Two-Term Dilogarithm Identities
Related to Conformal Field Theory

Andrei G. Bytsko

Vienna, Preprint ESI 785 (1999)  
November 4, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
Two-term dilogarithm identities related to conformal field theory

Andrei G. Bytsko

Steklov Mathematics Institute,
Fontanka 27, St.Petersburg 191011, Russia

Abstract

We study $2 \times 2$ matrices such that the corresponding TBA equations yield dilogarithm identities $L(x) + L(y) = c$ with $c$ being an effective central charge for certain conformal models. The properties of such matrices and the corresponding solutions of the TBA equations are discussed. Several continuous families and a 'discrete' set of admissible matrices $A$ are found. For these examples several new two-term dilogarithm identities are obtained and in some cases proven or shown to be equivalent to some previously known identities.

1 Introduction.

The (normalized) Rogers dilogarithm is a transcendental function defined for $x \in [0,1]$ as follows

$$L(x) = \frac{6}{x} \left( \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln x \ln (1 - x) \right).$$

(1.1)

It is a strictly increasing continuous function satisfying the following functional equations:

$$L(x) + L(1 - x) = 1,$$

(1.2)

$$L(x) + L(y) = L(xy) + L\left( \frac{x(1 - y)}{1 - xy} \right) + L\left( \frac{y(1 - x)}{1 - xy} \right).$$

(1.3)

Dilogarithm identities of the form

$$\sum_{k=1}^{r} L(x_k) = c,$$

(1.4)

where $c \geq 0$ is a rational number, and $x_k \in [0,1]$ are algebraic numbers (i.e. they are real roots of polynomial equations with integer coefficients) arise in different contexts in mathematics and theoretical physics (see e.g., [1] and references therein). In particular, they appear in the description of the asymptotic behaviour of an infinite series $\chi(q)$ of the form

$$\chi(q) = q^\text{const} \sum_{\vec{m}} \frac{q^{\vec{m} \cdot A \vec{n} + \vec{n} \cdot \vec{B}}}{(q)_{m_1} \cdots (q)_{m_r}},$$

(1.5)
where \((q)_n = \prod_{k=1}^n (1 - q^k)\) and \((q)_0 = 1\). Suppose that \(A\) and \(B\) are such that the sum in (1.5) involves only non-negative powers of \(q\) (hence \(\chi(q)\) is convergent for \(0 < |q| < 1\)). Let \(q = e^{2\pi i \tau}, \text{Im}(\tau) > 0\) and \(\hat{q} = e^{-2\pi i /\tau}\). The saddle point analysis (see e.g., [2, 3]) shows that the asymptotics of \(\chi(q)\) in the \(q \to 1\) limit is \(\chi(q) \sim \hat{q}^{-\frac{1}{\tau}}\) with \(c\) given by (1.4) and the numbers \(0 \leq x_i \leq 1\) satisfying the following equations

\[
x_i = \prod_{j=1}^r (1 - x_j)^{(A_{ij} + A_{ji})}, \quad i = 1, \ldots, r.
\]  

(1.6)

The principal aim of this work is to search for \(2 \times 2\) matrices \(A\) with rational entries such that \(c\) computed according to (1.4) and (1.6) has the form of the effective central charge, \(c_{st}\), of a minimal Virasoro model \(\mathcal{M}(s,t)\), i.e.

\[
c_{st} = 1 - \frac{6}{st},
\]

(1.7)

where \(s\) and \(t\) are co-prime numbers. Classification of the corresponding \(A\) matrices is a step towards classification of the dilogarithm identities (1.4) for \(r = 2\) (the complete classification is an open problem that appears to be quite involved). As seen from (1.6), we can consider only symmetric \(A\) matrices.

The physical motivation for the formulated mathematical task is twofold. First, equations (1.4) and (1.6) arise in the context of the thermodynamic Bethe ansatz (TBA) approach to the ultra-violet limit of (1+1)-dimensional integrable systems [4]. In this case the matrix \(A\) is related to the corresponding S-matrix, \(S(\theta)\), and \(c\) gives the value of the effective central charge of the ultra-violet limit of the model in question. Below we will refer to a system of equations of the type (1.6) as the TBA equations.

Second, equations (1.4) and (1.6) appear in the very conformal field theory. Namely, for certain \(A\) the series (1.5) can be identified (upon choosing specific \(B\)) and possibly imposing some restriction on the summation over \(\vec{m}\) as characters (or linear combinations of characters) of irreducible representations of the Virasoro algebra. In this case \(c\) is the value of the effective central charge of the conformal model to which the character \(\chi(q)\) belongs.

