On the Spectrum of Cartesian Powers
for the Classical Automorphisms

Oleg N. Ageev


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OLEG N. AGEEV
Department of Mathematics, Moscow State Technical University, 107005 Moscow, Russia e-mail: ageev@mx.bmstu.ru

Abstract

In this article the following results are proved: The set of all essential spectral multiplicities of $T^{(n)} = T \times \cdots \times T$ is $\{n, n(n-1), \ldots, n!\}$ for Chacon's transformation $T$, or, equivalently, the operator $T^{(n)}$ has simple spectrum on the subspace $C_{n,m}(n)$ of all functions that are invariant with respect to permutations of the coordinates. As immediate corollary of this fact, we have the disjointness of all convolution powers of the spectral measure for Chacon's transformation. If $n=2$, then $T \times T$ has a homogeneous spectrum of multiplicity 2 on $\{\text{const}\}^2$; this is a solution of Rokhlin's problem for Chacon's transformation. Analogous statements are considered for other classical automorphisms.

1. We consider piecewise affine orientation and Lebesque measure $l$ preserving transformations on a half-open interval $[0, 1) = I$. A transformation and an operator on $L_2(I) : T f(x) = f(Tx)$ are often called automorphisms and denoted by the same symbol $T$.

For the automorphism $T$, by a tower we mean a collection of disjoint half-open intervals $\{J_i\}_{i=1}^q$, where $|J_i| = |J_j|$ $(i, j \in \{1, \ldots, q\})$ and $T$ maps linearly $J_i$ onto $J_{i+1}$ $(i = 1, \ldots, q-1)$. The set $J_i$ is called the $i$-th level of the tower, $q$ its height, $J_0$ its top.

We will use the following modification of the step by step definition of the classical transformation $T$ of Chacon. For $n = 1$ we have the tower $C_1 = \{[0, 2/3]\}$. At step $n$ we have the tower

$$C_n = \{J_1, \ldots, J_{q_n}\}$$

and $T$ is not defined on $J_{q_n} \cup [1 - 1/3^n, 1)$, where

$$q_n = \frac{3^n - 1}{2}, \quad |J_k| = \frac{2}{3^n}, \quad J_1 \cup \ldots \cup J_{q_n} = [0, 1 - \frac{1}{3^n})$$

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Divide $J_k$ into half-open intervals $J_{k,0}, J_{k,1}, J_{k,2}$ of the same length. Then subtowers
\[ \{J_{1,i}, J_{2,i}, \ldots, J_{q_n,i}\} \quad i = 0, 1, 2 \]
are obtained by cutting $C_q$. A tower $C_{q+1}$ is obtained by stacking the $(i+1)$th subtower on top of the $i$th subtower, and placing only one level between the first and second subtowers. Therefore,
\[ C_{q+1} = \{J_{1,0}, \ldots, J_{2n,0}, J_{1,1}, \ldots, J_{q_n,1}, J', J_{1,2}, \ldots, J_{q_n,2}\} \]
and $T$ is not defined on $J_{q_n,2} \cup [1 - 1/3^{q+1}, 1)$, where $J' = [1 - 1/3^q, 1 - 1/3^{q+1})$.
Hence at this step we have defined $T$ on a set $J_{q_n,0} \cup J_{q_n,1} \cup J'$.

Note papers [1],[2],[3] studied this transformation and its generalizations. Now we consider its spectral properties.

Let $T$ be a unitary operator on $L_2(l)$. We use the following notation: the symbol $\rightarrow$ denotes the weak operator convergence; $C_T(f_i : i \in A)$ is the smallest closed subspace of $L_2(l)$ that contains the vectors $T^n f_i$ for all $n \in \mathbb{Z}, i \in A$; $T^{(n)} = T \times T \times \cdots \times T$ ($n$ times); $\sigma_T(f)$ is the unique Borel measure on $T = \{\lambda \in C : |\lambda| = 1\}$ with $\sigma_T(f)(n) = (T^n f, f)$; $\sigma_T$ is the measure of maximal spectral type (i.e. the maximum of $\sigma_T(f)$); $\tau(c)$ is the system of canonical conditional measures for $\sigma_T^f = \sigma_T \times \cdots \times \sigma_T$ on the fiber $C = \{(\lambda_1, \ldots, \lambda_n) : \lambda_1 \cdots \lambda_n = c\}$; $\sigma^{(n)}$ is the image of the measure $\sigma^n$ under the map $(\lambda_1, \ldots, \lambda_n) \rightarrow \lambda_1 \cdots \lambda_n$; $M(T)$ is the set of essential spectral multiplicities of the operator $T$; $m(T) = \max \{k : k \in M(T)\}$.

