A Global Theory of Algebras of Generalized Functions

M. Grosser
M. Kunzinger
R. Steinbauer
J. Vickers

Vienna, Preprint ESI 813 (1999)  
December 27, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
A global theory of algebras of generalized functions

M. Grosser, M. Kunzinger, R. Steinbauer
Universität Wien
Institut für Mathematik

J. Vickers
University of Southampton,
Department of Mathematics

Abstract. We present a geometric approach to defining an algebra \( \hat{G}(M) \) (the Colombeau algebra) of generalized functions on a smooth manifold \( M \) containing the space \( D'(M) \) of distributions on \( M \). Based on differential calculus in convenient vector spaces we achieve an intrinsic construction of \( \hat{G}(M) \). \( \hat{G}(M) \) is a differential algebra, its elements possessing Lie derivatives with respect to arbitrary smooth vector fields. Moreover, we construct a canonical linear embedding of \( D'(M) \) into \( \hat{G}(M) \) that renders \( C^\infty(M) \) a faithful subalgebra of \( \hat{G}(M) \). Finally, it is shown that this embedding commutes with Lie derivatives. Thus \( \hat{G}(M) \) retains all the distinguishing properties of the local theory in a global context.

2000 Mathematics Subject Classification. Primary 46F30; Secondary 46T30.

Key words and phrases. Algebras of generalized functions, Colombeau algebras, distributions on manifolds, calculus on infinite dimensional spaces.

1 Introduction

Colombeau’s theory of algebras of generalized functions ([6], [7], [8], [23]) is a well-established tool for treating nonlinear problems involving singular objects, in particular, for studying nonlinear differential equations in generalized functions. A Colombeau algebra \( \mathcal{G}(\Omega) \) on an open subset \( \Omega \) of \( \mathbb{R}^n \) is a differential algebra containing \( D'(\Omega) \) as a linear subspace and \( C^\infty(\Omega) \) as a faithful subalgebra. The embedding \( D'(\Omega) \hookrightarrow \mathcal{G}(\Omega) \) commutes with partial differentiation and the functor \( \Omega \to \mathcal{G}(\Omega) \) defines a fine sheaf of differential algebras. In view of L. Schwartz’s impossibility result ([26]) these properties of Colombeau algebras are optimal in a very precise sense (cf. [23], [24]). In the so-called “full” version of the theory the embedding \( D'(\Omega) \hookrightarrow \mathcal{G}(\Omega) \) is canonical, as opposed to the “special” or “simplified” version (cf. the remark below), where the embedding depends on a particular mollifier.

The main interest in the theory so far has come from the field of nonlinear partial differential equations (cf. e.g. [3], [4], [10], [11], [12], as well as
[23] and the literature cited therein), whereas the development of a theory of Colombeau algebras on manifolds proceeded at a much slower pace. [1] presents an approach which basically consists in lifting the sheaf \( \mathcal{G} \) from \( \mathbb{R}^n \) to a manifold \( M \). A more refined sheaf-theoretic analysis of Colombeau algebras on manifolds is given in [13]. It treats the “special” version of the algebra in the sense of [23], p. 169f, whose elements (termed ultrafunctions by the authors) depend on a real regularization parameter. In both approaches, as well as in the construction envisaged in [2], the canonical embedding \( \mathcal{C}^\infty(M) \hookrightarrow \mathcal{D}'(M) \hookrightarrow \mathcal{G}(M) \) is lost when passing from \( M = \Omega \subseteq \mathbb{R}^n \) to a general manifold (due to the fact that in these versions action of diffeomorphisms commutes with embedding only in the sense of association but not with equality in the algebra). Also, although Lie derivatives of elements of \( \mathcal{G}(M) \) are defined in [13] the operation of taking Lie derivatives does not commute with the embedding (again this property only holds in the sense of association).

To remedy the first of these defects, J. F. Colombeau and A. Meril in [9] (using earlier ideas of [6]) introduced an algebra of generalized functions on \( \Omega \) whose elements depend smoothly (in the sense of Silva-differentiability) on \((\varphi, x)\), where \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) and \( x \in \Omega \). The aim of [9] is to make the embedding of \( \mathcal{D}'(\Omega) \) (which is done basically by convolution with the \( \varphi \)'s: \( u \in \mathcal{D}'(\Omega) \mapsto ((\varphi, x) \mapsto \langle u(y), \varphi(y - x) \rangle) \)) commute with the action of diffeomorphisms. However, an explicit counterexample in [19] demonstrated that the construction given in [9] is in fact not diffeomorphism invariant. Also in [19], J. Jelínek gave an improved version of the theory clarifying a number of open questions but still falling short of establishing the existence of an invariant (local) Colombeau algebra. Finally, this aim was achieved in [16] and [17], where a complete construction of diffeomorphism invariant Colombeau algebras on open subsets of \( \mathbb{R}^n \) is developed. Several characterization results derived there will be crucial for the presentation in the following sections.

Summing up, the current state of the theory is that a diffeomorphism invariant local version (and, therefore, a version on manifolds) of Colombeau algebras providing canonical embeddings of smooth functions and distributions is available. Recent applications of Colombeau algebras to questions of general relativity (e.g. [5], [2], [21], [27], for a survey see [28]) have underscored the need for a theory of algebras of generalized functions on manifolds that enjoys two additional features: first, it should be geometric in the sense that its basic objects should be defined intrinsically on the manifold itself. Second, the object to be constructed should be a differential algebra with Lie derivatives commuting with the embedding of \( \mathcal{D}(M) \).

In this paper we construct an algebra \( \hat{\mathcal{G}}(M) \) satisfying both of these requirements. In particular, elements of \( \hat{\mathcal{G}}(M) \) possess a Lie derivative \( \hat{L}_X \)
with respect to arbitrary smooth vector fields $X$ on $M$. Moreover, each Lie derivative commutes with the canonical embedding of $\mathcal{D}'(M)$ into $\hat{G}(M)$, so that in fact all the distinguishing properties of Colombeau algebras on open subsets of $\mathbb{R}^n$ are retained in the global case. The key concept leading to a global formulation of the theory is that of smoothing kernels. Definition 3.3 (i) below is in a sense the diffeomorphism invariant ‘essence’ of the process of regularization via convolution and linear scaling on $\mathbb{R}^n$ while 3.3 (ii) is the invariant formulation of the interplay between $x$- and $y$-differentiation in the local context. Finally, a number of localization results in section 4 allow one to make full use of the well-developed local theory also in the global context.

2 Notation and terminology, the local theory

Throughout this paper, $M$ will denote an oriented paracompact $C^\infty$-manifold of dimension $n$. An atlas of $M$ will usually be written in the form $\mathfrak{A} = \{(U_\alpha, \psi_\alpha) : \alpha \in A\}$. By $\Omega^p_c(M)$ we mean the space of compactly supported (smooth) $n$-forms on $M$. Locally, for coordinates $y^1, \ldots, y^n$ on $U_\alpha$, elements of $\Omega^p_c(\psi_\alpha(U_\alpha))$ will be written as $\varphi \, d^\alpha y := \varphi \, dy^1 \wedge \cdots \wedge dy^n$. The pullback of any $\omega \in \Omega^p_c(\psi_\alpha(U_\alpha))$ under $\psi_\alpha$ is written as $\psi_\alpha^*(\omega)$. Then $\int_M \psi_\alpha^*(\varphi \, d^\alpha y) = \int_{\psi_\alpha(U_\alpha)} \varphi \, d^\alpha y$ for all $\varphi \, d^\alpha y \in \Omega^p_c(\psi_\alpha(U_\alpha))$. For $U$ (open) $\subseteq M$ open we will notationally suppress the embedding of $\Omega^p_c(U)$ into $\Omega^p_c(M)$, and similarly for the inclusion of $\mathcal{D}(U)$ (compactly supported smooth functions with support contained in $U$) into $\mathcal{D}(M)$. We generally use the following convention: if $B \subseteq \psi_\alpha(U_\alpha)$ then $\hat{B} := \psi_\alpha^{-1}(B)$ and if $f : \psi_\alpha(U_\alpha) \to \mathbb{R}$ resp. $\mathbb{C}$ then $\hat{f} := f \circ \psi_\alpha$. A similar “hat-convention” will be applied to the function spaces to be defined in the following section. Moreover, since $M$ is supposed to be oriented we can and shall identify $n$-forms and densities henceforth.

For an open subset $\Omega$ of $\mathbb{R}^n$, the space of distributions on $\Omega$ (i.e., the dual of the (LF)-space $\mathcal{D}(\Omega)$) will be denoted by $\mathcal{D}'(\Omega)$. For a diffeomorphism $\mu : \hat{\Omega} \to \Omega$, the pullback of any $u \in \mathcal{D}'(\Omega)$ under $\mu$ is defined by

$$\langle \mu^* (u), \varphi \rangle = \langle u(y), \varphi(\mu^{-1}(y)) \rangle \cdot |\det D\mu^{-1}(y)|$$  \hspace{1cm} (1)

Concerning distributions on manifolds we follow the terminology of [14] and [22], so the space of distributions on $M$ is defined as $\mathcal{D}'(M) = \Omega^p_c(M)'$. Observe that in this setting test objects no longer have function character but are $n$-forms. In the context of Colombeau algebras it is natural to regard smooth functions as regular distributions (which in the non-orientable case enforces the use of test densities; this is in accordance with [18] but has to be distinguished from the setting of [15]).
Operations on distributions are defined as (sequentially) continuous extensions of classical operations on smooth functions. In particular, for \( X \in \mathcal{X}(M) \) (the space of smooth vector fields on \( M \)) and \( u \in \mathcal{D}'(M) \) the Lie derivative of \( u \) with respect to \( X \) is given by \( \langle L_X u, \omega \rangle = -\langle u, L_X \omega \rangle \). If \( u \in \mathcal{D}'(M) \), \( (U_\alpha, \psi_\alpha) \in \mathcal{A} \) then the local representation of \( u \) on \( U_\alpha \) is the element \( (\psi_\alpha^{-1})^*(u) \in \mathcal{D}(\psi_\alpha(U_\alpha)) \) defined by

\[
\langle (\psi_\alpha^{-1})^*(u), \varphi \rangle = \langle u|_{\psi_\alpha}, \psi_\alpha^* (\varphi d^n y) \rangle \quad \forall \varphi \in \mathcal{D}(\psi_\alpha(U_\alpha))
\]

(2)

It should be clear from (1) and (2) that the character of test objects as \( n \)-forms is actually already built into the local theory of distributions on \( \mathbb{R}^n \) (i.e., on the right hand side of (1) \( u \) acts exactly on the coefficient function of \( (\mu^{-1})^*(\varphi d^n y) \)).