In the simplest case, \(r = 1\), there are only five algebraic numbers on the interval \([0, 1]\) such that \(c\) in (1.4) is rational,

\[
L(0) = 0, \quad L(1 - \rho) = \frac{2}{3}, \quad L(\frac{1}{2}) = \frac{1}{2}, \quad L(\rho) = \frac{3}{2}, \quad L(1) = 1.
\]

(1.8)

Here \(\rho = \frac{1}{2}(\sqrt{5} - 1)\) is the positive root of the equation \(x^2 + x = 1\). All the values of \(c = L(x)\) listed in (1.8) have the form (1.7) (with \((s,t) = (2, 3), (2, 5), (3, 4), (3, 5), \) and \(st = \infty\) for \(c = 1\)). They correspond, respectively, to

\[
A = \infty, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad 0
\]

(1.9)

All these \(A\) allow us to construct Virasoro characters of the form (1.5). In particular, \(A = \infty\) implies \(\chi(q) = 1\), which is the only character of the trivial \(\mathcal{M}(2, 3)\) model, and \(A = 0\) gives (for \(B = 0\)) the eta-function \(\eta(q)\). For the other \(A\) we have, for instance, (see e.g., [3] and references therein)

\[
\chi_{1,1}^{2.5} = q^{\frac{1}{10}} \sum_{m=0}^{\infty} \frac{q^{m^2 + m} (q_m)}{(q)_m}, \quad \chi_{1,2}^{3.4} = q^{\frac{3}{10}} \sum_{m=0}^{\infty} \frac{q^{m^2 + \frac{1}{2} m} (q_m)}{(q)_m}, \quad \chi_{1,2}^{3.5} + \chi_{1,3}^{3.5} = q^{\frac{1}{10}} \sum_{m=0}^{\infty} \frac{q^{m^2} (q_m)}{(q)_m}.
\]

(1.10)

The observation that all values of \(c\) obtained from the \(r = 1\) TBA equations are of the form (1.7) motivates our choice of \(c\) for the \(r = 2\) case. Notice however that in the latter case (1.4)
we will assume that and therefore does not lead to non-trivial. This leads to a statement would be desirable. Thus, the determinant of the functions properties of the \( r = 2 \) TBA equations are described (e.g., we find what \( A \) correspond to \( c = 1, c < 1 \) and \( c > 1 \), and several continuous families of \( A \) matrices are found. In section 3 a ‘discrete’ set of \( A \) matrices is presented and the corresponding dilogarithm identities are obtained. Some of these identities are proven or shown to be equivalent to previously known identities.

2 Properties of \( r = 2 \) TBA equations.

Our aim is to search for symmetric \( 2 \times 2 \) matrices \( A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \) with rational entries such that for \( 0 \leq x, y \leq 1 \) satisfying the equations

\[
\begin{align*}
x &= (1 - x)^{2a}(1 - y)^{2b} \\
y &= (1 - x)^{2b}(1 - y)^{2d}
\end{align*}
\]

the value of

\[ c = L(x) + L(y) \]

acquires the form (1.7) (\(|s| \) and \(|t| \) are to be co-prime numbers and \( st \) may be negative).

Since the summation in (1.5) is taken over non-negative numbers it is too restrictive to require \( A \) to be positive definite. Instead, we impose weaker conditions ensuring that the sum in (1.5) involves only non-negative powers of \( q \):

\[ a, d \geq 0, \quad b \geq -\min(a, d). \]

It is easy to see that these are sufficient conditions for (2.1) to have a solution on the interval \([0, 1]\).

For \( b = 0 \) equations (2.1) decouple. Then, taking the finite values of \( a \) and \( d \) from the list (1.9), we can obtain

\[ c = \frac{4}{5}, \frac{9}{10}, 1, \frac{11}{10}, \frac{5}{8}, \frac{7}{8}, \frac{3}{2}, \frac{8}{5}, 2. \]  

(2.4)

The first two values correspond to the \( M(5, 6) \) and \( M(5, 12) \) minimal models, whereas the last four values correspond to the \( Z_8, Z_{10}, Z_{13} \) and \( Z_{\infty} \) parafermionic models. Another possibility for the \( b = 0 \) case is to take \( a \) to be any positive (rational) number and put \( d = 1/(4a) \). As seen from (2.1), this leads to \( y = 1 - x \), and hence gives \( c = 1 \) due to the relation (1.2). In fact, it appears that the described set of values of \( c \) for \( b = 0 \) is exhaustive (a rigorous proof of this statement would be desirable). Thus, the \( b = 0 \) case does not produce any \( c \) different from (2.4) and therefore does not lead to non-trivial \( r = 2 \) dilogarithm identities. For the rest of the paper we will assume that \( b \neq 0 \).

For the further discussion we introduce the following notations. \( D := ad - b^2 \) stands for the determinant of \( A \). \( c[A] \) denotes the value of \( c \) computed for \( A \) according to (2.1)-(2.2). The functions \( \kappa(t) \) and \( \delta(t) \) are defined for \( t \geq 0 \) as follows:

\[ \kappa(t) = \xi, \quad \delta(t) = L(\xi), \quad \text{where} \quad \xi = (1 - \xi)^{2t}, \quad 0 \leq \xi \leq 1. \]  

(2.5)
Notice that the system (2.1) may have, in general, several solutions on the interval \([0,1]\). For example, if \(a > 0\), \(1/2 > b > 0\), \(d = 0\) (notice that \(\kappa(0) = 1\)), the system (2.1) possesses the extra solution \(x = 0, y = 1\). Such a situation is undesirable from the physical point of view \((x_i\) in the TBA equations (1.6) are physical entities which should be defined uniquely). Therefore, in the present paper we will deal mainly with such \(A\) matrices that the solution of (2.1) is unique.

