Representations in $L^2$-Spaces
on Infinite-Dimensional Symmetric Cones

Karl-Hermann Neeb
Bent Ørsted


Supported by Federal Ministry of Science and Transport, Austria
Available via http://www esi.ac.at
Representations in $L^2$-spaces on infinite-dimensional symmetric cones

Karl-Hermann Neeb, Bent Ørsted

Introduction

In a previous paper [NÖ98] we gave an algebraic classification of the unitary highest weight representations for the infinite-dimensional analogs of the automorphism groups of the classical bounded symmetric domains. We also realized these representations globally in certain Hilbert spaces of holomorphic functions on certain corresponding domains in (infinite-dimensional) Hilbert spaces. This was accomplished using the Harish-Chandra realizations as bounded domains, whereas the Cayley transformed Siegel domains did not enter. Since in the finite-dimensional case, it is exactly in the Siegel domains that one applies Laplace transform methods to study the finer analytical properties of the unitary highest weight representations, it is natural to see to what extent such tools make sense in the infinite-dimensional case as well. Hence in particular we want to understand as explicitly as possible the action of the affine part of the automorphism group of an infinite dimensional tube domain on $L^2$-realizations of unitary highest weight modules. In the finite-dimensional case the $L^2$-picture for unitary highest weight representations of automorphism groups of tube domains has first been described by J. L. Clerc in [Cl95]. It has been refined in several respects in [HN97].

We shall construct infinite-dimensional analogs of the Wishart distributions on symmetric cones, perhaps of independent interest in probability theory (see also [OV96]); the most explicit results in this paper are on the case of scalar measures, but there are extensions to the case of vector-valued measures as well (i.e. where the measure takes as values positive operators in a Hilbert space). The difficulties we have to face in this project are twofold. First, there are the subtleties of infinite-dimensional measure theory. There is no unique canonical dual space on which a measure with a given Laplace transform should live. This is a well known phenomenon in the theory of Gaussian measures. In our context we will see that there are three different realizations of the measures that we are interested in, and each of these three realizations has its different merits. The second problem is that the infinite-dimensionality of the domains we are working with causes a certain type of singularity of holomorphic functions in the sense that if they are naturally defined on a domain in a space of trace class operators, then they do not extend to the corresponding domain in the space of Hilbert-Schmidt operators. Nevertheless in our situations this type of singularity can always be removed by multiplying with a certain function, so that we obtain functions on domains in $L^2$-spaces.

Since the representations we consider are defined most naturally before the removal of the singularities takes place, the representations one obtains after this process become more complicated. Furthermore, we will see that this process often leads to multiplier representations which are non-trivial in the sense that, even though we obtain a projective representation of the small group, to obtain a unitary representation of the larger group, we have to pass to a non-trivial central extension.

The main achievements of this paper are the following:
1) the construction of the $L^2$-realization of all unitary highest weight representations of automorphism groups of classical infinite-dimensional tube domains; in particular the vector-valued $L^2$-spaces seem to be completely new.
A projective representation of the full identity component of the affine automorphism group of the Hilbert-Schmidt version $\Sigma_2$ of the tube domain. The cocycle is trivial on the subgroup corresponding to the trace class version $\Sigma_1$, but non-trivial on the large group. The structure of the cocycle gives a geometric explanation for some of the difficulties we encountered in [NØ98] (Section IV).

We show that the operator-valued measures corresponding to the vector valued highest weight representations have densities of a rather weak type with respect to Wishart distributions which makes it possible to determine their “supports” (Section VI).

In Section I we give the necessary background on measures and Laplace transforms in the infinite-dimensional setting, and in Section II we recall the Wishart distributions and the Euclidian Jordan algebra theory, which is then extended to the case of infinite-dimensional symmetric cones. Section III relates our previous construction of unitary highest weight representations and their reproducing kernels to positive definite functions on cones; this is done via the Cayley transform, and gives a correspondence between the highest weight representations of automorphisms groups of bounded domains and admissible representations of $L$, the structure group associated to the cone. The main technical result here is Theorem III.9, which asserts the existence of an extension of the representations to $L_1$, a certain trace-class completion of $L$, using results in [Ne88b]. Furthermore, we show that the corresponding projective representation even extends to the considerably larger group $L_0$ of all bounded invertible operators on the Hilbert space on which the Jordan algebra is represented in a natural way. In Section IV we construct the corresponding projective unitary representations of the affine automorphism group of the Hilbert-Schmidt version $\Sigma_2$ of the tube domain and get the existence of the measures giving the unitary structure. Finally in Sections V, VI, and VII we give some more details on the measures, in particular (in the scalar case) ergodicity for the “compact” structure group and hence irreducibility for the corresponding motion group, see Corollary V.2. Note the example in Section VI, where the vector-valued measure is made explicit in the case of the odd metaplectic representation.

One may think of our results as treating natural analogs of the metaplectic representation, and at the same time giving a way of understanding the role played by cocycles in the theory of representations of infinite-dimensional groups. At the same time we have seen ways of understanding the restrictions of the unitary highest weight representations to subgroups, viz. the affine groups of the Jordan algebra (a maximal parabolic), and the analog of the maximal compact subgroup (without making that completely explicit, though). It would be interesting to consider the spectrum of other subgroups in these representations. For example, one expects a unitary highest weight representation to decompose as a direct sum of such when restricted to a subgroup also of hermitian type.

I. Measures and Laplace Transforms in an Infinite Dimensional Setting

In this section we describe preliminary material on measures and Laplace transforms in the infinite-dimensional setting and discuss Gaussian measures as an important class of examples which plays a crucial role in this paper. The main result is Theorem I.3, an existence theorem for projective limits of operator-valued measures. In Theorem I.7 we explain the relation to the corresponding space of holomorphic functions.

**Definition I.1.** (a) Let $(J, \leq)$ be a directed set. A projective family of measurable spaces is a family $(X_j, \mathcal{E}_j)_{j \in J}$ of measurable spaces together with measurable maps $\varphi_{jk}: X_k \to X_j$, $j \leq k$, satisfying $\varphi_{jj} = \text{id}_{X_j}$ and $\varphi_{jk} \circ \varphi_{kl} = \varphi_{jl}$ for $j \leq k \leq l$. The projective limit measurable space $X := \lim_{\leftarrow} X_j$ is the set

$$X = \left\{ x \in \prod_{j \in J} X_j; (\forall j < k) \varphi_{jk}(x_k) = x_j \right\}$$

dowed with the smallest $\sigma$-algebra $\mathcal{E}$ for which all the projections $p_j: X \to X_j$ are measurable maps.
If the projective limit measurable space \((X, \mathcal{S}) := \lim_{\to}(X_j, \mathcal{S}_j)\) carries a \(\text{Herm}^+(H)\)-valued measure \(\mu\) for which the natural maps \(\varphi_j: X \to X_j\) satisfy \(\varphi_j^* \mu = \mu_j\) for all \(j \in J\), then it is unique. It exists if the following conditions are satisfied:

1. Each measure \(\mu_j\) is inner regular with respect to some topology \(\tau_j\) on \(X_j\).
2. If \(J\) is uncountable, then for every increasing sequence \(M := \{j_n: n \in \mathbb{N}\} \subseteq J\) the range of the natural mapping
   \[
   \varphi_M: X \to X_M := \lim_{n \to \infty} X_{j_n}
   \]
is thick in the sense that its complement contains no measurable subset of non-zero measure. Moreover, one of the following two conditions has to be satisfied:

1. \((X_j, \mathcal{E}_j)\) is countably separated for each \(j \in J\), or
2. the maps \(\varphi_{j,k}: (X_k, \tau_k) \to (X_j, \tau_j)\) are continuous for \(j \leq k\).

**Proof.** If \(H\) is one-dimensional, i.e., the \(\mu_j\) are probability measures, then the assertion follows from the following results in [Ya85]: Theorem 8.2 (reduction to the countable case) and Theorem 7.2 (the countable case).

For the general case, let \(S \in \text{Herm}^+(H)\) be a positive trace class operator. Then the measures \(\mu^S: E \mapsto \text{tr}(\mu_j(E)S)\) form a consistent family of measures on the \(X_j\) which are inner regular with respect to the topology \(\tau_j\) on \(X_j\). Hence the scalar case applies and we find a unique finite positive measure \(\mu^S\) on \(X\) with \(\varphi_j^*\mu^S = \mu^S_j\) for each \(j \in J\). Furthermore the assignment \(S \mapsto \mu^S\) is additive for each measurable subset \(E \in \mathcal{E}\). Hence we find for each \(E \in \mathcal{E}\) a unique operator \(\mu(E) \in \text{Herm}^+(H)\) with

\[
\text{tr}(\mu(E)S) = \mu^S(E)
\]

for all \(S \in \text{Herm}^+(H)\). It follows from [Ne98a, Prop. I.7, Th. I.10] that we thus obtain a \(\text{Herm}^+(H)\)-valued measure on \(X\). \(\blacksquare\)

**Remark 1.4.** (a) If \((X_i, \mathcal{E}_i)_{i \in I}\) is a family of measurable spaces and \(J\) the directed set of finite subsets of \(I\), then we assign to each \(F \in J\) the finite product measurable space \(X_F := \prod_{i \in F} X_i\). With the natural projections

\[
\varphi_{F_1, F_2}: X_{F_2} \to X_{F_1}, \quad F_1 \subseteq F_2
\]

we obtain a projective system of measurable spaces. If for each \(i \in I\) we are given a probability measure \(\mu_i\), then the projective family of measures \(\mu_F = \otimes_{i \in F} \mu_i\) defines a probability measure on the product space

\[
X = \lim X_F = \prod_{i \in I} X_i.
\]

The main point is that this requires no additional regularity condition ([Ya85, Th. 12.1]).

(b) Suppose that \(\mu\) is a measure on a projective limit measurable space \((X, \mathcal{E}) := \lim(X_j, \mathcal{E}_j)\) and write \(\mathcal{E}_0\) for the set algebra of all those subsets of \(X\) which can be written as \(\varphi_j^{-1}(E_j)\) for some \(E_j \in \mathcal{E}_j\). We claim that the characteristic functions \(\chi_E, E \in \mathcal{E}_0\), form a total subset of \(L^2(X, \mu)\).

