On Locally $q$–Complete Domains in Kähler Manifolds

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1 Introduction

Let \( \Omega \) be a relatively compact open set in a Kähler manifold \( X \) with Kähler form \( \omega \). Denote by \( \delta_\Omega \) its boundary distance function from the boundary, \( \delta_\Omega(x) = \text{dist}(x, \partial \Omega) \). If \( \Omega \) is locally Stein and \( X = \mathbb{P}^n \), Takeuchi [11] showed that \( -\log \delta_\Omega \) is strongly plurisubharmonic. This result has been extended to \( X \) having positive holomorphic bisectional curvature. (See [3], [10], and [4].) If, moreover, \( \Omega \) has smooth boundary of class \( C^2 \), then Ohsawa and Sibony [6] have shown that \( \Omega \) admits a negative exhaustion strongly plurisubharmonic function, i.e., \( \Omega \) is hyperconvex according to Stehlé terminology [9]. This is a parallel result to that of Diederich and Fornæss [2] where the case \( X = \mathbb{C}^n \) (or \( X \) = Stein manifold) is settled.

Here we consider the following generalization of Ohsawa and Sibony’s result which is recovered for \( q = 1 \), namely we prove the following theorem.

Theorem 1 Let \( \Omega \) be a relatively compact domain with smooth boundary of class \( C^2 \) in a Kähler manifold \( X \) which has positive holomorphic bisectional curvature. If \( \Omega \) is locally \( q \)-complete, then there exist: \( \varepsilon_0 \in (0, 1) \) small enough and a neighborhood \( U \) of \( \partial \Omega \) such that the function \( \varphi := -\delta_\Omega \) is \( q \)-convex on \( U \cap \Omega \) for \( 0 < \varepsilon \leq \varepsilon_0 \). More precisely, there is \( c_\varepsilon > 0 \) such that

\[
L(\varphi, \cdot) \geq c_\varepsilon |\varphi|\omega \text{ on } U \cap \Omega.
\]

Note that for every integer \( q > 1 \) there is a domain \( \Omega \subset \mathbb{P}^n \), \( n = 2q - 1 \), which has real-analytic boundary, is locally \( q \)-complete, and \( \Omega \) fails to be \( q \)-complete. On the other hand, in proving theorem 1, one cannot apply the usual techniques from Stein manifolds since there exists a Stein domain \( D \subset \mathbb{P}^5 \) such that there does not exist a strongly plurisubharmonic function near \( \partial D \).

2 Preliminaries

The definition of \( q \)-convexity is the same as in [1]: A real function \( \varphi \) on an open subset \( D \) of \( \mathbb{C}^n \) is \( q \)-convex if it is differentiable of class \( C^2 \) and the next quadratic form

\[
L(\varphi, z)(\xi, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \eta_j x, \xi, \eta \in \mathbb{C}^n,
\]
has at least $n - q + 1$ positive ($> 0$) eigenvalues. Define the Levi form of $\varphi$ at $z$ by

$$L(\varphi, z)\xi = L(\varphi)(x, \xi).$$

This definition can be also made for complex manifolds by local charts.

Let $X$ be a complex manifold, if there exists an exhaustion function $\varphi$ on $X$ which is $q$-convex.

3 Proof of theorem 1

Our proof is inspired from the case $q = 1$ given in [6], however with some changes which are done subsequently. Moreover, though more general, our proof is simpler.

We consider the continuous function $r : X \to \mathbb{R}$ given by $r(x) = \text{dist}(x, \partial \Omega)$ if $x \in \Omega$ and $r(x) = -\text{dist}(x, \partial \Omega)$ if $x \in X \setminus \Omega$.

Then, by standard arguments, there exists a tubular open neighborhood $V$ of $\partial \Omega$ on which $r$ is smooth of class $C^2$. By Matsumoto's paper [7], $-\log r$ is $q$-convex on $V \cap D$ for some tubular open neighborhood $V$ of $\partial \Omega$; in fact, her proof gives more, namely, there exists a constant $c > 0$, which depends only on the holomorphic bisectional curvature of $X$ and the closure of $\Omega$, and a family of complex vector subspaces $E_x \subset \mathcal{T}_xX, x \in V \cap \Omega$, of dimension $n - q + 1$ such that:

$$L(-\log r, x)(\xi) \geq c\|\xi\|_\omega^2, \xi \in E_x, x \in V \cap \Omega.$$

We note that, except for the particular case $q = 1$ which is treated in [6], this estimate and the method of Ohsawa and Sibony [6] are not sufficient to deduce that $-r'$, for some $c > 0$ small enough, is $q$-convex on a $V \cap \Omega$, even with a possible shrunk $V$ along the boundary of $\Omega$.

