On Cocycles with Values in the Group $SU(2)$

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Abstract

In this paper we introduce the notion of degree for $C^1$–cocycles over irrational rotations on the circle with values in the group $SU(2)$. It is shown that if a $C^1$–cocycle $\varphi : \mathbb{T} \to SU(2)$ over an irrational rotation by $\alpha$ has nonzero degree, then the skew product

$$\mathbb{T} \times SU(2) \ni (x, g) \mapsto (x + \alpha, g \varphi(x)) \in \mathbb{T} \times SU(2)$$

is not ergodic and the group of essential values of $\varphi$ is equal to the maximal Abelian subgroup of $SU(2)$. Moreover, if $\varphi$ is of class $C^2$ (with some additional assumptions) the Lebesgue component in the spectrum of the skew product has countable multiplicity. Possible values of degree are discussed, too.

1 Introduction

Assume that $T : (X, \mathcal{B}, \lambda) \to (X, \mathcal{B}, \lambda)$ is an ergodic measure–preserving automorphism of standard Borel space. Let $G$ be a compact Lie group, $\mu$ its Haar measure. For a given measurable function $\varphi : X \to G$ we study spectral properties of the measure–preserving automorphism of $X \times G$ (called skew product) defined by

$$T_\varphi : (X \times G, \lambda \otimes \mu) \to (X \times G, \lambda \otimes \mu), \quad T_\varphi(x, g) = (Tx, g \varphi(x)).$$

A measurable function $\varphi : X \to G$ determines a measurable cocycle over the automorphism $T$ given by

$$\varphi^{[n]}(x) = \begin{cases} 
\varphi(x)\varphi(Tx)\ldots\varphi(T^{n-1}x) & \text{for } n > 0 \\
\varepsilon & \text{for } n = 0 \\
(\varphi(T^{-1}x)\varphi(T^{-2}x)\ldots\varphi(T^{-n}x))^{-1} & \text{for } n < 0,
\end{cases}$$

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which we will identify with the function \( \varphi \). Then \( T_\varphi^n(x, g) = (T x, g \varphi^{(n)}(x)) \) for any integer \( n \). Two cocycles \( \varphi, \psi : X \to G \) are cohomologous if there exists a measurable map \( p : X \to G \) such that

\[
\varphi(x) = p(x)^{-1} \psi(x) p(Tx).
\]

In this case, \( p \) will be called a transfer function. If \( \varphi \) and \( \psi \) are cohomologous, then the map \( (x, g) \mapsto (x, p(x)g) \) establishes a metrical isomorphism of \( T_\varphi \) and \( T_\psi \).

By \( T \) we will mean the circle group \( \{ z \in \mathbb{C}; |z| = 1 \} \) which most often will be treated as the group \( \mathbb{R}/\mathbb{Z} \); \( \lambda \) will denote Lebesgue measure on \( T \). We will identify functions on \( T \) with periodic of period 1 functions on \( \mathbb{R} \). Assume that \( \alpha \in T \) is irrational. We will treat the case where \( T \) is the ergodic rotation on \( T \) given by \( Tx = x + \alpha \).

In the case where \( G \) is the circle and \( \varphi \) is a smooth cocycle, spectral properties of \( T_\varphi \) depend on the topological degree \( d(\varphi) \) of \( \varphi \). For example, in [5] A. Iwanik, M. Lemańczyk, D. Rudolph have proved that if \( \varphi \) is a \( C^2 \)-cocycle with \( d(\varphi) \neq 0 \), then \( T_\varphi \) is ergodic and it has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. On the other hand, in [3] P. Gabriel, M. Lemańczyk, P. Liardet have proved that if \( \varphi \) is absolutely continuous with \( d(\varphi) = 0 \), then \( T_\varphi \) has singular spectrum.

The aim of this paper is to find a spectral equivalent of topological degree in case \( G = SU(2) \).

## 2 Degree of cocycle

In this section we introduce the notion of degree in case \( G = SU(2) \). For a given matrix \( A = [a_{ij}]_{i,j=1,...,d} \in M_d(\mathbb{C}) \) define \( \|A\| = \sqrt{\frac{1}{d} \sum_{i,j=1}^d |a_{ij}|^2} \). Observe that if \( A \) is an element of the Lie algebra \( \mathfrak{su}(2) \), i.e.

\[
A = \begin{bmatrix}
  ia & b + ic \\
  -b + ic & -ia
\end{bmatrix},
\]

where \( a, b, c \in \mathbb{R} \), then \( \|A\| = \sqrt{\det A} \). Moreover, if \( B \) is an element of the group \( SU(2) \), i.e.

\[
A = \begin{bmatrix}
  z_1 & z_2 \\
  -\overline{z_2} & \overline{z_1}
\end{bmatrix},
\]

where \( z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \), then \( \text{Ad}_B A = BAB^{-1} \in \mathfrak{su}(2) \) and \( \|\text{Ad}_B A\| = \|A\| \).

Consider the scalar product of \( \mathfrak{su}(2) \) given by

\[
\langle A, B \rangle = -\frac{1}{8} \text{tr}(\text{ad} A \circ \text{ad} B).
\]
Then \( \|A\| = \sqrt{\langle A, A \rangle} \). By \( L^2(X, \mathfrak{su}(2)) \) we mean the space of all functions \( f : X \to \mathfrak{su}(2) \) such that
\[
\|f\|_{L^2} = \sqrt{\int_X \|f(x)\|^2 dx} < \infty.
\]
For two \( f_1, f_2 \in L^2(X, \mathfrak{su}(2)) \) set
\[
\langle f_1, f_2 \rangle_{L^2} = \int_X \langle f_1(x), f_2(x) \rangle dx.
\]
The space \( L^2(X, \mathfrak{su}(2)) \) endowed with the above scalar product is a Hilbert space.

By \( L^1(X, \mathfrak{su}(2)) \) we mean the space of all functions \( f : X \to \mathfrak{su}(2) \) such that
\[
\|f\|_{L^1} = \int_X \|f(x)\| dx < \infty.
\]
The space \( L^1(X, \mathfrak{su}(2)) \) endowed with the norm \( \|f\|_{L^1} \) is a Banach space.

For a given measurable cocycle \( \varphi : \mathbb{T} \to SU(2) \) consider the unitary operator
\[
U : L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2)), \quad Uf(x) = \text{Ad}_{\varphi(x)}f(Tx).
\]
Then \( U^n f(x) = \text{Ad}_{\varphi^n(x)}f(T^n x) \) for any integer \( n \).

**Lemma 2.1** There exists an operator \( P : L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2)) \) such that
\[
\frac{1}{n} \sum_{j=0}^{n-1} U^j f \to Pf \quad \text{in} \quad L^2(\mathbb{T}, \mathfrak{su}(2))
\]
for any \( f \in L^2(\mathbb{T}, \mathfrak{su}(2)) \) and \( U \circ P = P \). Moreover, \( \|Pf\| \) is constant \( \lambda \)-a.e..

**Proof.** First claim of the lemma follows from the von Neumann ergodic theorem. Since \( U \circ P = P \), we have \( \text{Ad}_{\varphi(x)}Pf(Tx) = Pf(x) \), for \( \lambda \)-a.e. \( x \in \mathbb{T} \). It follows that \( \|Pf(Tx)\| = \|Pf(x)\| \), for \( \lambda \)-a.e. \( x \in \mathbb{T} \). Hence \( \|Pf(x)\| \) is constant, for \( \lambda \)-a.e. \( x \in \mathbb{T} \), by the ergodicity of \( T \). \( \square \)

**Lemma 2.2** For every \( f \in L^2(\mathbb{T}, \mathfrak{su}(2)) \) the sequence \( \frac{1}{n} \sum_{j=0}^{n-1} U^j f \) converges \( \lambda \)-almost everywhere.

**Proof.** Let \( \tilde{f} \in L^2(\mathbb{T} \times SU(2), \mathfrak{su}(2)) \) be given by \( \tilde{f}(x, g) = \text{Ad}_g f(x) \). Then
\[
\tilde{f}(T^n \varphi(x, g)) = \text{Ad}_g(U^n f(x))
\]
for any integer \( n \). By the Birkhoff ergodic theorem, the sequence
\[
\frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(T^n \varphi(x, g)) = \text{Ad}_g\left( \frac{1}{n} \sum_{j=0}^{n-1} U^j f(x) \right)
\]
3
converges for \( \lambda \otimes \mu \)-a.e. \((x, g) \in \mathbb{T} \times SU(2)\). Hence there exists \(g \in SU(2)\) such that \(\text{Ad}_g(\frac{1}{n} \sum_{j=0}^{n-1} U^j f(x))\) converges for \(\lambda\)-a.e. \(x \in \mathbb{T}\), and the proof is complete. 

Recall that, if a function \(\varphi : \mathbb{T} \to SU(2)\) is of class \(C^1\), then \(D\varphi(x)\varphi(x)^{-1} \in \mathfrak{su}(2)\) for every \(x \in \mathbb{T}\).

**Lemma 2.3** For every \(C^1\)-cocycle \(\varphi : \mathbb{T} \to SU(2)\), there exists \(\psi \in L^2(\mathbb{T}, \mathfrak{su}(2))\) such that
\[
\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \quad \text{in} \quad L^2(\mathbb{T}, \mathfrak{su}(2)) \quad \text{and} \quad \lambda\text{-almost everywhere.}
\]
Moreover, \(\|\psi\|\) is a constant function \(\lambda\)-a.e. and \(\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)\) for \(\lambda\)-a.e. \(x \in \mathbb{T}\).

**Proof.** Since
\[
D\varphi^{(n)}(x) = \sum_{j=0}^{n-1} \varphi(x) \ldots \varphi(T^{j-1}x) D\varphi(T^jx) \varphi(T^{j+1}x) \ldots \varphi(T^{n-1}x),
\]
we have
\[
D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} = \sum_{j=0}^{n-1} \varphi(x) \ldots \varphi(T^{j-1}x) D\varphi(T^jx)\varphi(T^jx)^{-1} \varphi(T^{j+1}x)^{-1} \ldots \varphi(x)^{-1}
\]
\[
= \sum_{j=0}^{n-1} \varphi(j)(x) D\varphi(T^jx)\varphi(T^jx)^{-1}(\varphi(j)(x))^{-1}
\]
\[
= \sum_{j=0}^{n-1} U^j(D\varphi \varphi^{-1})(x),
\]
where \(U\) is the unitary operator given by (1). Applying Lemmas 2.1 and 2.2, we get \(\psi = P(D\varphi \varphi^{-1})\), and the proof is complete.

**Definition 1** The number \(\|\psi\|\) will be called the degree of the cocycle \(\varphi\) and denoted by \(d(\varphi)\).

Lemma 2.3 shows that
\[
\frac{1}{n} \|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1} \to d(\varphi).
\]
On the other hand, \(\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1}\) is the length of the curve \(\varphi^{(n)}\). Geometrically speaking, the degree of \(\varphi\) is the limit of length(\(\varphi^{(n)}\))/\(n\).

A measurable cocycle \(\delta : \mathbb{T} \to SU(2)\) is called diagonal if there exists a measurable function \(\gamma : \mathbb{T} \to \mathbb{T}\) such that
\[
\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \frac{1}{\gamma(x)} \end{bmatrix}.
\]
Theorem 2.4 Suppose that $\varphi : \mathbb{T} \to SU(2)$ is a $C^1$-cocycle with $d(\varphi) \neq 0$. Then $\varphi$ is cohomologous to a diagonal cocycle.