We say that $T$ has simple spectrum if $m(T) = 1$. Then $L_2(l) = C_T(\varphi)$ for some $\varphi(x)$. Denote
\[ \varphi_k(x) = \varphi(x_1) \cdots \varphi(x_k) \]
$C_{ain}(n)$ is the space of functions from $L_2(l^n, l^m)$ which are invariant with respect to any permutations of the coordinates;
\[ H_{i_1, \ldots, i_k} = C_E(T^{j_1} \varphi_{i_1}(x_{i_1}) \cdots T^{j_k} \varphi_{i_k}(x_{i_k}) : j_i \in \mathbb{Z}), \]
where $E$ is the identity operator,
\[ \varphi_0(x) = \varphi(x) - \int x \varphi(x) dl. \]

It is well known that for Chacon's transformation $T$, $m(T) = 1$ and $\sigma_T = \sigma_T(\varphi) = \sigma_T^{(n)} + \delta$, where $\sigma_T = \sigma_T(\varphi_0)$ is continuous and supp $\delta = \{1\}$.
In [4] it was proved that $\sigma_T^{(m)} - \sigma_T^{(n)}$ if $m \neq n$. 
Theorem 1 If \( T \) is Chacon’s transformation, then for any \( n \),
\[
M(T^{(n)}) = \{n, n(n-1), \ldots, n!\} \quad \text{on} \quad \{\text{const}\}^\perp.
\]

Corollary 1 For \( n = 2 \) we have a specific example of a transformation with homogeneous spectrum of multiplicity 2. (It was found out for the first time in [5] and this is a solution of Rokhlin’s problem).

Corollary 2 If \( m \neq n \), then
\[
\sigma_T^{(m)} \neq \sigma_T^{(n)}.
\]

Proof. Let \( n > m \). It is clear that the measure of the maximal spectral type of the operator \( T^{(n)} \) on \( H^{[1, \ldots, k]} \) is \( \sigma_T^{(k)} \). From continuity of \( \sigma_T \), we have (see [5])
\[
M \left( T^{(n)} \right|_{H^{[1, \ldots, k]}} = M \left( T^{(k)} \right|_{H^{[1, \ldots, k]}} \subseteq k! N.
\]

If \( \sigma_T^{(m)} \neq \sigma_T^{(n)} \), then there exists a measure \( \nu \) which is their common component. Therefore mutually orthogonal spaces \( H^{[1, \ldots, m]} \) and \( H^{[1, \ldots, n]} \) have \( m! \) and \( n! \) pairwise orthogonal subspaces \( C_{T^{(n)}}(f_i) \) respectively, where \( \sigma_T^{(n)}(f_i) = \nu \). Thus \( m(T^{(n)}) \geq m! + n! \). This contradicts Theorem 1.

Remark 1 In fact Property (\( * \)) will be proved below for any operator (not necessarily transformation) satisfying the conditions of Lemma 1 and (\( * * \)), where \( \Theta_k \) has \( n - 1 \) various values.

Remark 2 Property (\( * \)) was proved in [5] for a generic \( T \) and it was exactly the answer on Katok’s question [6]. But it is often more simply to verify some property for the element than to prove the fact that it is from a generic set having this property.

Lemma 1 If a unitary operator \( T \) has simple continuous spectrum on \( \{\text{const}\}^\perp \) \( \{\text{const}\}^\perp \) is invariant under \( T \), then the following properties are equivalent:
1). The operator \( T^{(n)} \) has simple spectrum on \( C_{\text{sim}}(n) \).
2). \( M(T^{(n)}) = \{n, n(n-1), \ldots, n!\} \quad \text{on} \quad \{\text{const}\}^\perp \).
3). \( \nu_n(\varphi) \), modulo coordinate permutation, consists of single point for a.e.c with respect to the measure \( \sigma_T^{(n)} \).
4). \( C_{\text{sim}}(n) = C_{T^{(n)}}(\varphi) \).