As in [16], [17] differential calculus in infinite dimensional vector spaces will be based on the presentation in [20]. The basic idea is that a map \( f : E \to F \) between locally convex spaces is smooth if it transports smooth curves in \( E \) to smooth curves in \( F \), where the notion of smooth curves is straightforward (via limits of difference quotients). The diffeomorphism invariant theory of Colombeau algebras on open subsets of \( \mathbb{R}^n \) introduced in [16] is based on this notion of differentiability. Of the two (equivalent) formalisms for describing the diffeomorphism invariant local Colombeau theory analyzed in [16], Section 5 only one (the so-called \( J \)-formalism which was employed by Jelínek in [19]) lends itself naturally to an intrinsic generalization on manifolds, as the embedding of distributions does not involve a translation (see below). We briefly recall the main features of this theory.

Let \( \Omega \subseteq \mathbb{R}^n \) open; then define

\[
\mathcal{A}_0(\Omega) = \{ \varphi \in \mathcal{D}(\Omega) | \int \varphi(\xi) d\xi = 1 \}
\]

\[
\mathcal{A}_q(\mathbb{R}^n) = \{ \varphi \in \mathcal{A}_0(\mathbb{R}^n) | \int \varphi(\xi) \xi^\alpha d\xi = 0, \ 1 \leq |\alpha| \leq q, \ \alpha \in \mathbb{N}_0^n \} \quad (q \in \mathbb{N}).
\]

The basic space of the diffeomorphism invariant local Colombeau algebra is defined to be \( \mathcal{E}(\Omega) = \mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega) \). The algebra itself is constructed as the quotient of the space of moderate modulo the ideal of negligible elements \( R \) of the basic space where the respective properties are defined by plugging scaled and transformed “test objects” into \( R \) and analyzing its asymptotic behavior on these “paths” as the scaling parameter \( \varepsilon \) tends to 0. Diffeomorphism invariance of the whole construction is achieved by diffeomorphism invariance of this process— termed “testing for moderateness resp. negligibility”— in [16], Section 9. We shall discuss this matter in some more detail at the end of this section but now proceed by defining the actual “test
objects.” Set \( I = (0, 1] \). Let \( \mathcal{C}_b^\infty (I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \) be the space of smooth maps \( \phi : I \times \Omega \to \mathcal{A}_0(\mathbb{R}^n) \) such that for each \( K \subset \subset \Omega \) (i.e., \( K \) a compact subset of \( \Omega \)) and any \( \alpha \in \mathbb{N}_0^n \), the set \( \{ \partial_x^a \phi(e, x) \mid e \in (0, 1], x \in K \} \) is bounded in \( \mathcal{D}(\mathbb{R}^n) \). For any \( m \geq 1 \) we set

\[
\mathcal{A}_m^\Omega(\Omega) := \{ \Phi \in \mathcal{C}_b^\infty (I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \mid \sup_{x \in K} \int_{\Omega} |\Phi(e, x)(\xi)\xi^a d\xi| = O(e^m) \}
\]

\[
\mathcal{A}_m^\Delta(\Omega) := \{ \Phi \in \mathcal{C}_b^\infty (I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \mid \sup_{x \in K} \int_{\Omega} |\Phi(e, x)(\xi)\xi^a d\xi| = O(e^{m+1-|\alpha|}) \}
\]

Elements of \( \mathcal{A}_m^\Omega(\Omega) \) are said to have asymptotically vanishing moments of order \( m \) (more precisely, in the terminology of [17], elements of \( \mathcal{A}_m^\Omega(\Omega) \) are of type \( [A_m] \), the abbreviation standing for asymptotic vanishing of moments globally, i.e., on each \( K \subset \subset \Omega \). The spaces \( \mathcal{A}_m^\Omega(\Omega) \) and \( \mathcal{A}_m^\Delta(\Omega) \) play a crucial role in the characterizations of the algebra ([16], Section 10) and in Section 4 below (cf. also the discussion of diffeomorphism invariance at the end of this section). Also for later use we note that \( \mathcal{A}_m^\Omega(\Omega) \subseteq \mathcal{A}_m^\Delta(\Omega) \subseteq \mathcal{A}_{m-1}^\Omega(\Omega) \). For \( x \in \mathbb{R}^n \), \( e \in I \) we define translation resp. scaling operators by \( T_x : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n) \), \( T_x(\varphi) = \varphi(\cdot - x) \) and \( S_x : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n) \), \( S_x \varphi = \varepsilon^{-n} \varphi(\cdot / \varepsilon) \). Now the subspaces of moderate resp. negligible elements of \( \mathcal{E}(\Omega) \) are defined by

\[
\mathcal{E}_m(\Omega) = \{ R \in \mathcal{E}(\Omega) \mid \forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^n \exists N \in \mathbb{N} \forall \phi \in \mathcal{C}_b^\infty (I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \mid \sup_{x \in K} |\partial^a (R(T_x S_x \phi(e, x), x))| = O(e^{-N}) \ (e \to 0) \}
\]

\[
\mathcal{N}(\Omega) = \{ R \in \mathcal{E}(\Omega) \mid \forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^n \forall r \in \mathbb{N} \exists m \in \mathbb{N} \forall \phi \in \mathcal{C}_b^\infty (I \times \Omega, \mathcal{A}_m(\mathbb{R}^n)) \mid \sup_{x \in K} |\partial^a (R(T_x S_x \phi(e, x), x))| = O(e^{-r}) \ (e \to 0) \}
\]

By [16], Th. 7.9 and [17], Corollaries 16.8 and 17.6, \( R \in \mathcal{E}_m(\Omega) \) is negligible iff \( \forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_0^n \forall r \in \mathbb{N} \exists m \in \mathbb{N} \forall \phi \in \mathcal{A}_m(\Omega), \exists C > 0 \exists \eta > 0 \forall \varepsilon (0 < \varepsilon < \eta) \forall x \in K:\n\]

\[
|\partial^a (R(T_x S_x \phi(e, x), x))| \leq C \varepsilon^r .
\]

The diffeomorphism invariant Colombeau algebra \( \mathcal{G}(\Omega) \) on \( \Omega \) is the quotient algebra \( \mathcal{E}_m(\Omega) / \mathcal{N}(\Omega) \). Partial derivatives in \( \mathcal{G}(\Omega) \) are defined as

\[
(D_i R)(\varphi, x) = -((d_i R)(\varphi, x))(\partial_i \varphi) + (\partial_i R)(\varphi, x)
\]

where \( d_i \) and \( \partial_i \) denote differentiation with respect to \( \varphi \) and \( x_i \), respectively. Note that formula (3) is exactly the result of translating the usual partial
differentiation from the C-formalism to the J-formalism (cf. [16], Section 5). With these operations, \( \Omega \to \mathcal{G}(\Omega) \) becomes a fine sheaf of differential algebras on \( \mathbb{R}^n \). For \( R \in \mathcal{E}_m(\Omega) \) we denote by \( \text{cl}[R] \) its equivalence class in \( \mathcal{G}(\Omega) \). The map
\[
\iota : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)
\]
\[
u \to \text{cl}[\langle \phi, x \rangle \rightarrow \langle u, \varphi \rangle]
\]
provides a linear embedding of \( \mathcal{D}'(\Omega) \) into \( \mathcal{G}(\Omega) \) whose restriction to \( \mathcal{C}^\infty(\Omega) \) coincides with the embedding
\[
\sigma : \mathcal{C}^\infty(\Omega) \to \mathcal{G}(\Omega)
\]
\[
f \to \text{cl}[\langle \phi, x \rangle \rightarrow f(x)]
\]
so \( \iota \) renders \( \mathcal{C}^\infty(\Omega) \) a faithful subalgebra and \( \mathcal{D}'(\Omega) \) a linear subspace of \( \mathcal{G}(\Omega) \). Moreover, \( \iota \) commutes with partial derivatives due to the specific form of (3). The pullback of \( R \in \mathcal{E}(\Omega) \) under a diffeomorphism \( \mu : \bar{\Omega} \to \Omega \) is defined as
\[
(\bar{\mu}R)(\bar{\phi}, \bar{x}) = R(\bar{\mu}(\phi, \bar{x}))
\]
where \( \bar{\mu}(\phi, \bar{x}) = \left( (\phi \circ \mu^{-1}) \cdot |\det D\mu^{-1}|, \mu(\bar{x}) \right) \). Pullback under diffeomorphisms then commutes with the embedding of \( \mathcal{D}'(\Omega) \) into \( \mathcal{G}(\Omega) \).