Proof. For every nonzero $A \in \mathfrak{su}(2)$ there exists $B_A \in SU(2)$ such that

$$B_A A(B_A)^{-1} = \begin{bmatrix} i \|A\| & 0 \\ 0 & -i \|A\| \end{bmatrix}.$$  

Indeed, if $A = \begin{bmatrix} ia & b + ic \\ -b + ic & -ia \end{bmatrix}$, then we can take

$$B_A = \begin{cases} \begin{bmatrix} -i \sqrt{\frac{1}{2} \frac{1}{|a|^2} \frac{b + ic}{a}} & -i \sqrt{\frac{1}{2} \frac{1}{|a|^2} \frac{b - ic}{a}} \\ i \sqrt{\frac{1}{2} \frac{1}{|a|^2} \frac{b - ic}{a}} & i \sqrt{\frac{1}{2} \frac{1}{|a|^2} \frac{b + ic}{a}} \end{bmatrix} & \text{if } |a| \neq |A| \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{if } a = -|A| \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } a = |A|. \end{cases}$$

Set $p(x) = B_{\psi(x)}$. Then $p : \mathbb{T} \to SU(2)$ is a measurable function and

$$\psi(x) = p(x)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x).$$

Since $\varphi(x) \psi(Tx) \varphi(x)^{-1} = \psi(x)$, we have

$$\varphi(x)p(Tx)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(Tx)\varphi(x)^{-1} = p(x)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x).$$

Hence

$$p(x)\varphi(x)p(Tx)^{-1} \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} = \begin{bmatrix} i d(\varphi) & 0 \\ 0 & -i d(\varphi) \end{bmatrix} p(x)\varphi(x)p(Tx)^{-1}.$$  

Since $d(\varphi) \neq 0$, we see that the cocycle $\delta : \mathbb{T} \to SU(2)$ defined by $\delta(x) = p(x)\varphi(x)p(Tx)^{-1}$ is diagonal. \qed

For a given $C^1$-cocycle $\varphi : \mathbb{T} \to SU(2)$ with nonzero degree let $\gamma = \gamma(\varphi) : \mathbb{T} \to \mathbb{T}$ be a measurable cocycle such that the cocycles $\varphi$ and $\begin{bmatrix} \gamma & 0 \\ 0 & (\gamma)^{-1} \end{bmatrix}$ are cohomologous. It is easy to check that the choice of $\gamma$ is unique up to a measurable cohomology with values in the circle and inverse.

Theorem 2.4 shows that if $d(\varphi) \neq 0$, then the skew product $T_\varphi$ is metrically isomorphic to a skew product of an irrational rotation on the circle and a diagonal cocycle. It follows that $T_\varphi$ is not ergodic. However, in the next sections we show that if $d(\varphi) \neq 0$, then $\varphi$ is not cohomologous to a constant cocycle. Moreover,
the skew product $T_\gamma : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable. We prove also that (with some additional assumptions on $\varphi$) the Lebesgue component in the spectrum of $T_\gamma$ has countable multiplicity. It follows that if $d(\varphi) \neq 0$, then:

- all ergodic components of $T_\varphi$ are metrically isomorphic to $T_\gamma$,
- the spectrum of $T_\varphi$ consists of two parts: discrete and mixing,
- (with some additional assumptions on $\varphi$) the Lebesgue component in the spectrum of $T_\varphi$ has countable multiplicity.

In case $G = \mathbb{T}$ the topological degree of each $C^1$-cocycle is an integer number. An important question is: what can one say on values of degree in case $G = SU(2)$? If a cocycle $\varphi$ is cohomologous to a diagonal cocycle via a smooth transfer function, then $d(\varphi) \in 2\pi \mathbb{N}_0 = 2\pi (\mathbb{N} \cup \{0\})$. We call a function $f : \mathbb{T} \to SU(2)$ absolutely continuous if $f_{ij} : \mathbb{T} \to \mathbb{C}$ is absolutely continuous for $i,j = 1,2$. Suppose that $\varphi$ is cohomologous to a diagonal cocycle via an absolutely continuous transfer function. Then $\varphi$ can be represented as $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$, where $\delta,p : \mathbb{T} \to SU(2)$ are absolutely continuous and $\delta$ is diagonal. Since $\varphi^{(n)}(x) = p(x)^{-1}\delta^{(n)}(x)p(T^nx)$, we have

$$
\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} = \frac{1}{n}(-p(x)^{-1}Dp(x) + \varphi^{(n)}(x)p(T^nx)^{-1}Dp(T^nx)(\varphi^{(n)}(x))^{-1}
+p(x)^{-1}D\delta^{(n)}(x)(\delta^{(n)}(x))^{-1}p(x)).
$$

On the other hand, $\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix}$, where $\gamma : \mathbb{T} \to \mathbb{T}$ is an absolutely continuous cocycle of the form $\gamma(x) = \exp 2\pi i(\hat{\gamma}(x) + kx)$, where $k$ is the topological degree of $\gamma$ and $\hat{\gamma} : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function. Then

$$
\frac{1}{n}D\gamma^{(n)}(x)(\gamma^{(n)}(x))^{-1} = 2\pi i \left( \frac{1}{n} \sum_{j=0}^{n-1} D\hat{\gamma}(T^jx) + k \right) \to 2\pi ik
$$
in $L^1(\mathbb{T},\mathbb{R})$, by the Birkhoff ergodic theorem. It follows that

$$
\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \to p(x)^{-1} \begin{bmatrix} 2\pi ik & 0 \\ 0 & -2\pi ik \end{bmatrix} p(x)
$$
in $L^1(\mathbb{T},\mathfrak{su}(2))$. Hence $d(\varphi) = 2\pi |d(\gamma)| \in 2\pi \mathbb{N}_0$.

In Section 7 it is shown that if $\alpha$ is the golden ratio, then the degree of every $C^2$-cocycle belongs to $2\pi \mathbb{N}_0$, too.
3 Notation and facts from spectral theory

Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$. By the cyclic space generated by $f \in \mathcal{H}$ we mean the space $\mathbb{Z}(f) = \text{span}\{U^n f; n \in \mathbb{Z}\}$. By the spectral measure $\sigma_f$ of $f$ we mean a Borel measure on $T$ determined by the equalities

$$\hat{\sigma}_f(n) = \int_T e^{2\pi i n x} d\sigma_f(x) = (U^n f, f)$$

for $n \in \mathbb{Z}$. Recall that there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \ldots.$$

Moreover, for any sequence $\{f'_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ satisfying (2) we have $\sigma_{f_1} \equiv \sigma_{f'_1}, \sigma_{f_2} \equiv \sigma_{f'_2}, \ldots$. The above decompositions of $\mathcal{H}$ are called spectral decompositions of $U$.

The spectral type of $\sigma_{f_1}$ (the equivalence class of measures) will be called the maximal spectral type of $U$. We say that $U$ has Lebesgue (continuous singular, discrete) spectrum if $\sigma_{f_1}$ is equivalent to Lebesgue (continuous singular, discrete) measure on the circle. An operator $U$ is called mixing if

$$\hat{\sigma}_f(n) = (U^n f, f) \to 0$$

for any $f \in \mathcal{H}$. We say that the Lebesgue component in the spectrum of $U$ has countable multiplicity if $\lambda \ll \sigma_{f_n}$ for every natural $n$ or equivalently if there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that the cyclic spaces $\mathbb{Z}(g_n)$ are pairwise orthogonal and $\sigma_{g_n} \equiv \lambda$ for every natural $n$.

For a skew product $T_\varphi$ consider its Koopman operator

$$U_{T_\varphi} : L^2(\mathbb{T} \times G, \lambda \otimes \mu) \to L^2(\mathbb{T} \times G, \lambda \otimes \mu), \quad U_{T_\varphi} f(x, g) = f(Tx, g \varphi(x)).$$

Denote by $\tilde{G}$ the set of all equivalence classes of unitary irreducible representations of the group $G$. For any unitary irreducible representation $\Pi : G \to \mathcal{U}(\mathcal{H}_\Pi)$ by $\{\Pi_{ij}\}_{i,j=1}^{d_\Pi}$ we mean the matrix elements of $\Pi$, where $d_\Pi = \dim \mathcal{H}_\Pi$. Let us decompose

$$L^2(\mathbb{T} \times G) = \bigoplus_{\Pi \in \tilde{G}} \bigoplus_{i=1}^{d_\Pi} \mathcal{H}_i^{\Pi},$$

where

$$\mathcal{H}_i^{\Pi} = \{ \sum_{j=1}^{d_\varphi} \Pi_{ij}(g)f_j(x); f_j \in L^2(\mathbb{T}, \lambda), j = 1, \ldots, d_\varphi \} \simeq L^2(\mathbb{T}, \lambda) \oplus \ldots \oplus L^2(\mathbb{T}, \lambda).$$

Observe that $\mathcal{H}_i^{\Pi}$ is a closed $U_{T_\varphi}$-invariant subspace of $L^2(\mathbb{T} \times G)$ and

$$U_{T_\varphi}^{n} \left( \sum_{j=1}^{d_\varphi} \Pi_{ij}(g)f_j(x) \right) = \sum_{j,k=1}^{d_\varphi} \Pi_{ik}(g)\Pi_{kj}(\varphi^n(x))f_j(T^n x).$$
Consider the unitary operator $M_i^I : \mathcal{H}_i^I \to \mathcal{H}_i^I$ given by

$$M_i^I \left( \sum_{j=1}^{d} \Pi_{ij}(g) f_j(x) \right) = \sum_{j=1}^{d} e^{2\pi i x} \Pi_{ij}(g) f_j(x).$$

Then

$$U_{T_i}^n M_i^I f = e^{2\pi i n} M_i^I U_{T_i}^n f$$

for any $f \in \mathcal{H}_i^I$. It follows that

$$\int_T e^{2\pi i x} d\sigma_{M_i^I} f(x) = (U_{T_i}^n M_i^I f, M_i^I f) = e^{2\pi i n} (U_{T_i}^n f, f) = \int_T e^{2\pi i x} d(T^* \sigma f)(x)$$

for any $f \in \mathcal{H}_i^I$. Hence $\sigma_{M_i^I} f = T^* \sigma f$.

**Lemma 3.1** For every $\Pi \in \tilde{G}$ and $i = 1, \ldots, d_\Pi$ if the operator $U_{T_i} : \mathcal{H}_i^I \to \mathcal{H}_i^I$ has absolutely continuous spectrum, then it has Lebesgue spectrum of uniform multiplicity.

**Proof.** Let $\mathcal{H}_i^I = \bigoplus_{n=1}^\infty \mathbb{Z}(f_n)$ be a spectral decomposition. Then

$$\mathcal{H}_i^I = (M_i^I)^m \mathcal{H}_i^I = \bigoplus_{n=1}^\infty \mathbb{Z}((M_i^I)^m f_n)$$

is a spectral decomposition for any integer $m$. Therefore $\sigma_{f_n} \equiv \sigma_{(M_i^I)^m f_n} \ll \lambda$ for every natural $n$ and integer $m$. Suppose that there exists a Borel set $A \subset \mathbb{T}$ such that $\sigma_{f_n}(A) = 0$ and $\lambda(A) > 0$. Then

$$\sigma_{f_n}(\bigcup_{m \in \mathbb{Z}} T^m A) = 0 \quad \text{and} \quad \lambda(\bigcup_{m \in \mathbb{Z}} T^m A) = 1,$$

by the ergodicity of $T$. It follows that $\sigma_{f_n} \equiv \lambda$ or $\sigma_{f_n} = 0$ for every natural $n$. ■

**Lemma 3.2** If

$$\sum_{n \in \mathbb{Z}} \left| \int_T \Pi_{ij}(\varphi^{(n)}(x)) dx \right|^2 < \infty$$

for $j = 1, \ldots, d_\Pi$, then $U_{T_i}$ has Lebesgue spectrum of uniform multiplicity on $\mathcal{H}_i^I$ for $i = 1, \ldots, d_\Pi$.