In fact, let \(V \subseteq L^2(X, \mu)\) denote the closed subspace generated by the characteristic functions \(\chi_E, E \in \mathcal{E}_0\). Then \(\mathcal{E} := \{E \subseteq X: \chi_E \in V\}\) is a monotone system containing \(\mathcal{E}_0\). Hence it contains the monotone system generated by the set algebra \(\mathcal{E}_0\) and therefore the \(\sigma\)-algebra \(\mathcal{E}\). We conclude that \(V = L^2(X, \mu)\) which proves our claim. It follows in particular that

\[
L^2(X, \mu) = \lim_{j \to \infty} L^2(X_j, \mu_j)
\]

is a direct limit of Hilbert spaces because the right hand side can be identified in a natural way with a subspace of \(L^2(X, \mu)\) which, in view of the observation above, is dense. This observation shows in particular that whether a measure exists or not, one always can consider the direct limit Hilbert space \(\lim_{j \to \infty} L^2(X_j, \mu_j)\) as a substitute of \(L^2(X, \mu)\) if \(\mu\) does not exist. \(\blacksquare\)

**Lemma 1.5.** If \(\varphi: E \to F\) is a continuous linear mapping between locally convex spaces, \(\varphi': F' \to E'\) its adjoint map, and \(\mu_E\) a probability measure on \(E\) (defined on the smallest \(\sigma\)-algebra for which all continuous functionals are measurable), then \(L(\varphi^\ast \mu_E) = L(\mu_E) \circ \varphi'\).

**Proof.** For \(\alpha \in F'\) we have

\[
L(\varphi^\ast \mu_E)(\alpha) = \int_F e^{-\alpha(x)} d(\varphi^\ast \mu_E)(x) = \int_E e^{-\alpha(\varphi(x))} d\mu_E(y) = L(\mu_E)(\varphi', \alpha).
\]

\(\blacksquare\)
Definition 1.6. (a) If \( X \) is a set and \( V \) is a complex Hilbert space, then a function \( K: X \times X \to B(V) \) is called a positive definite kernel if for each finite sequence \( (x_1, v_1), \ldots, (x_n, v_n) \in X \times V \) we have
\[
\sum_{i,j=1}^{n} \langle K(x_i, x_j), v_j, v_i \rangle \geq 0.
\]
This condition is equivalent to the existence of a Hilbert subspace \( \mathcal{H}_K \subseteq V^X \) such that the evaluation functions \( K_x: \mathcal{H}_K \to V \) satisfy \( K(x, y) = K_xK_y^* \) (cf. [Ne99, Sect. I]).
(b) If \( U \) is a real vector space, \( U_C \) its complexification, and \( \Omega \subseteq U \) a convex set, then we call \( T_\Omega := \Omega + iU \subseteq U_C \) the corresponding tube domain. Note that \( T_\Omega \) is invariant under the complex conjugation \( z = x + iy \mapsto \overline{z} := x - iy \). If \( V \) is a Hilbert space, then a function \( \varphi: T_\Omega \to B(V) \) is said to be positive definite if the kernel \( K(z, w) := \varphi(iz, \overline{w}) \) is positive definite which in turn is equivalent to the positive definiteness of the restriction to \( \Omega \times \Omega \) (cf. [Gl99]). If \( \varphi \) is positive definite, then we write \( \mathcal{H}_\varphi \subseteq V^{T_\Omega} \) for the corresponding Hilbert space.
(c) We say that a subset \( \Omega \) in the real vector space \( U \) is finitely open if for each finite-dimensional subspace \( U_0 \subseteq U \) the intersection \( U \cap \Omega \) is open in \( U \). If \( V \) is a Hilbert space, then a function \( f: T_{\Omega} \to V \) on the corresponding tube domain is called Gateaux holomorphic if the restriction to each tube domain \( T_{\Omega \cap \Omega} \) is holomorphic. We write \( \text{Hol}_{\varphi}(T_{\Omega}, V) \) for the space of all Gateaux holomorphic \( V \)-valued functions on \( T_{\Omega} \). For the theory of holomorphic functions on domains in infinite-dimensional spaces we refer to the book of Hervé [He89] (see also Appendix III in [Ne99]).

Theorem 1.7. Let \( U \) be a real vector space and \( \Omega \subseteq U \) a non-empty convex finitely open subset. Let \( V \) be a Hilbert space and \( \varphi: T_{\Omega} \to B(V) \) a positive definite \( (G) \)-holomorphic function. Then the following assertions hold:
(i) There exists a unique \( \text{Herm}^+(V) \)-valued measure \( \mu \) on the smallest \( \sigma \)-algebra on \( U^* \) for which all point evaluations are continuous such that \( \mathcal{L}(\mu) = \varphi \).
(ii) The map
\[
\mathcal{F}: L^2(U^*, \mu) \to \text{Hol}_{\varphi}(T_{\Omega}, V), \quad f \mapsto \hat{f}
\]
with \( \langle \hat{f}(z), v \rangle := \langle f, e^{-\frac{z}{2}}v \rangle \) is an isometry onto the reproducing kernel space \( \mathcal{H}_\varphi \) corresponding to the kernel associated to \( \varphi \).
(iii) Let \( U_0 \subseteq U \) be a subspace with \( \Omega_0 := \Omega \cap U_0 \neq \emptyset \) and \( \mu_0 \) the unique \( \text{Herm}^+(V) \)-valued measure on \( U_0^* \) with \( \mathcal{L}(\mu_0) = \varphi |_{\tau_{\Omega_0}}. \) Then the restriction map \( r: U^* \to U_0^* \) satisfies \( r^* \mu = \mu_0 \) and induces an isometric embedding
\[
\eta: L^2(U_0^*, \mu_0) \to L^2(U^*, \mu), \quad f \mapsto f \circ r.
\]
If, in addition, there exists a topology on \( U \) for which \( \Omega_0 \) is dense in \( \Omega \) and \( \varphi \) is continuous, then \( \eta \) is surjective.

Proof. (i) (cf. [Gl99]) The case where \( U \) is finite-dimensional so that \( \Omega \) is open and \( \varphi \) is holomorphic is covered by [Ne98a, Th. III.5]. For the general case we consider the directed family \( \mathcal{F} = \{ U_j: j \in J \} \) of all those finite-dimensional subspaces \( U_j \subseteq U \) with \( \Omega_j := U_j \cap \Omega \neq \emptyset. \) We apply the finite-dimensional case to each function \( \varphi_j := \varphi |_{\tau_{\Omega_j}} \) and obtain a unique \( \text{Herm}^+(V) \)-valued measure \( \mu_j \) on \( U_j^* \) with \( \mathcal{L}(\mu_j) = \varphi_j. \) The uniqueness of these measures implies that if \( \eta_{j,k}: U_k^* \to U_j^* \) are the restriction maps, then \( \eta_{j,k} \mu_k = \mu_j \) whenever \( U_j \subseteq U_k \).

The projective limit of the measurable spaces \( U_j^* \) is the algebraic dual space \( U^* \) endowed with the smallest \( \sigma \)-algebra \( \mathcal{S} \) for which all evaluations in elements of \( U \) are measurable. Since the conditions in Theorem 1.3 are trivially satisfied in this situation, there exists a unique \( \text{Herm}^+(V) \)-valued measure \( \mu \) on \( \mathcal{S} \) with \( \mathcal{L}(\mu) = \varphi \).

(ii) Here we simply observe that with the appropriate modifications, the proof of [Ne98a, Th. III.9] also applies in the infinite-dimensional situation.

(iii) First we note that the uniqueness of \( \mu_0 \) and \( \mathcal{L}(r^* \mu) = \varphi |_{\tau_{\Omega_0}} \) implies that \( \mu_0 = r^* \mu \) for \( f \in L^2(U_0^*, \mu_0) \) (cf. Definition 1.2(i)) and \( P_x := P_{x,r} \) we have
\[
\|f \circ r\|_2^2 = \int_{U^*} P_f(r(\alpha)) \, d\mu(\alpha) = \int_{U_0^*} P_{f(\beta)} \, d(r^* \mu)(\beta) = \|f\|_2^2.
\]
showing that the map $f \mapsto f \circ r$ induces an isometric embedding $\eta \colon L^2(U^*_0, \mu_0) \to L^2(U^*, \mu)$.

To prove the remaining assertion, we assume that $\varphi$ is continuous on $\Omega$ for a topology for which $\Omega_0$ is dense. For $z \in T_\Omega$ and $v \in V$ we consider the function $f_{z,v}(a) = e^{-\frac{i}{2} \langle z, a \rangle}$ on $U^*$ and for $z \in (U_0)_c$ we write $f_{z,v}^a$ for the corresponding function on $U_0^*$. Then $f_{z,v}^a \circ r = f_{z,v}$ shows that the range of the isometric map $\eta$ contains all functions $f_{z,v}, z \in T_\Omega, v \in V$ (cf. [Ne98a, Lemma 3.8]). Now for each $v \in V$ the function

$$\psi_v : T_\Omega \to L^2(U^*, \mu), \quad z \mapsto f_{z,v},$$

satisfies $\|\psi_v(z)\|^2 = \|f_{z,v}\|^2 = \langle \varphi(z), v, v \rangle$, hence is locally bounded (from [Ne99, Prop. A.110, 11] it even follows that it is holomorphic). Furthermore it is weakly continuous and $z \mapsto \|f_{z,v}\|$ is continuous. This proves that $\psi_v$ is continuous. Now $\psi_v(T_\Omega) \subseteq \text{im} \varphi$ implies $\psi_v(T_\Omega) \subseteq \text{im} \eta$, so that $\eta$ is surjective because the functions $f_{z,v}, z \in \Omega, v \in V$ form a total subset of $L^2(X, \mu)$, as we have observed in (ii).

