) and a one-form on \(\mathcal{W}\) which satisfy the generalised monopole equation
\[
s_h dV + (1/2) \nu dV = da,
\]
where \(s_h\) is taken with respect to \(h\). Then
\[
g = Vh \pm V^{-1} (dz + \alpha)^2
\]
is an ASD metric with an isometry \(K = \partial_z\). The minus sign in \((2.24)\) is chosen if \(h\) has signature \((+ - - -)\).

In what follows we shall consider ASD structures which are also (complexified) hyper-Hermitian.

A smooth manifold \(\mathcal{M}\) equipped with three almost complex structures \((I, J, K)\) satisfying the algebra of quaternions is called hyper-complex iff the almost complex structure \(J_\lambda = aI + bJ + cK\) is integrable for any \((a, b, c) \in S^2\). We use \(\lambda = (a + ib)/(c - 1)\), a stereographic coordinate on \(S^2\) which we view as a complex projective line \(\mathbb{CP}^1\). Let \(g\) be a Riemannian metric on \(\mathcal{M}\). If \((\mathcal{M}, J_\lambda)\) is hyper-complex and \(g(J_\lambda X, J_\lambda Y) = g(X, Y)\) for all vectors \(X, Y\) on \(\mathcal{M}\) then the triple \((\mathcal{M}, J_\lambda, g)\) is called a hyper-Kähler.

We will in practice be interested in complexified or indefinite hyper-Hermitian metrics with signature \((+ --)\) for which the tensors \((I, J, K)\) must necessarily be complex. In signature \((+ + - -)\) we can arrange that one be real and the other two be pure imaginary, in the latter case they determine a pair of transverse null foliations.

We shall restrict ourselves to oriented four manifolds. In four dimensions a hyper-complex structure defines a conformal structure, which in explicit terms is represented by a conformal orthonormal frame of vector fields \((X, IX, JX, KX)\), for any \(X \in T\mathcal{M}\). It is well known \([1]\) that this conformal structure is ASD with the orientation determined by the complex structures.

If there exists a choice of a conformal factor such that a two form \(\Sigma_\lambda\) defined by \(\Sigma_\lambda( X, Y) := g(X, J_\lambda Y)\) is closed (with fixed \(\lambda\)) for all \(\lambda \in \mathbb{CP}^1\) and all vectors \((X, Y)\) then \((\mathcal{M}, J_\lambda, g)\) is called hyper-Kähler.

We shall use the following characterisation of the hyper-Hermiticity condition:

**Proposition 2.6** \(([21, 6])\) Let \(\nabla_{AA'}\) be four independent real vector fields on a four-dimensional real manifold \(\mathcal{M}\), and let
\[
L_0 = \nabla_{00'} - \lambda \nabla_{01'} , \quad L_1 = \nabla_{10'} - \lambda \nabla_{11'}, \quad \text{where } \lambda \in \mathbb{CP}^1 .
\]
If
\[
[L_0 , L_1] = 0
\]
for every \(\lambda\), then \(\nabla_{AA'}\) is a null tetrad for a \((+ --)\) hyper-Hermitian metric on \(\mathcal{M}\). Every \((+ --)\) hyper-Hermitian metric arises in this way. Moreover, if the vectors \(\nabla_{AA'}\) preserve a volume form \(\text{vol}_g\) on \(\mathcal{M}\), then \(f^{-1}\nabla_{AA'}\) is a null tetrad for a \((+ --)\) hyper-Kähler metric on \(\mathcal{M}\). Here \(f^2 = \text{vol}_g(\nabla_{00'}, \nabla_{10'}, \nabla_{01'}, \nabla_{11'})\).

Now we shall use \((2.11)\) and Proposition 2.5 to construct ASD metrics out of solutions to the dKP equation, and Proposition 2.6 to show that they are hyper-Kähler.

Assume that \(h = \nu\) as in \((2.11)\). Taking the exterior derivative of the generalised monopole equation \((2.23)\) yields
\[
0 = \nabla_i \nabla^i V + (1/2) (\nabla^i \nu_i) V + \nu^i \nabla_i V = V_{yy} - V_{xx} + u V_{xx} + 2 u_x V_x + u_{xx} V\]
\[
(2.26)
\]
which is just a linearisation of the dKP equation \((2.9)\) (note that for \(u = 0\) \((2.26)\) is just the wave equation relative to the flat metric \(dy^2 - 4dz dt\)). One solution is \(V = u_x/2\). One could find a corresponding \(\alpha\) and write down a metric using formula \((2.24)\) (see the remarks after Proposition 2.7), but we shall present a different method based on the Lax operators.

Take the Lax operators \((2.10)\) and introduce a new spectral parameter \(\lambda := \tilde{\lambda} - z\) for some \(z\). The function \(u(x, y, t)\) does not depend on \(z\) so we can replace \(\partial_\lambda\) by \(\partial_z\). This yields (with dropped primes and added tildes)
\[
\tilde{L}_0 = \partial_x + u \partial_x + z \partial_z + u_y \partial_y - \lambda \partial_y,
\]
\[
\tilde{L}_1 = \partial_y - z \partial_z + u_x \partial_x - \lambda \partial_x.
\]
To obtain a pair of exactly commuting operators take
\[
L_1 := \tilde{L}_1 = \partial_y - z \partial_x + u_x \partial_z - \lambda \partial_x, \\
L_0 := \tilde{L}_0 + z \tilde{L}_1 = \partial_z - (u + z^2) \partial_x + (u_y + u_x z) \partial_z - \lambda (\partial_y + z \partial_x).
\]
If \(u(x, y, t)\) is a solution to (2.9) then these operators satisfy \([L_0, L_1] = 0\) and so, by Proposition 2.6, the vectors
\[
\nabla_{10}^1 = \partial_y - z \partial_x + u_x \partial_z, \quad \nabla_{11}^1 = \partial_x, \quad \nabla_{00}^1 = \partial_z - (u + z^2) \partial_x + (u_y + u_x z) \partial_z, \quad \nabla_{01}^1 = (\partial_y + z \partial_x),
\]
form a hyper-Hermitian frame. The vectors \(\nabla_{AA'}\) preserve the volume form \(\omega_g = dt \wedge dy \wedge dx \wedge dz\), and \(f^2 = u_x/2\). Therefore we have the following

**Proposition 2.7** Let \(u = u(x, y, t)\). The metric
\[
g = \frac{u_x}{2}(dy^2 - 4dzdt - 4udt^2) - \frac{u_x}{2}(dz - \frac{dx}{2} - u_y dt)^2
\]
(2.27)
is hyper-Kähler.

**Remarks:**
- The above metric has a Killing vector \(\partial_z\) with the dual
\[
K = - \frac{2}{u_x}(dz - \frac{dx}{2} - u_y dt),
\]
and the formulae (2.22) gives rise to the Einstein–Weyl structure (2.11). The self-dual part of \(dK\) is a simple two-form. In section 5 we shall show that all hyper-Kähler metrics with such symmetries are locally given by (2.27).
- Note that \(u_x \neq 0\) for (2.27) to be well defined. To obtain a flat metric take \(u = -x/t\) which is a special case of (2.12). The metric (2.27) becomes
\[
g = 2dx \frac{dt}{t} - 2x \frac{dt^2}{t^2} + 2tdx^2 + 2dzdy.
\]
Putting \(x = Xt + z^2t/2\), \(y = Y - zt\) yields the flat metric
\[
g = 2dXdt + 2dzdY.
\]
- The metric (2.27) could be found directly from the monopole equation (2.23) as follows: Rewrite the metric (2.11) in an orthonormal triad \(h = \epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2\), where
\[
\epsilon_1 = dy, \quad \epsilon_2 = dx + (u - 1) dt, \quad \epsilon_3 = dx + (u + 1) dt.
\]
The duality relations \(*_h \epsilon_1 = \epsilon_3 \wedge \epsilon_2, \quad *_h \epsilon_2 = \epsilon_1 \wedge \epsilon_3, \quad *_h \epsilon_3 = \epsilon_1 \wedge \epsilon_2\) yield
\[
*_h dt = dt \wedge dy, \quad *_h dx = 2dt \wedge dz, \quad *_h dz = dy \wedge dx + 2udx \wedge dt.
\]
Take \(V = u_x/2\), and use the above relations to write the monopole equation (2.23) as
\[
\frac{u_x}{2} dy \wedge dx + u_x dt \wedge dx + (u_x^2 + uu_x - \frac{u_x}{2}) dy \wedge dt = dv.
\]
Choosing the gauge in which \(a = a_1 dy + a_2 dt\) (this is always possible by redefining a coordinate \(z\) along the orbits of a Killing vector) gives
\[
(a_1)_x = -\frac{u_x}{2}, \quad (a_2)_x = -u_x y, \quad (a_1)_y = (a_1)_t = \frac{u_x}{2} - u_y dt. \quad (2.29)
\]
All solutions to this system of equations are gauge equivalent to
\[
a = -\frac{u_x}{2} dy - u_y dt.
\]
Substituting \(V, a\) and \(h\) to (2.24) yields (2.27).
• The Lax pair (2.10) can be obtained from the hyper-Kähler Lax pair by a symmetry reduction: The distribution $(\mathcal{K}, \hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1)$ is not integrable, as $[\mathcal{K}, \hat{\mathcal{L}}_0] = -\partial_y$ and $[\mathcal{K}, \hat{\mathcal{L}}_1] = -\partial_x$. To obtain an integrable distribution, one needs to lift $\mathcal{K}$ to the correspondence space by $\tilde{\mathcal{K}} = \mathcal{K} - \partial_\lambda$. Then $(\hat{\mathcal{K}}, \tilde{\mathcal{L}}_0, \tilde{\mathcal{L}}_1)$ is an integrable distribution, but $\tilde{\mathcal{K}}(\lambda) \neq 0$, which forces us to introduce an invariant spectral parameter $\lambda = \Lambda + z$. This implies that in the Lax pair we replace all $\partial_\lambda$ by $\tilde{\partial}_\lambda$. Now we restrict ourselves to invariant solutions to $\tilde{\mathcal{L}}_0 \Psi = \tilde{\mathcal{L}}_1 \Psi = 0$, and so we ignore $\tilde{\mathcal{K}}$ in the Lax pair. The reduced Lax pair is given by (2.10).