**Proof.** Fix $1 \leq i \leq d_\Pi$. Note that

$$\langle U_{T_i}^n \Pi_{ij}, \Pi_{ij} \rangle = \sum_{k=1}^{d_\Pi} \int_{\mathbb{T}} \int_{\mathbb{G}} \langle \Pi_{ik}(g) \Pi_{kj}(\varphi^{(n)}(x)), \Pi_{ij}(g) \rangle dg dx = \frac{1}{d_\Pi} \int_{\mathbb{T}} \Pi_{ij}(\varphi^{(n)}(x)) dx.$$
Since
\[ \sum_{n \in \mathbb{Z}} |\langle U_{T \varphi}^n \Pi_{ij}, \Pi_{ij} \rangle|^2 < \infty, \]
we have \( \sigma_{\Pi, ij} \ll \lambda \) for \( j = 1, \ldots, d_\Pi \). From (3) we get \( \sigma_{(M^\Pi)^j \Pi, ij} \ll \lambda \) for any integer \( m \). Since \( \{f \in \mathcal{H}^\Pi: \sigma_f \ll \lambda \} \) is a closed linear subspace of \( L^2(T \times G) \) and the set \( \{(M^\Pi)^m \Pi; j = 1, \ldots, d_\Pi, m \in \mathbb{Z}\} \) generates the space \( \mathcal{H}^\Pi \), it follows that \( U_{T \varphi} \) has absolutely continuous spectrum on \( \mathcal{H}^\Pi \). By Lemma 3.1, \( U_{T \varphi} \) has Lebesgue spectrum of uniform multiplicity on \( \mathcal{H}^\Pi \). ■

**Corollary 3.3** For every \( \Pi \in \hat{G} \), if
\[ \sum_{n \in \mathbb{Z}} \| \int_T \Pi(\varphi^{(n)}(x)) dx \|^2 < \infty, \]
then \( U_{T \varphi} \) has Lebesgue spectrum of uniform multiplicity on \( \bigoplus_{i=1}^{d_\Pi} \mathcal{H}^\Pi_i \). ■

Similarly one can prove the following result.

**Theorem 3.4** For every \( \Pi \in \hat{G} \), if
\[ \lim_{n \to \infty} \int_T \Pi(\varphi^{(n)}(x)) dx = 0, \]
then \( U_{T \varphi} \) is mixing on \( \bigoplus_{i=1}^{d_\Pi} \mathcal{H}^\Pi_i \). ■

**4 Representations of \( SU(2) \)**

In this section some basic information about the theory of representations of the group \( SU(2) \) are presented. By \( \mathcal{P}_k \) we mean the linear space of all homogeneous polynomials of degree \( k \in \mathbb{N}_0 \) in two variables \( u \) and \( v \). Denote by \( \Pi_k \) the representation of the group \( SU(2) \) in \( \mathcal{P}_k \) given by
\[ \Pi_k \left( \begin{bmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix} \right) f(u, v) = f(z_1 u - \overline{z_2} v, z_2 u + \overline{z_1} v). \]
Of course, all \( \Pi_k \) are unitary (under an appropriate inner product on \( \mathcal{P}_k \)) and the family \( \{\Pi_0, \Pi_1, \Pi_2, \ldots\} \) is a complete family of continuous unitary irreducible representations of \( SU(2) \). In the Lie algebra \( \mathfrak{su}(2) \), we choose the following basis:
\[ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]
Let $V_k$ be a $k+1$-dimension linear space. For every natural $k$ there exists a basis $v_0, \ldots, v_k$ of $V_k$ such that the corresponding representation $\Pi_k^*$ of $\mathfrak{su}(2)$ in $V_k$ has the following form:

$$
\begin{align*}
\Pi_k^*(e)v_i &= i(k-i+1)v_{i-1} \\
\Pi_k^*(f)v_i &= v_{i+1} \\
\Pi_k^*(h)v_i &= (k-2i)v_i
\end{align*}
$$

for $i = 0, \ldots, k$. Then

$$
\|A\| \leq \|\Pi_k^*(A)\| \leq k^2\|A\|
$$

for any $A \in \mathfrak{su}(2)$.

**Lemma 4.1**

$$
\det \Pi_{2k-1}^*(A) = ((2k-1)!!)^2(\det A)^k
$$

for any $A \in \mathfrak{su}(2)$ and $k \in \mathbb{N}$.

**Proof.** For every $A \in \mathfrak{su}(2)$ there exists $B \in SU(2)$ and $d \in \mathbb{R}$ such that $A = \text{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}$. Then

$$
\Pi_{2k-1}^*(A) = \Pi_{2k-1}^*(\text{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}) = \text{Ad}_{\Pi_{2k-1}^*(B)}\Pi_{2k-1}^*(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}).
$$

It follows that

$$
\det \Pi_{2k-1}^*(A) = \det \Pi_{2k-1}^*(\begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}) = ((2k-1)!!)^2d^{2k} = ((2k-1)!!)^2(\det A)^k.
$$

**Lemma 4.2** For any nonzero $A \in \mathfrak{su}(2)$ the matrix $\Pi_{2k-1}^*(A)$ is invertible. Moreover, for every natural $k$ there exists a real constant $K_k > 0$ such that

$$
\|\Pi_{2k-1}^*(A)^{-1}\| \leq K_k\|A\|^{-1}
$$

for every nonzero $A \in \mathfrak{su}(2)$.

**Proof.** First claim of the lemma follows from Lemma 4.1. Set $C = \Pi_{2k-1}^*(A)$. Then

$$
|[C]_{i,j}| \leq (2k)^{4k}(2k-1)!!\|A\|^{2k-1}
$$

for $i, j = 1, \ldots, 2k$. It follows that

$$
|(C^{-1})_{i,j}| = \frac{|[C]_{i,j}|}{\det \Pi_{2k-1}^*(A)} \leq \frac{(2k)^{4k}(2k-1)!!\|A\|^{2k-1}}{(2k-1)!!^2\|A\|^{2k}} = \frac{(2k)^{4k}(2k-1)!!}{((2k-1)!!)^2}\|A\|^{-1}.
$$

Hence

$$
\|C^{-1}\| \leq \frac{(2k)^{4k+1}(2k-1)!!}{((2k-1)!!)^2}\|A\|^{-1}.
$$
5 Ergodicity and mixing of $T_\gamma$

Lemma 5.1 Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $L^2(\mathbb{T}, \mathbb{C})$ such that $\int_0^x f_n(y)dy \to 0$ for any $x \in \mathbb{T}$. Let $g: \mathbb{T} \to \mathbb{C}$ be a bounded measurable function. Then

$$\lim_{n \to \infty} \int_\mathbb{T} f_n(y)g(T^n y)dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_0^x f_n(y)g(y)dy = 0$$

for any $x \in \mathbb{T}$.

**Proof.** By assumption, the sequence $\{f_n\}_{n \in \mathbb{N}}$ tends to zero in the weak topology in $L^2(\mathbb{T}, \mathbb{C})$, which implies immediately the second claim of the lemma. Since $\{f_n\}_{n \in \mathbb{N}}$ converges weakly to zero, for every integer $m$ we have

$$\lim_{n \to \infty} \int_\mathbb{T} f_n(T^{-n} y) \exp 2\pi i m y dy = \lim_{n \to \infty} \int_\mathbb{T} f_n(y) \exp 2\pi i m(y + n\alpha) dy = 0.$$ 

It follows that the sequence $\{f_n \circ T^{-n}\}_{n \in \mathbb{N}}$ converges weakly to zero. Therefore

$$\int_\mathbb{T} f_n(y)g(T^n y)dy = \int_\mathbb{T} f_n(T^{-n} y)g(y)dy = 0.$$

This gives immediately the following conclusion.

Corollary 5.2 Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $L^2(\mathbb{T}, M_k(\mathbb{C}))$ ($k$ is a natural number) such that $\int_0^x f_n(y)dy \to 0$ for any $x \in \mathbb{T}$. Let $\varphi: \mathbb{T} \to M_k(\mathbb{C})$ be a bounded measurable function. Then

$$\lim_{n \to \infty} \int_\mathbb{T} f_n(y)g(T^n y)dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_0^x f_n(y)g(y)dy = 0$$

for any $x \in \mathbb{T}$.

Theorem 5.3 Let $\varphi : \mathbb{T} \to SU(2)$ be a $C^1$-cocycle with nonzero degree. Then the skew product $T_{\gamma(\varphi)} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable.

**Proof.** By Theorem 3.4, it suffices to show that

$$\lim_{n \to \infty} \int_\mathbb{T} (\gamma^{(n)}(x))^k dx = 0$$

for every nonzero integer $k$. Fix $k \in \mathbb{N}$. Denote by $\psi : \mathbb{T} \to su(2)$ the limit (in $L^2(\mathbb{T}, su(2)))$ of the sequence $\left\{\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1} d\lambda\right\}_{n \in \mathbb{N}}$. Let $p : \mathbb{T} \to SU(2)$ be a measurable function such that

$$\begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix} = p(x) \varphi(x) p(Tx)^{-1} \quad \text{and} \quad \text{Ad}_{p(x)}(\psi(x)) = \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix},$$


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where $d$ is the degree of $\varphi$ (see the proof of Theorem 2.4). Then
\[
\begin{bmatrix}
(g^{(n)})^k & (g^{(n)})^{k-2} & 0 \\
& \ddots & \vdots \\
0 & & (g^{(n)})^{-k+2} & (g^{(n)})^{-k}
\end{bmatrix}
= \Pi_k(p)\Pi_k(\varphi^{(n)})\Pi_k(p\circ T^n)^{-1}
\]
for any natural $n$ and
\[
\text{(6)} \quad \text{Ad}_{\Pi_k(p(x))} \Pi_k^* (\psi(x)) = \Pi_k^*(\text{Ad}_{p(x)}\psi(x)) = \\
= \Pi_k^*(\begin{bmatrix}
id & 0 \\
0 & -id
\end{bmatrix}) = \\
= \begin{bmatrix}
kid & (k-2)id & 0 \\
& \ddots & \vdots \\
0 & & (-k+2)id & -kid
\end{bmatrix}.
\]
Recall that for any differentiable function $\xi : \mathbb{T} \to SU(2)$ and for any representation $\Pi$ of $SU(2)$ we have
\[
D(\Pi\xi(x))(\Pi\xi(x))^{-1} = \Pi^*(D\xi(x)\xi(x)^{-1}).
\]
Therefore
\[
\int_0^x \frac{1}{n} \Pi_k^*(D\varphi^{(n)}(y)(\varphi^{(n)}(y))^{-1})\Pi_k(\varphi^{(n)}(y))dy = \int_0^x \frac{1}{n} D(\Pi_k\varphi^{(n)}(y))dy = \\
= \frac{1}{n}(\Pi_k(\varphi^{(n)}(x)) - \Pi_k(\varphi^{(n)}(0)))
\]
tends to zero for any $x \in \mathbb{T}$. Since
\[
\frac{1}{n} \Pi_k^*(D\varphi^{(n)}(\varphi^{(n)})^{-1}) \to \Pi_k^* \psi
\]
in $L^2(\mathbb{T}, M_{k+1}(\mathbb{C}))$, it follows that
\[
\int_0^x \Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))dy \to 0
\]
for any $x \in \mathbb{T}$. By Corollary 5.2,
\[
\int_{\mathbb{T}} \Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^ny))^{-1}dy \to 0.
\]
On the other hand,
\[
\Pi_k(p(y))\Pi_k^{*}(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^{n}y))^{-1} = \begin{bmatrix}
  ikd(\gamma^{(n)}(y))^k & \cdots & 0 \\
  & \ddots & \vdots \\
  0 & \cdots & -ikd(\gamma^{(n)}(y))^{-k}
\end{bmatrix},
\]
by (5) and (6). Therefore
\[
\lim_{n \to \infty} \int_{\mathbb{T}} (\gamma^{(n)}(y))^m dy = 0
\]
for any nonzero \(m \in \{-k, -k + 2, \ldots, k - 2, k\}\), which completes the proof. \(\blacksquare\)

6 Spectral analysis of cocycles with nonzero degree

In this section it is shown that for every cocycle \(\varphi : \mathbb{T} \to SU(2)\) if \(d(\varphi) \neq 0\) and it satisfies some additional assumptions, then the Lebesgue component in the spectrum of \(T_{\varphi}\) has countable multiplicity.