**Example 1.8.** (Gaussian measures) (a) An interesting particular case, where Theorem 1.7 applies is the setting where $V$ is a real pre-Hilbert space, $\Omega = V$ and $\varphi(z) = e^{\frac{i}{2} \langle z, z \rangle}$. Here $\langle \cdot, \cdot \rangle$ denote the hermitian extension of the scalar product on $V$ to the complexification $V_C$, and $x + iy = x - iy, x, y \in V$. Then the corresponding measure on $V^*$ is called the Gaussian measure of $V$ and is denoted $\gamma_V$. It is uniquely determined by $L(\gamma_V)(z) = e^{\frac{i}{2} \langle \sigma, \gamma \rangle}$ for $z \in V_C$.

(b) If $V \to W$ is a dense isometric embedding, then $\varphi_V|_W = \varphi_W$, and the uniqueness of the measure implies that the restriction map $r : W^* \to V^*$ satisfies $r^* \gamma_W = \gamma_V$. Moreover, Theorem 1.7(ii) implies that

$$\eta : L^2(V^*, \gamma_V) \to L^2(W^*, \gamma_W), \quad f \mapsto f \circ r$$

is isometric. This means that the two $L^2$-spaces are essentially the same, even though they are modeled on different measurable spaces. These observations apply in particular to the case where $W$ is a completion of $V$.

(c) From now on we assume that $V$ is complete, i.e., a real Hilbert space. To each finite rank operator $A \in B_\text{fin}(V_C)$ we associate linearly a function $q_A$ on $V^*$ in such a way that for a rank-1-operator $P_{x,y}$ with $P_{x,y}(v) = \langle v, y \rangle x$ we have

$$q_{P_{x,y}}(a) = \frac{1}{2} a(x) \overline{a(y)}.$$

Let $\Sigma := \{X \in B(V_C); X + X^* \succcurlyeq 0 \}$. We claim that for $1 + A \in \Sigma$ and $A \in B_\text{fin}(A)$ we have $e^{-\frac{i}{2} A} \in L^1(V^*, \gamma_V)$ with

$$\int_V e^{a(z) - \frac{i}{2} \langle a, z \rangle} \, d\gamma_V(a) = \det (1 + A)^{-\frac{1}{2}} e^{\frac{i}{2} (1 + A)^{-1} \langle z, z \rangle}, \quad (1.1)$$

where $\det^{-\frac{1}{2}} : \Sigma \to \mathbb{C}^X$ corresponds to the unique branch of the square root on the open convex domain $\Sigma$ which is uniquely determined by $\det (1)^{-\frac{1}{2}} = 1$.

Since the real part of $q_A$ only depends on the hermitian part $\frac{1}{2}(A + A^*)$ of $A$, we may w.l.o.g. assume that $A$ is hermitian. Then the general assertion follows from the holomorphy of both sides as functions on $\Sigma$, and the assertion for a hermitian $A$. In view of the assumption that $\text{rk} A < \infty$, we even may assume that $\dim V < \infty$. Choosing an orthonormal basis of eigenvectors for $A$, we observe that both sides decompose as products. Hence we may assume that $\dim V = 1$. Then we have to show that

$$\int_{\mathbb{R}} e^{a(z) - \frac{i}{2} a^2} \, d\gamma_\mathbb{R}(a) = \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} e^{a(z) - \frac{i}{2} a^2} \, da = (1 + a)^{-\frac{1}{2}} e^{\frac{i}{2} (1 + a)^{-1} z^2}.$$ 

The integral exists if and only if $1 + a > 0$, and in this case the verification of the formula is a simple exercise. This proves (1.1).

The right hand side of (1.1) still makes sense for $1 + A \in \Sigma_1 := \Sigma \cap (1 + B_1(V))$. To prove the assertion in this case, one also reduces matters to the case where $A$ is hermitian. Then
we consider a sequence $A_n \rightarrow A$ with $A_n = A_n^* \in B_{\text{fin}} \cap \Sigma$. Furthermore we may assume that all eigenvector for $A$ are eigenvectors of each $A_n$. Now the corresponding formula for the finite rank case shows that the sequence $e^{-tA_n}$, $n \in \mathbb{N}$, is a convergent sequence in $L^1(V^*, \gamma_V)$. In this sense the function $e^{-tA}$ makes sense as an element of $L^1(V^*, \gamma_V)$ and (1.1) holds. By abuse of notation we also write $q_A(a) = \frac{1}{2} \langle A, a, a \rangle$ for $A \in \Sigma_1$, $a \in V^*$, even though $\langle A, a, a \rangle$ is not defined literally.

(d) We keep the assumption that $V$ is complete. We consider the action of $GL(V)$ (the group of continuous linear operators on $V$) on the algebraic dual $V^*$ by $g \cdot a := a \circ g^{-1}$.

It is shown in [Se58, Th.3] that the Gaussian measure $g^* \gamma_V$ on $V^*$ is absolutely continuous with respect to $\gamma_V$ if and only if $g$ is contained in the restricted general linear group

$$rGL(V) := \{g \in GL(V); g^Tg - 1 \in B_2(V)\},$$

the subgroup of $GL(V)$ consisting of those operator $g$ with polar decomposition $g = up$ and $p - 1 \in B_2(V)$. For $g \in rGL$ we write

$$X(g)(a) := \frac{d\gamma_V(g^{-1}.a)}{d\gamma_V(a)} \in L^1(V^*, \gamma_V)$$

for the corresponding Radon-Nikodym derivative. To obtain an explicit formula, one has to consider elements of a smaller subgroup. In the sense discussed in (c), we have for $gg^T \in GL_1(V)$ the explicit formula (using the notation $g^{-T}$ for the inverse transpose of $g$)

$$X(g)(a) = \det (gg^T)^{\frac{1}{2}} e^{\frac{1}{2}((1 - g^{-T}g^{-1}).a, a)}$$

([Se58, p.23], [Se65, pp.463-466]). We recall from [Sh62, Cor. 3.1.2] that the function

$$X: GL_2(V) \rightarrow L^1(V^*, \gamma_V)$$

is continuous. The natural unitary action of $rGL(V)$ on $L^2(V^*, \gamma_V)$ is given by

$$(\pi(g).f)(a) = \sqrt{X(g)(a)} f(g^{-1}.a),$$

and for $gg^T \in GL_1(V)$ we obtain in particular

$$(\pi(g).f)(a) = (\det gg^T)^{-\frac{1}{2}} e^{\frac{1}{2}((1 - g^{-T}g^{-1}).a, a)} f(g^{-1}.a)$$

(cf. [Sh62, Th.3.1]).

II. Wishart Distributions and Their Generalizations

In this section we will study certain $L^2$-spaces of measures living on the algebraic dual space of the infinite-dimensional classical euclidean Jordan algebra $\text{Herm}(J, \mathbb{K})$, the finite hermitian $J \times J$-matrices, where $J$ is an infinite set and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. These Jordan algebras are of infinite rank and direct limits of finite-dimensional euclidean Jordan algebras. First we will recall the construction of the Wishart distributions in the finite-dimensional case, and then we will see how we can use Gaussian measures of infinite-dimensional Hilbert spaces to see that the construction from the finite-dimensional case essentially carries over to the infinite-dimensional case.

In Section III we will see that the scalar-valued measures considered in this section correspond to the scalar type highest weight representations of the corresponding conformal Koecher-Tits groups. This implies that these representations have natural realizations in $L^2$-spaces on the set $\Omega_J^+$ of finite rank positive semidefinite matrices in the dual space $\text{Herm}(J, \mathbb{K})^*$ consisting of all hermitian $J \times J$-matrices. The measures showing up in this picture are natural infinite-dimensional analogs of Wishart distributions.

In Section IV we carry out the next step of this process which consists in extending this picture to obtain $L^2$-realizations of general unitary highest weight representations of the conformal groups in vector-valued $L^2$-spaces. In this case the measures we have to consider are operator-valued measures and we will use Theorem I.7 to prove their existence.
Wishart distributions

In this small subsection $U$ is a finite-dimensional euclidean Jordan algebra, $\Omega$ is the open cone of invertible squares in $U$, $L \subseteq \text{GL}(U)$ the identity component of the structure group, $\Delta = \Delta_U: U \to \mathbb{R}$ the determinant function of $U$, and $\Omega_+^k \subseteq \Omega^* \subseteq U^*$ the set of elements of rank $\leq k$ in the dual cone (cf. [FK94]). It is instructive for the construction in the infinite-dimensional case that we first recall the finite-dimensional results. The notation is the usual, $n$ denotes dimension, and $d$ the “field” dimension.

**Proposition II.1.** Let $U$ be a simple euclidean Jordan algebra of rank $r$. For
\[ a \in \{0, \ldots, (r-1)^2\} \quad \text{or} \quad a > (r-1)^2 \]
there exists a so called Riesz measure $R_a$ on $\Omega^*$ such that $L(R_a) = \Delta_U^a$. Moreover, the following assertions hold:
\begin{enumerate}[(i)]
  \item For $k \leq r-1$ the measure $R_k$ is supported by $\Omega_+^k$ and satisfies $R_k(\Omega_+^{k-1}) = 0$.
  \item The measures $R_a$ are $L$-semimart and satisfies $\| R_a \| L_a = \det(l) \frac{R_a}{\| R_a \|}$ for all $l \in L$.
  \item The function $\Delta_U^a$ on $\Omega$ is positive definite if and only if $a \in \{0, \ldots, (r-1)^2\} \quad \text{or} \quad a > (r-1)^2$.
\end{enumerate}

**Proof.** [FK94, Prop. VII.2.3] ■

**Definition II.2.** (a) Let $U$ be a euclidean Jordan algebra and $E$ a euclidean vector space. A symmetric representation of $U$ on $E$ is a linear map $\varphi: U \to \text{Sym}(E)$ satisfying
\[ \varphi(x \cdot y) = \frac{1}{2} (\varphi(x)\varphi(y) + \varphi(y)\varphi(x)). \]

(b) If $(\varphi, E)$ is a symmetric representation of the euclidean Jordan algebra, then the associated quadratic form $Q: E \to U^*$ is defined by
\[ \langle Q(v), x \rangle = \frac{1}{2} \langle \varphi(x), v \rangle \quad \text{for} \ x \in U, v \in E. \]