In the covariantly constant primed spin frame the null tetrad is

$$e^{0^\prime} = -u_x dt, \quad e^{1^\prime} = \frac{dz - u_y dt}{u_x},$$

$$e^{0^1} = dz - u_x dy + (u_y + zu_x) dt, \quad e^{1^1} = dx + u dt + \frac{dz - u_y dt}{u_x},$$

and the metric (2.27) is $2(e^{0^0} e^{1^1} - e^{0^1} e^{1^0})$. The basis of SD two form is in this frame given by

$$\Sigma^{0^0} = dz \wedge dt, \quad \Sigma^{0^1} = dz \wedge dy + d(u + z^2) \wedge dt,$$

$$\Sigma^{1^1} = u_x dx \wedge dy - u_y dy \wedge dt + u_y dx \wedge dt + d(uz) \wedge dt + dz \wedge (dx + zd y + z^2 dt).$$

They satisfy

$$-2 \Sigma^{0^0} \wedge \Sigma^{1^1} = \Sigma^{0^1} \wedge \Sigma^{0^1}, \quad d \Sigma^{0^0} = d \Sigma^{0^1} = d \Sigma^{1^1} = 0,$$

which again implies that the metric (2.27) is hyper-Kähler. Note that the Killing vector $K = \partial_z$ does not preserve the Kähler form $\Sigma^{0^1}$.

3 Examples

3.1 dKP EW spaces with $S^1$ symmetry

In this subsection we shall construct EW structures depending on one arbitrary function of one variable.

To find some explicit examples of (2.11) assume that $u$ is independent of $y$. Therefore it satisfies the simple equation $uu_x = u_t$, all solutions of which are given in an implicit form

$$u(x,t) = f(x + tu(x,t))$$

(more general hodograph transformations for dKP arising from its connection with equations of hydrodynamic type were studied in [17], and [12]).

Here $f$ is an arbitrary function of one variable $s := x + tu(x,t)$. The idea is to write the Einstein-Weyl structure (2.11) making use of this ‘hodograph transformation’. We have

$$h = dy^2 - 4dt(dx + u dt) = dy^2 - 4dt(ds - t du) = dy^2 - 4tds + 4tdt f(s)$$

where we performed a coordinate transformation $(x, y, t) \rightarrow (s, y, t)$. Defining $F(s) := df/ds$ and replacing $u_x$ by $F/(1 - t F)$ yields the EW structure

$$h = dy^2 + 4 (t F(s) - 1) dt ds, \quad \nu = 4 \frac{F(s)}{t F(s) - 1} dt,$$

which depends on one arbitrary function $F(s)$ (which we shall take to be strictly negative) of one variable. This structure has signature $(++-)$.

We shall now show that formulae (3.30) give a class of EW structures on principal $S^1$ bundles over Weyl manifolds.

**Proposition 3.1** Let $(N, [H], \nu_H)$ be a two-dimensional manifold with a Weyl structure of signature $(+-)$ and let $\pi : \mathcal{W} \rightarrow N$ be an $S^1$ bundle over $N$. If

$$h := dy^2 + \pi^* H, \quad \nu := \pi^* \nu_H$$

(where $y$ is a coordinate on a fibre) is an EW structure on $\mathcal{W}$ then it can be put in the form (3.30).
Proof. We can use isothermal coordinates \((\tilde{z}, t)\) on \(\mathcal{N}\) and choose a representative of a conformal class \([H]\) such that \(h\) and \(\nu\) are

\[
h = dy^2 + 2G(\tilde{z}, t)d\tilde{z}dt, \quad \nu = K(\tilde{z}, t)dt. \tag{3.31}
\]

Each EW structure of this form is equivalent to (3.30). This can be seen as follows: Equations \(\chi_{21} = 0, \chi_{22} = 0\) imply that \(K = \frac{4G}{G_f + f(t)}\). The function \(f(t)\) can be absorbed in the definition of \(G\). Then the vanishing of \(\chi_{23}\) (all remaining EW equations are satisfied trivially) yields \(G(\tilde{z}, t) = -2F_1(\tilde{z}) + 2F_2(\tilde{z})\) for arbitrary \(F_1\) and \(F_2\). Now we define a new coordinate \(s\) by \(ds := F_1(\tilde{z})d\tilde{z}\). Equivalence between (3.31) and (3.30) is finally obtained by putting \(F(s) := F_2(s)/F_1(s)\). The metric (3.30) is not Einstein as \(G_{22} \neq 0, G_{13} \neq 0\). 

To visualise the two-dimensional surface \(\mathcal{N}\) on which \(H\) is defined one can restrict a flat \((+ + +)\) metric on \(\mathbb{R}^4\), \(g = d\tilde{x}d\tilde{y} - d\tilde{z}dt\) to the intersection of the paraboloid \(w = t^2/2\) with the hypersurface \(f = f(s)\). \(\square\)

The hyper-Kähler structure corresponding to (3.30) has an additional null Killing vector \(\partial_\tilde{y}\) and is (with definitions \(dw := -Fd\tilde{s}, \tilde{F}(\tilde{w}) := \tilde{F}^{-1}\) given by

\[
g = d\tilde{x}dt + d\tilde{y}dy + (t - \tilde{F}(\tilde{w}))d\tilde{z}^2
\]

where \(\tilde{F}(\tilde{w})\) is arbitrary.

Other examples (without a Killing vector) can be obtained from

\[
u = t \frac{dA(t)}{dt} - x \frac{x}{t} + y \sqrt{\frac{x^2}{t} + A(t)},
\]

where \(A(t)\) is arbitrary.

### 3.2 dKP metrics which are hyper-CR

Let us recall that that an EW metric is called hyper-CR (or special) if it admits a two-parameter family of shear-free, divergence-free geodesic congruences [3]. All hyper-CR EW spaces arise as reductions of hyper-Kähler metrics by triholomorphic homotheties [5]. In this section we shall find all EW metrics in 2+1 dimensions which are both dKP and hyper-CR. This will lead to a class of solutions to the dKP equation depending on one arbitrary function of one variable.

**Proposition 3.2** All EW metrics which admit a constant weighted vector and a two parameter family of shear-free geodesic congruences with a vanishing divergence are either spaces of constant curvature or are locally of the form

\[
h = dy^2 - 4dzdt - 4\left(\frac{P(t)}{y} - \frac{x^2}{y^2}\right)dt^2, \quad \nu = \frac{8x}{y^3}dt, \tag{3.32}
\]

where \(P\) is an arbitrary function of \(t\).

**Proof.** The hyper-CR condition for a metric is characterised [9] by the existence of a scalar \(\rho\) of weight \(-1\) which (together with the Einstein–Weyl one form \(\nu\)) satisfies the monopole equation

\[
s_b (d\rho + \frac{1}{2} \rho \lambda) = d\nu, \tag{3.33}
\]

and the algebraic constraint

\[
\rho^3 = \frac{8}{3} W. \tag{3.34}
\]

We shall impose these conditions on the dKP metric (2.11). The monopole equation yields

\[
(4u_{xx} - 2\rho_y)dz \wedge dt + \rho_x dy \wedge dz + (2\rho_x u - \rho_t + 2\rho u_x + 4u_{xy})dy \wedge dt = 0
\]

which (together with (3.34)) gives four scalar equations:

\[
\rho_y = 2u_{xx}, \quad \rho_x = 0, \quad 2\rho u_x - \rho_t + 4u_{xy} = 0, \quad \rho^3 = -8u_{xx}. \tag{3.35}
\]