Finally we return to the issue of diffeomorphism invariance. Since the action of a diffeomorphism on \( R \in \mathcal{E}(\Omega) \) is defined in a functorial way diffeomorphism-invariance of the definition of the moderate resp. negligible elements of \( \mathcal{E}(\Omega) \) indeed is invariant if the class of scaled and transformed “test objects” is; more precisely if
\[
\phi(\varepsilon, x) = S^{-1}_\varepsilon \circ T_{-x} \circ \text{pr}_1 \circ \bar{\mu}(T_{\mu^{-1}} \circ S_x \tilde{\phi}(\varepsilon, \mu^{-1} x))(\xi), \mu^{-1} x)
\]
\[
= \tilde{\phi}(\varepsilon, \mu^{-1} x) \left( \mu^{-1}(\varepsilon \xi + x) - \mu^{-1} x \right) \varepsilon \cdot |\det D\mu^{-1}(\varepsilon \xi + x)| \quad (4)
\]
is a valid “test object” if \( \tilde{\phi}(\varepsilon, \bar{x}) \) was. However, if \( \tilde{\phi} \in \mathcal{C}^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \) in general \( \phi \) will neither be defined on all of \( I \times \Omega \) nor will its moments be vanishing up to order \( q \). To remedy these defects a rather delicate analysis of the testing procedure is required: Denote by \( \mathcal{C}^\infty_{k,w}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) \) the space of all \( \phi : D \to \mathcal{A}_0(\mathbb{R}^n) \) where \( D \) is some subset (depending on \( \varphi \)) of \( (0, 1] \times \Omega \) and for \( D, \varphi \) the following holds:

For each \( L \subset \subset \Omega \) there exists \( \varepsilon_0 \) and a subset \( U \) of \( D \) which is open in \( (0, 1] \times \Omega \) such that
\[
(0, \varepsilon_0] \times L \subseteq U(\subseteq D) \text{ and } \phi \text{ is smooth on } U \quad (5)
\]
\[
\{ \partial^\beta \phi(\varepsilon, x) \mid 0 < \varepsilon \leq \varepsilon_0, \ x \in L \} \text{ is bounded in } \mathcal{D}(\mathbb{R}^n) \forall \beta \in \mathbb{N}_0^n \quad (6)
\]
(the subscript \(w\) signifies the weaker requirements on the domain of definition of \(\phi\)). Then by [16], Th. 10.5 \(R \in \mathcal{E}(\Omega)\) is moderate iff \(\forall K \subseteq \Omega \ \forall \alpha \in \mathbb{N}_0^\alpha \ \exists N \in \mathbb{N} \ \forall \phi \in C^\infty_{h,w}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)), \ \phi : D \to \mathcal{A}_0(\mathbb{R}^n)) \exists C > 0 \ \exists \eta > 0 \ \forall \varepsilon (0 < \varepsilon < \eta) \ \forall x \in K: (\varepsilon, x) \in D \) and

\[
|\partial^\alpha (R(T_x S_x \phi(\varepsilon, x), x))| \leq C \varepsilon^{-N}
\]

Moreover, by [16], Th. 7.14, (4) is an element of \(C^\infty_{h,w}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))\) for every \(\phi \in C^\infty_{h,w}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))\). The subspace of \(C^\infty_{h,w}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))\) consisting of those \(\phi\) whose moments up to order \(m\) vanish asymptotically on each compact subset of \(\Omega\) will be written as \(\mathcal{A}^m_{m,w}(\Omega)\) (also, \(\mathcal{A}^m_{m,w}(\Omega)\) is defined analogously). By the proof of [16], Cor. 10.7 \(R \in \mathcal{E}_m(\Omega)\) is negligible iff \(\forall K \subseteq \Omega \ \forall \alpha \in \mathbb{N}_0^\alpha \ \forall r \in \mathbb{N} \ \exists m \in \mathbb{N} \ \forall \phi \in \mathcal{A}^m_{m,w}(\Omega), \ \phi : D \to \mathcal{A}_0(\mathbb{R}^n)) \exists C > 0 \ \exists \eta > 0 \ \forall \varepsilon (0 < \varepsilon < \eta) \ \forall x \in K: (\varepsilon, x) \in D \) and

\[
|\partial^\alpha (R(T_x S_x \phi(\varepsilon, x), x))| \leq C \varepsilon^{-r}
\]

and if \(\phi \in \mathcal{A}^m_{m,w}(\Omega)\) then (4) defines an element of \(\mathcal{A}^m_{m,w}(\Omega)\). These facts directly imply diffeomorphism invariance of local Colombeau algebras ([16], Thms. 7.15 and 7.16).

The diffeomorphism invariance of the scaled, transformed “test objects” demonstrated above suggests to choose an analogue of these as the main “test objects” on the manifold, where no natural scaling resp. translation operator is available. It is precisely this notion which is captured in the definition of the smoothing kernels below.

### 3 Smoothing kernels and basic function spaces

In this section we introduce the basic definitions and operations needed for an intrinsic definition of algebras of generalized functions on manifolds.

#### 3.1 Definition

\(\hat{\mathcal{A}}_0(M) := \{ \omega \in \Omega^\infty_0(M) : \int \omega = 1 \}\)

The basic space for the forthcoming definition of the Colombeau algebra on \(M\) is defined as follows:

#### 3.2 Definition

\(\hat{\mathcal{E}}(M) = C^\infty(\hat{\mathcal{A}}_0(M) \times M)\)
Then
\[(\psi_a^* \times \psi_a^{-1})(\hat{A}_0(\psi_a(U_a)) \times \psi_a(U_a)) \subseteq \hat{A}_0(M) \times M\]
and locally we have \(\hat{E}(\psi_a(U_a)) \cong E(\psi_a(U_a))\), the isomorphism being effected by \(\hat{E}(\psi_a(U_a)) \ni R \mapsto \{([\varphi, x] \mapsto \hat{R}(\varphi) d^n y, x]\). In what follows, we will therefore use \(\hat{E}(\psi_a(U_a))\) and \(E(\psi_a(U_a))\) interchangeably. Clearly the map
\[\psi_a^* \times \psi_a^{-1} : \Omega^n(\psi_a(U_a)) \times \psi_a(U_a) \to \Omega^n(\psi_a(U_a)) \times U_a \hookrightarrow \Omega^n(M) \times M\]
is smooth. Therefore, for any \(R \in \hat{E}(M)\) its local representation
\[(\psi_a^{-1})^\circ R := R \circ (\psi_a^* \times \psi_a^{-1})\]
is an element of \(\hat{E}(\psi_a(U_a))\). More generally, if \(\mu : M_1 \to M_2\) is a diffeomorphism and \(R \in \hat{E}(M_2)\) then its pullback \(\tilde{\mu}(R) \in \hat{E}(M_1)\) under \(\mu\) is defined as \(R \circ \tilde{\mu}\) where \(\tilde{\mu}(\omega, p) = ((\mu^{-1})^* \omega, \mu(p))\) and clearly \((\mu_1 \circ \mu_2)^* \tilde{\mu} = \tilde{\mu}_2 \circ \tilde{\mu}_1\) for diffeomorphisms \(\mu_1, \mu_2\).

Let \(f : M \times M \to \bigwedge^n T^* M\) be smooth such that for each \((p, q) \in M \times M\), \(f(p, q)\) belongs to the fiber over \(q\) or, equivalently, that for each fixed \(p \in M\), \(f_p := (q \mapsto f(p, q))\) represents a member of \(\Omega^n(M)\). Obviously, \(p \mapsto f_p \in C^\infty(M, \Omega^n(M))\) in this case. Given a smooth vector field \(X\) on \(M\), we define two notions of Lie derivatives of \(f\) with respect to \(X\) which, essentially, arise as Lie derivatives of \(q \mapsto f(p, q)\) resp. \(p \mapsto f(p, q)\):

On the one hand, viewing \(\Omega^n(M)\) together with the topology of pointwise (i.e., fiberwise) convergence as a locally convex space we define
\[(L_X f)(p, q) := L_X (p \mapsto f(p, q)) = \frac{d}{dt} \bigg|_0 (f(F^X_t)(p), q); \quad (7)\]
on the other hand, we set
\[(L_X f)(p, q) := L_X (q \mapsto f(p, q)) = \frac{d}{dt} \bigg|_0 ((F^X_t)^* f_p)(q) \quad (8)\]
where the latter symbol \(L_X\) denotes the usual Lie derivative on the bundle \(\Omega^n(M)\).

Now we are ready to introduce the space of smoothing kernels which will serve as an analogue for the (unbounded) sequences \((\varphi_\varepsilon)_\varepsilon\) used in [7].

3.3 Definition \(\Phi \in C^\infty(I \times M, \hat{A}_0(M)) \subseteq C^\infty(I \times M \times M, \Lambda^n T^* M)\) is called a smoothing kernel if it satisfies the following conditions

\[(i) \quad \forall K \subset \subset M \exists \varepsilon_0, C > 0 \quad \forall p \in K \quad \forall \varepsilon \leq \varepsilon_0 : \text{supp} \Phi(\varepsilon, p) \subseteq B_{C \varepsilon}(p)\]
(ii) \( \forall K \subset M \ \forall k, l \in \mathbb{N}_0 \ \forall X_1, \ldots, X_k, Y_1, \ldots, Y_l \in \mathcal{X}(M) \)
\[
\sup_{p \in K, q \in M} ||L_{Y_1} \cdots L_{Y_l}(L^l_{X_k} + L^l_{X_{k-1}}) \cdots (L^l_{X_1} + L^l_{X_0}) \Phi(\varepsilon, p)(q)|| = O(\varepsilon^{-(n+1)})
\]

The space of smoothing kernels on \( M \) is denoted by \( \tilde{A}_0(M) \).