Let $(\varphi, E)$ be a symmetric representation of $U$ on $E$ and $\gamma_E$ the Gaussian measure on $E$ given by
\[ d\gamma_E(v) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{4} ||v||^2} d\lambda_E(v), \]
where $N = \dim E$ and $\lambda_E$ is Lebesgue measure on $U$. Then the measure $Q^* \gamma_E$ on $U$ is called the Wishart distribution associated to the representation $\varphi$. ■

**Remark II.3.** It is easy to identify the Wishart distributions in terms of Riesz measures. To do this, we calculate the Laplace transforms in $y \in \Omega$ using formula (1.1):
\[ \mathcal{L}(Q^* \gamma_E)(y) = \int_U e^{-\langle x, y \rangle} \, d(Q^* \gamma_E)(y) = \int_E e^{-\langle Q(y), x \rangle} \, d\gamma_E(x) \]
\[ = \int_E e^{-\varphi(1+y)\cdot v} \, d\gamma_E(v) = \det(1 + \varphi(y))^{-\frac{1}{2}}. \]

If, in addition, $\varphi(1) = 1$, then [FK94, Prop. IV.4.2] shows that
\[ \det(1 + \varphi(y))^{-\frac{1}{2}} = \det(1 + y)^{-\frac{1}{2}} = \Delta(1+y)^{-\frac{1}{2}} = \mathcal{L}(e^{-1^* R_{\frac{y}{2}}})(y), \]
where $1^*: U^* \to \mathbb{R}$ denote the evaluation in the unit element $1 \in U$. Thus
\[ Q^* \gamma_E = e^{-1^* R_{\frac{y}{2}}}. \]

We note that one similarly shows that the image of $\lambda_E$ under $Q$ is a multiple of $R_{\frac{y}{2}}$ (cf. [FK94, Prop. VII.2.4]). ■
Example 11.4. Let \( r \in \mathbb{N} \) and \( U = \text{Herm}(r, \mathbb{K}) \). We consider \( E := \mathbb{M}(r, k; \mathbb{K}) \) as a euclidean vector space, where the scalar product is given by \( \langle A, B \rangle := \text{Re} \text{tr}(AB^*) \). Then we obtain a symmetric representation \( \varphi \) of \( U \) on \( E \) by \( \varphi(A)(B) := AB \). In this case
\[
\langle \varphi(A), B, B \rangle = \text{Re} \text{tr}(ABB^*) = \langle A, B B^* \rangle_U,
\]
so that the corresponding quadratic form \( Q: E \to U^* \) is given by \( Q(B) = \frac{1}{2} B B^* \).

Let \( d = \dim \mathbb{K} \). Then \( N = d rk \), so that \( N^2 = k d^2 \). This shows that for each Riesz measure \( R_{k \tilde{\varphi}} \), \( k \in \mathbb{N}_0 \), there exists a symmetric Jordan algebra representation \( \varphi \) with \( Q^*_E = e^{-1} R_{k \tilde{\varphi}} \).

This covers in particular all singular Riesz measures, i.e., those with \( k \leq r - 1 \).

Below we will see how the preceding example can be generalized to an infinite dimensional setting.

The infinite-dimensional setting

Now we turn to the infinite-dimensional versions of the classical euclidean Jordan algebras \( \text{Herm}(n, \mathbb{K}) \). We first introduce the notation that we will use throughout this paper.

**Notation:** Let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \), \( d = \dim \mathbb{K} \), and \( J \) be an arbitrary infinite set. We write \( U = \text{Herm}(J, \mathbb{K}) \) for the locally finite euclidean Jordan algebra of finite \( J \times J \)-matrices, i.e., finitely supported functions \( J \times J \to \mathbb{K} \). Here the Jordan product is given by \( x * y := \frac{1}{2}(xy + yx) \) and the scalar product by
\[
\langle x, y \rangle := \text{Re} \text{tr}(xy) = \frac{1}{d} \text{tr}_\mathbb{K}(xy),
\]
where \( \text{tr}_\mathbb{K}(x) \) is the trace of \( x \) as an operator on the real vector space \( \mathbb{K}(J) \cong \mathbb{R}^{(J^2)} \). We write \( H := H(J, \mathbb{K}) \) for the real Hilbert space of all square summable functions \( J \to \mathbb{K} \) with the scalar product
\[
\langle x, y \rangle := \sum_{j \in J} \text{Re}(x_j y_j^*).
\]

Then \( B(H, \mathbb{K}) \subseteq B(H) \) denotes the space of all \( \mathbb{K} \)-linear bounded operators on \( H \), \( \text{Sym}(H) \subseteq B(H) \) the space of symmetric operators on \( H \), \( U_b := \text{Herm}(H) := \text{Sym}(H) \cap B(H, \mathbb{K}) \) the space of bounded \( \mathbb{K} \)-hermitian operators (the “bounded version of \( U^* \)”) and \( U_p := \text{Herm}_p(H) := \text{Herm}(H) \cap B_p(H) \) the space of hermitian operators of Schatten class \( p \in [1, \infty] \). For \( p = 1 \) we obtain the trace class operators and for \( p = 2 \) the Hilbert-Schmidt operators.

In \( B(H) \) we have the Siegel tube \( \Sigma_b := \{ Z \in B(H); Z + Z^T \succ 0 \} \), its restricted versions \( \Sigma_p := \Sigma_b \cap (1 + B_p(H)) \), \( p \in [1, \infty] \), \( \Sigma := \Sigma_b \cap U_c \), as well as the convex domains \( \Omega := \Sigma_b \cap (1 + U) \). We have the natural inclusions
\[
U \subseteq U_1 \subseteq U_2 \quad \text{and} \quad \Omega \subseteq \Omega_1 \subseteq \Omega_2.
\]

The dual space \( U^* \) can be identified in a natural way with the space of all hermitian \( J \times J \)-matrices, where the real bilinear pairing \( U \times U^* \to \mathbb{R} \) is given by
\[
U \times U^* \to \mathbb{R}, \quad (x \alpha) \mapsto \text{Re} \text{tr}(x \alpha)
\]
which makes sense since the matrix \( x \alpha \) has finitely many non-zero rows. Even though we can identify \( U \) in a natural way with a subspace of \( U^* \), we prefer not to do so because this often leads to conceptual confusion and certain groups act in different ways.

Let \( \mathfrak{g}(J, \mathbb{K}) \) denote the Lie algebra of finite \( J \times J \)-matrices and \( \text{GL}(J, \mathbb{K}) \) be the group of all those invertible \( J \times J \)-matrices \( g \) for which \( g - 1 \) is finite, where \( 1 \) denotes the identity matrix. Further let
\[
L := \text{GL}(J, \mathbb{K})_0
\]
denote the identity component of the group \( \text{GL}(J, \mathbb{K}) \). Note that \( \text{GL}(J, \mathbb{K}) \) is connected for \( \mathbb{K} = \mathbb{C} \) and \( \mathbb{H} \), but not for \( \mathbb{K} = \mathbb{R} \). Here our notation is slightly inconsistent with the notation.
for the finite-dimensional case because the action of \( L \) on \( U \) is not faithful, but it will turn out that it is much more convenient to work directly with the group \( \text{GL}(J, \mathbb{K}) \) and its completions. For \( g \in \text{GL}(H) \) we write \( g^T \) for the adjoint operator on \( H \) (considered as a real Hilbert space) and observe that for \( g \in \text{GL}(H, \mathbb{K}) \) we have \( g^T = g^* \). The groups

\[
L_p := \{ g \in \text{GL}(H, \mathbb{K}) : g^T g - 1 \in B_p(H) \}, \quad p = 1, 2
\]

act in a natural way on \( U_p \) by \( g \cdot x = gxg^T \), and this action preserves the domains \( \Omega_p \) because for \( g \in L_p \) and \( A \in \Omega_p \) we have

\[
g \cdot A - 1 = gAg^T - 1 = g(A - 1)g^T + gg^T - 1 \in U_p + U_p \subseteq U_p.
\]

The action on \( U_p \) induces a natural action on the algebraic dual space \( U_p^* \) by \( (g \cdot A) (x) := A(g^{-1}x) \) for \( x \in U_p \). Likewise we have an action of the small group \( L \) on the algebraic dual space \( U^* \).

With respect to these actions we have \( \Omega_p = L_p \cdot 1 \) and \( \Omega = L \cdot 1 \). Moreover, the action of \( L \) on \( U^* \) preserves for each \( k \in \mathbb{N} \) the set

\[
\Omega_k := \{ a \in \Omega^* : \text{rk} a \leq k \} \subseteq U^*
\]

of matrices of rank \( \leq k \).

**Example II.5.** We generalize Example II.4 to the infinite-dimensional setting. We have a Jordan algebra representation of \( U \) on the space \( \mathbb{M}(J, k; \mathbb{K}) \) of all \( J \times k \)-matrices, but this space is in some sense too big to carry the structure of a euclidean vector space. On the subspace of all finite matrices we have the natural real bilinear scalar product

\[
\langle A, B \rangle := \text{Re \ tr}(AB^*)
\]

which also makes sense if only one of the two arguments has only finitely many non-zero entries.

Nevertheless, for each \( A \in \mathbb{M}(J, k; \mathbb{K}) \) and \( x \in U \) the matrix \( \varphi(x)A = xA \) has only finitely many non-zero rows, hence is a finite \( J \times k \)-matrix. Therefore

\[
\langle \varphi(x)A, A \rangle = \text{Re \ tr}(\langle \varphi(x)A, A^* \rangle)
\]

makes sense, and we thus obtain a quadratic map

\[
Q : \mathbb{M}(J, k; \mathbb{K}) \to U^*, \quad A \mapsto \frac{1}{2}AA^*
\]

with

\[
\langle Q(A), x \rangle = \frac{1}{2}\langle \varphi(x)A, A \rangle \quad \text{for} \quad x \in U, A \in \mathbb{M}(J, k; \mathbb{K})..
\]

Note that, even though the matrix product \( A^*A \) makes no sense, the product \( AA^* \) makes sense and is a hermitian \( J \times J \)-matrix.