If \(u_{xx} = 0\) then the last relation in (3.35) gives \(\rho = 0\). The monopole equation then implies that \(\nu\) is closed, and the Einstein–Weyl metric is conformal to Einstein. Therefore we assume \(u_{xx} \neq 0\).
Differentiating the third equation in (3.35) with respect to \(x\) (and using the first two equations) gives
\[
\rho = -2 \frac{u_{xxy}}{u_{xx}}.
\]
The integrability conditions to (the otherwise over-determined system) (3.35) are
\[
\begin{align*}
u_{xxx} &= 0, & u_{xxy} - u_{xxyy} u_{xx} &= u_{xxx}, & 4u_{xxy} &= \eta u_{xx}^3, \\
\eta u_{xxy} u_{xtt} &- u_{xyy} u_{xx} u_{xy} + 2 u_{x} u_{xx} u_{xyy} - 2 u_{xy} u_{xx}^2 &= 0.
\end{align*}
\] (3.36)
The first condition implies \(u(x, y, t) = ax^2 + bx + c\). Here \(a, b, c\) are functions of \(y\) and \(t\), which satisfy
\[
\begin{align*}
a_{yy} + 6a^2 &= 0, \\
b_{yy} - 2a_t + 6ab &= 0, \\
c_{yy} - b_t + 2ac + b^2 &= 0, \\
a_y^2 - a_{yy} - 2a^3 &= 0, \\
a_y^2 + 4a^3 &= 0, \\
a_{yy} a_t - a_y a_t - 2a a_t b + 2 b a^2 &= 0. \\
\end{align*}
\] (3.37, 3.38, 3.39)
Equations (3.37, 3.38, 3.39) follow from the dKP (2.9), and the other equations are the integrability conditions (3.36). Solve (3.41) to find \(a(y, t) = -\left(\frac{y}{G(t)}\right)\). We now perform the coordinate transformation (2.20) with \(f = -L/2\) and \(g = 0\) to set \(L(t) = 0\). One verifies that (3.37), and (3.41) are now also satisfied. Equation (3.38) gives \(b(y, t) = -M(t) y^{-2} + N(t) y^3\), but (3.42) implies \(N(t) = 0\). So far we have
\[
h = dy^2 - 4dxdt + 4 \left(c(y, t) - \frac{xM(t)}{y^2} - \frac{x^3}{y^3}\right) dt^2, \quad \nu = \frac{8x + 4M(t)}{y^2} dt.
\]
The function \(M(t)\) can be eliminated by the coordinate transformation (2.20) with \(g = M/2\). Imposing (3.39) yields \(c(y, t) = P(t)/y + R(t)y^2\), leaving
\[
h = dy^2 - 4dxdt + 4 \left(-\frac{x^2}{y^2} + \frac{P(t)}{y} + R(t)y^2\right) dt^2, \quad \nu = \frac{8x}{y^2} dt.
\]
We eliminate \(R(t)\) by performing the conformal rescaling and associated coordinate redefinitions of (2.21) with \(c(t)\) satisfying
\[
R = -\frac{c''''}{6c'^3} + \frac{1}{4} \left(\frac{c'''}{c'^2}\right)^2.
\]
This yields, dropping the tildes and with a redefinition of \(P\),
\[
u(x, y, t) = -\frac{x^2}{y^2} + \frac{P(t)}{y}.
\]
The Einstein–Weyl structure is therefore (3.32). The arbitrary function \(P(t)\) can not be eliminated. This can be seen by finding the symmetries (1.4) of the EW structure (3.32). We summarise our findings in the table below:

| \(i\)  | \(P(t) = 0\) | \(K_1, K_2, K_3, K_4\) |
| \(ii\) | \(P(t) = \text{const} \neq 0\) | \(K_1, K_2 + 3K_3, K_4\) |
| \(iii\) | \(P(t) = (bt + c)^{\frac{4}{3}}\) | \(cK_1 + aK_2 + bK_3\) |
| \(iv\)  | general \(P(t)\) | none |

where \(a, b, c\) are constants, and
\[
K_1 = \partial_t, \quad K_2 = (1/2)y\partial_y + x\partial_x, \quad K_3 = (1/2)y\partial_y + t\partial_t, \quad K_4 = ty\partial_y + (y^2 + 2xt)\partial_x + 3t^2\partial_t.
\]
Note that in case \((ii)\) we can redefine coordinates to set \(P(t) = 1\). The vector fields \(K_1, K_2 + 3K_3, K_4\) generate the Lie group of Bianchi type VIII, i.e. \(SU(1, 1)\), and the cases \((i)\) and \((ii)\) give homogeneous EW spaces. Case \((iii)\) can be reduced to \(P(t) = \Gamma^a, \quad K = K_3 + [(2a+1)/3]K_2\), where \(a = \text{const} \neq 0\). □
4 The twistor correspondences and Lax formulations

In this section we shall study the twistor theory of the EW spaces. We first discuss the twistor correspondence in the flat case. We then give a Lax formulation of the EW equations and derive from it the twistor correspondence. We study this correspondence in relation to reductions of the anti-self-duality equations on four-dimensional conformal structures. We then reformulate the Einstein–Weyl equations in terms of a certain two-form on the trivial $\mathbb{CP}^1$ bundle over a Weyl space.

4.1 The flat correspondence

Let us begin by recalling Ward’s approach [31] to twistors in (2+1)-dimensional flat space-times. Rearrange the space time coordinates $(x, y, t)$ as a symmetric two-spinor

$$ x^{A^B} := \begin{pmatrix} t \\ y/2 \\ x \end{pmatrix}, $$

such that the space-time metric and the volume form are:

$$ h = -2dx_{A^B}d\omega^{A^B}, \quad \text{vol}_h = dx_{A^B}^{A^I} \wedge dx_{A^B}^{B^J}. $$

The two-dimensional spinor indices are raised and lowered with the symplectic form $\varepsilon_{A^B}$, such that $\varepsilon_{B^A} = 1$ (see [24] for a full account of the two-spinor formalism). We shall use the abstract index convention $V^A = V(A^B) = V^A\pi^B$ based on an isomorphism $\mathfrak{t}^*\mathfrak{k} = S(\mathfrak{a} \otimes \mathfrak{b})$.

The projective mini-twistor space of $\mathbb{R}^{2+1}$ is the two-dimensional complex manifold $Z = T\mathbb{CP}^1$ which is the total space of the line bundle $O(2)$ of Chern class 2 over $\mathbb{CP}^1$. Points of $Z$ correspond to null 2-planes in $\mathbb{R}^{2+1}$ via the incidence relation

$$ x^{A^B} \quad \pi_{A^B} \quad \omega. \quad (4.43) $$

Here $(\omega, \pi_A, \pi_{\bar{A}})$ are homogeneous coordinates on $O(2)$: $(\omega, \pi_A) \sim (\rho^2 \omega, \rho \pi_A)$, where $\rho \in \mathbb{C}$. In the affine coordinates $\lambda := \pi_A / \pi_A, \xi := \omega / (\pi_A)^2$ equation (4.43) is $\xi = x + \lambda y + \lambda^2 t$. First fix $(\omega, \pi_A)$. If $(\xi, \lambda)$ are both real then (4.43) defines a null plane in $\mathbb{R}^{2+1}$. If both $\xi$ and $\lambda$ are complex then the solution to (4.43) is a time like curve in $\mathbb{R}^{2+1}$. We shall say that this curve is oriented to the future if $\text{Im} \lambda > 0$ and to the past otherwise. If $\lambda$ is real and $\xi$ is complex then (4.43) has no solutions for finite $x^{A^B}$.

An alternate interpretation of (4.43) is to fix $x^{A^B}$. This determines $\omega$ as a function of $\pi_A$, i.e., a section of $O(2) \to \mathbb{CP}^1$ when factorized out by the relation $(\omega, \pi_A) \sim (\rho^2 \omega, \rho \pi_A)$. These are embedded rational curves with normal bundle $O(2)$. Two rational curves $l_{p_1}$ and $l_{p_2}$ (corresponding to $(t_1, y_1, x_1)$ and $(t_2, y_2, x_2)$ respectively) intersect at two points

$$ \lambda_{1, 2} = \frac{2R_2 \pm \sqrt{h(R, R)}}{2R_1}, \quad \text{where} \quad R_i := (t_1 - t_2, y_1 - y_2, x_1 - x_2). $$

Therefore the incidence of curves in $Z$ encodes the causal structure of $\mathbb{R}^{2+1}$ in the following sense: $l_{p_1}$ and $l_{p_2}$ intersect at (a) one point, (b) two real points, (c) two complex points conjugates of each other, if $p_1, p_2$ are (a) null separated, (b) space-like separated, (c) time-like separated.

Examining the relevant cohomology groups shows that the moduli space of curves with normal bundle $O(2)$ in $Z$ is $\mathbb{C}^3$. The real space-time $\mathbb{R}^{2+1}$ arises as the moduli space of curves that are invariant under the conjugation $(\omega, \pi_A) \mapsto (\bar{\omega}, \bar{\pi}_A)$.

The correspondence space $F = \mathbb{C}^3 \times \mathbb{CP}^1 = \{ (p, Z) \in \mathbb{C}^3 \times Z \mid Z \in l_p \}$. By definition, it inherits fibrations over both $\mathbb{C}^3$ and $Z$ and the fibration of $F = \mathbb{C}^3 \times \mathbb{CP}^1$ over $Z$ has fibres spanned by the distribution $L_A = x^{B^D} \partial_{A^B^D}$, where $\partial_{A^B^D} x^{C^D} = 1/2(\varepsilon_{A^B} x^{C^D} + \varepsilon_{B^D} x^{A^C})$. In the affine coordinates $\pi_A = (1, -\lambda)$ this distribution is

$$ L_0 = \partial_t - \lambda \partial_y, \quad L_1 = \partial_y - \lambda \partial_x. $$

The use of primed (rather than unprimed) spinors in this section originates from the representation of Einstein–Weyl spaces as reductions of ASD (rather than SD) metrics in four dimensions. ASD structures (for which the covariantly constant self-dual spinors are conventionally denoted as having primed indices) are taken as basic because they arise from a natural choice of orientation and conformal structure on a Kähler manifold.
(we have ignored the constant factor $\pi_1$). Note that this $L_A$ is the special case $u(x, y, t) = 0$ of the Lax pair \((2.10)\) for the dKP equation.