In (i) the radius of the ball \( B_{\varepsilon C}(p) \) has to be measured with respect to the Riemannian distance induced by a Riemannian metric \( h \) on \( M \). By Lemma 3.4 below, (i) then holds in fact for the distance induced by any Riemannian metric \( h' \) with a new set of constants \( \varepsilon_0(h') \) and \( C(h') \). Similarly in (ii), \( \| \cdot \| \) denotes the norm induced on the fibers of \( \Omega^o_\varepsilon(M) \) by any Riemannian metric on \( M \) (i.e. convergence with respect to \( \| \cdot \| \) amounts to convergence of all components in every local chart). Thus both (i) and (ii) are independent of the Riemannian metric chosen on \( M \), hence intrinsic.

3.4 Lemma Let \( M \) be a smooth paracompact manifold and let \( h_1, h_2 \) be Riemannian metrics on \( M \). Then for all \( K \subset M \) there exist \( \varepsilon_0(K) \) and \( C > 0 \) such that \( \forall p \in K \ \forall \varepsilon \leq \varepsilon_0 \):
\[
B_{\varepsilon}^{(2)}(p) \subseteq B_{C\varepsilon}^{(1)}(p)
\]
where \( B_{\varepsilon}^{(i)}(p) = \{ q \in M : d_i(p, q) < \varepsilon \} \) and \( d_i \) denotes Riemannian distance with respect to \( h_i \).

Proof. We first show that \( \forall K \subset M \exists C \geq 0 \) such that
\[
h_1(p)(v, v) \leq Ch_2(p)(v, v) \quad \forall p \in K \ \forall v \in T_p M.
\]
Without loss of generality we may assume \( K \subset U_\alpha \) for some chart \((\psi_\alpha, U_\alpha)\). Denoting by \( h_\alpha^i \) \((i = 1, 2)\) the local representations of \( h_i \) in this chart it follows that
\[
f : (x, v) \rightarrow \frac{h_1^\alpha(x)(v, v)}{h_2^\alpha(x)(v, v)}
\]
is continuous (even smooth) on \( \psi_\alpha(U_\alpha) \times \mathbb{R}^n \setminus \{0\} \). Thus
\[
\sup_{x \in \psi_\alpha(K)} \sup_{x \in \mathbb{R}^n \setminus \{0\}} f(x, v) = \sup_{x \in \psi_\alpha(K)} \sup_{x \in \mathbb{R}^n \setminus \{0\}} f(x, v) < \infty
\]
with \( B_{\varepsilon}^{(\text{ucl})}(0) \) the Euclidian ball of radius 1 around 0.

Next we choose a geodesically convex (with respect to \( h_2 \)) relatively compact neighborhood \( U_p \) of \( p \in M \) (cf. e.g. [25], Prop. 5.7). Moreover, let \( C \) be the
square root of the constant in (9) with $K = \overline{U}_p$. Given any $q, q' \in U_p$ let $\alpha$ be the unique $h_\varepsilon$-geodesic in $U_p$ connecting $q$ and $q'$. Then $d_1(q, q') \leq L_1(\alpha) \leq C L_2(\alpha) = C d_2(q, q')$ where $L_i$ denotes the length of $\alpha$ with respect to $h_i$.

Now for each $p \in K$ there exist $U_p$ and $C_p$ as above and we choose $\varepsilon_p$ such that $B_{\varepsilon_p}^i(p) \subseteq U_p$. Then there exist some $p_1, \ldots, p_m$ in $K$ with $K \subseteq \bigcup_{i=1}^m B_{\varepsilon_p}^i(p_i) =: U$ and we set $\varepsilon_0 := \min(\text{dist}_2(K, \partial U), \frac{\varepsilon_{p_1}}{2}, \ldots, \frac{\varepsilon_{p_m}}{2})$ and $C := \max_{1 \leq i \leq m} C_{p_i}$. Let $p \in K$, $\varepsilon \leq \varepsilon_0$, $q \in B_2(p)$. There exists some $i$ with $d_2(p, p_i) \leq \frac{\varepsilon_{p_i}}{2}$ and by construction $d_2(p, q) \leq \frac{\varepsilon_{p_i}}{2}$, so $p, q \in B_2(p_i) \subseteq U_i$. Hence $d_1(p, q) \leq C_{p_i} d_2(p, q)$ and, finally, $q \in B_{C_2}^i(p)$. 

The next step is to introduce the following grading on the space of smoothing kernels.

3.5 Definition. For each $m \in \mathbb{N}$ we denote by $\tilde{\mathcal{A}}_m(M)$ the set of all $\Phi \in \tilde{\mathcal{A}}(M)$ such that $\forall f \in C^\infty(M)$ and $\forall K \subset M$

$$\sup_{p \in K} \left| f(p) - \int_M f(q) \Phi(\varepsilon, p)(q) \right| = O(\varepsilon^{m+1})$$

3.6 Remark. 3.5 is modelled with a view to reproducing the main technical ingredient for proving $\mathcal{A}_\infty = \sigma$ (i.e. the fact that the embedding of distributions into $\mathcal{G}$ coincides with the “identical” embedding on $C^\infty$, cf. e.g., [16], Th. 7.4 (iii)) in the local theory. Essentially, the argument in the local case consists in (substitution of $y' = \frac{y - x}{\varepsilon}$ and) Taylor expansion of $f(x) - f(y)\varepsilon^n \varphi\left(\frac{y - x}{\varepsilon}\right)dy$ ($\varphi \in \mathcal{A}_0$) yielding appropriate powers of $\varepsilon$ as to establish this term to be negligible. In fact, this argument is at the very heart of Colombeau’s construction and may be viewed as the main technical motivation for the concrete form of the sets $\mathcal{A}_\varphi$ and, a fortiori, of the ideal $\mathcal{N}$. In 3.5 we turn the tables and define the sets $\mathcal{A}_m(M)$ by the analogous estimate. Moreover, as we shall see shortly (cf. 4.2), locally there is a one-to-one correspondence between elements of $\mathcal{A}_m(M)$ and elements of $\mathcal{A}_m^\infty$, so the two approaches are in fact equivalent (although only one of them, namely the one occurring in 3.5 admits an intrinsic formulation on $M$).

We are now in a position to prove the nontriviality of the space of smoothing kernels as well as of the spaces $\tilde{\mathcal{A}}_m(M)$:

3.7 Lemma

(i) $\tilde{\mathcal{A}}_0(M) \neq \emptyset$. 

10
(ii) $\mathcal{A}_\varepsilon^m(M) \neq \emptyset$ ($m \in \mathbb{N}$).

**Proof.** (i) Let $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ be an oriented atlas of $M$ such that each $U_\alpha$ is relatively compact. Let $\{\chi_\alpha \mid \alpha \in A\}$ be a subordinate partition of unity and pick $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. For each $\alpha$, choose $\hat{\chi}^1_\alpha \in \mathcal{D}(U_\alpha)$ such that $\hat{\chi}^1_\alpha \equiv 1$ in an open neighborhood $W_\alpha \subset U_\alpha$ of $\text{supp} \, \hat{\chi}^1_\alpha$. Let $\chi_\alpha := \hat{\chi}^1_\alpha \circ \psi^{-1}_\alpha$, $\chi^1_\alpha := \hat{\chi}^1_\alpha \circ \psi^{-1}_\alpha$. Now set

$$\varphi^0_\alpha(\varepsilon, x)(y) := \varepsilon^{-n} \varphi\left(\frac{y-x}{\varepsilon}\right).$$

There exists some $\varepsilon^0_\alpha = \varepsilon^0_\alpha(\text{supp} \, \chi_\alpha)$ in $(0,1]$ such that for each $x \in \text{supp} \, \chi_\alpha$ and each $0 < \varepsilon \leq \varepsilon^0_\alpha$ we have $\text{supp} \, \varphi^0_\alpha(\varepsilon, x) \subset \subset \psi_\alpha(W_\alpha)$. Choose $\lambda_\alpha : \mathbb{R} \to [0,1]$ smooth, $\lambda_\alpha \equiv 1$ on $(0, \frac{\varepsilon^0_\alpha}{3}]$ and $\lambda_\alpha \equiv 0$ on $(\frac{\varepsilon^0_\alpha}{2},1]$. Finally, let $\omega \in \mathcal{A}_0(M)$.

Then we define our prospective smoothing kernel by

$$\Phi(\varepsilon, p)(q) := \sum_\alpha \hat{\chi}_\alpha(p) \left[ \chi^1_\alpha \left( \lambda_\alpha(\varepsilon) \psi_\alpha\left( \varphi^0_\alpha(\varepsilon, \psi_\alpha(p))(\cdot) \chi^1_\alpha(\cdot) \right) \right) d^n y + (1 - \lambda_\alpha(\varepsilon) )\omega \right](q).$$

By construction, $\Phi$ is smooth and $\int \Phi(\varepsilon, p) = 1$ for all $\varepsilon \in (0,1]$ and all $p \in M$. For any given $K \subset M$, set $\varepsilon_K := \min \frac{\varepsilon^0_\alpha}{3}$ where $\alpha$ ranges over the (finitely many) $\alpha$ with $K \cap U_\alpha \neq \emptyset$; then the terms in (10) containing $\omega$ vanish for $p \in K$ and $\varepsilon \leq \varepsilon_K$.