We think of the space \( \mathbb{M}(J, k; \mathbb{K}) \) as the dual of the euclidean vector space \( V \) of all finite \( J \times k \)-matrices with the real bilinear scalar product \( \langle A, B \rangle := \text{Re \ tr}(AB^*) \). We thus obtain a Gaussian measure \( \gamma_V \) on \( \mathbb{M}(J, k; \mathbb{K}) \) and \( \nu_k := Q^* \gamma_V \) is a certain probability measure on \( U^* \) for which the subset \( \Omega_k^* \) of all those positive semidefinite hermitian matrices in \( U^* \) of rank \( \leq k \) is thick in the sense that its complement contains no measurable subset of non-zero measure.

If \( 1 + x \in \Omega \) is positive definite, then we obtain as in the finite-dimensional case:

\[
\mathcal{L}(\nu_k)(x) = \int_{U^*} e^{-\varphi(x)A} d(\gamma_V)(A) = \int_{U^*} e^{-\langle Q(A), x \rangle} d\gamma_V(A)
\]

\[
= \int_{V^*} e^{-\frac{1}{2}\langle \varphi(x)A, A \rangle} d\gamma_V(A) = \det_V (1 + \varphi(x))^{-\frac{k}{2}} = \det_V (\varphi(1 + x))^{-\frac{k}{2}}.
\]

In our case we have \( \det_V (\varphi(1 + x)) = \det_\mathbb{R}(1 + x)^k \) and therefore

\[
\mathcal{L}(\nu_k)(x) = \det_\mathbb{R}(1 + x)^{-\frac{k}{2}}.
\]
On the other hand \( \Delta_x(1 + x) = \Delta(1 + x)^d \) holds for the Jordan determinant of \( U \) (see below) and each \( x \in U \). We thus conclude in particular that for each \( k \in \mathbb{N}_0 \) the function

\[
\Delta_{-\frac{i}{d}} : \Omega \to \mathbb{R}
\]

is the Laplace transform of a positive measure on \( \Omega_k^+ \).

If we replace \( U \) by the bigger Banach Jordan algebra \( U_1 \), then the determinant function still makes sense on the domain \( \Omega_1 \), where we have \( \Delta(x)^d = \det_x(x) \). Using Theorem I.7, we obtain a probability measure \( \nu_k \) on \( U_1^* \) with

\[
L(\nu_k)^{(x)}(x) = \Delta(1 + x)^{-\frac{i}{d}}
\]

for \( x \in \Omega_1 \). Moreover, the natural map \( r : U_1^* \to U^* \) satisfies \( r^* \nu_k = \nu_k \) and induces an isometric bijection

\[
L^2(U^*, \nu_k) \to L^2(U_1^*, \nu_k), \quad f \mapsto f \circ r
\]

(Theorem I.7).

Applying Proposition IV.1 below to the representation of \( L \) given by \( \rho(g) = \det_x(g)^{-\frac{i}{d}} \), we see that we obtain a unitary representation of the semidirect product group \( U_1 \rtimes L_1 \) on the Hilbert space \( L^2(U_1^*, \nu_k) \) by

\[
((u, g), f)(\alpha) = e^{i\alpha(u)} e^{i\alpha(1-gg^\top)} \det_x(gg^\top)^{-\frac{i}{d}} f(g^{-1} \, \alpha).
\]

We note that the isometric embedding

\[
L^2(U^*, \nu_k) \to L^2(V^*, \gamma_V), \quad f \mapsto f \circ Q
\]

is equivariant with respect to the action of the group \( L \) on \( L^2(U^*, \nu_k) \cong L^2(U_1^*, \nu_k) \) because we have for \( A \in V^* \) and \( g \in L \) the relation \( Q(g.A) = Q(gg^{-\top}A) = \frac{1}{2}gg^{-\top}AA^*g^{-1} = g.Q(A) \), where the action on the right hand side refers to the action of \( L \) on \( U^* \). Hence

\[
(g.f)(Q(A)) = \det_x(gg^\top)^{-\frac{i}{d}} e^{i\alpha(1-gg^\top)} f(g^{-1} \cdot Q(A)) = \det_x(gg^\top)^{-\frac{i}{d}} e^{i\alpha(1-gg^\top)A} f(Q(g^{-1} \cdot A)) = (g \cdot (f \circ Q))(A),
\]

where the last term has to be understood in the sense of Example I.8(d). \( \square \)

**Appendix II: The quaternionic case**

For the convenience of the reader we briefly recall the convenient description of the Hilbert space \( H \) and the operators thereon for the quaternionic case \( \mathbb{K} = \mathbb{H} \). Quaternionic Hilbert spaces are most conveniently described as follows. Let \( H_0 \) be a complex Hilbert space endowed with an antilinear isometric involution \( v \mapsto \overline{v} \). Then the antilinear operator on \( H_0 \oplus H_0 \) given by \( \sigma(x, y) = (-\overline{y}, \overline{x}) \) satisfies \( v^2 = -1 \). We thus obtain on \( H := H_0 \oplus H_0 \) the structure of a quaternionic Hilbert space with \( H^\mathbb{H} \cong H_0 \oplus H_0 \) (by restriction of scalars to \( \mathbb{C} \)).

We write a complex linear operator \( X \) on \( H \) as a \( 2 \times 2 \)-matrix of the form

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in B(H_0)
\]

and for \( A \in B(H_0) \) we define the operator \( \overline{A} \) by \( \overline{\sigma} \, v := \overline{A(v)} \). We see that \( X \sigma = \sigma X \) is equivalent to

\[
-\overline{A} \overline{x} + B \overline{y} = -C \overline{x} + D \overline{y}, \quad -C \overline{x} + D \overline{y} = A \overline{x} + B \overline{y} = \overline{A} \overline{x} + \overline{B} \overline{y}
\]
for $x, y \in H_0$. This means that $D = \overline{A}$ and $C = -B$. Therefore the $\mathbb{H}$-linear operators on $H$
correspond to the matrices of the form

$$X = \begin{pmatrix} A & B \\ -B & \overline{A} \end{pmatrix}, \quad A, B \in B(H_0).$$

Let $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and observe that $JX = \overline{X}J$ if and only if $X$ is $\mathbb{H}$-linear. We identify $B(H_0)$ with a subspace of $B(H)$ by the mapping $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$. In this sense an operator $X = A + JB$ as above is hermitian if and only if $A + JB = X = X^* = A^* + B^*J^* = A^* + B^*(-J) = A^* - JB^T$.

In this case we put $B^T := B^*$. Hence

$$\text{Herm}(H, \mathbb{H}) = \{ A + JB : A^* = A, B^T = -B \} \cong \text{Herm}(H_0, \mathbb{C}) \oplus J \text{Skew}(H_0).$$

We have $-J(A + JB) = B - JA$, and

$$(B - JA)^T = B^T - A^TJ^T = B^T + A^TJ = B^T + JA^*.$$ We conclude that $-J(A + JB)$ is skew-symmetric if and only if $B$ is skew-symmetric and $A$ is hermitian. This shows that

$$J\text{Herm}(H, \mathbb{H}) \subseteq \text{Skew}(H).$$

One easily checks that $J\text{Herm}(H, \mathbb{H})$ is a real form of the space $\text{Skew}(H)$, i.e., $\text{Skew}(H) = J\text{Herm}(H, \mathbb{H}) \oplus iJ\text{Herm}(H, \mathbb{H})$.

### III. Relations to Unitary Highest Weight Representations

The main objective of this paper is to provide an $L^2$-realization of the unitary highest weight representations of the automorphism groups of infinite-dimensional Hilbert domains which have been classified in [N08]. In this section we explain how one associates to a unitary highest weight representation of such a group a positive definite function on the domain $\Omega_1$, resp. $T_{\mathbb{R}}$. Using the tools described in Section I, we can represent these positive definite functions by operator-valued measures on the algebraic dual space $U_1^*$. In view of the finite-dimensional situation, one expects to obtain an explicit picture for the action of the group $U_1 \rtimes L_1$ acting by affine automorphisms of the domain $T_{\mathbb{R}}$ on the corresponding Hilbert spaces. For the highest weight representations of scalar type, this construction leads to the measures $\nu_k^1$ on $U_1^*$ that we have discussed in Example II.5.

On the other hand, the construction of the measure $\nu_k^1$ as a pushforward of a Gaussian measure shows that not only the group $L_1$ but also the group $L_2$ acts on the corresponding Hilbert space. To make the action of $L_2$ and also of the affine group $U_2 \rtimes L_2$ more explicit, we first use the regularized determinant $\det_2$ on the space $1 + B_2(H)$ first to modify the positive definite function on the domain $\Omega_1$ in such a way that it extends to a positive definite function on the bigger domain $\Omega_2$ and thus to obtain a measure on $U_2^*$. Using this new realization of our Hilbert space, we obtain an explicit formula for the representation of the group $L_2$. It turns out that this representation extends in a natural way to a multiplier representation of the group $U_2 \rtimes L_2$, but that the corresponding cocycle is not trivial, so that this representation can be viewed as a unitary representation of a non-trivial central extension, but not as a unitary representation of the group $U_2 \rtimes L_2$ itself.
Unitary highest weight representations

In this subsection we consider the conformal Lie algebra \( g \otimes \mathbb{R} \) associated to the Jordan algebra \( U \), resp. its complexifications \( g \) which is the conformal Lie algebra of the complex Jordan algebra \( U_{\mathbb{C}} \). We explain how the unitary highest weight representations of the conformal algebra \( g \) are related to unitary highest weight representations of the Lie algebra \( \mathfrak{f} \) of \( L \).