**Proposition 1.** Suppose that \(A\) satisfies (2.3) and

\[
D \geq -\frac{1}{2} \max \left\{ d \left( \frac{1}{\kappa(a)} - 1 \right), a \left( \frac{1}{\kappa(d)} - 1 \right) \right\}.
\]

(2.6)

Then the system (2.1) possesses a unique solution on the interval \([0,1]\).

The proof of this and of the other propositions in this section is given in the Appendix. Equation (2.6) involves the function \(\kappa(t)\) which cannot be expressed in terms of elementary functions. It can be reduced to more explicit (although weaker) estimates. For instance, employing the Bernoulli and a Jensen-type inequalities to estimate \(\kappa(t)\), we derive that (2.6) holds if \(D \geq -ad\) for \(d \leq \frac{1}{2}\), \(b > 0\) and \(D \geq -(2ad)/(2d+1)\) for \(d > \frac{1}{2}\), \(b > 0\).

**Proposition 2.** Suppose that \(A\) is a symmetric invertible \(r \times r\) matrix such that the corresponding solution of (1.6) on the interval \([0,1]\) is unique. Then

\[
c[A] + c \left[ \frac{1}{4} A^{-1} \right] = r.
\]

(2.7)

This proposition explains why it makes sense to allow \(st\) in (1.7) to be negative. If \(c[A] = 1 - \frac{a}{2t} > 1\), then \(c \left[ \frac{1}{4} A^{-1} \right] = 1 + \frac{a}{2t} < 1\). Furthermore, Proposition 2 shows also that we can consider only \(b > 0\). Indeed, if \(b < 0\), then \(D > 0\) to satisfy (2.3). Therefore, for the ‘dual’ matrix \(\frac{1}{4} A^{-1}\) the \(b\) entry is positive.

**Proposition 3.** Suppose that \(A\) satisfies (2.3). Then

\[
c[A] > 1 \quad \text{if and only if} \quad b < \frac{1}{2} \quad \text{and} \quad ad < \left( \frac{1}{2} - b \right)^2;
\]

(2.8)

\[
c[A] = 1 \quad \text{if and only if} \quad b \leq \frac{1}{2} \quad \text{and} \quad ad = \left( \frac{1}{2} - b \right)^2;
\]

(2.9)

\[
c[A] < 1 \quad \text{otherwise}.
\]

(2.10)

Equation (2.9) implies that the solution of (2.1) satisfies the relation \(x + y = 1\) if and only if the matrix \(A\) has the form

\[
A = \left( \begin{array}{c} \frac{1}{2} - a \sqrt{ad} \\ \frac{1}{2} - \sqrt{ad} \end{array} \right), \quad a, d \geq 0.
\]

(2.11)

Notice that here \(D = \sqrt{ad} - 1/4\) and Proposition 1 cannot guarantee uniqueness of the solution of (2.1) for sufficiently small values of \(ad\). However, as seen from the proof, even if (2.1) has several solutions all they satisfy the relation \(x + y = 1\).

**Proposition 4.** Suppose that the matrix \(A\) is such that the corresponding solution of (2.1) is unique. Then this solution satisfies the relation \(x = y\) if and only if \(a = d\).

This proposition implies that the value of \(c[A]\) for a matrices of the form

\[
A = \left( \begin{array}{cc} a & b \\ b & a \end{array} \right)
\]

(2.12)

depends only on \((a + b)\). Indeed, for \(x = y\) and \(a = d\) the system (2.1) turns into the pair of coinciding equations for one variable. Therefore, \(x = y = \kappa(a + b)\) and (2.2) yields \(c[A] = 2\delta(a + b)\).

4
Thus, for the matrix $A$ of the form (2.12) the corresponding $r = 2$ dilogarithm identity reduces to an $r = 1$ identity. Therefore, the only values of $(a + b)$ in (2.12) that correspond to rational value of $c$ are given by the set (1.9). Namely, for $(a + b) = 1$, $\frac{1}{2}$, $\frac{5}{2}$, we obtain, respectively,

$$c = \frac{1}{3}, \quad 1, \quad \frac{5}{3}, \quad 2. \quad (2.13)$$

The $c = 1$ case here is just a particular case of the family of matrices (2.11) corresponding to $d = a$, $b = \frac{1}{2} - a$. The value $c = \frac{1}{3}$ is the effective central charge for the $M(5,6)$, $M(3,10)$ and $M(2,15)$ minimal models. The existence of the family of matrices (2.12) yielding this value of $c$ was observed in [6]. The following realizations of (1.5) (with certain restriction on the summation) as Virasoro characters are known for this family: $a = \frac{2}{3}$, $b = \frac{1}{3}$ gives $\chi_{1,3}$ and $\chi_{1,4} + \chi_{1,5}$ [3]; $a = b = \frac{1}{3}$ gives $\chi_{1,1}$, $\chi_{1,4}$, $\chi_{2,2}$ and $\chi_{2,4}$ [6]; $a = 1$, $b = 0$ gives $\chi_{1,5}$ [6]. Let us remark that, according to Proposition 1, the solution of (2.1) for the $a + b = 1$ case of (2.12) is unique at least for $a > 0.25$. Numerical computations show that it becomes non-unique for $a < a_0 \approx 0.1$.