Proof. It is convenient to consider a spectral representation of the operator \( T^{(n)} \). The operator \( T^{j_1} \times T^{j_2} \times \cdots \times T^{j_n} \) on \( L_2(I^n, I^n) \) is spectrally isomorphic to the operator \( \lambda_1^{j_1} \cdots \lambda_n^{j_n} \cdot E \) on \( L_2(T^n, \sigma_T^n) \). The spectral multiplicity function of the operator \( T^{(n)} = \lambda_1 \cdots \lambda_n \cdot E \) at the point \( c \) is equal to the number of the points in the support of \( \nu(\cdot | c) \). The function \( \varphi_n \) will correspond to 1, the function \( \varphi_k \) will correspond to a function with the support in the set \( \{\lambda_1, \ldots, \lambda_n\} : \lambda_{k+1} = \ldots = \lambda_n = 1 \} \). \( T^{(n)} \) has analogous representation on \( H^{[1, \ldots, k]} \) if we replace \( \sigma_T^n \) by \( \sigma_T^0 \).
Clearly, 
\[ C_{T^{(n)}}(\varphi_n) \subseteq C_{sim}(n), \quad \sigma_{T^{(n)}}(\varphi_n) = \sigma_{T}^{(m)}. \]

Thus 1) \(\Leftrightarrow\) 4).

By 3), it follows that any symmetric function \(\varphi_{sim}(\lambda_1, \ldots, \lambda_n)\) is equal to some function \(\tilde{\varphi}\) for \(a.e. (\lambda_1, \ldots, \lambda_n)\) with respect to the measure \(\sigma_T\), where \(\tilde{\varphi}\) is constant on any fiber \(C = \{(\lambda_1, \ldots, \lambda_n) : \lambda_1 \cdot \ldots \cdot \lambda_n = c\}\). Thus \(\varphi_{sim} \in C_{\tilde{\varphi}(1)}\) and this implies 4).

Suppose that 3) does not hold. Then it is not difficult to construct two symmetric functions \(f_1, f_2\) with disjoint supports such that
\[ \int_C f_i d\nu_n(\cdot | c) = \psi(c) \neq 0. \]

Hence, 
\[ \sigma_{T^{(n)}}(f_1) = \sigma_{T^{(n)}}(f_2). \]

Taking into account that \(C_{\tilde{\varphi}(1)}(f_1)\) and \(C_{\tilde{\varphi}(1)}(f_2)\) are orthogonal, we obtain that 1) does not hold.

The measure \(\sigma_T^{(k)}\) has a component equivalent to \(\sigma_T^{(k)} \times \delta^{n-k}\), where \(\text{supp} \delta = \{1\}\).

Therefore,
\[ \text{supp } \nu_n(\cdot | c) \cap \{(\lambda_1, \ldots, \lambda_n) : \lambda_{k+1} = \ldots = \lambda_n = 1\} \supseteq \text{supp } \nu_K(\cdot | c). \]

Hence from 3) it follows 3) for \(\nu_k(\cdot | c)\), where \(k = 1, 2, \ldots, n\). Therefore,
\[ M\left(T^{(n)} | H_{\{n, \ldots, k\}}\right) = M\left(T^{(k)} | H_{\{n, \ldots, k\}}\right) \subseteq M(T^{(k)}) \subseteq \{1, \ldots, k!\} \quad (2) \]

for any collection \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\). Using (1), we obtain
\[ M\left(T^{(n)} | H_{\{n, \ldots, i_k\}}\right) = \{k!\}. \quad (3) \]

Using (2), as in the proof of Corollary 2, we obtain
\[ \sigma_{T}^{(k)} = \sigma_{T}^{(m)}. \]

Therefore from 3) it follows 2).

From 2), using (1), we have (3) and the property \(\sigma_{T}^{(k)} = \sigma_{T}^{(m)}\) \((k \neq m)\).