In order to prove 3.3 (i), let $K \subset M$. Since for $p \in K$ and $\varepsilon \leq \varepsilon_K$, supp $\Phi(\varepsilon, p)$ is contained in a finite union of supports of $q \mapsto [\psi_\alpha^*(\varphi^0_\alpha(\varepsilon, \psi_\alpha(p))(\cdot) \chi^1_\alpha(\cdot) d^n y)](q)$

it suffices to consider only one term of the latter form. Clearly, there exist $C$ and $(0 <) \varepsilon_0 (\leq \varepsilon_K)$ such that supp $\varphi^0_\alpha(\varepsilon, x) \subset \subset B^\varepsilon_C(x)$ for $x \in \text{supp} \, \chi_\alpha$ and all $\varepsilon \leq \varepsilon_0$. Extending the pullback under $\psi_\alpha$ of a suitable cut-off of the Euclidian metric on $\psi_\alpha(U_\alpha)$ to a Riemannian metric on $M$, the result follows from 3.4.

Concerning 3.3 (ii), since $K = \bigcup_{i=1}^m K_{\alpha_i}, K_{\alpha_i} \subset \subset U_{\alpha_i}$, it suffices to estimate $\Phi$ on each $K_{\alpha_i} \times (0, \varepsilon_i]$ for some $\varepsilon_i > 0$. Thus we may assume that $K \subset \subset U_{\alpha_0}$ for some fixed $\alpha_0$. Let $L$ be a compact neighborhood of $K$ in $U_{\alpha_0}$ and let $\varepsilon \leq \varepsilon_L$ in what follows. Each of the terms in (10) for which supp $\hat{\chi}^1_\alpha$ does not intersect $K$ vanishes in some open neighborhood of $K$ and can therefore be neglected. Each of the finitely many remaining terms vanishes outside supp $\hat{\chi}^1_\alpha$. Since $K \subseteq (K \setminus \text{supp} \, \hat{\chi}^1_\alpha) \cup (K \cap \overline{W_\alpha})$ it is sufficient to let $p$ range over $K \cap \overline{W_\alpha}$ in the estimate. Now for $p \in L$, $\chi^1_\alpha$ can be omitted from the
corresponding term in (10). Thus, using local coordinates of \((U_{a_0}, \psi_{a_0})\) we have to estimate a finite number of Lie derivatives of terms of the form

\[
(\mu^{-1})^s \begin{pmatrix} y' 
\end{pmatrix} \left( y \mapsto \varepsilon^{-n} \varphi \left( \frac{y - x}{\varepsilon} \right) \right) d^n y
= y \mapsto \varepsilon^{-n} \varphi \left( \frac{\mu^{-1}(y) - \tilde{x}}{\varepsilon} \right) \cdot \det D\mu^{-1}(y) d^n y,
\]

(11)
each for \(\tilde{x} \in \psi_a(K \cap \Pi a)\), respectively, where \(\mu = \psi_{a_0} \circ \psi_{a}^{-1}\) and \(\mu(\tilde{z}) = z\).

Note that for \(\tilde{x}\) in a compact subset of \(\psi_a(U_{a_0} \cap U_{a})\) and sufficiently small \(\varepsilon\) (say, \(\varepsilon \leq \varepsilon_1(\leq \varepsilon_K)\)) the support of

\[
\tilde{y} \mapsto \varepsilon^{-n} \varphi \left( \frac{\tilde{y} - \tilde{x}}{\varepsilon} \right) d^n \tilde{y}
\]
is contained in \(\psi_a(U_{a_0} \cap U_{a})\), rendering (11) well-defined. Setting

\[
\phi(\varepsilon, x)(y) := \varphi \left( \frac{\mu^{-1}(x + \varepsilon y) - \mu^{-1}(x)}{\varepsilon} \right) \cdot \det D\mu^{-1}(x + \varepsilon y),
\]
the coefficient of the right hand side of (11) can be written as \(T_x S_\varepsilon \phi(\varepsilon, x)\).

By [16], Th. 7.14 and the remark following it, for each compact set \(L\) there exists \(\varepsilon_2 (\leq \varepsilon_1)\) such that \(\{\partial^\beta e \phi(\varepsilon, x)\} (\beta | 0 < \varepsilon < \varepsilon_2, x \in L)\) is (defined and) bounded in \(D\) for each \(\beta \in \mathbb{N}_0^n\). Of course (12) corresponds to (2) in [9] resp. to (42) in [19], the only difference being that since we use an oriented atlas and our test objects are forms the determinants are automatically positive. Since \(L_X (\rho d^n y) = (L_X \rho) d^n y + \rho L_X (d^n y)\), in order to verify 3.3 (ii) we have to estimate terms of the form

\[
L_{Y_1} \ldots L_{Y_{i'}} (L'_{X_{1_1}} + L_{X_{1_1}}) \ldots (L'_{X_{l'}} + L_{X_{l'}}) T_x S_\varepsilon \Phi(\varepsilon, x)(y)
\]
(13)
where

\[
X_i = \sum_{r_i=1}^n a^i_{r_i} \partial_{r_i}, \quad Y_j = \sum_{s_j=1}^n b^j_{s_j} \partial_{s_j}
\]
are local representations in \(\psi_a(U_{a_i})\) of vector fields on \(M\) and \(0 \leq k' \leq k, 0 \leq l' \leq l\). Explicitly, (13) is given by

\[
\prod_{j=1}^{l'} \left( \sum_{s_j=1}^n b^j_{s_j}(y) \frac{\partial}{\partial y^s_j} \right) \prod_{i=1}^{k'} \left( \sum_{r_i=1}^n \left( a^i_{r_i}(x) \frac{\partial}{\partial x^r_i} + a^i_{r_i}(y) \frac{\partial}{\partial y^r_i} \right) \right) T_x S_\varepsilon \phi(\varepsilon, x)(y).
\]
Note that $\frac{\partial}{\partial y^j} T_xS_x \phi = \varepsilon^{-1} T_xS_x \frac{\partial}{\partial y^j} \phi$ and
\[
\left( a_i^t(x) \frac{\partial}{\partial x^{r_i}} + a_i^r(y) \frac{\partial}{\partial y^{r_i}} \right) T_xS_x \phi =
T_xS_x \left( a_i^t(x) \frac{\partial}{\partial x^{r_i}} + \frac{1}{\varepsilon} (a_i^t(x + \varepsilon y) - a_i^t(x)) \frac{\partial}{\partial y^{r_i}} \right) \phi.
\]

Each of the maps
\[
(\varepsilon, x, y) \rightarrow \begin{cases} 
\frac{a_i^t(x + \varepsilon y) - a_i^t(x)}{\varepsilon} & \text{for } \varepsilon \neq 0, \\
D a_i^t(x)y & \text{for } \varepsilon = 0
\end{cases}
\] (14)
is smooth (hence uniformly bounded) on each relatively compact subset of its domain of definition. Due to the boundedness of $\{\partial_x^2 \phi(\varepsilon, x)(\cdot) | 0 < \varepsilon \leq \varepsilon_2, x \in \psi_0(K \cap \Omega)\}$ only values of $y$ from a bounded region of $\mathbb{R}^n$ are relevant in (14). Thus one further restriction of the range of $\varepsilon$ establishes the claim.

(ii) Choose the $\phi^0_a$ as in (i), yet this time additionally requiring $\varphi \in \mathcal{A}_m(\mathbb{R}^n)$. To estimate $\sup_{p \in K} |f_M(\Phi(\varepsilon, p)(q) \tilde{f}(q) - \tilde{f}(p))|$ for some $K \subset \subset M$ and $\tilde{f} \in \mathcal{C}^{\infty}(M)$ again only finitely many terms of (10) have to be taken into account; also, for small $\varepsilon$, the terms involving $\omega$ vanish and $\chi^1$ can be neglected. Then
\[
\sup_{p \in K \cap \text{supp} \chi_a} |\int_M \psi_a^1(\phi^0_a(\varepsilon, \psi_a(p))(\cdot)) d^n y(q) \tilde{f}(q) - \tilde{f}(p)| =
= \sup_{x \in \psi_a(\Omega \cap \text{supp} \chi_a)} |\int_{\psi_a(U_a)} \frac{1}{\varepsilon} \varphi(\frac{y - x}{\varepsilon}) f(y) d^n y - f(x)| = O(\varepsilon^{m+1})
\]

Next, we introduce the appropriate notion of Lie derivative for elements of $\hat{\mathcal{E}}(M)$.

3.8 Definition For any $R \in \hat{\mathcal{E}}(M)$ and any $X \in \mathfrak{X}(M)$ we set
\[
(\hat{L}_XR)(\omega, p) := -d_t R(\omega, p)(L_X \omega) + L_X(R(\omega, \cdot))|_p
\] (15)
Here, $d_t R(\omega, x)$ denotes the derivative of $\omega \rightarrow R(\omega, x)$ in the sense of [16], section 4. In order to obtain a structural description of this Lie derivative (which, at the same time, entails $\hat{L}_XR \in \hat{\mathcal{E}}(M)$ for $R \in \hat{\mathcal{E}}(M)$), for any $X \in \mathfrak{X}(M)$ define $X_A \in \mathfrak{X}(\hat{\mathcal{A}}_0(M))$ by
\[
X_A : \hat{\mathcal{A}}_0(M) \rightarrow \hat{\mathcal{A}}_0(M)
X_A(\omega) = -L_X \omega.
\]
\[
(\hat{L}_X R)(\omega, p) = (L_{(X_A, X)} R)(\omega, p) = (L_{(-L_X, X)} R)(\omega, p)
\]

i.e. \(\hat{L}_X\) is the Lie derivative of \(R\) with respect to the smooth vector field \((X_A, X)\) on \(\mathcal{A}_0(M) \times M\). Thus, indeed \(\hat{L}_X R \in \mathcal{E}(M)\).