We also define the correspondence space $F_W = \mathbb{R}^{2+1} \times \mathbb{C}P^1$ for $\mathbb{R}^{2+1}$. Let $Z_{\mathbb{B}}$ be the sub-manifold of $Z$ preserved by the conjugation

$$(\omega, \pi_B, \pi_{1B}) \mapsto (\omega, \pi_B, \pi_{1B}),$$

and let $l_p$ be the real line in $Z_{\mathbb{B}}$ that corresponds to $p \in W$ and let $Z \in l_p$. The totally real correspondence space is a four-dimensional real manifold defined by $F_{\mathbb{R}}^4 := Z_{\mathbb{B}} \times (\mathbb{R}^{2+1})_{\pi \in l_p}$ and can be represented as the set $\tilde{\lambda} = \tilde{\lambda}$ or $\pi_A^B \neq \pi_{1B}$. The distribution $L_{A^1} \cap L_{A^2}$ is one dimensional, spanned by $\pi_B^0 \pi^B \partial_{A^1} \partial_{B^1}$, on the complement of $F_{\mathbb{R}}^4$. On $F_{\mathbb{R}}^4$ $L_{A^1} \cap L_{A^2}$ is two real dimensional as here $L_{A^1} = L_{A^2}$. The real correspondence space $F_{\mathbb{R}}$ divides $F_W = \mathbb{R}^{2+1} \times \mathbb{C}P^1$ into two halves.

### 4.2 The Lax formulation and twistor correspondence

**Proposition 4.1** Let $V_1, V_2, V_3$ be three independent holomorphic vector fields on a three-dimensional complex manifold $W$ such that

$$L_0^i = V_i - \tilde{\lambda}V_3 + f_0^i \partial_{\tilde{\lambda}}, \quad L_{1i} = V_2 - \tilde{\lambda}V_3 + f_1^i \partial_{\tilde{\lambda}},$$

is an integrable distribution for some functions $f_0^i, f_1^i$, which are third-order polynomials in $\tilde{\lambda} \in \mathbb{C}P^1$. Then there exists a one form $\nu$ such that the contravariant metric $V_3 \otimes V_3 - 1/2(V_1 \otimes V_3 + V_3 \otimes V_1)$ and $\nu$ give an EW structure on $W$. Each EW structure arises in this way.

**Remarks:**

- The Lax pair \((2.10)\) for the dKP equation is of course a special case of \((4.44)\).
- The Lax formulations are widely applicable in the theory of integrable systems and so the above proposition can be applied outside twistor theory. It is however much easier to prove Proposition 4.1 using the twistor geometry, rather than an explicit calculation. This justifies adopting the spinor notation

$$\nabla_A^B = \begin{pmatrix} V_1 & V_2 \\ V_2 & V_3 \end{pmatrix}, \quad f_A^i = (f_0^i, f_1^i), \quad \pi^A = (1, -\tilde{\lambda}),$$

in which the Lax pair has the compact form $L_{A^1} = \pi_{B^1} \nabla_A^B + f_A^i \partial_{\tilde{\lambda}}$. We shall use this notation in the proof of Proposition 4.1.

- The third-order polynomials $f_A^i$ contain eight functions not depending on $\tilde{\lambda}$. These can be reduced to four functions by choice of a suitable spin frame for which $f_A^i$ become linear in $\tilde{\lambda}$. In this frame there exists a vector formula for $\nu$ in terms of $\Gamma_{;jkl}$ and $f_A^i$.
- Proposition 4.1 holds for complex solutions and for any choice of signature for real space time.

**Proof of Proposition 4.1.** Assume that $h = V_2 \otimes V_2 - 1/2(V_1 \otimes V_3 + V_3 \otimes V_1)$ and $\nu$ gives an EW structure. Let $V(\tilde{\lambda}) = V_1 - 2\tilde{\lambda}V_2 + \lambda V_3$. Then $g(V(\tilde{\lambda}), V(\tilde{\lambda})) = 0$ for all $\tilde{\lambda} \in \mathbb{C}P^1$ so $V(\tilde{\lambda})$ determines a sphere of null vectors. Choose $l_0^i = V_1 - \tilde{\lambda}V_3, l_1^i = V_2 - \tilde{\lambda}V_3$ as a basis of the orthogonal complement of $V(\tilde{\lambda})$. For each $\tilde{\lambda} \in \mathbb{C}P^1$ the vectors $l_0^i, l_1^i$ give a null two-surface. It is well known [4, 15, 25] that the EW equations on $(h, \nu)$ are equivalent to the integrability conditions of null, totally geodesic planes. Therefore the Frobenius theorem implies that the horizontal lifts

$$L_0^i = V_1 - \tilde{\lambda}V_3 + f_0^i \partial_{\tilde{\lambda}}, \quad L_{1i} = V_2 - \tilde{\lambda}V_3 + f_1^i \partial_{\tilde{\lambda}}$$

of $l_0^i, l_1^i$, to $T(W \times \mathbb{C}P^1)$ span an integrable distribution. The functions $f_0^i$ and $f_1^i$ are third order in $\tilde{\lambda}$, because the M"obius transformations of $\mathbb{C}P^1$ are generated by vector fields quadratic in $\tilde{\lambda}$, and $l_0^i, l_1^i$ are linear $\tilde{\lambda}$.

The above argument can be made more explicit in spinor notation: let $L_A^i$ be horizontal lift of $l_A^i = \pi^B \nabla_A^B$ to the weighted spin bundle (i.e. $L_A^i \pi_{ci} = 0$). This yields

$$L_A^i = \pi_B^i \nabla_A^B + \Gamma_{A^B} \pi^B \pi^D, \quad \frac{1}{2} \pi_B^i \frac{\partial}{\partial \pi^A} - \frac{1}{2} \pi_A^i \frac{\partial}{\partial \pi^B} - \varepsilon_{A^B} \pi^C \cdot \frac{\partial}{\partial \pi^i},$$

(4.45)
where $\Gamma_{\alpha\beta\gamma}^{\mu}$ is spinor Levi–Civita connection defined by $\nabla_{\alpha}^{\mu} \pi_{\gamma} = -\Gamma_{\alpha\beta\gamma}^{\mu} \pi_{\beta}$. The integrability conditions imply $[L_{\alpha}, L_{\beta}] = 0$ (mod $L_{\alpha}$). The distribution $L_{\alpha}$, when projected to $\mathcal{J}_{\mathcal{W}}$ is given by (4.44), where

$$f_{\alpha} = \Gamma_{\alpha\beta\gamma}^{\mu} \pi^{\beta} \pi^{\gamma} + \frac{1}{4} \pi_{\alpha} \nu^{\beta} \pi^{\mu} \pi^{\nu},$$

The twistor space $Z$ for a solution to the EW equations on $(\mathcal{W}, h, \nu)$ associated to the Lax system on $L_{\alpha}$ as above is obtained by factoring the spin bundle $\mathcal{W} \times \mathbb{CP}^{1}$ by the twistor distribution (Lax pair) $L_{\alpha}$. This clearly has a projection $q : \mathcal{W} \times \mathbb{CP}^{1} \to Z$ and we have a double fibration:

$$\begin{array}{cc}
\mathcal{W} \times \mathbb{CP}^{1} & \mathcal{W} \\
\mathcal{W} \times \mathbb{CP}^{1} & Z
\end{array}$$

Each point $p \in \mathcal{W}$ determines a sphere $l_{p}$ made up of all the null totally geodesic two-surfaces through $p$. The normal bundle of $l_{p}$ in $Z$ is $N = T\mathcal{Z}|_{p}/Tl_{p}$. This is a rank one vector bundle over $\mathbb{CP}^{1}$, therefore it has to be one of the standard line bundles $\mathcal{O}(n)$.

Lemma 4.2 The holomorphic curves $l_{p} := q(\mathbb{CP}^{1}_{p})$ where $\mathbb{CP}^{1}_{p} = r^{-1}(p)$, $p \in \mathcal{W}$, have normal bundle $N = \mathcal{O}(2)$.

Proof. To see this, note that $N$ can be identified with the quotient $r^*(T_{\mathcal{W}})/[\text{span} \ L_{\alpha}, L_{\beta}]$. In their homogeneous form the operators $L_{\alpha}$ have weight one, so the distribution spanned by them is isomorphic to the bundle $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. The definition of the normal bundle as a quotient gives a sequence of sheaves over $\mathbb{CP}^{1}$:

$$0 \to \mathbb{C}^2 \otimes \mathcal{O}(-1) \to \mathbb{C}^2 \to N \to 0$$

and we see that $N = \mathcal{O}(2)$, because the last map, in the spinor notation, is given explicitly by $V^{\alpha\beta} \mapsto V^{\alpha\beta} \pi_{\alpha} \pi_{\beta}$ clearly projecting onto $\mathcal{O}(2)$. □

A generalisation of the flat mini-twistor correspondence to the 2+1 EW spaces is given by the following proposition.

Proposition 4.3 ([15]) Any solution to the EW equations (1.5) is equivalent to a complex surface $Z$ with a family of rational curves with normal bundle $\mathcal{O}(2)$.

Points of $\mathcal{W}$ correspond to curves in $Z$ with self-intersection number 2. The Kodaira theorem [18] applied to deformations preserving the real structure of $Z$ guarantees the existence of a three-dimensional complex family of such curves. Points of $Z$ correspond to totally geodesic hypersurfaces in $\mathcal{W}$. Non-null geodesics in $\mathcal{W}$ consist of all the curves in $Z$ which intersect at two fixed points in $Z$. Null geodesics correspond to curves passing through one point with a given tangent direction. Thus the projective and conformal structures can be reconstructed. □

### 4.3 Mini-twistor spaces from twistor spaces

**Proposition 4.4** All Einstein–Weyl spaces arise as symmetry reductions of hyper-Hermitian metrics (or indefinite hyper-Hermitian metrics) in four-dimensions.