To complete the general discussion of the properties of solutions to the system (2.1) let us find some estimates for $c[A]$.

Proposition 5. Suppose that $A$ is such that (2.3) holds and $a \geq d > 0$. Then the following lower and upper bounds on $c[A]$ hold:

$$\delta(b + d) + I\left(\kappa(d) \frac{c + h}{a + b}\right) \leq c[A] \leq \delta(a + b) + \delta(d), \quad \text{for} \quad d \leq b, \quad (2.14)$$

$$\delta(b + d) + I\left(\kappa\left(\frac{D}{a + b}\right) \frac{c + d}{b}\right) \leq c[A] \leq \delta(a + b) + \delta\left(\frac{D}{a + b}\right), \quad \text{for} \quad d \geq b > 0, \quad (2.15)$$

$$\delta(a + b) + \delta\left(\frac{D}{a + b}\right) \leq c[A] \leq 2\delta(b + d), \quad \text{for} \quad b < 0. \quad (2.16)$$

As an application of this proposition, we notice that for matrix $A$ such that $a + b > \xi_3 \approx 5.2$, $d > \xi_3$ and $b > 0$ the value of $c[A]$ cannot be the effective central charge, $c_{st} = 1 - \frac{b}{a + b}$, of a minimal model (recall that $s$ and $t$ are to be co-prime). Indeed, the smallest non-zero value of $c_{st}$ is $2/5$, whereas $c[A] \leq 2\delta(\xi_3) < 2/5$.

3 Solutions of $r = 2$ TBA equations and corresponding dilogarithm identities.

Eqs. (2.12) (for $a + b = 0, 1, 1, 2$) and (2.11) are examples of continuous families of matrices $A$ with rational value of $c[A]$. Now we will present ‘discrete’ series of matrices $A$ having $c[A]$ in the form (1.7). For completeness, the previously known examples are also listed. Let us remind that, according to Proposition 2, the list of the matrices $A$ below can be doubled by including their duals, $\frac{1}{A} A^{-1}$, but they do not give new dilogarithm identities.

Proposition 6. Among the matrices of the form

$$A = \frac{1}{2} \begin{pmatrix} 2a & 1 \\ 1 & 1 \end{pmatrix}, \quad a \geq 0 \quad (3.1)$$

only those with $a = 0$, $\frac{1}{2}$, $1$, $2$, $\infty$ have rational value of $c[A]$. These values are, respectively, $c = 1$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{10}$, $\frac{1}{2}$.

Proof. Denote $a = 1 - x$, $v = 1 - y$. In these variables the system (2.1) corresponding to (3.1) can be rewritten as follows:

$$v = 1 - av, \quad 1 - a^2 = (a^2)^x. \quad (3.2)$$
Using the first of these relations and employing the formulae (1.2)-(1.3), we obtain

\[ L(x) + L(y) = 2 - L(u) - L(v) = 2 - L(1 - v) - L(u^2) - L(1 - u) = 2 - L(x) - L(y) - L(u^2), \quad (3.3) \]

and hence

\[ c[A] = L(x) + L(y) = 1 - \frac{1}{2}L(u^2). \quad (3.4) \]

Thus, \( c[A] \) is rational only if \( L(u^2) \) belongs to the list (1.8), i.e., \( u^2 = 0, 1 - \rho, \frac{1}{2}, \rho, 1 \). Noticing that for \( w = u^2 \) the second equation in (3.2) takes the form \( w = (1 - w)^{1/3} \), we obtain the possible values of \( a \) as inverse to these in (1.9) (cf. Proposition 2).

For \( a = 0 \) the matrix (3.1) is a particular case of (2.11). For \( a = \infty \) the corresponding series (1.5) contains no summation over the first variable and thus reduces to the \( r = 1 \) case giving the characters of the \( M(3, 4) \) minimal model (for instance, the second character in (1.10)). For \( a = 1/2 \) the matrix (3.1) is a particular case of (2.12). It allows to construct several characters of the \( M(5, 6) \) minimal model [6], e.g.,

\[ \chi_{5,6}^{2,2+2Q} = q^{1/30} \sum_{m_2 = Q \mod 2}^{\infty} \frac{q^{2(m_1^2 + m_2^2) + 3m_1 + m_2 + \frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}, \quad Q = 0, 1. \quad (3.5) \]

The corresponding dilogarithm identity is \( 2L(1 - \rho) = \frac{4}{5} \).

For \( a = 1 \) the matrix (3.1) allows us to construct all characters of the \( M(3, 8) \) (see [6], the case of \( \chi^{3,8} \) was found earlier in [3]). For instance,

\[ \chi_{1,4}^{3,8} = q^{1/30} \sum_{m = 0}^{\infty} \frac{q^{m_2^2 + \frac{1}{2}m_1^2 + m_1 + m_2 + \frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}. \quad (3.6) \]

The corresponding dilogarithm identity is

\[ L(1 - \frac{1}{\sqrt{2}}) + L(\sqrt{2} - 1) = \frac{3}{4}, \quad (3.7) \]

or, equivalently, \( L(\frac{1}{\sqrt{2}}) - L(\sqrt{2} - 1) = \frac{1}{4} \). The latter relation is just a particular case, \( x = \frac{1}{\sqrt{2}} \), of the Abel’s duplication formula (which follows from (1.3))

\[ \frac{1}{2}L(x^2) = L(x) - L(\frac{x}{1 + x}). \quad (3.8) \]