We fix \(K \subseteq \{1, \ldots, n\}, \quad |K| = k\). Consider a measure \(\sigma_K = \sigma_1 \times \ldots \times \sigma_n\), where \(\sigma_j = \sigma_T^{(k)}\) for \(j \in K\) and \(\sigma_j = \delta\) otherwise. Let \(\nu_K(\cdot | c)\) be a system of canonical conditional measures for the measure \(\sigma_K\) on fibers \(C = \{(\lambda_1, \ldots, \lambda_n) : \lambda_1 \cdot \ldots \cdot \lambda_n = c\}\) \(\{d\sigma_K = \nu_K(\cdot | c) \cdot d\sigma_T^{(k)}\}\). Clearly,
\[ \sigma_T^{(k)} \{(\lambda_1, \ldots, \lambda_k) : \lambda_i = \lambda_j\} = 0 \quad (i \neq j). \]

Therefore, using (3), we have that \(a.e. c\) with respect to the measure \(\sigma_T^{(k)}\), the set \(\text{supp } \nu_K(\cdot | c)\) consists of \(k!\) points, obtained from \(\lambda(c)\) by permutations of the coordinates inside \(K\), where \(\lambda(c) = (\lambda_1(c), \ldots, \lambda_n(c))\), \(\lambda_j(c) \neq 1\) for \(j \in K\) and \(\lambda_j(c) = 1\) otherwise.
Using 
\[ \sigma_T^{(k)} - \sigma_T^{(m)} \quad (k \neq m), \]
we have 
\[ \nu_n (\cdot |c) \sim \sum_{k:|k|=k} \nu_K (\cdot |c) \]
for a.e. c with respect to the measure \( \sigma_T^{(k)} \). Hence \( \text{supp} \nu_n (\cdot |c) \) contains only the points obtained from \( \lambda(c) \) by permutation of \( n \) coordinates for a.e. c with respect to the measure \( \sigma_T^{(k)} \). Taking into account that
\[ \sigma_T^{(n)} \sim \delta + \sum_{k=1}^n \sigma_T^{(k)}, \]
we conclude that 3) holds. Lemma 1 is proved. \( \square \)

**Remark 3.** If Property (3) holds for some \( n \), then
\[ \sigma_T^{(m)} - \sigma_T^{(k)} \quad (k \neq m, \quad k, m, \leq n). \]
Indeed, \( C_{sim}(n) \) is a direct sum of invariant subspaces \( C_i \) \( (i = 0, n) \), where \( \sigma_T^{(i)} \) is the measure of maximal spectral type of the operator \( T^{(n)} \) on \( C_i \) \( (i = 1, n) \).

2. To prove Theorem 1, we need the following fact.

**Lemma 2** For any \( p \in \mathbb{Z}_+ \),
\[ T^{2n+q_{n-p}} \to \frac{(3^p+1)E + 2 \cdot (3^p+1)T^{4\frac{1}{2}}}{4 \cdot 3^p+1} \quad \text{as} \quad n \to \infty. \]

The proof is by direct calculations as the proof of the well-known relation
\[ T_n \to 0.5(E + T) \quad \text{as} \quad n \to \infty. \]

**Proof of Theorem 1.** Consider the operator \( \hat{T}^{(n)} \) (as in the proof of Lemma 1). Fix a "generic" with respect to measure \( \sigma_T^{(n)} \) fiber \( C = \{ (\lambda_1, ..., \lambda_n) : \lambda_1, ..., \lambda_n = c \} \). Now we prove Property 3) in Lemma 1. Put
\[ P_n(\lambda_1, ..., \lambda_n, \Theta) = (1 + \Theta \lambda_1 + \lambda_1^2) \cdot \ldots \cdot (1 + \Theta \lambda_n + \lambda_n^2). \]
If \( T^{ki} \to Q \) then \( T^{(n)} \to Q^{(n)} \). Then for any \( f \)
\[ Q^{(n)} f \in C_{\hat{T}^{(n)}}(f). \]
Using Lemma 2, we have
\[ P_n(\lambda_1, ..., \lambda_n, \Theta) \in C_{\hat{T}^{(n)}}(1) \]
for different \( \Theta_p \in \mathbb{R} \), i.e. \( P_n(\lambda_1, ..., \lambda_n, \Theta_p) \) are constants on the fiber \( C \) for a.e. \( (\lambda_1, ..., \lambda_n) \) with respect to the measure \( \nu_n (\cdot |c) \). The function \( P_n(\lambda_1, ..., \lambda_n, \Theta) \) is a polynomial of degree \( n \) in the variable \( \Theta \) with constant
leading coefficient on the fiber $C$. Since the number of various $\Theta$ at least $n-1$, this polynomial is uniquely defined, i.e., the rest of its coefficients are constants for a.e. $(\lambda_1, \ldots, \lambda_n)$ with respect to the measure $\nu_0(\cdot|c)$ on the fiber $C$. Therefore it has only one collection of zeros for a.e. $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \cdot \ldots \cdot \lambda_n = c$. Using the equality