**Proof.** Consider an EW structure with the corresponding Lax pair (4.44). Choose a spin frame in which $f_{\alpha}$ is linear in $\lambda$: $f_{\alpha} = U_{\alpha} + \lambda W_{\alpha}$ (this is always possible by making a suitable Möbius transformation of $\mathbb{CP}^{1}$ and choosing an appropriate conformal scale), and introduce a new spectral parameter $\lambda := \lambda - z$ for some $z$. Nothing in the $L_{\alpha}$ depends on $z$ so we can replace $\partial_{\lambda}$ by $\partial_{z}$. This yields (with a dropped prime)

$$L_{\alpha} = \nabla_{\alpha} - \lambda \nabla_{\alpha},$$

where

$$\begin{align*}
\nabla_{\alpha} \psi &= \nabla_{\alpha} \psi + z \nabla_{\alpha} \psi + (U \psi + z W \psi) \partial_{z}, \\
\nabla_{\alpha} \dot{\psi} &= \nabla_{\alpha} \dot{\psi} + z \nabla_{\alpha} \dot{\psi} + (U \dot{\psi} + z W \dot{\psi}) \partial_{z}, \\
\nabla_{\alpha} - \lambda \nabla_{\alpha} 
\end{align*}$$

and we have the following solutions.

13
where $U_{1}, U_{2}, W_{1}, W_{2}$ are four functions not depending of $\lambda$. One is left with a Lax pair for a hyper-Hermitian four manifold because $L_{A}$ can be made to commute exactly (as in Proposition 2.6) by choosing two solution to the background coupled neutrino equation (see [6] for details). This Lax pair has an obvious symmetry $\partial_{x}$. □

**Remark:** All EW spaces arise as symmetry reductions of a pair of coupled PDEs [6], [13] associated to hyper-Hermitian four manifolds. In [2] Proposition 4.44 was proven using different methods for EW spaces of Riemannian signature.

The twistor construction of Hitchin can be viewed as a reduction of Penrose’s Nonlinear Graviton construction. It follows from [16] (compare Proposition 2.5) that the mini-twistor space $E$ corresponding to $W$ is a factor space $P\mathcal{T}/K$ where $P\mathcal{T}$ is the twistor space of $(M, g)$ and $K$ is a holomorphic vector field on $P\mathcal{T}$ corresponding to a conformal Killing vector $K$.

Below we shall state the Penrose result extended to the Einstein and hyper-Hermitian cases:

**Proposition 4.5** Let $P\mathcal{T}$ be a three-dimensional complex manifold with a four-dimensional family of rational curves (invariant under a complex conjugation with fixed points) with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then the moduli space $\mathcal{M}$ of these sections is equipped with an ASD conformal structure $[g]$ of signature $(+, +, -)$. Conversely given an ASD four-manifold there will always exists a corresponding twistor space. Moreover $\mathcal{M}$ is:

- **Hyper-Kähler** if there exists a projection $\mu : P\mathcal{T} \to \mathbb{P}^{3}$, and each fibre of this projection is equipped with an $\mu^{*}\mathcal{O}(2)$ valued symplectic form [23] (equivalently, we can require that the canonical bundle $\kappa$ of $P\mathcal{T}$ is $\kappa = \mu^{*}\mathcal{O}(4)$).
- **Hyper-Hermitian** if there is a projection $\mu : P\mathcal{T} \to \mathbb{P}^{1}$ [6].
- **Einstein** $(\mathcal{R}_{\text{ab}} = \mathcal{A}_{\text{ab}})$ if there exists a contact structure $\tau \in \Lambda^{2}(T^{*}P\mathcal{T}) \otimes \mathcal{O}(2)$, where now $\mathcal{O}(2) = \kappa^{-1/2}$, and $\kappa$ is the canonical bundle $\Omega^{3}$, such that $\tau \wedge d\tau = \lambda^{*} \xi$ where $\xi \in \Omega^{3} \otimes \kappa^{-1}$ [30].

### 4.3.1 Construction of the two-form

Consider an ASD four-manifold $(M, [g])$. Define the non-projective twistor space, $\mathcal{T}$, to be the total space of the line bundle $\kappa^{-1/2} \to P\mathcal{T}$ where $\kappa = \Omega^{3}$ is the canonical bundle. In the conformally-flat case $\mathcal{T}$ is the tautological line bundle $\mathcal{O}(-1)$, i.e. $\mathbb{C}^{4} \to \mathbb{P}^{3}$, and we will also use this notation, $\mathcal{T} = \mathcal{O}(-1)$ in the curved case. The nonprojective spin bundle $S_{\mathbb{R}} \to \mathcal{M}$ is defined to be the total space of the pullback of this line bundle to the correspondence space $\mathcal{F} = \mathcal{M} \times \mathbb{P}^{1}$. Clearly $S_{\mathbb{R}} = M \times \mathbb{C}^{2}$. The fibration $q : S^{4} \to \mathcal{T}$ is spanned by a lift of the twistor distribution or Lax pair. The non-projective spin bundle is the total space of a line bundle, which we will also denote by $\mathcal{O}(-1)$, over $\mathcal{F}$. (Note that in the hyper-Hermitian case the line bundles $\mathcal{O}(n)$ just defined will not be the same as $\mu^{*}\mathcal{O}(n)$ unless $(M, [g])$ is in fact hyper-Kähler)

The space $\mathcal{T}$ admits an Euler vector field $\mathcal{V}$ being the total space a of line bundle, and a tautological three-form, $\xi$ the pullback of the tautological three-form on $\kappa$. These satisfy $L_{\mathcal{V}} \xi = 4 \xi$. Let $\phi = d\xi$, then $\xi = 4\phi(\mathcal{V}, ..., ...)$. $\xi$ can be thought of as a form on $P\mathcal{T}$ with values in the dual canonical bundle $\kappa^{*}$.

We now impose a symmetry: let $K, \bar{K}$, and $\mathcal{K}$ be respectively: a conformal Killing vector on $\mathcal{M}$, its lift to the correspondence space $\mathcal{M} \times \mathbb{P}^{1}$, and the holomorphic vector field on $\mathcal{T}$ which is the push-forward of $\bar{K}$.

**Proposition 4.6** The two form $\mathcal{S} := q^{*}\phi(\mathcal{K}, \mathcal{V}, ..., ...) \in \Lambda^{2}(T^{*}S^{4})$ satisfies

\[ \mathcal{S} \wedge \mathcal{S} = 0, \quad d\mathcal{S} = \beta \wedge \mathcal{S} \quad L_{\mathcal{K}} \mathcal{S} = 0 \quad (4.46) \]

for some one-form $\beta$ homogeneous of degree 0 in $\tau^{4}$.

**Proof:** It follows from the definition of $\mathcal{S}$ that the integrable twistor distribution belongs is the kernel of $\mathcal{S}$. Therefore equations (4.46) follow from Frobenius’ theorem. The one-form $\beta$ is defined up to the addition of $d(ln \varphi)$ where $\varphi$ is a twistor function homogeneous of degree 0. □

From $L_{\mathcal{V}} \mathcal{S} = 4 \mathcal{S}$ and $\mathcal{V} \cdot \mathcal{S} = 0$ it follows that $\mathcal{S}$ descends to $\mathcal{F}$ where it takes values in $\mathcal{O}(4)$. Note however that $d\mathcal{S}$ does not descend as $\mathcal{V} \cdot d\mathcal{S} = L_{\mathcal{V}} S \neq 0$. To differentiate $\mathcal{S}$ on $\mathcal{F}$ we need a nonzero section of $\mathcal{O}(4)$ in order to dehomogenise $\mathcal{S}$. When $(\mathcal{M}, g)$ is ASD Einstein or vacuum we can find a section of $\mathcal{O}(4)$ to dehomogenise $\mathcal{S}$. This section necessarily has zeroes, and so equivalently, this requires the existence of a divisor description of the dual canonical bundle. This can be seen from the twistor construction.
**Vacuum case:** The twistor space fibres over $\mathbb{P}^1$ and so we can pull back $\pi \cdot d\tau$ to $\mathcal{P}$. Let $\mathcal{K}$ be a holomorphic vector field on $\mathcal{P}$ such that $\mathcal{L}_g \cdot \Sigma = \eta \Sigma$ (where $\mathcal{K}$ corresponds to a Homothetic Killing vector on $\mathcal{M}$). The function $D := K \cdot \pi \cdot d\tau$ is a section of $\mathcal{O}(2)$ and the two-form $D^{-2} \mathcal{K} \cdot \mathcal{J} \cdot d\tau$ descends to the mini-twistor space $Z$.