What we need, in view of the computations we made, is a precise relation of $E_x$ with respect to the normal direction of the hypersurface $\{r = r(x)\}$ at $x$.

In fact, by shrinking $V$ along $\partial \Omega$, if necessary, we may assume that for every point $x \in V \cap \Omega$ there are: a point $y \in \partial \Omega$ and a minimal geodesic $\gamma : [0, b] \to M$ with $b = r(x)$ such that $\gamma(0) = x, \gamma(b) = y$, and $\gamma([0, b]) \subset \Omega$. If $T_y(\partial \Omega)$ denotes the complex tangent space to the real hypersurface $\partial \Omega$ and if $N_x$ is the parallel translate of $T_y(\partial \Omega)$ at $x$ along $\gamma$, then the gradient $\nu_x$ of $r$ at $x$ (computed with respect to $\omega$) is orthogonal to $N_x$. If $F_y \subset T_y(\partial \Omega)$ is a complex vector subspace, we let $E'_x$ denotes the parallel translate of $F_y$ at $x$ along $\gamma$. Clearly, if $E_x = E'_x \oplus \nu_x \mathbb{C}$, then $\dim E_x = 1 + \dim E'_x$.

The key observation which follows by a careful analysis of Matsumoto's proof [5] is contained in the next lemma.

**Lemma 1** After further shrinking of $V$ along $\partial \Omega$ there exist: a constant $c > 0$ and a family of $(n - q)$ dimensional complex vector spaces $\{F_y\}_{y \in \partial \Omega}$, $F_y \subset T_y(\partial \Omega)$, such that the eigenvalues of the Levi form of $-\log r$ restricted to $E_x$ are at least $c$.

**Remark.** It is crucial here that $E_x$ is written in the form $E'_x \oplus \nu_x \mathbb{C}$ with $\nu_x - E'_x$ and that the eigenvalues on $E'_x$ are bounded from below by a positive constant. Second, to proceed with the proof of the theorem, we choose coordinates near $y_0 \in \partial \Omega$ such that
\(x_{2n} = r_., \epsilon_i(r) = 0, i = 1, \ldots, n - 1,\) where \(\{\epsilon_i\}\) is an orthonormal basis for the complex tangent space to \(\partial \Omega\) near \(y_0\).

Now, in order to conclude the proof of theorem 1, by standard arguments we reduce ourselves to the next lemma. Note that the computations can be made with respect to an arbitrary Kähler metric, but for simplicity we prefer to state it for the euclidean metric.

**Lemma 2** Let \(D \subset \mathbb{C}^n\) be an open set and \(\Omega \subset D\) open with smooth boundary of class \(C^2\) and \(r\) defined as above. Let \(U\) be a tubular neighborhood of \(\partial \Omega\) on which \(r\) is of class \(C^2\). Assume that there are: a non empty compact set \(K \subset \partial \Omega\), an open neighborhood \(W\) of \(K, W \subset U\), a constant \(c > 0\), and a family of complex vector spaces \(\{E'_z\}_{z \in W \cap \Omega}\) of \(\mathbb{C}^n\) with \(\nu_z := \overline{\partial}r(z) - E'_z\) for all \(z \in W \cap \Omega\), such that if \(E_z = E'_z \oplus \nu_z\mathbb{C}\), then

\[
(1) \quad -r(z)L(r(z))(\xi) + |< \partial r(z), \xi>|^2 \geq cr^2(z)\|\xi\|^2, \quad z \in W \cap \Omega, \xi \in E_z.
\]

Then there exist: a neighborhood \(V\) of \(K\), \(V \subset W\), \(\epsilon_\alpha \in (0,1)\) sufficiently small, such that for every \(\epsilon, 0 < \epsilon \leq \epsilon_\alpha\) there is a constant \(c_\epsilon > 0\) such that

\[
(2) \quad -r(z)L(r(z))(\xi) + (1 - \epsilon)|< \partial r(z), \xi>|^2 \geq c_\epsilon r^2(z)\|\xi\|^2,
\]

for all \(z \in V \cap \Omega\) and \(\xi \in E_z\).