$$P_0(\lambda_1, \ldots, \lambda_n, \Theta) = \lambda_1 \cdot \ldots \cdot \lambda_n \cdot (\Theta + \lambda_1 + \tilde{\lambda}_1) \cdot \ldots \cdot (\Theta + \lambda_n + \tilde{\lambda}_n),$$

we have a system $\lambda_i + \tilde{\lambda}_i = c_i$, modulo coordinate permutation of $(c_1, \ldots, c_n)$, for a.e. $(\lambda_1, \ldots, \lambda_n)$ on the fiber $C$.

Clearly, $c_i \in \mathbb{R}$. Hence $\lambda_i \in \{\lambda_i^{(0)}, \tilde{\lambda}_i^{(0)}\}$, modulo coordinate permutation of $(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})$, for a.e. $(\lambda_1, \ldots, \lambda_n)$ on the fiber $C$.

Therefore supp $\nu_0(\cdot|c)$ is contained in the finite set of points, obtained by permutations of the coordinates of $A = \{(\lambda_1, \ldots, \lambda_n) : \lambda_i \in \{\lambda_i^{(0)}, \tilde{\lambda}_i^{(0)}\}, i = 1, n\}$.

Since $\nu_0(\cdot|c)$ is symmetric, it is sufficient to prove that $\nu_0(\cdot|c)(a) > 0$, only one element $a$ from $A$.

Suppose that $\nu_0(\cdot|c)(a_i) > 0$, where $a_i \in A, (i = 1, 2, a_1 \neq a_2)$. Since $a_i \in C$,

$$a_i \in \{(\lambda_1, \ldots, \lambda_n) : \lambda_j, \ldots, \lambda_{j_k} \pm \pm 1\} = B_{\pm} \{j_1, \ldots, j_k\}$$

for some collection $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, where $k \geq 2$.

Since the Borel measure $\sigma_T$ has an atom only at the point 1, it is not difficult to conclude that

$$\sigma_T^{(0)}(B_{\pm}(j_1, \ldots, j_k) \setminus \{(1, \ldots, 1)\}) = 0.$$

Therefore on the generic fiber $C$ we have

$$\nu_0(\cdot|c)(B_{\pm}(j_1, \ldots, j_k) \cap C \setminus \{(1, \ldots, 1)\}) = 0.$$

We obtain a contradiction with our assumptions. Theorem 1 is proved.

3. The classical generalization of Chacon’s transformation is the transformation of Rudolph - del Junco [7]. Here each tower $C_{n+1}$ is obtained by cutting $C_n$ into $2^{n+1}$ subtowers and stacking the $(i+1)$th subtower on top of the $i$th, placing one level between the $(2^n)$th and $(2^n + 1)$th subtowers. In this case Property (1) was proved in [5], since

$$T_k(\Theta) = \Theta E + (1 - \Theta)T$$

for any $\Theta \in [0, 1]$.

4. Another natural generalization of Chacon’s transformation is the following staircase construction. Fix $r \in \{3, 4, \ldots\}$. Each tower $C_{n+1}$ is obtained by cutting $C_n$ into $r$ subtowers and then stacking the $(i+1)$th subtower on top of the $i$th, placing $i - 1$ levels between them.

**Theorem 2** For any staircase $T$, and for any $n$,

$$M(T^{(n)}) = \{n, n(n-1), \ldots, n!\} \cdot \{\text{const}\}^4.$$
REFERENCES