**Einstein case:** Let $\mathcal{P} \cdot B$ be the projective twistor space corresponding to a solution of the ASD Einstein equations. It is equipped with a contact structure $\mathcal{K} \in \Lambda^2(T^*\mathcal{P} \cdot B \cdot F) \otimes \mathcal{O}(2)$ such that $\mathcal{K} \cdot \mathcal{L} \cdot d\tau = \Delta \zeta$. $d\tau$ defines a holomorphic symplectic structure on the non-projective twistor space $\mathcal{P} \cdot B$. If $\mathcal{K}$ is a Killing vector on an ASD Einstein manifold then the corresponding holomorphic vector field on the non-projective twistor space is Hamiltonian with respect to $d\tau$. To see this, define a section of $\mathcal{O}(2)$ by $D := K \cdot \mathcal{J} \cdot d\tau$. We have $dD = \mathcal{L}_G \cdot \mathcal{K} \cdot \mathcal{J} \cdot d\tau = -\mathcal{K} \cdot \mathcal{J} \cdot d\tau$ as $\mathcal{K}$ is a symmetry.

On the projective spin bundle $\mathcal{F}$ define

$$\Pi := D^{-2} \Sigma.$$

We have the following result:

**Proposition 4.7** The two-form $\Pi$ is well defined on the Einstein–Weyl correspondence space $\mathcal{F} \cdot W$. It satisfies

$$d\Pi = 0, \quad \Pi \cdot \Pi = 0, \quad (4.47)$$

where $d = d\lambda \circ \partial_x + d\lambda \circ \partial_y$ is the exterior derivative on $\mathcal{F} \cdot W$. Any two linearly independent vectors $L_A \cdot \mathcal{J} \cdot S = 0$ form a Lax pair for the EW equations.

**Proof.** The simplicity follows from $\Sigma \bigwedge \Sigma = 0$. In the vacuum case the two form

$$\Pi = q^* \cdot K \cdot \mathcal{J} \cdot \mathcal{K} \cdot (\pi \cdot d\tau) \quad (4.48)$$

is a pull back of a closed and simple form on $\mathcal{P} \cdot B$. In the Einstein case

$$\Pi = D^{-2} q^* \cdot K \cdot \mathcal{J} \cdot (\Lambda \tau \cdot d\tau) = d(\Lambda \tau / D).$$

15
Therefore Einstein-Weyl metrics which come from ASD Einstein and hyper-Kähler four manifolds give rise to the same structure on the reduced spin bundle. The form $\Pi$ descends to $\mathcal{F}_W$ because $K \cdot d\Pi = 0$ and $d(K \cdot \Pi) = 0$. \hfill \Box

**Remark.** In [28] certain dispersionless integrable systems were expressed in terms of $\Pi$ satisfying (4.47).

The two form $\bar{\Sigma}$ can be equivalently constructed from the data on $\mathcal{M}$ as follows. Let $K$ be a Killing vector on a general ASD conformal manifold $(\mathcal{M}, [g])$, and let $\Xi$ be a volume form on the non-projective primed spin bundle $S^4\bar{\mathcal{M}}$. Define the two form on $S^4\bar{\mathcal{M}}$

$$\bar{\Sigma} := \Xi(\bar{L}_0, L_1, \bar{K}, \Gamma_\Xi, \ldots). \quad (4.49)$$

Here $\Gamma_\Xi = \pi^4/\partial x^4$ is the Euler vector field on $S^4\bar{\mathcal{M}}$, $L_0$ is the twistor distribution, and $\bar{K}$ is a Lie lift of $K$ to $S^4\bar{\mathcal{M}}$. Now assume that $(\mathcal{M}, [g])$ is also vacuum. Consequently $\nabla_{A\bar{A}} K^A_{B\bar{B}} = \text{const}$ and the spin bundle is equipped with a canonical divisor $D := \pi^4 \pi^B \nabla_{A\bar{A}} K^A_{B\bar{B}} \in \mathcal{O}(2)$ which descends to the reduced spin bundle $\bar{\mathcal{M}}$ (Figure 1). It is easy to prove that now

$$\bar{\Sigma} = \pi_{A\bar{A}} \pi_{B\bar{B}}, \Sigma^{B\bar{B}} \pi_{C\bar{C}} \pi_{D\bar{D}} \phi^{A\bar{A}} \Sigma^{C\bar{C}} D_{A\bar{A}} + \pi_{A\bar{A}} \pi_{B\bar{B}} \pi_{C\bar{C}} \pi_{D\bar{D}} \phi^{A\bar{A}} \Sigma^{C\bar{C}} D_{B\bar{B}} \wedge (K \cdot \Sigma^A B_{B\bar{B}}),$$

$$\beta = \frac{4\phi_{A\bar{A}} \phi_{B\bar{B}} \phi^{A\bar{A}} \phi^{B\bar{B}}}{\pi_{A\bar{A}} \pi_{B\bar{B}} \phi^{A\bar{A}} \phi^{B\bar{B}}} = d\ln D^2$$

$$\Pi = d\lambda \wedge \frac{K \cdot \Sigma(\lambda)}{D^2} = \frac{\Sigma(\lambda)}{D}, \quad \text{where} \quad \Sigma(\lambda) = \pi_{A\bar{A}} \pi_{B\bar{B}} \Sigma^{A\bar{A}} B_{B\bar{B}}. \quad (4.50)$$

From the last formula it follows that to construct $\Pi$ one should rewrite $\Sigma(\lambda)/D$ in the coordinates in which $K = \partial_\lambda$, and then replace all $d\lambda$s by the differentials of a suitably defined invariant spectral parameter.

**Example.** We shall now illustrate the construction of $\Pi$ with a simple example. Let $2d\mu d\bar{\mu} - 2 d\lambda d\bar{\lambda}$ be a flat metric on $\mathbb{R}^2,^2$ and let $K = z \partial_z - \bar{z} \partial_{\bar{z}}$ be a Killing vector. The flat twistor distribution and the lifted symmetry are:

$$L_0 = \partial_{\bar{\mu}} - \lambda \partial_{\bar{\lambda}}, \quad L_1 = \partial_\mu - \lambda \partial_\lambda, \quad \bar{K} = z \partial_z - \bar{z} \partial_{\bar{z}} + \lambda \partial_{\bar{\lambda}}.$$

The volume form on $\mathcal{F}$ and the two-form $\Sigma(\lambda)$ are given by

$$\Xi = d\lambda \wedge d\mu \wedge d\bar{\lambda} \wedge d\bar{\mu}, \quad \Sigma(\lambda) = -\lambda^2 d\bar{\mu} \wedge d\bar{\lambda} + \lambda (dw \wedge d\bar{u} - dz \wedge d\bar{z}) + dw \wedge dz.$$  

In the covariantly constant frame we introduce $2 r := \ln(z/\bar{z}), \ 2 \phi := \ln(z/\bar{z})$, so that $\bar{K} = \partial_\phi + \lambda \partial_\lambda$.

In these coordinates

$$\Sigma(\lambda) = -\lambda^2 e^{-\phi} d\bar{u} \wedge (dr - d\phi) + \lambda (dw \wedge d\bar{u} + 2 e^{2r} dr \wedge d\phi) + e^{r+\phi} dw \wedge (dr + d\phi)$$

and (from (4.50))

$$\Pi = e^{r} (d\bar{u} \wedge d\lambda - \bar{\lambda}^2 dw \wedge d\bar{\lambda} - \bar{\lambda} dw \wedge dr - \lambda^{-1} dw \wedge dr) + 2 \bar{\lambda}^{-1} e^{2r} dr \wedge d\bar{\lambda} + \lambda dw \wedge d\bar{\mu} \quad (4.51)$$

where $\bar{\lambda} = \lambda e^{-\phi}$ is an invariant spectral parameter.

The two form $\Pi$ can be also obtained as a pull-back from $\mathcal{PT}$. Local inhomogeneous coordinates on $\mathcal{PT}$ pulled back to $\mathcal{F}$ are given by $(\lambda, \mu^4 = \lambda \bar{u} + z, \mu^5 = \lambda \bar{z} + w)$. The holomorphic vector field on $\mathcal{PT}$ is $K = \mu^6 \partial_{\mu^6} + \lambda \partial_\lambda$. From (4.48) we have

$$q^*(K \cdot \mathcal{J} (d\lambda \wedge d\mu^0 \wedge d\mu^1) = (\mu^0 d\lambda - \lambda d\mu^1) \wedge d\mu^1 = \lambda^2 d\mu^1 \wedge d(\mu^0 / \lambda).$$

Thus

$$\Pi = d\mu^1 \wedge (d(\mu^0 / \lambda) = dP \wedge dQ$$

which agrees with (4.51). Here $P = \bar{u} + \bar{\lambda}^{-1} e^{\phi}$ and $Q = \bar{\lambda} e^{\phi} + w$ are coordinates on mini-twistor space pulled back to the reduced spin bundle.

$^2$We assume that $\nabla_{A\bar{A}} K^A_{B\bar{B}} = 0$. If $\nabla_{A\bar{A}} K^A_{B\bar{B}} = 0$ then $K$ is triholomorphic and a section of $\mathcal{O}(2)$ which descends to the reduced spin bundle is $(\xi \cdot \bar{\xi})^2$ where $\xi_{A\bar{A}}$ is any constant spinor.

$^3$By the reduced spin bundle (correspondence space) we mean the space of orbits of $\bar{K}$ in $S^4\bar{\mathcal{M}}$ (in $\mathcal{F}$).
5 Twistor theory of the dKP Einstein-Weyl structures

Here we give an account of the twistor theory of the dKP EW metrics, and the dKP equation (some connections between a twistor theory and the dKP equations have been discussed in [14]). We shall also characterise all four dimensional hyper-Kähler and ASD Einstein metrics that give rise to the dKP EW structures.