Now we introduce a notation, which is necessary to prove the main theory. Let \(f, g : \mathbb{T} \to M_k(\mathbb{C})\) be functions of bounded variation (i.e. \(f_{ij}, g_{ij} : \mathbb{T} \to \mathbb{C}\) have bounded variation for \(i, j = 1, \ldots, k\)) and let one of them be continuous. We will use the symbol \(\int_{\mathbb{T}} f \, dg\) to denote the \(k \times k\)-matrix given by
\[
(\int_{\mathbb{T}} f \, dg)_{ij} = \sum_{i=1}^{k} \int_{\mathbb{T}} f_{ii} \, dg_{ij}
\]
for \(i, j = 1, \ldots, d\). It is clear that if \(g\) is absolutely continuous, then
\[
\int_{\mathbb{T}} f \, dg = \int_{\mathbb{T}} f(x) \, Dg(x) dx.
\]
Moreover, applying integration by parts, we have
\[
\int_{\mathbb{T}} f \, dg = - (\int_{\mathbb{T}} g^{T} \, df^{T})^{T}.
\]

**Theorem 6.1** Let \(\varphi : \mathbb{T} \to SU(2)\) be a \(C^2\)-cocycle with \(d(\varphi) \neq 0\). Suppose that the sequence \(\{\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\}\) is uniformly convergent and \(\{D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\}\) is bounded in \(L^1(\mathbb{T}, \mathfrak{su}(2))\). Then the Lebesgue component in the spectrum of \(T_{\varphi}\) has countable multiplicity. Moreover, the Lebesgue component in the spectrum of \(T_{\gamma(\varphi)}\) has countable multiplicity, too.
Proof. First observe that it suffices to show that for every natural $k$ there exists a real constant $C_k > 0$ such that

$$
\left\| \int_T \Pi_{2k-1} (\varphi^{(n)}(x)) \, dx \right\| \leq \frac{C_k}{n}
$$

for large enough natural $n$. Indeed, let $p : \mathbb{T} \to SU(2)$ be a measurable function such that

$$
p(x)\varphi(x)p(Tx)^{-1} = \delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \gamma(x) \end{bmatrix}.
$$

Consider the unitary operator $V : \mathcal{H}^{\Pi_{2k-1}}_1 \to \mathcal{H}^{\Pi_{2k-1}}_1$ given by

$$
V \left( \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_1 (g) f_i (x) \right) = \sum_{i,j=1}^{d_{\Pi_{2k-1}}} \Pi_{1j} (g) \Pi_{ji} (p(x)^{-1}) f_i (x).
$$

Then

$$
V^{-1} U_{\mathcal{H}_1} V \left( \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_1 (g) f_i (x) \right) = \sum_{i,j,m=1}^{d_{\Pi_{2k-1}}} \Pi_{1m} (g) \Pi_{mij} (p(x)) \Pi_{ji} (p(Tx)^{-1}) f_i (T x)
$$

$$
= \sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i} (g) \Pi_{ii} (\delta(x)) f_i (T x).
$$

From (9), $U_{\mathcal{H}_1} : \mathcal{H}^{\Pi_{2k-1}}_1 \to \mathcal{H}^{\Pi_{2k-1}}_1$ has Lebesgue spectrum of uniform multiplicity, by Corollary 3.3. Hence $V^{-1} U_{\mathcal{H}_1} V$ has Lebesgue spectrum of uniform multiplicity and it is the product of the operators $U_j : L^2 (\mathbb{T}, \mathbb{C}) \to L^2 (\mathbb{T}, \mathbb{C})$ given by $U_j f(x) = (\gamma(x))^{2k-2j+1} f(Tx)$ for $j = 1, \ldots, 2k$. Therefore $U_j$ has absolutely continuous spectrum for $j = 1, \ldots, 2k$. By Lemma 3.1, $U_j$ has Lebesgue spectrum for all $j = 1, \ldots, 2k$ and $k \in \mathbb{N}$. It follows that the Lebesgue component in the spectrum of $T_{\mathcal{H}(\varphi)}$ has countable multiplicity.

By assumption,

$$
\left\| \frac{1}{n} D_{\mathcal{H}(\varphi)}^{(n)} (\varphi^{(n)})^{-1} \right\| \to d(\varphi)
$$

uniformly. Therefore

$$
\left\| \frac{1}{n} D_{\mathcal{H}(\varphi)}^{(n)} (\varphi^{(n)}(x))^{-1} \right\| \geq d(\varphi)/2
$$

for large enough natural $n$. For all $A, B \in M_k (\mathbb{C})$ we have $\|AB\| \leq \sqrt{k} \|A\| \|B\|$. Applying these facts, (7) and (8) we get

$$
\left\| \int_T \Pi_{2k-1} (\varphi^{(n)}(x)) \, dx \right\| = \left\| \int_T \Pi_{2k-1} (\varphi^{(n)}(x)) (D_{\mathcal{H}(\varphi)}^{(n)} (\varphi^{(n)}(x)))^{-1} d\Pi_{2k-1} (\varphi^{(n)}(x)) \right\|
$$

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\[ \begin{align*}
&= \left\| \int_T (\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1} d\Pi_{2k-1} (\varphi^{(n)}(x))) \right\| \\
&= \left\| \int_T (\Pi_{2k-1}^* (\varphi^{(n)}(x)))^T d((\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1}))^T \right\| \\
&= \left\| \int_T [(\Pi_{2k-1}^* (\varphi^{(n)}(x)))^T ((\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1}))^T \right.

\left( \Pi_{2k-1}^* D (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1})) \right)^T \\

\left. \left( ((\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1}))^T \right) \right) \right\| dx \\
&\leq 2k \int_T \left\| (\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1}) \right\|^2 \left\| (\Pi_{2k-1}^* D (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1})) \right\| dx.
\end{align*} \]

By Lemma 4.2, we have 
\[ \left\| (\Pi_{2k-1}^* (D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}))^{-1}) \right\| \leq K_k \left\| D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \right\|^{-1}. \]

From this, (4) and (10) we obtain 
\[ \begin{align*}
&\left\| \int_T \Pi_{2k-1} (\varphi^{(n)}(x)) dx \right\| \\
&\leq \frac{K_k^2 (2k)^3}{n} \int_T \left\| \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}) \right\|^2 \left\| D\left( \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x)^{-1}) \right) \right\| dx \\
&\leq \frac{1}{n} \left( \frac{8K_k k^2}{d(\varphi)} \right)^2 \left\| D\left( \frac{1}{n} D\varphi^{(n)}(x)^{-1} \right) \right\|_{L^1}.
\end{align*} \]

for large enough natural \( n \). By assumption, there exists a real constant \( M > 0 \) such that \( \left\| D\left( \frac{1}{n} D\varphi^{(n)}(x)^{-1} \right) \right\|_{L^1} \leq M \). Then 
\[ \left\| \int_T \Pi_{2k-1} (\varphi^{(n)}(x)) dx \right\| \leq \frac{C_k}{n} \]

for large enough natural \( n \), where \( C_k = \left( \frac{8K_k k^2}{d(\varphi)} \right)^2 M \). \( \blacksquare \)

In this section we also present a class of cocycles satisfying the assumptions of Theorem 6.1. We will need the following lemma.

**Lemma 6.2** Let \( \{f_n : T \rightarrow \mathbb{C}^1 ; n \in \mathbb{N} \} \) be a sequence of absolutely continuous functions. Assume that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges in \( L^1(T, \mathbb{R}^d) \) to a function \( f \) and it is bounded for the sup norm. Suppose that there is a sequence \( \{h_n\}_{n \in \mathbb{N}} \) convergent in \( L^2_+(T, \mathbb{R}) \) and a sequence \( \{k_n\}_{n \in \mathbb{N}} \) bounded in \( L^2_+(T, \mathbb{R}) \) such that
\[ \| Df_n(x) \| \leq h_n(x)k_n(x) \] for \( \lambda \)-a.e. \( x \in T \)

and for any natural \( n \). Then \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) uniformly.
Proof. Denote by $h \in L^2_+(\mathbb{T}, \mathbb{R})$ the limit of the sequence $\{h_n\}_{n \in \mathbb{N}}$. Let $M > 0$ be a real number such that $\|k_n\|_{L^2} \leq M$ for all natural $n$. First observe that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is equicontinuous. Fix $\varepsilon > 0$. Take $n_0 \in \mathbb{N}$ such that $\|h_n - h\|_{L^2} < \varepsilon/2M$ for any $n \geq n_0$. Then for all $x, y \in \mathbb{T}$ and $n \geq n_0$ we have

$$\|f_n(x) - f_n(y)\| = \| \int_x^y Df_n(t) dt \| \leq \int_x^y \| Df_n(t) \| dt \leq \int_x^y h_n(t) k_n(t) dt \leq \|k_n\|_{L^2} \sqrt{\int_x^y h_n^2(t) dt} \leq M(\sqrt{\int_x^y h^2(t) dt} + \|h_n - h\|_{L^2}) \leq M(\sqrt{\int_x^y h^2(t) dt} + \frac{\varepsilon}{2M}).$$

Choose $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies $\int_x^y h^2(t) dt < (\varepsilon/2M)^2$. Hence if $|x - y| < \delta_1$, then $\|f_n(x) - f_n(y)\| < \varepsilon$ for any $n \geq n_0$. Next choose $0 < \delta < \delta_1$ such that $|x - y| < \delta$ implies $\|f_n(x) - f_n(y)\| < \varepsilon$ for any $n \leq n_0$. It follows that if $|x - y| < \delta$, then $\|f_n(x) - f_n(y)\| < \varepsilon$ for every natural $n$.

By the Arzela–Ascoli theorem, for any subsequence of $\{f_n\}_{n \in \mathbb{N}}$ there exists a subsequence convergent to $f$ uniformly. Consequently, the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ uniformly. ■

This gives the following corollary.

**Corollary 6.3** Let $\{f_n : \mathbb{T} \to \mathbb{C}^d ; n \in \mathbb{N}\}$ be a sequence of absolutely continuous functions. Assume that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges in $L^1(\mathbb{T}, \mathbb{R}^d)$ to a function $f$ and it is bounded for the sup norm. Suppose that there is a sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent in $L^2_+(\mathbb{T}, \mathbb{R})$, a sequence $\{k_n\}_{n \in \mathbb{N}}$ bounded in $L^2_+(\mathbb{T}, \mathbb{R})$ and a sequence $\{l_n\}_{n \in \mathbb{N}}$ convergent in $L^1_+(\mathbb{T}, \mathbb{R})$ such that

$$\|Df_n(x)\| \leq l_n(x) + h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}$$

and for any natural $n$. Then $\{f_n\}_{n \in \mathbb{N}}$ converges to $f$ uniformly. ■

We will denote by $BV^2(\mathbb{T}, SU(2))$ the set of all functions $f : \mathbb{T} \to SU(2)$ of bounded variation such that $Df(f)^{-1} \in L^2(\mathbb{T}, su(2))$.

**Lemma 6.4** Let $\varphi : \mathbb{T} \to SU(2)$ be a $C^2$–cocycle. Suppose that $\varphi$ is cohomologous to a diagonal cocycle with a transfer function in $BV^2(\mathbb{T}, SU(2))$. Then the sequence $\{\frac{1}{n} D\varphi^{(n)}((\varphi^{(n)})^{-1})\}_{n \in \mathbb{N}}$ is uniformly convergent and $\{D(\frac{1}{n} D\varphi^{(n)}((\varphi^{(n)})^{-1}))\}_{n \in \mathbb{N}}$ is bounded in $L^1(\mathbb{T}, su(2))$.