**3.9 Remark.** (Local description of \(\hat{L}_X\)) Let \(X \in \mathfrak{X}(M)\) and set \(X_a = (\psi_a^{-1})^*(X|_{U_a})\). Since the derivative of the (linear and continuous) map \(\psi_a^*: \Omega^r_a(U_a) \to \Omega^r_a(U_a)\) in any point equals the map itself, on \(U_a\) we obtain (writing \(R\) in place of \(R|_{U_a}\)):

\[
((\psi_a^{-1})^*(\hat{L}_X R))(\varphi \, d^n y, x) = (\hat{L}_X R)(\psi_a^*(\varphi \, d^n y), \psi_a^{-1}(x))
\]

\[
= L_X (R(\psi_a^*(\varphi \, d^n y), \ldots))(\psi_a^{-1}(x)) - (d_1 R)(\psi_a^*(\varphi \, d^n y), \psi_a^{-1}(x))(L_X(\psi_a^*(\varphi \, d^n y)))
\]

\[
= [L_{X_a}(\psi_a^{-1})^*(R)](\varphi \, d^n y, x) - [d_1((\psi_a^{-1})^*)^R(\varphi \, d^n y, x)](L_{\psi_a^*(\varphi \, d^n y)})
\]

In particular, if \(X_a = \partial_{y^i}\) \((1 \leq i \leq n)\) then this exactly reproduces the local algebra derivative with respect to \(y^i\) given in (3).

Finally we are in a position to define the subspaces of moderate and negligible elements of \(\mathcal{E}(M)\).

**3.10 Definition** \(R \in \hat{\mathcal{E}}(M)\) is moderate if \(\forall K \subset M \ \forall k \in \mathbb{N}_0 \ \exists N \in \mathbb{N} \ \forall X_1, \ldots, X_k \in \mathfrak{X}(M) \ \forall \Phi \in \hat{\mathcal{A}}_0(M)\)

\[
\sup_{p \in K} |L_{X_1} \ldots L_{X_k}(R(\Phi(\varepsilon, p), p))| = O(\varepsilon^{-N})
\]  \hspace{1cm} (16)

The subset of moderate elements of \(\mathcal{E}(M)\) is denoted by \(\hat{\mathcal{E}}_m(M)\).

**3.11 Definition** \(R \in \hat{\mathcal{E}}(M)\) is called negligible if it satisfies

\[
\forall K \subset M \ \forall k, l \in \mathbb{N}_0 \ \exists m \in \mathbb{N} \ \forall X_1, \ldots, X_k \in \mathfrak{X}(M) \ \forall \Phi \in \hat{\mathcal{A}}_m(M) \ \sup_{p \in K} |L_{X_1} \ldots L_{X_k}(R(\Phi(\varepsilon, p), p))| = O(\varepsilon^l)
\]  \hspace{1cm} (17)

The set of negligible elements of \(\mathcal{E}(M)\) will be denoted by \(\hat{\mathcal{N}}(M)\).
4 Construction of the algebra, localization

The following result is immediate from the definitions:

4.1 Theorem

(i) $\mathcal{E}_m(M)$ is a subalgebra of $\mathcal{E}(M)$.

(ii) $\hat{N}(M)$ is an ideal in $\mathcal{E}_m(M)$.

For the further development of the theory it is essential to achieve descriptions of both moderateness and negligibility that relate these concepts to their local analogues \cite{19,16}, thereby making available the host of local results already established for open subsets of $\mathbb{R}^n$ also in the global context.

As a first step, we are going to examine localization properties of smoothing kernels:

4.2 Lemma Denote by $(U_a, \psi_a)$ a chart in $M$.

(A) Transforming smoothing kernels to local test objects.

(i) Let $\Phi$ be a smoothing kernel. Then the map $\phi$ defined by

$$\phi(\varepsilon, x)(y)dy := \varepsilon^n ((\psi_a^{-1})^*\Phi(\varepsilon, \psi_a^{-1}x))(\varepsilon y + x) \quad (x \in \psi_a(U_a), y \in \mathbb{R}^n)$$

(18)

is an element of $C^\infty_{b,w}(I \times \psi_a(U_a), \mathcal{A}_0(\mathbb{R}^n))$.

(ii) If, in addition, $\Phi \in \mathcal{A}_m(M)$ for some $m \in \mathbb{N}$ then $\phi \in \mathcal{A}_{m,w}(\psi_a(U_a))$, i.e.,

$$\int \phi(\varepsilon, x)(y)y^\beta dy = O(\varepsilon^{m+1-|\beta|}) \quad (1 \leq |\beta| \leq m)$$

(19)

uniformly on compact sets. In particular, if $\Phi \in \mathcal{A}_{2m-1}(M)$ then $\phi \in \mathcal{A}_{m,w}(\psi_a(U_a))$.

(B) Transporting local test objects onto the manifold.

(i) Let $\phi \in C^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ and $\Phi_1 \in \mathcal{A}_0(M)$. Let $K \subset \subset \psi_a(U_a)$, $\chi, \chi_1 \in \mathcal{D}(\psi_a(U_a))$ with $\chi \equiv 1$ on an open neighborhood of $K$ and $\chi_1 \equiv 1$ on an open neighborhood of $\text{supp} \chi$.

$$\Phi(\varepsilon, p) := (1 - \hat{\chi}(p)\lambda(\varepsilon))\Phi_1(\varepsilon, p)$$

$$+ \hat{\chi}(p)\lambda(\varepsilon)\psi_a^*(\frac{1}{\varepsilon^n}\phi(\varepsilon, \psi_a p)(y - \psi_a p)\chi_1(y)dy)$$

(20)

is a smoothing kernel (the smooth cut-off function $\lambda$ is defined in the proof).
(ii) If, in addition, \( \phi \in \mathcal{A}_m^1(\psi_a(U_\alpha)) \) and \( \Phi_1 \in \tilde{\mathcal{A}}_m(M) \) then \( \Phi \in \tilde{\mathcal{A}}_m(M) \). In particular, if \( \phi \in \mathcal{A}_m^1(\psi_a(U_\alpha)) \) and \( \Phi_1 \in \tilde{\mathcal{A}}_m(M) \) then \( \Phi \in \tilde{\mathcal{A}}_m(M) \).

**Proof.**

(A) (i) Let \( \hat{\mathcal{D}}_1 := \{(\varepsilon, p) \in I \times U_\alpha : \supp(\Phi(\varepsilon, p) \subseteq U_\alpha) \} \) and set \( D = \text{int}(\{(\varepsilon \times \psi_a)(\hat{\mathcal{D}}_1)\}) \). Then evidently \( \phi \) is smooth on \( D \). Furthermore (5) is obvious from (i) in 3.3. Concerning (6), set

\[
\phi_0(\varepsilon, x) \equiv (\psi_\alpha^{-1})^* (\Phi(\varepsilon, \psi_\alpha^{-1}(x))) \quad ((\varepsilon, x) \in \mathcal{D}_1)
\]

Then we have

\[
\partial^\beta \partial_\varepsilon \phi = \partial^\beta \partial_\varepsilon \partial_x S_x^{-1} T_x^{-1} \phi_0 = \varepsilon|\beta| S_x^{-1} T_x^{-1} \partial^\beta (\partial_x + \partial_y) \phi_0.
\]

Let now \( K \subset \subset \psi_a(U_\alpha), x \in K \) and \( \varepsilon \leq \varepsilon(K) \). By Definition 3.3 (i) and Lemma 3.4 there exists a constant \( C \) such that \( \text{diam}(\supp(\phi_0(\varepsilon, x))) \leq \varepsilon C \) for all \( x \in K \). But then \( \text{diam}(\supp(\phi(\varepsilon, x))) \leq C \) for all such \( x \). Hence \( \bigcup_x \cup_{x \in K} \partial^2_x \phi(\varepsilon, x) \) is bounded uniformly in \( \varepsilon \). Moreover, observing that \( L^\beta_{\partial_\varepsilon} (L_{\partial_x} + L_{\partial_y}) \phi_0(\varepsilon, x)(d^a y) = (L^\beta_{\partial_\varepsilon} (L_{\partial_x} + L_{\partial_y}) \phi_0(\varepsilon, x)(d^a y), \) from Definition 3.3 (ii) we obtain \( \partial^\beta y (\partial_x + \partial_y) \phi_0 = O(\varepsilon^{-|\beta|+n}) \) which together with equation (21) gives the desired boundedness property.

(ii) Now suppose that \( \Phi \in \tilde{\mathcal{A}}_m(M) \) and for each \( |\beta| \leq m \) choose \( f_\beta \in \mathcal{D}(U_\alpha) \subset \subset \mathcal{D}(M) \) such that \( f_\beta \circ \psi_\alpha^{-1}(x) = x^\beta \) in a neighborhood of \( K \subset \subset \psi_a(U_\alpha) \). Then by assumption we have

\[
\sup_{x \in K} \left| \int_M f_\beta(q)\Phi(\varepsilon, \psi_\alpha^{-1}(x))(q) - f_\beta(\psi_\alpha^{-1}(x)) \right| = O(\varepsilon^{m+1})
\]

For \( \varepsilon \) sufficiently small this implies

\[
\sup_{x \in K} \left| \int_{\mathbb{R}^n} \varepsilon^{-n} \phi(\varepsilon, x)(\frac{y - x}{\varepsilon}) y^\beta d y - x^\beta \right| = O(\varepsilon^{m+1})
\]

or, upon substituting \( z = \varepsilon^{-1}(y - x) \) and expanding:

\[
\sup_{x \in K} \left| \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} |z|^{|\beta| - \gamma} \int_{\mathbb{R}^n} \phi(\varepsilon, x)(z) z^\gamma d z \right| = O(\varepsilon^{m+1})
\]