For \( a = 2 \) the matrix (3.1) allows us to construct some characters of the \( M(4, 5) \) [6]. For instance,

\[ \chi_{1,2}^{4,5} = q^{1/30} \sum_{m = 0}^{\infty} \frac{q^{2m_1^2 + \frac{1}{2}m_1^2 + m_1 + m_2 + \frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}. \quad (3.9) \]

The corresponding dilogarithm identity is \( L(1 - \sqrt{9}) + L(1 - \frac{1}{1 + \sqrt{9}}) = \frac{7}{10}, \) or, equivalently,

\[ L(\sqrt{9}) + L(1 + \sqrt{9}) = \frac{13}{10}. \quad (3.10) \]

This identity was found in [6] as a consequence of the formula (3.9). The proof of Proposition 6 provides an algebraic derivation for (3.10) based on the functional relation (1.3).

There exists the well-known representation of the type (1.5) for the characters of \( M(2, 2k+1) \) model with rank \( A = k-1 \) (it provides the sum side of the Andrews-Gordon identities [7]). In the \( k = 3 \) case the corresponding matrix \( A \) is

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad c[A] = 2/7. \quad (3.11) \]
The corresponding dilogarithm identity is \((\lambda = 2 \cos \frac{2\pi}{7})\)

\[
L\left(\frac{1}{\lambda}\right) + L\left(\frac{1}{\lambda^{-1}}\right) = \frac{4}{7}.
\]  
(3.12)

The following matrix allows us to construct all characters of the \(\mathcal{M}(3, 7)\) (see [6], the case of \(\chi_{1,2}^{3,7}\) was found earlier in [3])

\[
A = \frac{1}{4} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad c[A] = 3/7.
\]  
(3.13)

For instance,

\[
\chi_{1,3+Q}^{3,7} = \frac{1}{8} \sum_{\substack{n = 0 \\ m_2 = Q \mod 2}}^{\infty} \frac{q^{2m_1^2 + \frac{2}{5}m_2^2 + 2m_1m_2 + m_1 - \frac{1}{2}m_2}}{(q)m_1(q)m_2}, \quad Q = 0, 1.
\]  
(3.14)

The corresponding dilogarithm identity is \((\lambda = 2 \cos \frac{2\pi}{7})\)

\[
L\left(\frac{1}{\lambda}\right) + L\left(\frac{1}{\lambda^{-1}}\right) = \frac{4}{7}.
\]  
(3.15)

Let us remark that both (3.12) and (3.15) can be derived from the Watson identities [8]

\[
L(\alpha) - L(\alpha^2) = \frac{1}{7}, \quad L(\beta) + \frac{1}{5}L(\beta^2) = \frac{5}{7}, \quad L(\gamma) + \frac{1}{2}L(\gamma^2) = \frac{4}{7},
\]  
(3.16)

where \(\alpha, -\beta\) and \(-\gamma^{-1}\) are roots of the cubic

\[
t^3 + 2t^2 - t - 1 = 0
\]  
(3.17)
such that \(\lambda = 1 + \alpha = \beta^{-1} = (1 - \gamma)^{-1}\). The equivalence of (3.12) (rewritten in terms of \(\beta\)) and the second equation in (3.16) was shown in [1]. To establish the equivalence of (3.15) to the second equation in (3.16), we exploit the Abel’s duplication formula:

\[
L\left(\frac{1}{\lambda}\right) + L\left(\frac{1}{\lambda^{-1}}\right) = L(\beta^2) + L\left(\frac{\beta}{1 + \beta}\right) = L(\beta^2) + L(\beta) - \frac{1}{2}L(\beta^2) = L(\beta) + \frac{1}{2}L(\beta^2).
\]  
(3.18)

Now we present a list of matrices \(A\) yielding \(c[A]\) in the form (1.7) that have not appeared in the literature before. These are results of computer based search performed bearing in the mind the general properties of \(r = 2\) TBA equations discussed in the previous section. For some of the corresponding dilogarithm identities we give an explicit algebraic proof or show that they are equivalent to certain known identities. The cases where such a proof is lacking were checked numerically (with a precision of order \(10^{-15}\)).

The effective central charge of the \(\mathcal{M}(3, 5)\) model is produced by

\[
A = \frac{1}{4} \begin{pmatrix} 5 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}, \quad c[A] = 3/5.
\]  
(3.19)

One can derive from (2.1) that \(x = 1 - \delta^2\) and \(y = (1 + \delta)^{-2}\) where \(\delta\) is the positive root of the quartic

\[
\delta^4 + 2\delta^3 - \delta - 1 = 0.
\]  
(3.20)

Applying the Ferrari’s method, we reduce this equation to

\[
\delta^2 + \delta = \rho + 1.
\]  
(3.21)