**Proof.** By Corollary 6.3, it suffices to show that there exist a sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent in $L^2_+(\mathbb{T}, \mathbb{R})$, a sequence $\{k_n\}_{n \in \mathbb{N}}$ bounded in $L^2_+(\mathbb{T}, \mathbb{R})$ and a sequence $\{l_n\}_{n \in \mathbb{N}}$ convergent in $L^1_+(\mathbb{T}, \mathbb{R})$ such that

$$\|D(\frac{1}{n} D\varphi^{(n)}((\varphi^{(n)})^{-1}))\| \leq l_n(x) + h_n(x)k_n(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{T}.$$
By assumption, there exist $\delta, p \in BV^2(\mathbb{T}, SU(2))$ such that $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$, where $\delta$ is a diagonal cocycle. Then

$$D\varphi(x)\varphi(x)^{-1} = -p(x)^{-1}Dp(x) + p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x) + \varphi(x)p(Tx)^{-1}Dp(Tx)\varphi(x)^{-1}$$

for $\lambda$-a.e. $x \in \mathbb{T}$. Set

$$\tilde{\varphi}(x) = D\varphi(x)\varphi(x)^{-1}, \quad \tilde{p}(x) = p(x)^{-1}Dp(x) \quad \text{and} \quad \tilde{\delta}(x) = p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x).$$

Then $\tilde{\varphi}(x) = -\tilde{p}(x) + U\tilde{p}(x) + \tilde{\delta}(x)$, where $\tilde{p}, \tilde{\delta} \in L^2(\mathbb{T}, \mathfrak{su}(2))$. Since

$$\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1} = \frac{1}{n} \sum_{k=0}^{n-1} \varphi^{(k)} \circ T^k (\varphi^{(k)})^{-1},$$

we have

$$D\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} (\text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^j) \text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^k) - \text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^k) \text{Ad}_{\varphi^{(j)}}(\tilde{\varphi} \circ T^j))$$

$$+ \frac{1}{n} \sum_{k=0}^{n-1} \text{Ad}_{\varphi^{(k)}}(D\tilde{\varphi} \circ T^k)$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\varphi}, U^k \tilde{\varphi}] + \frac{1}{n} \sum_{k=0}^{n-1} U^k (D\tilde{\varphi}).$$

However,

$$\sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\varphi}, U^k \tilde{\varphi}] = \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j+1} \tilde{p} - U^j \tilde{p} + U^j \tilde{\delta}, U^k \tilde{\varphi}]$$

$$= \sum_{k=0}^{n-1} [U^k \tilde{p} - \tilde{p}, U^k \tilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\delta}, U^{j+1} \tilde{p} - U^j \tilde{p} + U^k \tilde{\delta}]$$

$$= \sum_{k=0}^{n-1} [U^k \tilde{p} - \tilde{p}, U^k \tilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \tilde{\delta}, U^k \tilde{\delta}] + \sum_{j=0}^{n-2} [U^j \tilde{\delta}, U^j \tilde{p} - U^{j+1} \tilde{p}].$$

Since

$$U^j \tilde{\delta}(x) = \text{Ad}_{\varphi^{(n)}(x)p(T^n x)^{-1}}(D\delta(T^n x)\delta(T^n x)^{-1})$$

$$= \text{Ad}_{p(x)^{-1}\varphi^{(n)}(x)}(D\delta(T^n x)\delta(T^n x)^{-1})$$

$$= \text{Ad}_{p(x)^{-1}}(D\delta(T^n x)\delta(T^n x)^{-1}),$$
we have \([U_j^j \tilde{\delta}, U_k^k \tilde{\delta}] = 0\) for any integers \(j, k\). Observe that \(||[A, B]|| \leq 2 \|A\| \|B\||\)
for any \(A, B \in \mathfrak{su}(2)\). It follows that
\[
\left\| D \left( \frac{1}{n} D \varphi^{(n)} (\varphi^{(n)})^{-1} \right) \right\| \\
\leq \frac{2}{n} \sum_{k=0}^{n-1} (||D \tilde{\varphi} \circ T^k|| + ||\tilde{p} \circ T^k|| \| \tilde{\varphi} \circ T^k \| + ||\tilde{\varphi} \circ T^k|| \| \tilde{\varphi} \circ T^k \| + ||\tilde{\varphi} \circ T^k|| \| \tilde{\varphi} \circ T^k \| ) \\
+ \| \tilde{p} \circ T^n \| \frac{2}{n} \sum_{k=0}^{n-1} \| \tilde{\varphi} \circ T^k \| .
\]

Set
\[
h_n = \frac{2}{n} \sum_{k=0}^{n-1} \| \tilde{\varphi} \circ T^k \| \\
k_n = \| \tilde{p} \circ T^n \| \\
l_n = \frac{2}{n} \sum_{k=0}^{n-1} (||D \tilde{\varphi} \circ T^k|| + ||\tilde{p} \circ T^k|| \| \tilde{\varphi} \circ T^k \| + ||\tilde{p} \circ T^k|| \| \tilde{\varphi} \circ T^k \| + ||\tilde{p} \circ T^k|| \| \tilde{p} \circ T^k \| ) .
\]

By the Birkhoff ergodic theorem, the sequence \(\{h_n\}_{n \in \mathbb{N}}\) converges in \(L^2_+(T, \mathbb{R})\) and the sequence \(\{l_n\}_{n \in \mathbb{N}}\) converges in \(L^1_+(T, \mathbb{R})\). This completes the proof. \(\blacksquare\)

Theorem 6.1 and Lemma 6.4 lead to the following conclusion.

**Corollary 6.5** Let \(\varphi : T \to SU(2)\) be a \(C^2\)-cocycle with \(d(\varphi) \neq 0\). Suppose that \(\varphi\) is cohomologous to a diagonal cocycle with a transfer function in \(BV^2(T, SU(2))\). Then the Lebesgue component in the spectrum of \(T_\varphi\) has countable multiplicity. Moreover, the Lebesgue component in the spectrum of \(T_{\gamma(\varphi)}\) has countable multiplicity, too.

The following result will be useful in the next section of the paper.

**Proposition 6.6** For every \(C^2\)-cocycle \(\varphi : T \to SU(2)\), the sequence
\[
\frac{1}{n^2} D \left( D \varphi^{(n)} (\varphi^{(n)})^{-1} \right)
\]
converges to zero in \(L^1(T, \mathfrak{su}(2))\).

The following lemmas are some simple generalizations of some classical results. Their proofs are left to the reader.

**Lemma 6.7** Let \(\{a_n\}_{n \in \mathbb{N}}\) be an increasing sequence of natural numbers and let \(\{f_n\}_{n \in \mathbb{N}}\) be a sequence in the Banach space \(L^2(T, M_2(\mathbb{C}))\). Then
\[
\frac{f_{n+1} - f_n}{a_{n+1} - a_n} \to g \text{ in } L^2(T, M_2(\mathbb{C})) \implies \frac{f_n}{a_n} \to g \text{ in } L^2(T, M_2(\mathbb{C})).
\]
Lemma 6.8 Let \( \{g_k^n; n \in \mathbb{N}, 0 \leq k < n\} \) be a triangular matrix of elements from \( L^2(\mathbb{T}, M_2(\mathbb{C})) \) such that \( \|g_k^n\| = O(1/n) \) and

\[
\sum_{k=0}^{n-1} g_k^n f_k \to g f \quad \text{and} \quad \sum_{k=0}^{n-1} f_k g_k^n \to f g \quad \text{in} \quad L^2(\mathbb{T}, M_2(\mathbb{C})).
\]

Then \( f_n \to f \) in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \) implies

\[
\sum_{k=0}^{n-1} g_k^n f_k \to g f \quad \text{and} \quad \sum_{k=0}^{n-1} f_k g_k^n \to f g \quad \text{in} \quad L^1(\mathbb{T}, M_2(\mathbb{C})).
\]

Proof of Proposition 6.6. First recall that

\[
\frac{1}{n^2} D(D\varphi^{(n)}(\varphi^{(n)})^{-1}) = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^j \varphi^{(n)} U^k \varphi^{(n)}] + \frac{1}{n^2} \sum_{k=0}^{n-1} U^k (D\varphi),
\]

where \( \varphi = D\varphi(\varphi)^{-1} \) and

\[
\frac{1}{n} \sum_{k=0}^{n-1} U^k \varphi \to \psi \quad \text{in} \quad L^2(\mathbb{T}, \mathfrak{su}(2)).
\]

Since \( \frac{1}{n^2} \sum_{k=0}^{n-1} U^k (D\varphi) \) uniformly converges to zero, it suffices to show that

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \varphi^{(n)} U^k \varphi^{(n)} = \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k \varphi^{(n)} U^j \varphi^{(n)} = \frac{1}{2} \psi \varphi \quad \text{in} \quad L^2(\mathbb{T}, M_2(\mathbb{C})).
\]

Set \( f_n = \sum_{k=0}^{n-1} (n - k) U^k \varphi \) and \( a_n = n^2 \). Then

\[
\frac{f_{n+1} - f_n}{a_{n+1} - a_n} = \frac{\sum_{k=0}^{n} (n + 1 - k) U^k \varphi - \sum_{k=0}^{n-1} (n - k) U^k \varphi}{(n + 1)^2 - n^2} = \frac{\sum_{k=0}^{n} U^k \varphi}{2n + 1} \to \frac{1}{2} \psi
\]

in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \). Applying Lemma 6.7, we get

\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (n - k) U^k \varphi \to \frac{1}{2} \psi \quad \text{in} \quad L^2(\mathbb{T}, M_2(\mathbb{C})),
\]

Therefore

\[
\frac{1}{n^2} \sum_{k=0}^{n-1} k U^k \varphi = \frac{1}{n} \sum_{k=0}^{n-1} U^k \varphi - \frac{1}{n^2} \sum_{k=0}^{n-1} (n - k) U^k \varphi \to \psi - \frac{1}{2} \psi = \frac{1}{2} \psi
\]

in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \). Applying Lemma 6.8 with \( g_k^n = \frac{k}{n} U^k \varphi \) and \( f_k = \frac{1}{k} \sum_{j=0}^{k-1} U^j \varphi \), we conclude that

\[
\sum_{k=0}^{n-1} g_k^n f_k = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k \varphi U^j \varphi \to \frac{1}{2} \psi \psi
\]

and

\[
\sum_{k=0}^{n-1} f_k g_k^n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \varphi U^k \varphi \to \frac{1}{2} \psi \psi
\]

in \( L^2(\mathbb{T}, M_2(\mathbb{C})) \), which completes the proof. \( \blacksquare \)
7 Possible values of degree

One may ask what we know about the set of possible values of degree. For $G = \mathbb{T}$ the degree of each smooth cocycle is an integer number. Probably, in the case of cocycles with values in $SU(2)$ the set of possible values of degree is more complicated. However, in this section we show that if $\alpha$ is the golden ratio, then the degree of each smooth cocycle belongs to $2\pi \mathbb{N}_0$. The idea of renormalization, which is used to prove this result is due to Rychlik [8].

Let $\alpha$ be the golden ratio (i.e. the positive root of the equation $\alpha^2 + \alpha = 1$). It will be advantageous for our notation to consider the interval $[-\alpha^2, \alpha)$ to be the model of the circle. Then the map $T : [-\alpha^2, \alpha) \to [-\alpha^2, \alpha)$ given by

$$T(x) = \begin{cases} x + \alpha & \text{for } x \in [-\alpha^2, 0) \\ x - \alpha^2 & \text{for } x \in [0, \alpha) \end{cases}$$

is the rotation by $\alpha$. Let $X = [-\alpha^2, \alpha^3)$. Then the first return time to $X$, which we call $\tau$, satisfies the following formula

$$\tau(x) = \begin{cases} 1 & \text{for } x \in [0, \alpha^3) \\ 2 & \text{for } x \in [-\alpha^2, 0) \end{cases}$$

and the first return map $T_X : X \to X$ is equal to $T$ up to a linear scaling. Indeed, if $M : \mathbb{T} \to X$ is the map given by $M(x) = -\alpha x$, then $T_X \circ M = M \circ T$.