From this, we prove (19) by induction with respect to \( |\beta| \): for \( \beta = e_k \) (the \( k \)-th unit vector) we obtain \( \left| \int_{\mathbb{R}^n} \phi(\varepsilon, x)(z) z_k d z \right| = O(\varepsilon^m) \). Suppose now that (19) has already been proved for \( |\beta| - 1 \), the result for
\(|\beta|\) follows from
\[
\sup_{x \in \hat{K}} \left| \sum_{0 < \gamma < \beta} \left( \frac{\beta}{\gamma} \right) x^{\beta - \gamma} \int_{\mathbb{R}^n} \phi(\varepsilon, x)(z) z^\gamma dz \right| + \varepsilon^{|\beta|} \int_{\mathbb{R}^n} \phi(\varepsilon, x)(z) z^\beta dz \right|_{\mathcal{O}(\varepsilon^{m+1})} = \mathcal{O}(\varepsilon^{m+1})
\]

(B) (i) First note that by the boundedness assumption all supports of \(\phi(I \times \text{supp}(\chi))\) are in a fixed compact subset of \(\mathbb{R}^n\). By [16], Lemma 6.2 we conclude that there exists \(\eta > 0\) such that for all \(\varepsilon \leq \eta\) and \(x \in \text{supp} \chi \text{supp} \left( \frac{1}{\varepsilon} \phi(\varepsilon, x)(\frac{x}{\varepsilon}) \right) \subseteq \{\chi_1 \equiv 1\}\). Now we are in the position to define \(\lambda\) as follows. Let \(\lambda \in C^\infty(\mathbb{R})\), \(0 < \lambda \leq 1\) and \(\lambda \equiv 1\) on \((-\infty, \eta/3)\) and \(\lambda \equiv 0\) on \((\eta/2, \infty)\). This actually implies \(\int \Phi(\varepsilon, p) = 1\) \(\forall p, \varepsilon\). Smoothness again is evident while condition 3.3 (i) is an easy consequence of [16], Lemma 6.2. 3.3 (ii) then follows exactly as (13) is proved in Lemma 3.7.

(ii) Let \(\hat{f} \in C^\infty(M)\) and \(p \in \hat{K} \subset U_\alpha\); then
\[
\left| \int_M \Phi(\varepsilon, p) (\hat{f})(q) - \hat{f}(p) \right| \leq 1 - \hat{\chi}(p) \lambda(\varepsilon) \left| \int_M \Phi_1(\varepsilon, p) (\hat{f})(q) - \hat{f}(p) \right| + 1 - \hat{\chi}(p) \lambda(\varepsilon) \left| \int_M \psi^{-1}_\alpha(\varepsilon - n \phi(\varepsilon, \psi_\alpha(p))(\frac{y - \psi_\alpha(p)}{\varepsilon}) \chi_1(y) d^n y) (q)(\hat{f})(q) - \hat{f}(p) \right|
\]

The first summand is of order \(\varepsilon^{m+1}\) by assumption and the second one (apart from \(\hat{\chi}\) and \(\lambda\)) for sufficiently small \(\varepsilon\) and setting \(f = \hat{f} \circ \psi^{-1}_\alpha\) equals
\[
\int_{\mathbb{R}^n} \phi(\varepsilon, x)(y) f(x + \varepsilon y) d^n y - f(x) = \sum_{0 < |\beta| \leq m} \frac{\partial^|\beta| f(x)}{\beta!} \varepsilon^{|\beta|} \int_{\mathcal{O}(\varepsilon^{m+1})} \phi(\varepsilon, x)(y) y^\beta dy
\]
so the claim follows.

\(\square\)

Restriction of an element \(R\) of \(\mathcal{E}(M)\) to an open subset \(U\) of \(M\) is defined by \(R|_U := R|_{A_0(U) \times U}\).

4.3 Theorem (Localization of moderateness) Let \(R \in \hat{\mathcal{E}}(M)\). Then
\(R \in \hat{\mathcal{E}}_m(M) \Leftrightarrow (\psi^{-1}_\alpha)^* (R|_{U_\alpha}) \in \mathcal{E}_m(\psi_\alpha(U_\alpha)) \forall \alpha\).
Proof. (⇒) For \( R \in \hat{\mathcal{E}}_m(M) \) and for \((U_\alpha, \psi_\alpha)\) some chart in \(M\) let \(R' := (\psi_\alpha^{-1})^\wedge (R|_{U_\alpha})\). Let \( K \subset \subset \psi_\alpha(U_\alpha) =: V_\alpha, \beta \in \mathbb{N}_0\) and let \( \phi \in \mathcal{C}^\infty_k(I \times \psi_\alpha(U_\alpha), \mathcal{A}_0(\mathbb{R}^n)) \). We have to show that

\[
\sup_{x \in K} |\partial^\beta (R'(T_x S_\varepsilon \phi(\varepsilon, x), x))| = O(\varepsilon^{-N})
\]  
(22)

for some \( N \in \mathbb{N} \). To this end we fix some \( \Phi_1 \in \tilde{\mathcal{A}}_0(M) \) and define \( \Phi \in \tilde{\mathcal{A}}_0(M) \) by (20). Let \( \partial^\beta = \partial_{i_1} \ldots \partial_{i_\ell} \) and for \( 1 \leq i_j \leq n \) choose \( X_{i_j} \in \mathfrak{X}(M) \) such that the local expression of \( X_{i_j} \) coincides with \( \partial_{i_j} \) on a neighborhood of \( K \). Then for \( \varepsilon \) sufficiently small (22) equals

\[
\sup_{p \in K} |L_{X_{i_1}} \ldots L_{X_{i_\ell}} R(\Phi(\varepsilon, p), p)|,
\]
so we are done.

(⇐) Let \( X_1, \ldots, X_k \in \mathfrak{X}(M) \) and (without loss of generality) \( K \subset \subset U_\alpha \) for some chart \((U_\alpha, \psi_\alpha)\). Let \( \Phi \in \tilde{\mathcal{A}}_0(M) \) and define \( \phi \) by (18). Since \( \phi : D \to \mathcal{A}_0(\mathbb{R}^n) \) belongs to \( \mathcal{C}^\infty_k(I \times \psi_\alpha(U_\alpha), \mathcal{A}_0(\mathbb{R}^n)) \) it follows from [16], Th. 10.5 that given \( \beta \in \mathbb{N}_0 \) there exists some \( N \in \mathbb{N} \) and some \( \varepsilon_0 > 0 \) such that for \( x \in K \) and \( \varepsilon \leq \varepsilon_0 \) we have \((\varepsilon, x) \in D \) and \(|\partial^\beta (\psi_\alpha^{-1})^\wedge (R|_{U_\alpha})(T_x S_\varepsilon \phi(\varepsilon, x), x))| = O(\varepsilon^{-N})\). Inserting this into the local representation of (16) immediately gives the result.

\[ \square \]

4.4 Theorem (Localization of negligibility) Let \( R \in \hat{\mathcal{E}}_m(M) \). Then

\[ R \in \hat{\mathcal{N}}(M) \iff (\psi_\alpha^{-1})^\wedge (R|_{U_\alpha}) \in \mathcal{N}(\psi_\alpha(U_\alpha)) \quad \forall \alpha. \]

Proof. (⇒) Let \( R \in \hat{\mathcal{E}}_m(M), (U_\alpha, \psi_\alpha) \) some chart in \(M\) and set \( R' := (\psi_\alpha^{-1})^\wedge (R|_{U_\alpha}) \). Let \( K \subset \subset \psi_\alpha(U_\alpha) =: V_\alpha, l \in \mathbb{N} \) and \( \beta \in \mathbb{N}_0 \). Set \( k = |\beta| \) and choose \( m \in \mathbb{N}_0 \) such that

\[
\sup_{p \in K} |L_{X_1} \ldots L_{X_k} (R(\Phi(\varepsilon, x), x))| = O(\varepsilon^l)
\]  
(23)

for all \( X_1, \ldots, X_k \in \mathfrak{X}(M) \) and all \( \Phi \in \tilde{\mathcal{A}}_m(M) \). Now let \( \phi \in \mathcal{A}_m^\infty(V_\alpha) \) and construct \( \Phi \) from \( \phi \) according to (20). By 4.2, \( \Phi \in \mathcal{A}_m(M) \), so (with \( X_{i_j} \) as in the proof of 4.3)

\[
\sup_{x \in K} |\partial^\beta (R'(T_x S_\varepsilon \phi(\varepsilon, x), x))| = \sup_{p \in K} |L_{X_{i_1}} \ldots L_{X_{i_\ell}} (R(\Phi(\varepsilon, x), x))| = O(\varepsilon^l)
\]

Thus the claim follows from the characterization of negligibility following the definition of \( \mathcal{N}(\Omega) \) (section 2).
Let $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $X_1, \ldots, X_k \in \mathcal{X}(M)$ and $\hat{K} \subset \subset U_\alpha$. By the discussion at the end of section 2 there exists $m'$ such that (with $R' = (\psi^{-1}_\alpha)^\wedge R|_{U_\alpha}$)

$$
\sup_{x \in \hat{K}} |\partial^\beta (R'(T_x S_x \phi(\varepsilon, x), x))| = \mathcal{O}(\varepsilon^l)
$$

(24)

for all $|\beta| \leq k$ and all $\phi \in \mathcal{A}^{\mathbb{E}}_{m', \omega}(V_\alpha)$. Now set $m = 2m' - 1$ and let $\Phi \in \mathcal{A}_m(M)$. Then $\phi$ defined by (18) is in $\mathcal{A}^{\mathbb{E}}_{m', \omega}(V_\alpha)$ by 4.2 (A) (ii). Hence inserting local representations of $X_1, \ldots, X_k$, (24) immediately implies the validity of (23), thereby finishing the proof. \hfill $\square$

It was shown in [17], sec. 13 that for all variants of (local) Colombeau algebras membership of any element $R$ of $\mathcal{E}_m$ to the ideal $\mathcal{N}$ can be tested on the function $R$ itself, without taking into account any derivatives of $R$. As a first important consequence of the above localization results we note that this rather surprising simplification also holds true for the global theory:

4.5 Corollary Let $R \in \hat{\mathcal{E}}_m(M)$. Then

$$
R \in \hat{\mathcal{N}}(M) \Leftrightarrow (17) \text{ holds for } k = 0.
$$

Proof. This follows directly from Th. 4.4 by taking into account [16], Th. 7.13 and [17], Th. 13.1. \hfill $\square$

Moreover, stability of $\hat{\mathcal{E}}_m(M)$ and $\hat{\mathcal{N}}(M)$ under Lie derivatives also follows from the local description:

4.6 Theorem Let $X \in \mathcal{X}(M)$. Then

(i) $\hat{L}_X \hat{\mathcal{E}}_m(M) \subseteq \hat{\mathcal{E}}_m(M)$.