The solution is \(\delta = \frac{1}{2}(\sqrt{3} + 2\sqrt{5} - 1) = \frac{1}{2}(\sqrt{4\rho + 5} - 1)\). The corresponding dilogarithm identity reads

\[
L(1 - \delta^2) + L\left(\frac{1}{\delta^{-1}}\right) = L\left(\frac{1}{4\sqrt{4\rho + 5} - 1 - \rho}\right) + L\left(\frac{1}{2} + \frac{1}{2}\rho - \frac{1}{2}\sqrt{4\rho + 2}\right) = \frac{3}{7}.
\]  
(3.22)
Gordon and McIntosh proved in [9] for the same $\delta$ the following identity

$$L(\delta) - L(\delta^3) = \frac{1}{6}.$$  \hspace{1cm} (3.23)

Let us show that (3.22) and (3.23) are equivalent. Using (1.2) and (3.8) several times, we find

$$L(1 - \delta^2) + L\left(\frac{1}{1+\delta}\right) = 1 - L(\delta^2) + L\left(\frac{1}{1+\delta}\right) = 1 - 2L(\delta)+2L\left(\frac{\delta}{1+\delta}\right) + 2L\left(\frac{1}{1+\delta}\right) - 2L\left(\frac{1}{2+\delta}\right)$$

$$= 1 - 2L(\delta) - 2L(1 - \delta^3) = 1 - 2\left(L(\delta) - L(\delta^3)\right) = \frac{3}{6}.$$  

In the last line we used that $(2 + \delta)^{-1} = 1 - \delta^3$ holds due to (3.21).

The solution is $x = \frac{1}{4}(3 - \sqrt{5}) = \frac{1}{2}(1 - \rho)$, $y = \sqrt{5} - 2 = 2\rho - 1$ and the corresponding dilogarithm identity reads

$$L\left(\frac{1}{2} - \frac{1}{2}\rho\right) + L(2\rho - 1) = \frac{1}{2}.$$  \hspace{1cm} (3.26)

To prove it we introduce $u = 1 - x$, $v = 1 - y$ and notice that $u = \frac{1}{2}(1 + \rho) = 1/(2\rho)$ and $uv = 2u - 1$, $1 - uv = 2(1 - u)$, $u(1 - v) = 1 - u$. Employing now (1.2) and (1.3), we obtain:

$$L(u) + L(v) = L(2u - 1) + L\left(\frac{1}{2}\right) + L\left(\frac{5}{2}\right) = L(2u - 1) + L\left(1 - \frac{1}{2u}\right) + \frac{1}{2} = L(\rho) + L(1 - \rho) + \frac{1}{2} = \frac{3}{6},$$  \hspace{1cm} (3.27)

which is equivalent to (3.26) due to (1.2).

For (3.25) the equations (2.1) can be transformed to the form:

$$x^4 - 6x^3 + 13x^2 - 10x + 1 = 0, \hspace{1cm} y^4 + 6y^3 - 11y^2 + 6y - 1 = 0$$  \hspace{1cm} (3.28)

and $y(3 - 2x) = (1 - x)$. Applying the Ferrari’s method to these equations, we reduce them to

$$x^2 + \left(\sqrt{2} - 3\right)x = 2\sqrt{2} - 3, \hspace{1cm} y^2 + 3(\sqrt{2} + 1)y = \sqrt{2} + 1.$$  \hspace{1cm} (3.29)

The solution is $x = \frac{1}{2}(3 - \sqrt{2}) - \frac{1}{2}\sqrt{2}\sqrt{2} - 1$, which leads to the following dilogarithm identity:

$$L\left(\frac{3}{2} - \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\sqrt{2} - 1\right) + L\left(\frac{1}{2} + \sqrt{2}\right)\sqrt{2}\sqrt{2} - 1 - \frac{3}{2} - \frac{3}{2}\sqrt{2} \right) = \frac{1}{2}.$$  \hspace{1cm} (3.30)

The effective central charge of the $\mathcal{M}(2,5)$ model is produced by

$$A = \frac{1}{2}\left(\begin{array}{cc} 8 & 5 \\ 5 & 4 \end{array}\right), \hspace{1cm} c[A] = 2/5.$$  \hspace{1cm} (3.31)

It can be derived from (2.1) that $x = 1 - u_+$ and $y = u_-(u_- - 1)^{-1}$, where $u_+ > 0$ and $u_- < 0$ are the real roots of the quartic

$$u^4 + u^3 + 3u^2 - 3u - 1 = 0.$$  \hspace{1cm} (3.32)

Applying the Ferrari’s method to this equation, we reduce it to

$$u^2 - \rho u = 2\rho - 1.$$  \hspace{1cm} (3.33)
The solution is 
\[
\frac{w}{\sqrt{\rho - 1}} = \frac{1}{2} \rho \pm \frac{1}{2} \sqrt{i \rho - 3},
\]
which leads to the following dilogarithm identity:
\[
L \left( 1 - \frac{1}{2} \rho - \frac{i}{2} \sqrt{i \rho - 3} \right) + L \left( \frac{1}{2} \sqrt{28 \rho + 45} - 2 \rho - \frac{3}{2} \right) = \frac{3}{5}.
\] (3.34)

The central charge of the \(M(6, 7)\) minimal model is produced by (this was noticed earlier by M. Terhoeven (unpublished))
\[
A = \frac{1}{2} \begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}, \quad c[A] = 6/7.
\] (3.35)

One can derive that in this case the solution of (2.1) is given by \(x = \mu^{-1}\) and \(y = 1 - \nu\), where \(0 < \nu < 1\) and \(\mu > 1\) are the real roots of the following equation
\[
t^6 - 7t^5 + 19t^4 - 28t^3 + 20t^2 - 7t + 1 = 0.
\] (3.36)
The corresponding dilogarithm identity reads \(L(\mu^{-1}) + L(1 - \nu) = \frac{\pi}{2}\), or equivalently
\[
L(\nu) - L(\frac{1}{\mu}) = \frac{\pi}{4}.
\] (3.37)

It would be interesting to clarify whether this identity is related to the Watson identities.