**Proof.** Let \(\xi \in E_z\). Write it uniquely in the form

\[
(3) \quad \xi = \xi'/r(z) + t\nu_z, \xi' \in E'_z, t \in \mathbb{C}.
\]

Since \(\|\xi\|^2 = \|\xi'/r^2(z)\|^2 + t^2\|\nu_z\|^2 \geq \|\xi'/r^2(z)\|^2\), inequality (1) gives after expanding:

\[
(4) \quad -L(r,z)(\xi')/r(z) + A(z, \xi) + \|\partial r(z)\|^2 |t|^2 \geq c\|\xi'/r^2(z)\|^2, \xi' \in E'_z, t \in \mathbb{C}.
\]

Here we set for \(z \in W \cap \Omega\) and \(\xi \in E_z\) (as in (3)):

\[
A(z, \xi) := -2\text{Re}\left(tL(r,z)(\xi', \nu_z)\right) - t^2 r(z)L(r,z)(\nu_z).
\]

Inequality (4) gives for \(t = 0, -L(r,z)(\xi') \geq 0\) for \(\xi' \in E'_z\). On the other hand, there is a constant \(M > 0\) such that

\[
\left|L(r,z)(\nu_z)\right| \leq M, \left|L(r,z)(\xi', \nu_z)\right| \leq M,
\]

for \(z \in U \cap \Omega, \xi' \in E'_z, \|\xi'\| = 1\). Denote

\[
B(z, \xi) := -L(r,z)(\xi')/r(z) + A(z, \xi).
\]

We claim now that there are constants \(\epsilon_\alpha > 0\) small enough such that for every \(\epsilon, 0 < \epsilon \leq \epsilon_\alpha\) there is \(c'_\epsilon > 0\) \((c'_\epsilon = o(\epsilon))\) such that

\[
(*) \quad B(z, \xi) + (1 - \epsilon)\|\partial r(z)\|^2 \cdot |t|^2 \geq c'_\epsilon |t|^2, \quad z \in U \cap \Omega, \xi \in E_z.
\]
for $z \in U \cap \Omega$, $\xi' \in E'_z$, and $t \in \mathbb{C}$ such that $\|\xi'\|^2 + \|t\|^2 = 1$. (Of course, after further shrinking of $U$ along $\partial \Omega$).

In particular, this gives trivially that $B(z, \xi) + (1 - \epsilon)\|\partial r(z)\|^2\|t\|^2 \geq 0$ for every $\xi \in E'_z$ and $t \in \mathbb{C}$. Adding this to (1) and dividing by 2 we get immediately the statement of theorem 1.

So we are left with the proof of the claim. For this let $\alpha \in (0,1)$ to be chosen later in proof. We want to estimate from below $B(z, \xi)$ and for this we consider two cases:

1) If $\|\xi\| \leq \alpha$, then $|t|^2 \geq 1 - \alpha^2$. Since $B(z, \xi) \geq A(z, \xi)$, we get easily:

$$B(z, \xi) \geq -M|t|^2\left(2\alpha/\sqrt{1 - \alpha^2} + r(z)\right).$$

Therefore, for every $\mu \in (0,1)$ one deduces:

$$B(z, \xi) + (1 - \mu)\|\partial r(z)\|^2|t|^2 \geq |t|^2\left((1 - \mu)\|\partial r(z)\|^2 - Mr(z) - 2\alpha M/\sqrt{1 - \alpha^2}\right).$$

II) If $\|\xi\| \geq \alpha$, then $\|\xi\|^2 \geq |t|^2\alpha^2/(1 - \alpha^2)$; hence (4) furnishes

$$B(z, \xi) + (1 - \mu)\|\partial r(z)\|^2|t|^2 \geq |t|^2\left(c\alpha^2/(1 - \alpha^2) - \mu\|\partial r(z)\|^2\right).$$

Now, consider in both cases $\mu = \alpha^2\beta$ with $\alpha > 0$ and $\beta > 0$ small enough and after further shrinking of $U$ along $\partial \Omega$ the claim results easily.

Remark. If, additionally in theorem 1, on every relatively compact open subset of $X$ there are $p$-convex functions, then $\Omega$ results $(p + q - 1)$-complete with a negative exhaustion function. As an example [10] for such $X$ one can take every open set of $\mathbb{P}^n \setminus A$, where $A \subset \mathbb{P}^n$ is an analytic set whose irreducible components are of dimension at least $n - p$.