**Definition III.1.** (a) An involutive Lie algebra is a pair \((g, *)\) of a complex Lie algebra \( g \) and an antilinear involutive antiautomorphism \( g \to g, X \mapsto X^* \). The subspace
\[ g_{\mathbb{R}} := \{ X \in g; X^* = -X \} \]
is a real form of \( g \) which determines the involution uniquely.
(b) We say that the involutive Lie algebra \((g, *)\) has a root decomposition if there exists a maximal abelian \(*\)-invariant subalgebra \( \mathfrak{h} \) such that \( g = \mathfrak{h} + \sum_{\alpha \in \Delta} g^\alpha \), where
\[ g^\alpha = \{ Z \in g; (\forall X \in \mathfrak{h})[X, Z] = \alpha(X)Z \}, \]
and \( \Delta = \{ \alpha \in \mathfrak{h}^* \setminus \{ 0 \}; g^\alpha \neq \{ 0 \} \} \) is the corresponding root system.
(c) A subset \( \Delta^+ \subseteq \Delta \) is called a positive system if \( \Delta = \Delta^+ \cup -\Delta^+ \) and no sum of positive roots vanishes.
(d) Let \( V \) be a \( g \)-module and \( v \in V^\lambda \) an \( \mathfrak{h} \)-weight vector of weight \( \lambda \). We say that \( v \) is a primitive element of \( V \) (with respect to the positive system \( \Delta^+ \)) if \( g^\alpha.v = \{ 0 \} \) holds for all \( \alpha \in \Delta^+ \). A \( g \)-module \( V \) is called a highest weight module with highest weight \( \lambda \) (with respect to \( \Delta^+ \)) if it is generated by a primitive element of weight \( \lambda \).
(e) We call a hermitian form \( \langle \cdot, \cdot \rangle \) on a \( g \)-module \( V \) contravariant if \( \langle X.v, w \rangle = \langle v, X^*.w \rangle \) holds for all \( v, w \in V \), \( X \in g \). A \( g \)-module \( V \) is said to be unitary if it carries a contravariant positive definite hermitian form.

**Proposition III.2.** Let \( g \) be an involutive complex Lie algebra with root decomposition and \( \Delta^+ \) a positive system. Then the following assertions hold:

1. For each \( \lambda \in \mathfrak{h}^* \) there exists a unique irreducible highest weight module \( L(\lambda) \) and each highest weight module of highest weight \( \lambda \) has a unique maximal submodule.
2. Each unitary highest weight module is irreducible.
3. If \( L(\lambda) \) is unitary, then \( \lambda = \lambda^* \).
4. If \( \lambda = \lambda^* \) and \( v_\lambda \in L(\lambda) \) is a primitive element, then \( L(\lambda) \) carries a unique contravariant hermitian form \( \langle \cdot, \cdot \rangle \) with \( \langle v_\lambda, v_\lambda \rangle = 1 \). This form is non-degenerate.

**Proof.** [Ne99, Props. IX.1.10/14]

**Example III.3.** A typical example of an involutive complex Lie algebra is the Lie algebra \( g := \mathfrak{gl}(J, \mathbb{C}) \) endowed with the natural involution given by \( E^*_{jk} = E_{kj} \) for the matrix units \( E_{jk} \), \( j, k \in J \). The diagonal operators in \( g \) form an abelian subalgebra \( \mathfrak{h} \) and we have a root decomposition
\[ g = \mathfrak{h} \oplus \bigoplus_{i \neq j \in J} g^{\varepsilon_i - \varepsilon_j}, \]
where \( \varepsilon_i(E_{jj}) = \delta_{ij} \) and \( g^{\varepsilon_i - \varepsilon_j} \cong \mathbb{C} E_{ij} \). The positive systems in \( \Delta \) are in one-to-one correspondence with the linear partial orderings \( \leq \) of the set \( J \). This correspondence is established by assigning to \( \leq \) the positive system
\[ \Delta^+_\leq := \{ \varepsilon_j - \varepsilon_k; j < k \} \]
(cf. [Ne98b, Lemma II.1]). A linear functional \( \lambda \in \mathfrak{h}^* \) can be identified in a natural way with the function \( \lambda; J \to \mathbb{C}, j \to \lambda_j := \lambda(E_{jj}) \). It has been shown in [Ne98b, Lemma II.4] that the highest weight module \( L(\lambda, \Delta^+_{\leq}) \) of \( \mathfrak{gl}(J, \mathbb{C}) \) is unitary if and only if \( \lambda \) is real and dominant integral in the sense that \( \lambda_j - \lambda_k \in \mathbb{N}_0 \) for \( j < k \).
Let \( \mathfrak{g}_\mathbb{R} \) denote the conformal algebra of \( U \) and note that for \( U = \text{Herm}(J, \mathbb{K}) \) we have

\[
\mathfrak{g}_\mathbb{R} = \begin{cases} 
\mathfrak{sp}(J, \mathbb{R}) & \text{for } \mathbb{K} = \mathbb{R}, \\
\mathfrak{su}(J, J) & \text{for } \mathbb{K} = \mathbb{C}, \\
o^*(2J) & \text{for } \mathbb{K} = \mathbb{H}.
\end{cases}
\]

Accordingly we have \( \mathfrak{g} := (\mathfrak{g}_\mathbb{R})_C \cong \mathfrak{sp}(J, \mathbb{C}), \mathfrak{sl}(J, \mathbb{C}), \mathfrak{o}(2J, \mathbb{C}) \). This is a complex involutive Lie algebra with the following properties. There exists a maximal abelian subalgebra \( \mathfrak{h} \) contained in the subalgebra \( \mathfrak{t} := \mathfrak{l}_C \subseteq \mathfrak{g} \) so that we obtain a root decomposition \( \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha} \). We call \( \Delta_0 := \{ \alpha \in \Delta; \mathfrak{g}^{\alpha} \subseteq \mathfrak{t} \} \) the set of compact roots and \( \Delta_\mathfrak{g} := \Delta \setminus \Delta_0 \) the set of non-compact roots. In [N098] we have classified all the unitary highest weight representations of these Lie algebras.

For a characterization of the Lie algebras \( \mathfrak{g}_\mathbb{R} \) as the hermitian real forms of simple complex locally finite Lie algebras with a root decomposition, we refer to [NeSt99]. In the following we explain how this classification looks like and what this means for the corresponding measures on \( U^* \).

One important observation is that if \( \Delta^+ \subseteq \Delta \) (the root system of \( \mathfrak{g} \)) is a positive system for which there exists a non-trivial unitary highest weight module \( L(\lambda) \), then the set \( \Delta^+_\lambda \) of positive non-compact roots is invariant under the Weyl group \( W_0 \) of \( \mathfrak{t} \), and for the simple hermitian Lie algebras this condition determines \( \Delta^+_\lambda \) up to sign. A necessary condition for the unitarity of \( L(\lambda, \Delta^+) \) is that \( \lambda \) is dominant integral with respect to \( \Delta^+_\lambda \) and antidominant with respect to \( \Delta^-_\lambda \) (cf. [NO98, Prop. I.5]). In the following the number

\[
k := -\frac{2}{d} \sup \{ \lambda(\gamma); \gamma \in \Delta^+_\lambda \}
\]

that we associate to a unitary highest weight module \( L(\lambda, \Delta^+) \) of \( \mathfrak{g} \) will play a crucial role because it describes the location of the corresponding operator-valued measure (cf. Section V).

**Theorem III.4.** The classification of the unitary highest weight modules for the Lie algebras \( \mathfrak{g} \) and the corresponding numbers \( k \in \mathbb{N}_0 \) are as follows. We assume that \( \lambda \in \mathfrak{h}^* \) is real-valued on hermitian elements, that it is dominant integral with respect to \( \Delta^+_\lambda \) and antidominant with respect to \( \Delta^-_\lambda \).

1. \( \mathbb{K} = \mathbb{R} \): Here \( \mathfrak{g}_\mathbb{R} = \mathfrak{sp}(J, \mathbb{R}) \) and \( \mathfrak{t} \cong \mathfrak{gl}(J, \mathbb{C}) \) so that we may identify \( \lambda \) with a function on \( J \). We assume that \( \Delta^+_\lambda = \{ \varepsilon_i - \varepsilon_j; i \neq j \in J \} \) and put \( M := \sup \{ \lambda(j); j \in J \} \). Then the highest weight module \( L(\lambda, \Delta^+) \) is unitary if and only if

\[
2M + \{ |\{ j \in J; \lambda(j) < M \} | \} + \{ |\{ j \in J; \lambda(j) < M \} | \} \leq 0,
\]

and in this case \( k = -2M \).

2. \( \mathbb{K} = \mathbb{C} \): Here \( \mathfrak{g}_\mathbb{R} = \mathfrak{u}(J_-, J_+) \), where \( J_{\pm} \) are copies of one set \( J \), and we furthermore have \( \mathfrak{t} \cong \mathfrak{gl}(J_-, \mathbb{C}) \oplus \mathfrak{gl}(J_+, \mathbb{C}) \). We assume that \( \Delta^+_\lambda = \{ \varepsilon_i - \varepsilon_j; i \in J_-, j \in J_+ \} \) and put \( m := \sup \lambda^- \) and \( M := \inf \lambda^+ \). Then the highest weight representation \( L(\lambda, \Delta^+) \) is unitary if and only if

\[
M - m \geq \{ |\{ j \in J_-; \lambda(j) \neq m \} | \} + \{ |\{ j \in J_+; \lambda(j) \neq M \} | \},
\]

and in this case \( k = M - m \).

3. \( \mathbb{K} = \mathbb{H} \): Here \( \mathfrak{g}_\mathbb{R} = \mathfrak{o}^*(2J_+ \cup J_-, \mathbb{C}) \), where \( J_{\pm} \) are copies of the set \( J \), and \( \mathfrak{t} \cong \mathfrak{gl}(J_+ \cup J_-, \mathbb{C}) \). We assume that \( \Delta^+_\lambda = \{ \varepsilon_i - \varepsilon_j; i \in J_- \cup J_+ \} \) and put \( M := \sup \lambda \). Then the highest weight module \( L(\lambda, \Delta^+) \) is unitary if and only if

\[
M + \{ |\{ j \in J_+ \cup J_-; \lambda(j) < M \} | \} \leq 0,
\]

and in this case \( k = -M \).