Define the non-projective twistor space, $\mathcal{Y}$ corresponding to a Weyl space $\mathcal{W}$, to be the total space of the line bundle $\kappa^{1/4} \to \mathcal{Z}$ where $\kappa = \Omega^2$ is the canonical bundle of $\mathcal{Z}$. The nonprojective spin bundle $S_A \to \mathcal{W}$ is the rank two vector bundle defined to be the total space of the pullback of this line bundle to the correspondence space $\mathcal{W} \times \mathbb{CP}^1$. The fibration $q: S^4 \to \mathcal{Y}$ is spanned by a lift of the mini-twistor distribution $L_A; (4.44)$

Any shear-free null geodesic congruence of the Einstein-Weyl structure determines a one-dimensional sub-manifold in $\mathcal{Z}$ (this is a reduction of the 4-dimensional Kerr theorem). A codimension-one submanifold determines a line bundle $[D]$ by the divisor construction; $[D]$ admits a section $D$ that vanishes precisely on the given submanifold.

When the Einstein-Weyl geometry arises from a solution of the dKP equation the dual canonical bundle $\kappa^{-1}$ of the mini-twistor space admits a fourth root that is given by the divisor construction, just that it admits a section $D$ that vanishes on a codimension-one subset. In general, as seen above, if the Einstein-Weyl geometry is a reduction of an ASD Einstein, or hyper-Kähler four-manifold, then $\kappa^{-1/2}$ admits a section whose zero set will generally have two components in the neighbourhood of a line. For an Einstein-Weyl dKP solution, the two ‘divisor curves’ in Fig. 1 degenerate to one curve. This observation gives rise to a twistor characterisation of solutions to the dKP equation.

**Proposition 5.1** There is a one to one correspondence between Einstein-Weyl spaces obtained from solutions to the dKP equation and two-dimensional complex manifolds with

- A three parameter family of rational curves with normal bundle $\mathcal{O}(2)$.
- A global section $l$ of $\kappa^{-1/4}$, where $\kappa$ is the canonical bundle.

In order to obtain a real Einstein-Weyl structure, we require an anti-holomorphic involution fixing a real slice, leaving a rational curve invariant and leaving the section of $\kappa^{-1/4}$ above invariant.

**Proof.** The global section $l$ of $\kappa^{-1/4}$, when pulled back to $S_A$, determines a homogeneity degree one function on each fibre of $A_S$ and so must, by globality, be given by $l = i^A \tau_A$; and since $l$ is pulled back from twistor space, it must satisfy $L_A l = 0$. This implies $D_{ABC}^\epsilon l^C = 0$, and (after some algebraic manipulations)

$$D_{AB} l^A = 0,$$

where $D$ is a covariant weighted derivative.

Therefore the null vector field $l^A = i^A \tau_A$ is covariantly constant. The Lemma 2.3 implies that the conformal weight of $i^A$ is $-1/4$ and hence that of $l^A$ is $-1/2$. This weight can be deduced from the correspondence as follows: the two form $\Sigma = \pi_A \epsilon^A B \eta \epsilon^C D \pi_C \eta \epsilon_D$ has conformal weight 0 on $S^4$. $\epsilon^A B \eta$ has weight 0, and $\epsilon^A B \eta$ weight $-1$ so $\pi_A$ has weight $1/2$. The global section $\pi_A i^A$ is weightless so the weight of $i^A$ is $-1/4$. Hence by Proposition 2.2 the corresponding Einstein-Weyl space arises from a solution to the dKP equation.

Conversely, given a solution to (2.9) one can obtain $\mathcal{Z}$ as a factor space of $\mathcal{W} \times \mathbb{CP}^1$ by the distribution (2.10) and the covariant constant weighted null vector $l^\epsilon = i^A l^A$ gives rise to the section $l = i^A \tau_A$ of $\kappa^{-1/4}$.

**Remark:** Note that there is not a $1:1$ correspondence between such twistor spaces and solutions to the dKP equation on account of the coordinate freedom (2.20) and (2.21). The coordinate choices implicit in a solution to the dKP equation can be encoded on the twistor space in the choice of the coordinates near the divisor as follows.

Let $P, Q$ be local coordinates on a neighborhood of the divisor in $\mathcal{Z}$ such that $Q = 0$ on the divisor and, setting $Q = Q^{-1}, P = \dot{P}/\dot{Q}^3$ on the complement of the divisor, we have

$$\Pi = dP \wedge dQ = -Q^{-4} d\dot{P} \wedge d\dot{Q}.$$

Consider a graph of a rational curve $\tilde{P}(\dot{Q})$. Parametrise the curve by $(t, y, x)$ as follows:

$$t := \tilde{P}_{\dot{Q}=0}, \quad y := \frac{d\tilde{P}}{d\dot{Q}}_{\dot{Q}=0}, \quad x := \frac{1}{2} \frac{d^2\tilde{P}}{d\dot{Q}^2} |_{\dot{Q}=0}.$$

17
Therefore the local coordinates \( P, Q \) have the following expansion near \( \lambda = \infty \)
\[
Q := \lambda + \sum_{i=1}^{\infty} u_i \lambda^{-i}, \quad P = \sum_{i=1}^{\infty} u_i Q^{-i} + x + Qy + Q^2 t
\]
(after performing an \( SL(2, \mathbb{C}) \) transformation and choosing a spin frame such that the constant term in the Laurent expansion of \( Q \) vanishes). When we pull the mini-twistor coordinates back to \( F \), then \( u_1, u_t \) become functions of \((x, y, t)\). The functions \( P \) and \( Q \) are solutions of Lax equations \( L_A' P = L_A' Q = 0 \). They form a local Darboux atlas as \( \Pi = \Pi P \wedge dQ \), where \( \Pi \) is given by (2.8).

\[
\Pi = dx \wedge d\lambda + dy \wedge d\left( \frac{\lambda^2}{2} + u_1 \right) + dt \wedge d\left( \frac{\lambda^3}{3} + \lambda u_1 + u_1 \right).
\]

The poles of \( \Pi \) occur on the divisor. Now \( \Pi \) is a pull back of a two-form from a two-dimensional manifold. Therefore it satisfies \( \Pi \wedge \Pi = 0 \), which yields \( w_{1x} = u_{1y} \) and the dKP equation (2.9) for \( u_1 \).

Thus, a solution to the dKP equation corresponds to a EW mini-twistor space as described in Proposition 5.1 together with a Darboux coordinate system as above on the third formal neighbourhood of the divisor. [It seems likely that the Benney hierarchy will similarly correspond to the EW dKP mini-twistor space as above together with the Darboux coordinate system on a neighbourhood of the divisor defined now to all orders.]

Now we are in a position to give a characterisation of the hyper-Kähler metrics (2.27).

**Proposition 5.2** Let \( g \) be an indefinite hyper-Kähler metric with a symmetry \( K \) satisfying \( dK_+ \wedge dK_+ = 0 \). Then \( g \) is locally of the form (2.27).

**Proof.** Let \( K \) be a vector field (corresponding to \( K \)) on a twistor space of \((M, g)\). The divisor
\[
K \cdot \pi \cdot d\pi = \pi A_1' B_1' \phi A'B'
\]
descends to the minitwistor space. If \( dK_+ \) is null then \( \phi A_1' B_1' = (1/2)\nabla_{AA'} K_0' = \iota_{A_1} \iota_{B_1} \) for some constant spinor \( \iota A' \). Therefore \( \pi \cdot t \) on \( \mathcal{P} \) defines a divisor in \( \mathcal{Z} \). It takes values in \( \kappa^{-1/4} \) because the canonical bundle of \( \mathcal{P} \) is the square of the pullback of the canonical bundle of \( \mathbb{C} \mathbb{P}^1 \). The assumptions of Proposition 5.1 are satisfied and so the EW structure corresponding to \( \mathcal{Z} \) is of the form (2.11). Therefore it follows from Proposition 2.5 that the metric \( g \) is given by
\[
g = \Omega\left( \tilde{V}(d\tilde{y}^2 - 4d\tilde{x}d\tilde{t} - 4d\tilde{t}^2) - \tilde{V}^{-1}(d\tilde{z} + \tilde{a})^2 \right) = \Omega \tilde{g},
\]
where \( \tilde{z}, \tilde{y}, \tilde{t} \) is a solution to dKP \((\tilde{V}, \tilde{a}) \) is a solution to the monopole equation (2.23), and \( \Omega \) is a conformal factor. Calculating the scalar curvature of the metric \( \tilde{g} \) yields
\[
\tilde{R} = 8(\tilde{V}_{\tilde{y}\tilde{y}} - \tilde{V}_{\tilde{x}\tilde{t}} + (\tilde{u}\tilde{V})_{\tilde{z}\tilde{z}})\tilde{V},
\]
and so \( \tilde{R} = 0 \) because \( \tilde{V} \) satisfies (2.26). However the metric \( g \) is hyper-Kähler, therefore its scalar curvature also vanishes. As a consequence we deduce that \( \Omega = \Omega(\tilde{t}) \). Now we can use the coordinate freedom (2.21) to absorb \( \Omega \) in the solution to the dKP equation. This yields
\[
g = (V(dy^2 - 4dxdt - 4dt^2) - V^{-1}(dz + a)^2) = \Omega \tilde{g},
\]
where \((V, a)\) is another solution to the monopole equation. In section 2.1 we showed that this metric is hyper-Kähler metric if \( V \) is a multiple of \( u_x \).