By $W^1$ we mean the space of all cocycle $\varphi : \mathbb{T} \to SU(2)$ such that the functions $\varphi : [-\alpha^2, 0) \to SU(2)$, $\varphi : [0, \alpha) \to SU(2)$ are both of class $C^1$ and

$$\lim_{x \to \alpha} D\varphi(x) \varphi(x)^{-1} \text{ and } \lim_{x \to 0^-} D\varphi(x) \varphi(x)^{-1}$$

exist. The topology of $W^1$ is induced from $C^1((-\alpha^2, 0) \cup (0, \alpha))$. Consider the renormalization operator $\Phi : W^1 \to W^1$ defined by

$$\Phi \varphi(x) = \varphi^{(\tau(Mx))}(Mx),$$

Then

$$\Phi^n \varphi(x) = \begin{cases} \varphi^{(\tau^{n+1})}(M^n x) & \text{for } x \in [-\alpha^2, 0) \\ \varphi^{(\tau^{n+2})}(M^n x) & \text{for } x \in [0, \alpha) \end{cases}$$

for any natural $n$, where $\{q_n\}_{n \in \mathbb{N}}$ is the Fibonacci sequence. By $W^1_0$ we mean the set of all cocycles $\varphi \in W^1$ such that $\varphi^{(2)}$ is continuous at 0. The set $W^1_0$ is a closed subset of $W^1$ and

$$\Phi(W^1_0) \subset W^1_0$$

(see [8]). It is easy to check that $\|D(\Phi \varphi)(\Phi \varphi)^{-1}\|_{L^1} \leq \|D\varphi(\varphi)^{-1}\|_{L^1}$ for any $\varphi \in W^1$. The following result is due to M. Rychlik [8].
Proposition 7.1 If \( \| D(\Phi^k \varphi)(\Phi^k \varphi)^{-1} \|_{L^1} = \| D\varphi(\varphi)^{-1} \|_{L^1} \) for all natural \( k \), then
\[
D\varphi(x)(\varphi(x))^{-1} = \alpha \text{Ad}_{\varphi(x)}[D\varphi(Tx)(\varphi(Tx))^{-1}]
\]
for every \( x \in [-\alpha^2,0) \). □

Lemma 7.2 Let \( \varphi : T \to SU(2) \) be a \( C^2 \)-cocycle. Assume that
\[
\frac{1}{n} D\varphi^{(n)}(0)(\varphi^{(n)}(0))^{-1} \to H \in \mathfrak{su}(2)
\]
and there is an increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of even numbers such that
\[
\lim_{k \to \infty} \alpha^n \int_0^{\alpha^n} |D(D\varphi^{(q_{n+1})}(x)(\varphi^{(q_{n+1})}(x))^{-1})|dx = 0
\]
for \( i = 1,2 \). Then \( \|H\| \in 2\pi \mathbb{N}_0 \).

Proof. First note that
\[
D\Phi^n \varphi(x)(\Phi^n \varphi(x))^{-1} = \begin{cases} 
\alpha^n D\varphi^{(q_{n+1})}(M^n x)(\varphi^{(q_{n+1})}(M^n x))^{-1} & \text{for } x \in [-\alpha^2,0) \\
\alpha^n D\varphi^{(q_{n+2})}(M^n x)(\varphi^{(q_{n+2})}(M^n x))^{-1} & \text{for } x \in [0,\alpha)
\end{cases}
\]
for any even \( n \). Since
\[
\frac{1}{q_{n+i}} D\varphi^{(q_{n+i})}(M^n x)(\varphi^{(q_{n+i})}(M^n x))^{-1} - \frac{1}{q_{n+i}} D\varphi^{(q_{n+i})}(0)(\varphi^{(q_{n+i})}(0))^{-1} \leq \frac{1}{q_{n+i}} \int_0^{\alpha^n} |D(D\varphi^{(q_{n+i})}(x)(\varphi^{(q_{n+i})})^{-1})|dx \leq \frac{1}{q_{n+i}} \int_0^{\alpha^n} |D(D\varphi^{(q_{n+i})}(x)(\varphi^{(q_{n+i})}))^{-1})|dx
\]
for all even \( n, i = 1,2 \) and
\[
\lim_{n \to \infty} \alpha^n q_{n+1} = 1/(1 + \alpha^2), \quad \lim_{n \to \infty} \alpha^n q_{n+2} = 1/(\alpha + \alpha^3),
\]
it follows that
\[
\lim_{k \to \infty} D\Phi^n \varphi(x)(\Phi^n \varphi(x))^{-1} = \lim_{k \to \infty} \alpha^n q_{n+1} \frac{1}{q_{n+1}} D\varphi^{(q_{n+1})}(0)(\varphi^{(q_{n+1})}(0))^{-1} \leq \frac{1}{1 + \alpha^2} H
\]
uniformly on \([-\alpha^2,0)\) and
\[
\lim_{k \to \infty} D\Phi^n \varphi(x)(\Phi^n \varphi(x))^{-1} = \lim_{k \to \infty} \alpha^n q_{n+2} \frac{1}{q_{n+2}} D\varphi^{(q_{n+2})}(0)(\varphi^{(q_{n+2})}(0))^{-1} \leq \frac{1}{\alpha + \alpha^3} H
\]
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uniformly on \([0, \alpha]\). Therefore we can assume that there exists \(v \in W^1\) such that

\[
\Phi^{n_k} \varphi \to v \quad \text{and} \quad D\Phi^{n_k} \varphi (\Phi^{n_k} \varphi)^{-1} \to Dv^{-1}
\]

uniformly. Then

\[
Dv(x)(v(x))^{-1} = \begin{cases} 
\alpha A & \text{for} \quad x \in [-\alpha^2, 0) \\
A & \text{for} \quad x \in [0, \alpha),
\end{cases}
\]

where \(A = 1/(\alpha + \alpha^3) H \in \mathfrak{su}(2)\). Therefore

\[
v(x) = \begin{cases} 
e^{\alpha x} B & \text{for} \quad x \in [-\alpha^2, 0) \\
\ne^{x} C & \text{for} \quad x \in [0, \alpha),
\end{cases}
\]

where \(B = v_-(0)\) and \(C = v_+(0)\). Since the set \(W_0^1 \subset W^1\) is closed and \(\Phi\)-invariant, \(v \in W_0^1\). It follows that

\[
(12) \quad Ce^{-\alpha A} B = B e^{\alpha A} C.
\]

Since \(v\) is a limit point of the sequence \(\{\Phi^n \varphi\}_{n \in \mathbb{N}}\), we have \(\|D\Phi^k v (\Phi^k v)^{-1}\|_{L^1} = \|Dv^{-1}\|_{L^1}\) for any natural \(k\). By Proposition 7.1,

\[
\lim_{x \to 0^-} Dv(x)(v(x))^{-1} = \alpha \text{Ad}_{v_-(0)} \lim_{x \to 0^-} Dv(x)(v(x))^{-1}.
\]

Hence

\[
\alpha A = \alpha \text{Ad}_B(A)
\]

and finally \(AB = BA\). Therefore

\[
\Phi v(x) = \begin{cases} 
e^{-\alpha x} C & \text{for} \quad x \in [-\alpha^2, 0) \\
\ne^{-x} BC & \text{for} \quad x \in [0, \alpha).
\end{cases}
\]

By Proposition 7.1,

\[
\lim_{x \to 0^-} D\Phi v(x)(\Phi v(x))^{-1} = \alpha \text{Ad}_{v_-(0)} \lim_{x \to 0^-} D\Phi v(x)(\Phi v(x))^{-1}.
\]

Hence

\[
-\alpha A = \alpha \text{Ad}_C(-A)
\]

and finally \(AC = CA\). It follows that \(B\) and \(C\) commute, by (12). From (12), we obtain \(e^{(\alpha + \alpha^3)A} = \text{Id}\). Therefore \(\|H\| = \|(\alpha + \alpha^3) A\| \in 2\pi \mathbb{N}_0\).

**Theorem 7.3** Suppose that \(\alpha\) is the golden ratio. Then for every \(C^2\)-cocycle \(\varphi : \mathbb{T} \to SU(2)\), we have \(d(\varphi) \in 2\pi \mathbb{N}_0\).
**Proof.** Fix \( n \in \mathbb{N} \) such that \( 2\alpha^{2n}[1/2\alpha^{2n}] \geq 4/5 \). Set \( I_j = [2(j-1)\alpha^{2n}, 2j\alpha^{2n}] \) for \( j \in E = \{1, \ldots, [1/2\alpha^{2n}]\} \) and \( \varepsilon_n = \frac{1}{\pi} \int_{\mathbb{T}} |D(\varphi_0(\varphi(t^{-1})) - \frac{1}{2}\int_{\mathbb{T}} |D(\varphi(t^{-1}))|d\lambda. \) By Proposition 6.6, \( \varepsilon_n \) tends to zero. For \( i = 1, 2 \) define

\[
E_i = \{ j \in E; \frac{1}{2\alpha^{2n}q_{2n+i}^2} \int_{I_j} |D(\varphi(t^{-1}))t^{-1})|d\lambda \leq 10\varepsilon_{2n+i} \}.
\]

Then

\[
\varepsilon_{2n+i} = \frac{1}{q_{2n+i}} \int_{\mathbb{T}} |D(\varphi(t^{-1}))t^{-1})|d\lambda
\]

\[
\geq \frac{1}{q_{2n+i}} \sum_{j \in E \setminus E_i} \int_{I_j} |D(\varphi(t^{-1}))t^{-1})|d\lambda
\]

\[
\geq 20\alpha^{2n}\varepsilon_{2n+i}([1/2\alpha^{2n}] - \#E_i).
\]

Hence

\[
\#E_i \geq [1/2\alpha^{2n}](1 - \frac{1}{10} [1/2\alpha^{2n}]) \geq \frac{7}{8}[1/2\alpha^{2n}]
\]

for \( i = 1, 2 \). Therefore

\[
\#(E_1 \cap E_2) \geq \#E_1 + \#E_2 - \#E \geq \frac{3}{4}[1/2\alpha^{2n}].
\]

Define

\[
G_n = \bigcup_{j \in E_1 \cap E_2} [(2j - 2)\alpha^{2n}, (2j - 1)\alpha^{2n}].
\]

Observe that \( y \in G_n \) implies

\[
\frac{1}{2\alpha^{2n}q_{2n+i}^2} \int_{y + \alpha^{2n}}^{y + \alpha^{2n}} |D(\varphi(t^{-1}))t^{-1})|d\lambda \leq 10\varepsilon_{2n+i}
\]

for \( i = 1, 2 \) and

\[
\lambda(G_n) \geq \alpha^{2n}\#(E_1 \cap E_2) \geq \frac{3}{8}2\alpha^{2n}[1/2\alpha^{2n}] \geq \frac{3}{10}.
\]

Set \( G' = \bigcap_{n \in \mathbb{N}} \bigcup_{k > n} G_k \). Then \( \lambda(G') \geq 3/10 \). Since \( \frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \) almost everywhere, we see that the set

\[
G = \{ x \in G' ; \frac{1}{n}D\varphi^{(n)}(\varphi^{(n)}(x))^{-1} \to \psi(x) \}
\]

has positive measure.