**Proof.** (1) We recall that

\[
\Delta_\mathfrak{h} = \{ \varepsilon_i - \varepsilon_j; i \neq j \in J \} \quad \text{and} \quad \Delta_\mathfrak{g} = \{ \pm(\varepsilon_i + \varepsilon_j); i, j \in J \}.
\]

A positive system is given by a partial order \( \preceq \) on the set \( J \) and for this order we may w.l.o.g. assume that

\[
\Delta^+_\mathfrak{h} = \{ \varepsilon_i - \varepsilon_j; i < j \} \quad \text{and} \quad \Delta^+_\mathfrak{g} = \{ \varepsilon_i + \varepsilon_j; i, j \in J \}
\]
(cf. [NÖ98]).

Now the antidominance of $\lambda$ with respect to $\Delta_\pm$ implies that $\lambda_j \leq 0$ for each $j \in J$, so that $M = \max\{\lambda_j; j \in J\}$ exists because the integrality with respect to $\Delta_k$ means that all differences $\lambda_j - \lambda_j$ are integral. The characterization of the unitary modules has been obtained in [NÖ98, Prop. 1.9]. In view of $d = 1$, we further have

$$-\frac{k}{2} = -\frac{dk}{2} = \sup\{\lambda(\hat{a}); a \in \Delta_k^+\} = M$$

because there exists an $a = 2\varepsilon_j$ with $M = \lambda_j = \lambda(\hat{a})$. This proves (1).

(2) We have

$$\Delta_k = \{\varepsilon_i - \varepsilon_j; i \neq j \in J_\pm\} \quad \text{and} \quad \Delta_k^+ = \{\pm(\varepsilon_i - \varepsilon_j); i \in J_-, j \in J_+\}.$$

In this case a positive system is given by a pair of partial orders $\preceq$ on the sets $J_+$ and $J_-$. For this order we may w.l.o.g. assume that

$$\Delta_k^+ = \{\varepsilon_i - \varepsilon_j; i \preceq j\} \quad \text{and} \quad \Delta_k^+ = \{\varepsilon_i - \varepsilon_j; i \in J_-, j \in J_+\}$$

(cf. [NÖ98]). The characterization of the unitary modules has been obtained in [NÖ98, Prop. 1.7]. We further have

$$-k = -\frac{dk}{2} = \sup\{\lambda(\hat{a}); a \in \Delta_k^+\} = m - M,$$

and this proves (2).

(3) We have

$$\Delta_k = \{\varepsilon_i - \varepsilon_j; i \neq j \in J_+ \cup J_-\} \quad \text{and} \quad \Delta_k^+ = \{\pm(\varepsilon_i + \varepsilon_j); i \neq j \in J_+ \cup J_-\}.$$

In this case a positive system is given by a partial order $\preceq$ on the set $J_+ \cup J_-$ and for this order we may w.l.o.g. assume that

$$\Delta_k^+ = \{\varepsilon_i - \varepsilon_j \in \Delta_k; i \preceq j\} \quad \text{and} \quad \Delta_k^+ = \{\varepsilon_i - \varepsilon_j; i \in J_-, j \in J_+\}.$$

The characterization of the unitary modules has been obtained in [NÖ98, Prop. 1.11]. If the highest weight representation $L(\lambda, \Delta_k^+)$ is unitary, then $|\{j \in J; \lambda_j = M\}| \geq 2$, so that

$$-2k = -\frac{dk}{2} = \sup\{\lambda(\hat{a}); a \in \Delta_k^+\} = 2M,$$

and this proves (3).

\section*{Admissible representations of $L$}

To relate unitary highest weight representations of the conformal algebras to positive definite functions on $\Omega$, resp. $\Omega_1$, we first use the correspondence to positive definite kernels on the bounded domains explained in [NÖ98] and then an appropriate Cayley transform to obtain a correspondence between the bounded and the unbounded picture.

\begin{proposition} \textbf{(The Cayley transform)}\ Let $H$ be a real Hilbert space and define the Cayley transform

$$c: GL(H) - \mathbf{1} \rightarrow B(H), \quad Z \mapsto (1 - Z)(1 + Z)^{-1}.$$

Then the following assertions hold:

\begin{itemize}
\item[(i)] \quad $c^2 = \text{id}_{GL(H) - \mathbf{1}}$, i.e., $c$ is a biholomorphic involution of the domain $GL(H) - \mathbf{1}$.
\item[(ii)] \quad $c(-Z) = c(Z)^{-1}$ and $c(Z^*) = c(Z)^*$ for $1 + Z \in GL(H)$.
\end{itemize}
\end{proposition}
(iii) For $Z, W \in \text{GL}(H) - 1$ we have
\[
\frac{c(Z) + c(W^*)}{2} = (1 + W^*)^{-1}(1 - W^*Z)(1 + Z)^{-1} = (1 + Z)^{-1}(1 - ZW^*)(1 + W^*)^{-1}.
\]

(iv) The Cayley transform maps the ball $D_b := \{ Z \in B(H); \| Z \| < 1 \}$ diffeomorphically onto the domain
\[
\Sigma_b := \{ W \in B(H); W + W^* \gg 0 \}.
\]
Moreover, for $p \in [1, \infty]$ and $D_p := D_b \cap B_p(H)$, $D := U_C \cap D_b$ we have $\Sigma_p := c(D_p) = \Sigma_b \cap (1 + B_p(H))$ and $c(D) = \Sigma$.

**Proof.**
(i) For $1 + Z \in \text{GL}(H)$ and $W := c(Z)$ the operator $W + 1 = 2(1 + Z)^{-1}$ is invertible, showing that $c(\text{GL}(H) - 1) \subseteq \text{GL}(H) - 1$. One easily checks that $c^2(Z) = Z$ for $Z \in \text{GL}(H) - 1$, and this proves (i).

(ii) These are trivial verifications.

(iii) The first equality follows from
\[
(1 + W^*)(c(Z) + c(W^*)) (1 + Z) = (1 + W^*)(1 - Z) + (1 - W^*)(1 + Z) = 2(1 - W^*Z).
\] The second one follows from the observation that the left hand side does not change if we exchange $Z$ and $W^*$.

(iv) For $\| Z \| < 1$ the operator $1 - Z^*Z$ is positive definite, so that (iii) implies that the same holds for $c(Z) + c(Z^*)$, i.e., $c(Z) \in \Sigma_b$.

Next we show that $1 + \Sigma_b \subseteq \text{GL}(H)$. Since $1 + \Sigma_b \subseteq \Sigma_b$, it suffices to show that $\Sigma_b \subseteq \text{GL}(H)$. So let $W \in \Sigma_b$. Then there exists $c > 0$ with $\langle (W + W^*).v, v \rangle \geq c \| v \|^2$ for all $v \in H$. Hence
\[
\| c v \|^2 \leq \langle W.v, v \rangle + \langle v, W.v \rangle \leq 2 \| W \| \| v \|^2
\]
implies that $W.v = 0$ entails $v = 0$, hence that $W$ is injective. The same argument shows that $W^*$ is injective, so that $(W.H)^* = \ker W^* = \{0\}$ implies that $W$ has dense range. Further (3.1) implies that $W$ has closed range so that $W$ is surjective, hence invertible. We conclude in particular that for each $W \in \Sigma_b$ the operator $1 + W$ is invertible. Now Proposition III.1 shows that $Z := c(W)$ satisfies
\[
1 - Z^*Z = (1 + Z^*) \frac{(W + W^*)}{2} (1 + Z) > 0,
\]
i.e., $\| Z \| < 1$, and therefore $c(D_b) = \Sigma_b$. Finally
\[
c(Z) - 1 = (1 - Z)(1 + Z)^{-1} - 1 = (1 - Z - 1 - Z)(1 + Z)^{-1} = 2Z(1 + Z)^{-1}
\]
shows that $Z \in B_p(H)$ is equivalent to $c(Z) - 1 \in B_p(H)$, and this proves the second part. ■

Now we consider the real Hilbert space $H = l^2(J, K)$ for an infinite set $J$ and $K \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. For $K = \mathbb{H}$ we write $H^C$ for the $\mathbb{H}$-vector space $H$ viewed as a complex vector space with respect to the inclusion $\mathbb{C} \hookrightarrow \mathbb{H}$. We define
\[
L_C := \begin{cases} 
\text{GL}(J, C) \subseteq \text{GL}(H_C) & \text{for } K = \mathbb{R}, \\
\text{GL}(J, C) \times \text{GL}(J, \mathbb{C}) & \text{for } K = \mathbb{C}, \\
\text{GL}(2J, C) \subseteq \text{GL}(H^C) & \text{for } K = \mathbb{H}.
\end{cases}
\]

**Definition III.6.**
(a) If $G$ is a group endowed with an involutive antiautomorphism $\tau$, then an **involutive representation** of $G$ on a Hilbert space $V$ is a group homomorphism $\rho: G \to \text{GL}(V)$ with $\rho(\tau g) = \rho(g)^*$ for all $g \in G$.
(b) On the group $L$ we consider the involutive antiautomorphism $\tau(g) = g^T$. We say that an involutive representation $\rho: L = \text{GL}(J, K) \to \text{GL}(V)$ on the Hilbert space $V$ is a **unitary highest weight representation** if there exists a dense subspace $V_0 \subseteq V$ on which the derived representation of $L_C$ is a highest weight representation in the sense of Definition III.1.

(c) An involutive representation $\rho$ of $L$ is said to be **admissible** if it is a unitary highest weight representation and if the restriction $\rho|_{\Omega}: \Omega \to B(V)$ is a positive definite function. ■
Example III.7. We have already seen in Example II.5 above that for the one-dimensional representations \( \rho_k \) of \( L \) given by

\[
\rho_k(g) = \det_k(gg^\top)^{-\frac{k}{2}} = |\det_k(g)|^{-\frac{k}{2}}
\]

the corresponding function on \( \Omega \to \mathbb{R}, x \mapsto \det_k(x)^{-\frac{k}{2}} \) is positive definite, so that the representations \( \rho_k, k \in \mathbb{Z}_+ \), are admissible since one-dimensional representations are trivially highest weight representations.