In the same circle of ideas we add the following completion. First we recall that a continuous function $\varphi$ on a complex manifold $X$ is said to be $q$-convex with corners if locally it may be written in the form $\varphi = \max(f_1, \ldots, f_k)$ with $f_i$ be $q$-convex. Then we say that $X$ is $q$-complete with corners if it admits an exhaustion function which is $q$-convex with corners. See [2] and [7].

Now we state:

**Proposition 1** Under the hypothesis of theorem 1, there is a continuous exhaustion function $\varphi : \Omega \to (-\infty, 0)$ which is $q$-convex with corners, smooth and $q$-convex on the set $\{\varphi > -c\}$ for some $c > 0$ sufficiently small. In particular, $\Omega$ is $\bar{q}$-complete, where $\bar{q} = n - [n/q] + 1$ and $n$ denotes the complex dimension of $X$.

**Proof.** As a first step in showing this we note a consequence of [5] and [12]:

**Lemma 3** Let $X$ be a Kähler with positive holomorphic bisectional curvature and $D \subset X$ a domain locally $q$-complete with corners. Then for every $\lambda \in \mathbb{R}$, $\lambda > 1$, there is $\tilde{\delta} \in C^0(D, \mathbb{R})$, $\tilde{\delta} > 0$, such that $1/\lambda < \tilde{\delta}/\delta < \lambda$ and $-\log \tilde{\delta}$ is $q$-convex with corners.
Using this lemma, we define $\varphi$ in the following way: Take constants $a, b$, $0 < a < b$ such that $\{ \delta < b \} \subset U$. (Here $U$ is as in the proof of theorem 1.) Then consider $\tilde{\delta}$ given by the above lemma such that $\tilde{\delta} > \delta$ on $\{ \delta = a \}$, $\tilde{\delta} < \delta$ on $\{ \delta = b \}$.

Put $\varphi := -\delta^\epsilon$ on $\{ \delta \leq a \}$, $\varphi = \max(-\delta^\epsilon, -\delta^\eta)$ on $\{ a \leq \delta \leq b \}$, and $\varphi := -\delta^\eta$ on $\{ \delta \geq b \}$. This is well-defined for every $\epsilon > 0$. It remains to see that $\varphi$ has the required properties for $\epsilon$ small enough. But this is a simple consequence of the fact that given a complex manifold $Z$ and $\psi$ be $q$-convex with corners on $Z$, then for every open set $W \subset Z$, there is $\epsilon > 0$ sufficiently small such that $-\exp(-\epsilon \psi)$ is $q$-convex with corners on $W$. To conclude one applies [2].

As immediate consequence to theorem 1 one has:

**Corollary 1** Let $\Omega \subset \mathbb{P}^n$ be a proper domain with smooth boundary of class $C^2$. If $\Omega$ is locally $q$-complete, then $\Omega$ admits a smooth negative exhaustion function which is $q$-convex near the boundary of $\Omega$ in $\Omega$.

**Remark.** It is easily seen that for every $q > 1$ there are examples of proper domains $\Omega$ of $\mathbb{P}^n$ $(2q < n)$ with real-analytic boundary and which are locally $q$-complete and fails to be $q$-complete.

For instance, let integers $q$ and $n$ such that $q > 1$ and $n < 2q$. Let $A, B$ be two disjoint copies of $\mathbb{P}^{n-1}$ which sit in $\mathbb{P}^n$. Then $D := \mathbb{P}^n \setminus (A \cup B)$ is $q$-convex. Let $\varphi : D \to \mathbb{R}$ be a smooth exhaustion function which is $q$-convex outside a compact set of $D$. Then $\Omega := \{ z \in D : \varphi(z) < C \}$ with $C > 0$ large enough is our example. It remains to check that $\Omega$ is not $q$-complete. For this, since obviously $\partial \Omega$ is not connected, we use the next simple lemma whose proof is a standard consequence of Serre's duality theorem.

**Lemma 4** Let $X$ be a connected non-compact complex manifold of dimension $n$ and $\Omega \subset X$ an open set such that $H^{n-1}(\Omega, \mathcal{K}_X) = 0$, where $\mathcal{K}_X$ is the canonical sheaf of $X$. Then $\partial \Omega$ is connected.

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**References**


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