For every \( y \in \mathbb{T} \) denote by \( \varphi_y : \mathbb{T} \to SU(2) \) the \( C^2 \)-cocycle \( \varphi_y(x) = \varphi(x + y) \). Suppose that \( y \in G \). Then \( \frac{1}{n}D\varphi^{(n)}(\varphi^{(n)}(0))^{-1} \to \psi(y) \) and there exists an
increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of natural numbers such that \( y \in G_{n_k} \) for any natural \( k \). Hence
\[
\alpha^{2n_k} \int_0^{\alpha^{2n_k}} D(D \varphi_{y}^{(q_{2n_k+i})}) \left( (\varphi_{y}^{(q_{2n_k+i})})^{-1} \right) d\lambda \leq 20(\alpha^{2n_k} q_{2n_k+i})^2 \varepsilon_{2n_k+i}
\]
for \( i = 1, 2 \). Since the sequence \( \{\alpha^n q_{n+i}\}_{n \in \mathbb{N}} \) converges for \( i = 1, 2 \) and \( \varepsilon_n \) tends to zero, letting \( k \to \infty \) we have
\[
\lim_{k \to \infty} \alpha^{2n_k} \int_0^{\alpha^{2n_k}} |D(D \varphi_{y}^{(q_{2n_k+i})}) \left( (\varphi_{y}^{(q_{2n_k+i})})^{-1} \right)| d\lambda = 0
\]
for \( i = 1, 2 \). By Lemma 7.2, \( \|\psi(y)\| \in 2\pi\mathbb{N}_0 \) for every \( y \in G \). Since \( d(\varphi) = \|\psi(y)\| \) for a.e. \( y \in T \), we conclude that \( d(\varphi) \in 2\pi\mathbb{N}_0 \). □

8 2–dimensional case

In this section we will be concerned with properties of smooth cocycles over ergodic rotations on the 2–dimensional torus with values in \( SU(2) \). By \( T^2 \) we will mean the group \( \mathbb{R}^2/\mathbb{Z}^2 \). We will identify functions on \( T^2 \) with periodic of period 1 in each coordinates functions on \( \mathbb{R}^2 \). Suppose that \( T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta) \) is an ergodic rotation on \( T^2 \). Let \( \varphi : T^2 \to SU(2) \) be a \( C^1 \)-cocycle over the rotation \( T \). Analysis similar to that in Section 2 shows that there exists \( \psi_i \in L^2(\mathbb{T}^2, \mathfrak{su}(2)) \), \( i = 1, 2 \) such that
\[
\frac{1}{n} \frac{\partial}{\partial x_i} \varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi_i \quad \text{in} \quad L^2(\mathbb{T}^2, \mathfrak{su}(2)).
\]
Moreover, \( \|\psi_i\| \) is a \( \lambda \otimes \lambda \)-a.e. constant function and \( \varphi(\bar{x})\psi_i(T\bar{x})\varphi(\bar{x})^{-1} = \psi_i(\bar{x}) \) for \( \lambda \otimes \lambda \)-a.e. \( \bar{x} \in T \times T \) for \( i = 1, 2 \).

**Definition 2** The pair
\[
(\|\psi_1\|, \|\psi_2\|) = \lim_{n \to \infty} \frac{1}{n} \left( \left\| \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1}, \left\| \frac{\partial}{\partial x_2} \varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1} \right)
\]
will be called the **degree** of the cocycle \( \varphi : T^2 \to SU(2) \) and denoted by \( d(\varphi) \).

Similarly, one can prove the following

**Theorem 8.1** If \( d(\varphi) \neq 0 \), then \( \varphi \) is cohomologous to a diagonal cocycle
\[
T^2 \ni \bar{x} \mapsto \begin{bmatrix} \gamma(\bar{x}) & 0 \\ 0 & \overline{\gamma(\bar{x})} \end{bmatrix} \in SU(2), \quad \text{where} \quad \gamma : T^2 \to T \quad \text{is measurable. Moreover,}
\]
the skew product \( T_\gamma : T^2 \times T \to T^2 \times T \) is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first two variables. □
Analysis similar to that in the proof of Theorem 6.1 gives

**Theorem 8.2** Let \( \varphi : \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle with \( d(\varphi) \neq 0 \). Suppose that the sequence \( \{ \frac{1}{n} \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \}_{n \in \mathbb{N}} \) is uniformly convergent and \( \{ \frac{\partial}{\partial x_1} \left( \frac{1}{n} \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \right) \}_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{T}^2, \mathfrak{su}(2)) \) for \( i = 1, 2 \). Then the Lebesgue component in the spectrum of \( T_\varphi \) has countable multiplicity. ■

By \( BV^R(\mathbb{T}^2, SU(2)) \) we mean the set of all measurable functions \( f : \mathbb{T}^2 \to SU(2) \) such that

- the functions \( f(x, \cdot), f(\cdot, x) : \mathbb{T} \to SU(2) \) are of bounded variation for any \( x \in \mathbb{T} \);
- the functions \( \frac{\partial}{\partial x_1} f(f)^{-1}, \frac{\partial}{\partial x_2} f(f)^{-1} : \mathbb{T}^2 \to \mathfrak{su}(2) \) are Riemann integrable for \( i = 1, 2 \).

Then we immediately get the following

**Lemma 8.3** Let \( \varphi : \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle. Suppose that \( \varphi \) is cohomologous to a diagonal cocycle with a transfer function in \( BV^R(\mathbb{T}^2, SU(2)) \). Then the sequence \( \{ \frac{1}{n} \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \}_{n \in \mathbb{N}} \) is uniformly convergent and \( \{ \frac{\partial}{\partial x_1} \left( \frac{1}{n} \frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1} \right) \}_{n \in \mathbb{N}} \) is uniformly bounded for \( i = 1, 2 \). ■

It is easy to check that if \( \varphi \) is cohomologous to a diagonal cocycle via a \( C^1 \) transfer function, then \( d(\varphi) \in 2\pi(\mathbb{N}_0 \times \mathbb{N}_0) \). However, in the next section we show that for every ergodic rotation \( T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta) \) there exists a smooth cocycle whose degree is equal to \( 2\pi(\|\beta\|, |\alpha|) \).

9 **Cocycles over flows**

Let \( \omega \) be an irrational number. By \( S : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T} \) we mean the ergodic flow defined by

\[
S_t(x_1, x_2) = (x_1 + t\omega, x_2 + t).
\]

Let \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) be a smooth cocycle over \( S \), i.e.

\[
\varphi_{t+s}(\vec{x}) = \varphi_t(\vec{x}) \varphi_s(S_t\vec{x})
\]

for all \( t, s \in \mathbb{R}, \vec{x} \in \mathbb{T}^2 \) or equivalently, \( \varphi \) is the fundamental matrix solution for a linear differential system

\[
\frac{d}{dt} y(t) = y(t)A(S_t\vec{x}),
\]

where \( A : \mathbb{T}^2 \to \mathfrak{su}(2) \), i.e. \( \varphi \) satisfies

\[
\begin{align*}
\frac{d}{dt} \varphi_t(\vec{x}) &= \varphi_t(\vec{x})A(S_t\vec{x}) \\
\varphi_0(\vec{x}) &= \text{Id}.
\end{align*}
\]
Then
\[ \frac{\partial}{\partial x_i} \varphi_{t+s}(\bar{x}) \varphi_{t+s}(\bar{x})^{-1} = \frac{\partial}{\partial x_i} \varphi_t(\bar{x}) \varphi_t(\bar{x})^{-1} + \text{Ad}_{\varphi_t(x)} \frac{\partial}{\partial x_i} \varphi_t(S_i \bar{x}) \varphi_t(S_i \bar{x})^{-1}. \]

Hence
\[ \left\| \frac{\partial}{\partial x_i} \varphi_{t+s}(\varphi_{t+s})^{-1} \right\|_{L^1} \leq \left\| \frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} + \left\| \frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} \]

It follows that the limit
\[ \lim_{i \to \infty} \frac{1}{|t|} \left\| \frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} \]
exists for \( i = 1, 2. \)

**Definition 3** The pair
\[ \lim_{i \to \infty} \frac{1}{|t|} \left( \left\| \frac{\partial}{\partial x_1} \varphi_t(\varphi_t)^{-1} \right\|_{L^1}, \left\| \frac{\partial}{\partial x_2} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} \right) \]
will be called the \textit{degree} of the cocycle \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) and denoted by \( d(\varphi). \)

For a given cocycle \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) over the flow \( S \), by \( \tilde{\varphi} : \mathbb{T} \to SU(2) \) we will mean the cocycle over the rotation \( T x = x + \omega \) defined by \( \tilde{\varphi}(x) = \varphi_1(x, 0). \) Then \( \tilde{\varphi}^{(n)}(x) = \varphi_n(x, 0). \)

**Lemma 9.1** \( d(\varphi) = (1, |\omega|)d(\tilde{\varphi}). \)

**Proof.** First observe that
\[ \varphi_{x_2}(x_1 - x_2 \omega, 0) \varphi_n(x_1, x_2) = \varphi_{n+x_2}(x_1 - x_2 \omega, 0) = \varphi_n(x_1 - x_2 \omega, 0) \varphi_{x_2}(x_1 - x_2 \omega + n \omega, 0). \]
Hence
\[ \varphi_n(x_1, x_2) = \varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \tilde{\varphi}^{(n)}(x_1 - x_2 \omega) \varphi_{x_2}(x_1 - x_2 \omega + n \omega, 0) \]
for all \( x_1, x_2 \in \mathbb{R} \) and \( n \in \mathbb{N}. \) Fix \( (x_1, x_2) \in [0, 1] \times [0, 1]. \) Then
\[
\frac{\partial}{\partial x_1} \varphi_n(x_1, x_2) \varphi_n(x_1, x_2)^{-1} = -\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2 \omega, 0) \\
+ \text{Ad}_{\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1}} (D \tilde{\varphi}^{(n)}(x_1 - x_2 \omega) \tilde{\varphi}^{(n)}(x_1 - x_2 \omega)^{-1}) \\
+ \text{Ad}_{\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \tilde{\varphi}^{(n)}(x_1 - x_2 \omega)} \left( \frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2 \omega + n \omega, 0) \varphi_{x_2}(x_1 - x_2 \omega + n \omega, 0)^{-1} \right).
\]
It follows that
\[
\left\| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} - \left\| D\varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1}
\]
\[
= \left\| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} - \int_0^1 \int_0^1 \left\| D\varphi^{(n)}(x_1 - x_2\omega)\varphi^{(n)}(x_1 - x_2\omega)^{-1} \right\| dx_1 dx_2
\]
\[
\leq 2 \int_0^1 \int_0^1 \left\| \frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0) \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \right\| dx_1 dx_2.
\]
Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} = \lim_{n \to \infty} \frac{1}{n} \left\| D\varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1} = d(\varphi).
\]
Similarly,
\[
\frac{\partial}{\partial x_2} \varphi_n(x_1, x_2) \varphi_n(x_1, x_2)^{-1} = -\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \frac{\partial}{\partial t} \varphi_{x_2}(x_1 - x_2\omega, 0)
\]
\[
+ \omega \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0)
\]
\[
- \omega \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}}(D\varphi^{(n)}(x_1 - x_2\omega)\varphi^{(n)}(x_1 - x_2\omega)^{-1})
\]
\[
+ \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}}(D\varphi^{(n)}(x_1 - x_2\omega + n\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)^{-1})
\]
\[
- \omega \text{Ad}_{\varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}}(D\varphi^{(n)}(x_1 - x_2\omega + n\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)^{-1}).
\]
It follows that
\[
\left\| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} \leq |\omega| \left\| D\varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1}
\]
\[
\leq 2 \int_0^1 \int_0^1 \left\| \frac{\partial}{\partial t} \varphi_{x_2}(x_1 - x_2\omega, 0) \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \right\| dx_1 dx_2
\]
\[
+ 2 |\omega| \int_0^1 \int_0^1 \left\| \frac{\partial}{\partial x_1} \varphi_{x_2}(x_1 - x_2\omega, 0) \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \right\| dx_1 dx_2.
\]
Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} = |\omega| \lim_{n \to \infty} \frac{1}{n} \left\| D\varphi^{(n)}(\varphi^{(n)})^{-1} \right\|_{L^1} = |\omega| d(\varphi),
\]
and the proof is complete. ■

Lemma 9.2 For any $C^2$-cocycle $\psi : \mathbb{T} \to SU(2)$ over the rotation $T$ there exists a $C^2$-cocycle $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ over the flow $S$ such that $\hat{\varphi} = \psi$. 