The list is completed with two matrices \(A\) such that \(d = 0\). As was remarked above, in such a case equations (2.1) have an extra solution \(x = 0, y = 1\). We however will focus on the 'regular' solution, \(0 < x, y < 1\).
\[
A = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad c[A] = 8/7.
\] (3.38)

The corresponding equations (2.1) can be transformed to the form:
\[
x = 1 - y^2, \quad y^3 + 2y^2 - y - 1 = 0.
\] (3.39)
The latter equation is exactly the cubic (3.17) and \(y = \alpha\). Therefore the dilogarithm identity yielding the value of \(c\) in (3.38) is equivalent to the first equation in (3.16):
\[
L(x) + L(y) = L(1 - \alpha^2) + L(\alpha) = L(1) + L(\alpha) - L(\alpha^2) = \frac{\pi}{2}.
\] (3.40)

This value is the central charge of the \(Z_5\) parafermionic model. Let us remark that, according to Proposition 2, the matrix \(\frac{1}{4}A^{-1}\) would give \(c = 6/7\) (corresponding to the \(M(6, 7)\) minimal model) but it does not satisfy (2.3).
\[
A = \frac{1}{18} \begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix}, \quad c[A] = 6/5.
\] (3.41)

The corresponding equations (2.1) can be transformed to the form:
\[
x = 1 - y^3, \quad y^4 + 2y^3 - y - 1 = 0.
\] (3.42)
The latter equation is exactly the quartic (3.20) and \(y = \delta\). Therefore the dilogarithm identity yielding the value of \(c\) in (3.38) is equivalent to the Gordon-McIntosh identity (3.23):
\[
L(x) + L(y) = L(1 - \delta^3) + L(\delta) = L(1) + L(\delta^3) - L(\delta) = \frac{6}{5}.
\] (3.43)

Let us remark that, according to Proposition 2, the matrix \(\frac{1}{4}A^{-1}\) would give \(c = 4/5\) (corresponding to the \(M(5, 6)\) minimal model) but it does not satisfy (2.3).
To summarize, we studied $2 \times 2$ matrices $A$ such that the corresponding TBA equations (2.1) yield the ‘effective central charge’ of the form $c_{\text{eff}} = 1 - \frac{a}{\beta}$ (where $|s|$ and $|t|$ are co-prime numbers). Certain properties of such matrices have been established. Several continuous families and a ‘discrete’ set of admissible matrices $A$ have been found. For these examples several new two-term dilogarithm identities have been obtained and in some cases proven or shown to be equivalent to some previously known identities.

The question whether the found set of $2 \times 2$ matrices is exhaustive remains open. If the set is complete (or can be completed), it can be used for a classification of massive $1+1$-dimensional integrable models with diagonal scattering by the admissible values of the effective central charge for the corresponding S-matrices.

It would be interesting to extend the obtained results to the rapidity dependent TBA equations [4] and, further, to the $Y$-systems [10].

Acknowledgments: I am grateful to K. Kokhas for helpful discussions. This work has been completed during the workshop “Applications of integrability” at the Erwin Schrödinger Institute, Vienna. I thank the organizers of the workshop and the members of the ESI for warm hospitality.

This work was supported in part by the grant RFFI-99-01-00101.

Appendix.

Proof of Proposition 1. Eliminating $x$ in (2.1), we obtain

$$y^{\frac{a}{\beta}} (1 - y)^{-\frac{a}{\beta}} + y^{\frac{a}{\beta}} (1 - y)^{-2D} = 1.$$  \hspace{1cm} (A.1)

Let $f(y)$ denote the l.h.s. of (A.1). For $D \ge 0$ the uniqueness of the solution is obvious since $f(y)$ is monotonic (strictly increasing for $b > 0$ and strictly decreasing for $b < 0$) on the interval $[0, 1]$. Consider now the case of $D < 0$ (which implies $b > 0$ because of (2.3)). We have $f(0) = 0$, $f(1) = \infty$ and $f(y)$ is a smooth (but not necessarily monotonic) function for $0 < y < 1$. Eq. (A.1) can have several solutions if $f'(y) \equiv df(y)/dy$ has roots on this interval. The explicit form of $f'(y)$ shows that this can occur only for $y > y_{\text{min}} = a(a - 2D)^{-1}$. Furthermore, if (A.1) has several solutions, then among the roots of $f'(y)$ there must be at least one, denote it $y_0$, such that $f(y_0) < 1$. As seen from (A.1), the necessary condition for this is $y_0 < \kappa(d)$. If this relation is incompatible with the condition $y > y_{\text{min}}$, i.e. $2D \ge -a(\frac{1}{\kappa(d)} - 1)$, then the solution of (A.1) and hence of (2.1) is unique. Considering in the same way the analogue of (A.1) for $x$, we obtain the condition $2D \ge -d(\frac{1}{\kappa(d)} - 1)$. Clearly, we can take the lowest of the two bounds.