Consider the metric (5.52) with an arbitrary monopole \( V \) (an arbitrary solution to the linearised dKP equation 2.26). The self-dual derivative of the isometry \( K = \partial_z \) is given by \( \phi A_1' = (u_x/V)\iota_{A_1} \iota_{B_1} \), for some constant spinor \( \iota_{A_1} \). The well known identity \( \nabla_a \nabla_b K_a = R_{bcad} K_d \) and the vacuum condition yield \( \nabla_a \phi B_{1C} = 0 \). Therefore (5.52) is hyper-Kähler iff \( u_x/V = \text{const} \).

**Remarks:**

- This Proposition corrects an omission made in the classification [8] of complexified hyper-Kähler spaces with symmetry. In the Appendix we shall demonstrate explicitly that the dKP equation is a reduction of the second heavenly equation considered in [8].

- Metrics (5.52) with \( V \neq \text{const} \times u_x \) are not vacuum, but they admit a covariantly constant real spinor. The full characterisation of these metrics will be given in our subsequent paper.

18
Proposition 5.3  All EW structures which arise from indefinite ASD Einstein metric with a symmetry $K$ satisfying $dK_+ \wedge dK_+ = 0$ are locally of the form (2.11).

Proof. The canonical divisor $D := K \cdot \tau$ (where $\tau$ is the contact structure) descends to a mini-twistor space. Because $dK_+$ is null the square root of $D$ exists and takes its values in $\kappa^{-1/4}$.

6 Symmetry reductions of hyper-Kähler metrics in 2+2 signature

Symmetry reductions of the hyper-Kähler condition on a real four-dimensional Riemannian metric have been completely classified:

- If the symmetry is tri-holomorphic, then the corresponding metric belongs to the Gibbons-Hawking class [10], and is given by a solution to the Laplace equation in three dimensions. The resulting Einstein-Weyl structures are trivial, and their mini-twistor space is $\mathbb{CP}^1$.

- Hyper-Kähler metrics with non-tri-holomorphic Killing vectors are given by solutions to the $SU(\infty)$ Toda equation [8]. The corresponding EW structures [32] are characterised by the existence of a shear-free, twist-free geodesic congruence [29]. Mini-twistor spaces are in this case equipped with a canonical divisor (two one-dimensional complex sub-manifolds) taking its values in $O(2)$ [19].

- Hyper-Kähler metrics with tri-holomorphic conformal symmetries yield a class of EW structures (called hyper-CR EW structures) characterised by the existence of a shear-free, divergence-free geodesic congruences [9]. The corresponding mini-twistor spaces are fibred over $\mathbb{CP}^1$.

- Hyper-Kähler metrics with non-tri-holomorphic, conformal symmetry (and the resulting EW structures) are given by solutions to a certain second order integrable equation in three dimensions [7]. This equation gives $SU(\infty)$-Toda and hyper-CR Einstein-Weyl structures as limiting cases. The EW structures arising from conformal, non-tri-holomorphic reductions are characterised by the existence of a shear-free geodesic congruence for which the twist is a constant multiple of the divergence [2].

The above list is not complete if one considers Hyper-Kähler metrics in $(+ + - -)$ signature. The existence of null structures of various kinds allows two additional types of symmetries:

- Hyper-Kähler metrics for which the self-dual part of a derivative of a Killing vector is null correspond to solutions of the dispersionless Kadomtsev-Petviashvili equation (2.9). The corresponding EW structures are characterised by the existence of a constant weighted vector. The mini-twistor spaces are such that the line bundle $\kappa^{-1/4}$ admits a section, where $\kappa$ is the canonical line bundle. The above statements have been proved in this paper.

- Hyper-Kähler metrics with conformal Killing vectors for which the self-dual part of a derivative of a conformal Killing vector is null.

The last possibility has not yet been investigated. The EW spaces will be given by a generalisation of the dKP equation. We intend to study this generalisation, and the corresponding EW geometries in a subsequent paper.

7 Outlook: a twistor theory for the full KP equation?

A combination of the dispersive limit of dKP with the twistor picture suggests a candidate for a twistor space for the full KP equation (2.6) (cf the similar proposal in [27]).

Let $x$ be a coordinate on a configuration space $Q$, and let $\lambda$ be the corresponding momentum. The extended six-dimensional phase-space $T^*(Q \times \mathbb{R}^2)$ is co-ordinated by $x^i = (x, y, t), \pi_i = (\lambda, H_2, H_3)$. Restrict the symplectic form $\Pi$ on $T^*(Q \times \mathbb{R}^2)$ to the four-dimensional correspondence space $\mathcal{F}^4$ obtained by putting $H_r := H_r(x^i, \lambda), r = 2, 3$. The (complexified) space $\mathcal{F}^4$ is foliated by sub-manifolds whose tangent vectors annihilate the symplectic form, which gives rise to a projection $p : \mathcal{F} \longrightarrow \mathcal{Z}$ such that $\Pi$ descends to a symplectic form on $\mathcal{Z}$. The two-dimensional complex manifold $\mathcal{Z}$ is the mini-twistor space for the extended configuration space $Q \times \mathbb{R}^2$ with its dKP Einstein-Weyl structure. It is believed that the Moyal quantisation of $T^*(Q \times \mathbb{R}^2)$ gives rise to the
full KP equation. This suggests the conjecture that there exists a correspondence between solutions to the full KP equation and the Moyal deformations of \( \mathcal{Z} \).

It will be instructive to compare this approach to the twistor constructions for the full KP equations described in [20], and §12.6 of [22].

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9 Appendix

Here we shall demonstrate (by an explicit calculation) that the dKP equation (2.9) is a reduction of the second heavenly equation by a Killing vector with a null self-dual derivative.

Let \( \Theta(z, t, q, y) \) satisfy (26).

\[
\Theta_{z y} - \Theta_{t q} + \Theta_{q y} \Theta_{yy} - \Theta_{q y}^2 = 0.
\]  

(9.53)

Then

\[
g = 2(dz dy + dq dt - \Theta_{q y} dz^2 - \Theta_{y y} dt^2 + 2 \Theta_{y q} dz dt)
\]

(9.54)

is a hyper-Kähler metric. All hyper-Kähler metrics can locally be put in the form (9.54).

Let \( K \) be a Killing vector such that \( dK_+ \wedge dK_+ = 0 \). There is no loss of generality [8] in choosing \( K = \partial_2 - 2z \partial_2 \), in which case \( dK_+ = 2 dt \wedge dz \).

The Killing equations yield \( (L_K \Theta)_{yy} = (L_K \Theta)_{q y} = 0, (L_K \Theta)_{y y} = 1 \). They integrate to

\[
\Theta = z q y + y A(z, t) + q B(z, t) + C(z, t) + G(y, t, q + z^2).
\]

(9.55)

The function \( C \) is pure gauge and can be set to zero without loss of generality. Imposing (9.53) gives two equations: the first is \( A_2 + B_2 = 2z^2 \), and we can deduce, without loss of generality, that \( A = z^2, B = -z^2 t \), and the second is

\[
-u - G_{uu} + G_{y y} G_{uu} - G_{y u}^2 = 0, \quad \text{where} \quad u = -(q + z^2).
\]

(9.56)

The last equation is equivalent to the dKP equation. To see this write (9.56) as a closed system

\[
dG = G_u du + G_t dt + G_y dy,
\]

(9.57)

Now rewrite the first equation as \( d(G - u G_u) = G_t dt + G_y dy - u dG_u \), and perform a Legendre transform

\[
x := G_u, \quad u = u(t, y, x), \quad H(t, y, x) := -G(t, y, u(t, y, x)) + xu(t, y, x).
\]

The relation \( dH = H_t dt + H_x dx + H_y dy \) implies \( H_t = -G_t, H_y = -G_y, H_x = u \). Equation (9.57) yields

\[-H_x dy \wedge dt \wedge dH_x + dx \wedge dy \wedge dH_x + dH_y \wedge dx \wedge dt = 0,
\]

which is equivalent to

\[
H_x H_y - H_x x + H_y y = 0.
\]

(9.58)

Taking the \( x \) derivative of the above equation and using \( H_x = u \) yields

\[
u_{xx} - u u_{xx} - u_x^2 = u_{yy}
\]
which is the dKP equation. To calculate the metric differentiate the relation \( x = G_u \) with respect to \( x \) and \( H_y = -G_y \) with respect to \( y \),

\[
1 = G_{uu} u_x, \quad 0 = G_{uy} + G_{uu} u_y, \quad 0 = G_{ut} + G_{uu} u_t, \quad G_{yy} = \frac{u_y^2}{u_x} + u u_x - u_t
\]

(we also used (9.58)). Therefore (from (9.55)) we have

\[
\Theta_{yy} = \frac{u_y^2}{u_x} + u u_x - u_t, \quad \Theta_{yq} = \frac{u_y}{u_x} + x, \quad \Theta_{qq} = \frac{1}{u_x}.
\]

The metric (9.54) in terms of \( u(x, y, t) \) is

\[
g = 2(-u_x dz dt + dz dy + 2 \frac{u_y}{u_x} dz dt - u_y dy dt - (u u_x + \frac{u_y^2}{u_x}) dt^2 - \frac{1}{u_x} dz^2)
\]

\[
= \frac{u_x}{2} (dz^2 - 4 dz dt - 4 u dt^2) - 2 \frac{u_y}{u_x} (dz - \frac{u_x dy}{2} - u_y dt)^2
\]

which is (2.27).

References


22