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**Proof.** Since the fundamental group of $SU(2)$ is trivial, we can choose a $C^2$-homotopy $\psi : [0, 1] \times \mathbb{T} \to SU(2)$ such that

$$
\psi(t, x) = \begin{cases} 
\text{Id} & \text{for } t \in [0, 1/4] \\
\psi(x) & \text{for } t \in [3/4, 1]. 
\end{cases}
$$

By $\psi : \mathbb{R} \times \mathbb{T} \to SU(2)$ we mean the $C^2$-function determined by

$$
\psi(n + t, x) = \psi(n)(x)\psi(t, x + n\omega)
$$

for any $t \in [0, 1]$ and $n \in \mathbb{Z}$. Then it is easy to check that

$$
(14) \quad \psi(n + t, x) = \psi(n)(x)\psi(t, x + n\omega)
$$

for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Let $\varphi : \mathbb{R} \times \mathbb{R}^2 \to SU(2)$ be defined by

$$
\varphi_t(x_1, x_2) = \psi(x_2, x_1 - x_2\omega)^{-1}\psi(t + x_2, x_1 - x_2\omega).
$$

It is easy to see that $\varphi_t(x_1 + 1, x_2) = \varphi_t(x_1, x_2)$ and $\varphi_t(x_1, x_2 + 1) = \varphi_t(x_1, x_2)$, by (14). Then $\varphi : \mathbb{R} \times \mathbb{T} \to SU(2)$ is a $C^2$-function and

$$
\varphi_{t+s}(\bar{x}) = \psi(x_2, x_1 - x_2\omega)^{-1}\psi(t + s + x_2, x_1 - x_2\omega) = \psi(x_2, x_1 - x_2\omega)^{-1}\psi(t + x_2, x_1 - x_2\omega) = \psi(x_2, x_1 - x_2\omega)^{-1}\psi(s + (x_2 + t), (x_1 + t\omega) - (x_2 + t)\omega)
$$

$$
= \varphi_t(\bar{x})\varphi_s(S_t\bar{x}).
$$

Moreover,

$$
\hat{\varphi}(x) = \varphi_1(x, 0) = \psi(0, x)^{-1}\psi(1, x) = \psi(x),
$$

which completes the proof. ■

Suppose that $\alpha, \beta, 1$ are independent over $\mathbb{Q}$. Set $\omega = \alpha/\beta$.

**Theorem 9.3** For every ergodic rotation $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ and for every natural $k$ there exists a $C^2$-cocycle over $T$ whose degree is equal to $2\pi k(1/\alpha, 1/\beta)$.

**Proof.** Let $S$ denote the ergodic flow given by (13). Suppose that $\varphi : \mathbb{R} \times \mathbb{T} \to SU(2)$ is a $C^2$-cocycle over $S$ such that $d(\hat{\varphi}) = 2\pi k$. Consider the cocycle $\varphi_\beta : \mathbb{T} \to SU(2)$ over the rotation $T = S_\beta$. Then $\varphi_\beta^{(n)} = \varphi_{\beta n}$ and

$$
\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_1} \varphi_\beta^{(n)}(\varphi_\beta^{(n)})^{-1} \right\|_{L^1} = |\beta| \lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_1} \varphi_{\beta n}(\varphi_{\beta n})^{-1} \right\|_{L^1}.
$$

It follows that

$$
d(\varphi_\beta) = |\beta|d(\varphi) = |\beta|(1, |\omega|)d(\hat{\varphi}) = (|\beta|, |\alpha|)d(\hat{\varphi}),
$$

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which proves the theorem. ■

Suppose that \( \beta \in (0,1) \). Let \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle over \( S \) such that \( \hat{\varphi} \) is a diagonal \( C^2 \)-cocycle with nonzero degree. Set \( T = S_\beta \) and \( \psi = \varphi_\beta \). Let \( p : \mathbb{T}^2 \to SU(2) \) be a \( BV_R \)-function such that

\[
p(x_1, x_2) = \varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1}
\]

for \( (x_1, x_2) \in \mathbb{R} \times [0,1) \). Then

\[
p(T(x_1, x_2)) = \begin{cases} 
\varphi_{x_2 + \beta}(x_1 - x_2 \omega, 0)^{-1} & \text{for } x_2 \in [0,1 - \beta) \\
\varphi_{x_2 + \beta - 1}(x_1 - (x_2 - 1) \omega, 0)^{-1} & \text{for } x_2 \in [1 - \beta, 1),
\end{cases}
\]

Moreover,

\[
\varphi_{x_2 + \beta}(x_1 - x_2 \omega, 0) = \varphi_{x_2}(x_1 - x_2 \omega, 0) \varphi_{\beta}(x_1, x_2)
\]

and

\[
\varphi_{x_2 + \beta - 1}(x_1 - (x_2 - 1) \omega, 0) = \varphi_{-1}(x_1 - (x_2 - 1) \omega, 0) \varphi_{x_2 + \beta}(x_1 - x_2 \omega, 0) = \varphi_{1}(x_1 - x_2 \omega, 0)^{-1} \varphi_{x_2 + \beta}(x_1 - x_2 \omega, 0).
\]

It follows that \( p(\bar{x}) \delta(\bar{x}) p(T \bar{x})^{-1} = \psi(\bar{x}) \), where \( \delta : \mathbb{T}^2 \to SU(2) \) is the diagonal \( BV_R \)-cocycle given by

\[
\delta(x_1, x_2) = \begin{cases} 
\text{Id} & \text{for } x_2 \in [0,1 - \beta) \\
\hat{\varphi}(x_1 - x_2 \omega) & \text{for } x_2 \in [1 - \beta, 1),
\end{cases}
\]

**Lemma 9.4** Let \( \phi : \mathbb{T}^2 \to \mathbb{T} \) be a cocycle over the rotation \( T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta) \). Suppose that \( \phi|\mathbb{T} \times [0,\gamma), \phi|\mathbb{T} \times [\gamma,1) \) are \( C^1 \)-functions, where \( \gamma \) is irrational. If \( d(\phi(\cdot,0)) \neq d(\phi(\cdot,\gamma)) \), then \( \phi \) is not a coboundary.

**Proof.** Set \( I_1 = [0,\gamma), I_2 = [\gamma,1) \), \( a_1 = d(\phi(\cdot,0)) \) and \( a_2 = d(\phi(\cdot,\gamma)) \). Then there exists a function \( \phi : \mathbb{T}^2 \to \mathbb{R} \) such that \( \phi|\mathbb{T} \times I_j \) is of class \( C^1 \) for \( j = 1,2 \) and \( \phi(x_1, x_2) = \exp 2\pi i (\bar{\phi}(x_1, x_2) + a_j x_1) \) for any \( (x_1, x_2) \in \mathbb{T} \times I_j \).

Clearly, it suffices to show that

\[
\int_{\mathbb{T}^2} \phi^{(n)}(x_1, x_2) dx_1 dx_2 \to 0.
\]

Next note that

\[
\phi^{(n)}(x_1, x_2) = \exp 2\pi i (\bar{\phi}^{(n)}(x_1, x_2) + (a_1 S_1^n(x_2) + a_2 S_2^n(x_2)) x_1 + c_n(x_2)),
\]

where \( S_1^n(x) = \sum_{k=0}^{n-1} \mathbf{1}_k(x + k\beta) \) and \( c_n(x) = \sum_{k=0}^{n-1} k a_2 (a_1 \mathbf{1}_k + a_2 \mathbf{1}_{\ell_k})(x + k\beta) \).

Since the rotation by \( \beta \) is uniquely ergodic,

\[
\frac{1}{n}(a_1 S_1^n + a_2 S_2^n) \to a_1 \gamma + a_2 (1 - \gamma)
\]

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uniformly. Since \( a_1 \neq a_2 \) and \( \gamma \) is irrational, there exists \( S > 0 \) and \( n_0 \in \mathbb{N} \) such that \( |a_1 S_1^n(x) + a_2 S_2^n(x)| \geq nS \) for all \( x \in \mathbb{T} \) and \( n \geq n_0 \). Applying integration by parts, we get

\[
\left| \int_{\mathbb{T}^2} \tilde{\phi}^{(n)}(x_1, x_2) dx_1 dx_2 \right|
\leq \int_0^1 \int_0^1 e^{2\pi i \tilde{\phi}^{(n)}(x_1, x_2)} dx_1 \left| \int_0^1 e^{2\pi i \tilde{\phi}^{(n)}(x_1, x_2)} dx_2 \right|
\leq \int_0^1 \int_0^1 e^{2\pi i \tilde{\phi}^{(n)}(x_1, x_2)} dx_1 \left| \int_0^1 e^{2\pi i \tilde{\phi}^{(n)}(x_1, x_2)} dx_2 \right|
\leq \frac{1}{n} \int_\mathbb{T} \frac{\partial}{\partial x_1} \tilde{\phi}^{(n)}(x_1, x_2) dx_1 dx_2.
\]

Since \( \frac{\partial}{\partial x_1} \tilde{\phi} \in L^1(\mathbb{T}^2, \mathbb{C}), \)

\[
\frac{1}{n} \frac{\partial}{\partial x_1} \tilde{\phi}^{(n)} \to \int_{\mathbb{T}^2} \frac{\partial}{\partial x_1} \tilde{\phi}(x_1, x_2) dx_1 dx_2 = 0
\]
in \( L^1(\mathbb{T}^2, \mathbb{C}), \) by the Birkhoff ergodic theorem, and the proof in complete. \( \blacksquare \)

This leads to the following conclusion.

**Corollary 9.5** For every ergodic rotation \( T \) on \( \mathbb{T}^2 \) there exists a \( C^2 \)-cocycle \( \psi \) with nonzero degree such that the Lebesgue component in the spectrum of \( T_\psi \) has countable multiplicity and \( \psi \) is not cohomologous to any diagonal \( C^1 \)-cocycle.

**Proof.** Let \( \hat{\varphi} : \mathbb{T} \to \mathbb{T} \) be a \( C^2 \)-function with nonzero topological degree. Let \( \varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2) \) be a \( C^2 \)-cocycle over \( S \) such that \( \hat{\varphi} = \left[ \begin{array}{cc} \varphi & 0 \\ 0 & (\varphi)^{-1} \end{array} \right] \). Define \( \psi = \varphi \beta \). Then \( d(\psi) = 2\pi |\beta| |\alpha| d(\varphi) \neq 0 \). Moreover, \( \psi \) and the diagonal cocycle \( \delta : \mathbb{T}^2 \to SU(2) \) given by

\[
\delta(x_1, x_2) = \left\{ \begin{array}{ll}
\text{Id} & \text{for } x_2 \in [0, 1 - \beta) \\
\varphi(x_1 - x_2 \omega) & \text{for } x_2 \in [1 - \beta, 1)
\end{array} \right.
\]

are cohomologous with a transfer function in \( BV^R(\mathbb{T}^2, SU(2)) \). Applying Theorem 8.2 and Lemma 8.3, we get the first part of our claim.

Next suppose that \( \psi \) is cohomologous to a diagonal \( C^1 \)-cocycle. Then it is easy to see that the cocycle \( \eta : \mathbb{T}^2 \to \mathbb{T} \) given by

\[
\eta(x_1, x_2) = \left\{ \begin{array}{ll}
\text{Id} & \text{for } x_2 \in [0, 1 - \beta) \\
\varphi(x_1 - x_2 \omega) & \text{for } x_2 \in [1 - \beta, 1)
\end{array} \right.
\]

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is cohomologous to a $C^1$-cocycle $g: \mathbb{T}^2 \to \mathbb{T}$. Applying Lemma 9.4 for $\phi = \eta g^{-1}$ and $\gamma = 1 - \beta$ we find that $\eta g^{-1}$ is not a coboundary, which completes the proof.

References