Proof of Proposition 2. Taking logarithm of the equations in (1.6), multiplying the resulting system with $\frac{1}{2} A^{-1}$ from the left, taking exponents of the new equations, and replacing all $x_i$ by $(1 - x_i)$, we obtain exactly equations (1.6) for $\frac{1}{2} A^{-1}$. Exploiting the property (1.2), we infer that $c[\frac{1}{2} A^{-1}] = \sum_{i=1}^{r} L(1 - x_i) = r - c[A]$.

Proof of Proposition 3. In the case of $b \ge \frac{1}{2}$ we have $x = (1 - x)^{2a}(1 - y)^{2b} \le (1 - x)^{2a}(1 - y) \le 1 - y$, where the equality holds only if $b = \frac{1}{2}$ and $a = 0$ or $d = 0$. Therefore $c[A] = L(x) + L(y) \le L(1 - y) + L(y) = 1$.

Consider now the $b < \frac{1}{2}$ case. Let $4ad = (2b - 1)^2$. Divide the first equation in (2.1) by $(1 - y)$ and take its $(2b - 1)$-th power. Divide the second equation in (2.1) by $(1 - x)$ and take its $2a$-th power. The r.h.s. of the resulting equations coincide. Thus, we obtain

$$\left(\frac{1 - y}{x}\right)^{1 - 2b} = \left(\frac{y}{1 - x}\right)^{2a},$$  \hspace{1cm} (A.2)
where the powers on both sides are positive. An assumption that \(1 - y > x\) and hence \(y < 1 - x\) leads to a contradiction since then the l.h.s. and the r.h.s. of (A.2) are, respectively, greater and smaller than 1. Analogously, an assumption that \(1 - y < x\) also leads to a contradiction. Thus, we conclude that \(1 - y = x\). Moreover, any matrix \(A\) such that \(e[A] = 1\) necessarily satisfies (2.9). Indeed, \(e = 1\) implies the relation \(x + y = 1\). Substituting it into (2.1), we obtain the conditions \(4ad = (2b - 1)^2\) and \(b \leq 1/2\) (the latter one guarantees existence of a solution on the interval \([0, 1])\).

The hyperbola \(4ad = (2b - 1)^2\) divides the quadrant \(a \geq 0, d \geq 0\) into two disjoint parts. Since \(e[A]\) is continuous function of \(a\) and \(d\), we infer that \(e[A] < 1\) for \(4ad > (2b - 1)^2\) (because \(x\) and \(y\) are small for large \(a\) and \(d\)) and \(e[A] > 1\) for \(4ad < (2b - 1)^2\) (because \(x \approx 1\) and \(y \approx 1\) for small \(a\) and \(d\)).

**Proof of Proposition 4.** Equation (A.1) in the \(a = d\) case coincides with its \(x\) analogue, that is \(x\) and \(y\) obey the same equation. This implies \(x = y\) since we required the uniqueness of the solution. The ‘only if’ part of the proposition is obvious, it suffices to substitute the relation \(x = y\) into (2.1).

**Proof of Proposition 5.** Notice that \(a \geq d\) implies \(x \leq y\). Indeed, for \(d\) finite and \(a \gg d\), it follows from (2.1) that \(x \approx 0\) whereas \(y\) is finite. Together with Proposition 4 this implies that \(x < y\) for all \(a > d\) since \(x\) and \(y\) are continuous functions of \(a, b, d\) (cf. (A.1)). Thus, we have \(1 - x \geq 1 - y\). Substituting this inequality into (2.1), we obtain (assuming \(b > 0\))

\[
(1 - y)^2(a+b) \leq x \leq \kappa(a+b), \quad \kappa(b+d) \leq y \leq (1 - x)^2(b+d).
\]

(A.3)

This provides the upper bound for \(x\) and the lower bound for \(y\). In order to find an upper bound for \(y\) we can simply notice that the second equation in (2.1) implies \(y < \kappa(d)\). Alternatively, we can first employ (2.1) to express \(y\) as follows: \(y = (1 - y)^{2D/ax^{b/a}}\). Together with \(x < y\) this yields \(y < \kappa(D_x)\). Comparing the values of \(D_x\) and \(d\), we infer that the first upper bound for \(y\) is better if \(d \leq b\). Now, if \(y < \kappa(t)\), then the definition (2.5) implies also that \(1 - y > \kappa(t)^{2}\). Substituting this relation (with \(t = d\) or \(t = \frac{D_x}{ax^{b/a}}\)) into the first inequality in (A.3), we obtain the corresponding lower bounds for \(x\). Having found the upper and lower bounds for \(x\) and \(y\), we obtain the estimates (2.14) and (2.15) simply exploiting that \(L(t)\) and hence \(\delta(t)\) are strictly monotonic.

The estimates in (2.16) are derived by similar considerations in the \(b < 0\) case.

**References**