(ii) $\hat{L}_X \hat{\mathcal{N}}(M) \subseteq \hat{\mathcal{N}}(M)$.

Proof. Let $R \in \hat{\mathcal{E}}_m(M)$, $X \in \mathcal{X}(M)$. By Th. 4.3 for any chart $(U_\alpha, \psi_\alpha)$ we have $(\psi^{-1}_\alpha)^\wedge (R|_{U_\alpha}) \in \mathcal{E}_m(\psi_\alpha(U_\alpha))$. Thus by [16], Th. 7.10 also $L_{X_\alpha} (\psi^{-1}_\alpha)^\wedge (R|_{U_\alpha}) = (\psi^{-1}_\alpha)^\wedge (\hat{L}_X R|_{U_\alpha}) \in \mathcal{E}_m(\psi_\alpha(U_\alpha))$ (where $X_\alpha$ denotes the local representation of $X$), which, again by Th. 4.3 gives the result. The claim for $\hat{\mathcal{N}}(M)$ follows analogously from [16], Th. 7.11. \hfill $\square$

Finally we are in a position to define our main object of interest:

4.7 Definition

$$
\hat{\mathbb{G}}(M) := \hat{\mathcal{E}}_m(M)/\hat{\mathcal{N}}(M)
$$

is called the Colombeau algebra on $M$. 

19
By construction, every $\hat{L}_X$ induces a Lie derivative (again denoted by $\hat{L}_X$) on $\hat{G}(M)$, so $\hat{G}(M)$ becomes a differential algebra. If $\hat{R} \in \hat{E}(M)$, its class in $\hat{G}(M)$ will be denoted by $\text{cl}[\hat{R}]$.

4.8 Theorem $\hat{G}(M)$ is a fine sheaf of differential algebras on $M$.

Proof. This is a straightforward consequence of [16], Th. 8.1. \hfill \Box

5 Embedding of distributions and smooth functions

In this section we show that in the global context $\hat{G}(M)$ displays the same set of (optimal) embedding properties as the local versions do on open sets of $\mathbb{R}^n$. Most importantly, we shall see that taking Lie derivatives with respect to arbitrary smooth vector fields commutes with the embedding.

To begin with, let $u \in \mathcal{D}'(M)$. The natural candidate for the image of $u$ in $\hat{G}(M)$ is $R_u(\omega, x) = \langle u, \omega \rangle$. We first show that $R_u \in \hat{E}_m(M)$ using Th. 4.3. Let $\omega \in \mathcal{D}(\psi_\alpha(U_\alpha))$; then

$$\left((\psi_\alpha^{-1})^*\left(R_u \mid \psi_\alpha\right)\right)(\omega, x) = (R_u \mid \psi_\alpha)(\psi_\alpha^*(\omega d^n y), \psi_\alpha^{-1}(x)) =$$

$$\langle u, \psi_\alpha^*(\omega d^n y) \rangle = \langle (\psi_\alpha^{-1})^*(u \mid \psi_\alpha), \omega \rangle$$

Since $(\psi_\alpha^{-1})^*(u \mid \psi_\alpha) \in \mathcal{D}'(\psi_\alpha(U_\alpha))$ it follows from the local theory that indeed $(\psi_\alpha^{-1})^*(R_u \mid \psi_\alpha) \in \mathcal{E}_m(\psi_\alpha(U_\alpha))$. Suppose now that $R_u \in \hat{N}(M)$. By the same reasoning as above $(\omega, x) \rightarrow \langle (\psi_\alpha^{-1})^*(u \mid \psi_\alpha), \omega \rangle \in \mathcal{N}(\psi_\alpha(U_\alpha))$ for each $\alpha$. Thus again by the respective local result $(\psi_\alpha^{-1})^*(u \mid \psi_\alpha) = 0$ for each $\alpha$, i.e. $u = 0$. Therefore

$$\iota : \mathcal{D}'(M) \rightarrow \hat{G}(M)$$

$$\iota(u) = \text{cl}[(\omega, x) \rightarrow \langle u, \omega \rangle]$$

is a linear embedding. What is more, as a direct consequence of (15) (noting that distributions are linear and continuous, hence equal to their differential in any point) we obtain

$$\iota(L_X u)(\omega, p) = \iota((\omega, p) \rightarrow -\langle u, L_X \omega \rangle) = -d_1 R_u(\omega, p)(L_X \omega)$$

$$+ L_X \left( R_u(\omega, \cdot) \right)_{|p} \mid_{\omega} \right) = \left( L_X R_u(\omega, p) = \hat{L}_X (\iota(u))(\omega, p) \right)$$

i.e., $\iota$ commutes with arbitrary Lie derivatives.
The natural operation for embedding smooth functions into $\hat{G}(M)$ is given by

$$\sigma : C^\infty(M) \to \hat{G}(M)$$
$$\sigma(f) = \text{cl}[(\omega, x) \to f(x)]$$

Obviously, $\sigma$ is an injective algebra homomorphism that commutes with Lie derivatives by (15). Moreover, $\iota$ coincides with $\sigma$ on $C^\infty(M)$. Making use of Th. 4.4 this again follows directly from the local result. Summing up, we have

5.1 Theorem $\iota : \mathcal{D}'(M) \to \hat{G}(M)$, is a linear embedding that commutes with Lie derivatives and coincides with $\sigma : C^\infty(M) \to \hat{G}(M)$ on $C^\infty(M)$.

Thus $\iota$ renders $\mathcal{D}'(M)$ a linear subspace and $C^\infty(M)$ a faithful subalgebra of $\hat{G}(M)$.

The following commutative diagram illustrates the compatibility properties of Lie derivatives with respect to embeddings established in this section:

\[\begin{array}{ccc}
\mathcal{C}^\infty & \xrightarrow{L_X} & \mathcal{C}^\infty \\
\sigma & \downarrow & \sigma \\
\mathcal{D} & \xrightarrow{L_X} & \mathcal{D} \\
\hat{\mathcal{G}} & \xrightarrow{\hat{L}_X} & \hat{\mathcal{G}}
\end{array}\]

6 Association

The concept of association or coupled calculus is one of the distinguishing features of local Colombeau algebras. It introduces a (linear) equivalence relation on the algebra identifying those elements which are “equal in the sense of distributions”, thereby allowing to identify “distributional shadows” of certain elements of the algebra. This construction amounts to determining the macroscopic aspect of the regularization procedure encoded in them. Phrased more technically, a linear quotient of $\mathcal{E}_m$ resp. $\mathcal{G}$ is formed containing
as a subspace. Especially in physical modelling this notion provides a useful tool for analyzing nonlinear problems involving singularities (cf. e.g. [5], [21], [27], [28]). In what follows we extend the notion of association to \( \mathcal{G}(M) \).

### 6.1 Definition

An element \([R]\) of \( \mathcal{G}(M) \) is called associated to \( 0 \) \( ([R] \approx 0) \) if for some (hence every) representative \( R \) of \([R]\) we have: \( \forall \omega \in \Omega^\mu_c(M) \) \( \exists m > 0 \) with

\[
\lim_{\varepsilon \to 0} \int_M R(\Phi(\varepsilon, p), p)\omega(p) = 0 \quad \forall \Phi \in \mathcal{A}_m(M) \quad (25)
\]

Two elements \([R], [S]\) of \( \mathcal{G}(M) \) are called associated \( ([R] \approx [S]) \) if \([R-S] \approx 0\). We say that \([R] \in \mathcal{G}(M)\) admits \( u \in \mathcal{D}'(M) \) as an associated distribution if \([R] \approx \iota(u)\), i.e. if \( \forall \omega \in \Omega^\mu_c(M) \) \( \exists m > 0 \) with

\[
\lim_{\varepsilon \to 0} \int_M R(\Phi(\varepsilon, p), p)\omega(p) = \langle u, \omega \rangle \quad \forall \Phi \in \mathcal{A}_m(M) \quad (26)
\]

Finally, by the same methods as in the local theory we obtain consistency in the sense of association of classical multiplication operations with multiplication in the algebra:

### 6.2 Proposition

(i) If \( f \in \mathcal{C}^\infty(M) \) and \( u \in \mathcal{D}'(M) \) then

\[
\iota(f)\iota(u) \approx \iota(fu) \quad (27)
\]

(ii) If \( f, g \in \mathcal{C}(M) \) then

\[
\iota(f)\iota(g) \approx \iota(fg) \quad (28)
\]

### References


**Electronic Mail:**
M.G.: michael@mat.univie.ac.at
M.K.: Michael.Kunzinger@univie.ac.at
R.S.: Roland.Steinbauer@univie.ac.at
J.V.: jav@maths.soton.ac.uk