The following proposition can be applied in all those cases that we have in mind because if \( \varphi: \Omega \to B(V) \), \( x \mapsto \rho(x) \) is positive definite, then the condition below is automatically satisfied ([HN67, Lemma V.11, Cor. V.13] or [Cl95]). We call a function on a finitely open domain \( \varphi \) in a finite-dimensional vector space *finitely continuous*, if its restrictions to all intersections of \( \Omega \) with finite-dimensional subspaces are continuous.

The following proposition is a special case of far more general extension results for positive definite functions by H. Glöckner (cf. [Gl93]).

**Proposition III.8.** Each finitely continuous positive definite function \( \varphi: \Omega \to B(V) \) satisfying

\[
\varphi(x + y) \leq \varphi(x) \quad \text{for } x \in \Omega, \quad y \in C := \text{Herm}^+(J, \mathbb{K})
\]

extends in a unique way to a Gateaux-holomorphic positive definite function on the tube domain \( T_\Omega \).

**Proof.** Instead of \( \Omega \), we may also consider the finitely open domain \( \Omega_U := \Omega - 1 \subseteq U \) which contains in particular the cone \( C \). We note that since \( U \) is a directed union of the subspaces \( U_F = \text{Herm}(F, \mathbb{K}), \ F \subseteq J \) a finite subset, it suffices to prove the assertion for the domains \( \Omega_U \cap U_F = \{ x \in U_F: 1 + x \gg 0 \} = \Omega_F - \{ \} \). This reduces the assertion to the case where \( J \) is finite and \( \Omega \subseteq U \) is the open cone of positive definite operators. This case is covered by [Ne98a, Th. IV.2].

Our next step consists in a description of the admissible representations of the group \( L \). This will be achieved by showing that they correspond exactly to the unitary highest weight representations of the locally finite hermitian Lie algebras of tube type that have been classified in [N098].

The following result will be one of our main tools throughout the remainder of this paper.

**Theorem III.9.** (Factorization Theorem) If \( \rho \) is an admissible representation of \( L \), then there exist \( k \in \mathbb{N}_0, \ c \in \mathbb{Z} \), and a continuous involutive representation \( \rho_0: L_0 = GL(H, \mathbb{K}) \to GL(V) \) such that

\[
\rho(g) = \det_k(gg^\top)^{-\frac{k}{2}} \det_k(gg^\top)^c \rho_0(g) \quad \text{for all } g \in L.
\]

It follows in particular that \( \rho \) extends uniquely to a continuous representation \( \rho \) of the Banach Lie group \( L_1 \). For \( \mathbb{K} = \mathbb{R}, \mathbb{H} \) we always have \( c = 0 \).

**Proof.** For the proof we consider the three different cases for \( \mathbb{K} \) separately. (1) \( \mathbb{K} = \mathbb{R} \). Here \( K = GL(J, \mathbb{C}) \) and the kernel on \( D = D_0 \cap U_C \) associated to the holomorphic highest weight representation \( \rho_K \) of \( K \) in [N098] is given by

\[
K(Z, W) = \rho_K(1 - ZW^*).
\]

Here we simply have \( K = L_C \) and we put \( \rho_L := \rho_K \). In view of Proposition III.5(iii) we have for \( Z, W \in D \) the relation

\[
\rho_L \left( \frac{C(Z) + C(W)^*}{2} \right) = \rho_L(1 + Z)^{-1} \rho_L(1 - ZW^*) \rho_L(1 + W)^{-*}.
\]

This implies that the function \( \rho_L \Sigma \) is positive definite if and only if the kernel \( K \) on \( D \) is positive definite.
Here \( K = \text{GL}(J, \mathbb{C}) \times \text{GL}(J, \mathbb{C}) \) and the kernel on \( \mathcal{D} \) associated to the holomorphic highest weight representation \( \rho_K = \rho_- \otimes \rho_+ \) of \( K \) in \([NO98]\) is given by

\[
K(Z, W) = \rho_-(1 - Z W^*) \otimes \rho_+(1 - W^* Z)^{-1}.
\]

The natural inclusion \( L \to K \) is given by \( g \mapsto (g, g^{-*}) \), so that the corresponding map \( \Omega \to K \) is given by \( Z \mapsto (Z, Z^{-1}) \). This map in turn extends to a holomorphic map \( \Sigma \to K \) given by the same formula. In this sense we have

\[
\rho_L(g) = \rho_K(g, g^{-*}) = \rho_-(g) \rho_+(g^*)^{-1}, \quad g \in L.
\]

Using Proposition III.5(iii), we now obtain

\[
\rho_L \left( \frac{c(Z) + c(W)^*}{2} \right) = \rho_L \left( (1 + Z)^{-1}(1 - Z W^*)(1 + W^*)^{-1} \right) \otimes \rho_+(1 + W^*)^{-1}(1 + Z)
\]

\[
= \rho_-(1 + Z)^{-1}(1 - Z W^*)(1 + W^*)^{-1} \otimes \rho_+(1 + W^*)^{-1}(1 + Z)
\]

\[
= \rho_L(1 + Z)^{-1}(1 - Z W^*) \otimes \rho_+(1 - W^* Z)^{-1} \rho_L(1 + W^*)^{-1}.
\]

(3) \( K = \mathbb{H} \). Here \( K = \text{GL}(2J, \mathbb{C}) \) and the kernel on \( \mathcal{D} \) associated to the holomorphic highest weight representation \( \rho_K \) of \( K \) in \([NO98]\) is given by

\[
K(Z, W) = \rho_K(1 - (JZ)(JW)^*) = \rho_K(1 - JZW^*J^*) = \rho_K(1 - JZW^*J^{-1}) = \rho_K(1 - Z W^*).
\]

Here the group \( K \) acts on \( JU_{\mathbb{C}} \subseteq \text{Skew}(H) \) by \( k, Z = kZ k^T \). On the other hand \( L \) acts on \( U \) by \( g, Z = gZ g^{-*} \). In view of \( J = g, Z = g g^{*} = g(JZ) g^* \), this means that the natural embedding \( L \to K \) is given by \( g \mapsto g, Z \), and this shows that \( \rho_L(g) = \rho_K(g) \), \( g \in L \), defines the representation \( \rho_L \) of \( L \) corresponding to the representation \( \rho_K \) of the complex group \( K = L_{\mathbb{C}} \).

Finally we obtain with Proposition III.5(iii) the relation

\[
\rho_L \left( \frac{c(Z) + c(W)^*}{2} \right) = \rho_L(1 + Z)^{-1} \rho_L(1 - Z W^*) \rho_L(1 + W^*)^{-1}
\]

\[
= \rho_L(1 + Z)^{-1} \rho_K(1 - Z W^*) \rho_L(1 + W^*)^{-1}.
\]

This proves that the function \( \rho_L \) on \( T_{\Omega} \) is positive definite if and only if the kernel \( K \) is positive definite on \( \mathcal{D} \).

Thus we have seen that in all three cases the function \( \rho \) on \( \Omega \) is positive definite if and only if the kernel \( K \) on \( \mathcal{D} \) is positive definite. If \( \rho \) is a unitary highest weight representation with highest weight \( \lambda \), this is equivalent to the unitarity of the highest weight module \( L(\lambda, \Delta^+) \) of the conformal Lie algebra \( \mathfrak{g} \) of \( U_{\mathbb{C}} \).

Again we have a look at all three cases to see that the assertions of the theorem hold if \( K \) is positive definite. We use the description of the unitary highest weight module of \( \mathfrak{g} \) in Theorem III.4 to see that in all cases the highest weight functionals \( \lambda \) on the Lie algebra \( \mathfrak{gl}(J, \mathbb{C}) \) can be written as \( \lambda = \lambda^0 + \text{ctr} \), where \( \lambda^0 \) has finitely many non-zero entries.

(1) \( K = \mathbb{R} \): Here we have for \( g \in K \):

\[
\rho_{\lambda}(g) = \rho_{\lambda^0}(g) \det(g)^M = \rho_{\lambda^0}(g) \det(g)^{-\frac{M}{2}}
\]

and for \( g \in L \) we have \( \det_{\mathbb{R}}(g) = \det(g) \).

(2) \( K = \mathbb{C} \): Here we have for \( g = (g_+, g_-) \in K \):

\[
\rho_{\lambda}(g_-, g_+) = \det(g_-)^{m} \det(g_+)^{M} \rho_{\lambda^0}(g_-) \otimes \rho_{\lambda^0}(g_+)
\]

\[
= \det(g_-g_+^{-1}) \frac{m}{2M} \det(g_+g_-)^{\frac{M}{2}} \rho_{\lambda^0}(g_-) \otimes \rho_{\lambda^0}(g_+).
\]
Note that to assign the appropriate sense to the first two factors separately, we have to consider an appropriate covering group of $K$ or simply the universal covering group $\text{GL}(J, \mathbb{C}) \times \text{GL}(J, \mathbb{C})$. For $g \in L$ the corresponding element in $K$ is $(g, g^{-1})$, so that

$$\det(g)^m \det(g^{-1})^M = \det((gg^*)^{\frac{m+M}{2}}) \det((gg^{-1})^\sim)^{-\frac{m+M}{2}} = \det((gg^T)^{-\frac{m+M}{2}}).$$

Here both factors make sense since the first one is defined on $L$, and so that the second one is also defined. It follows in particular that $m + M \in 2\mathbb{Z}$ whenever we start with a representation $\rho_L$ of $L$.

(3) $\mathbb{K} = \mathbb{C}$: First we note that for $g \in L = \text{GL}(J, \mathbb{K})$ we have

$$\det_\mathbb{K}(g) = \det_\mathbb{C}(gg^*) = |\det_\mathbb{C}(g)|^2.$$

Hence we obtain for $g \in K$ the relation $\rho_\lambda(g) = \det(g)^M \rho_\lambda^*(g)$ and for $g \in L$ we have

$$\det(g)^M = |\det_\mathbb{C}(g)|^M = \det_\mathbb{K}(g)^\sim = \det_\mathbb{K}(g)^{-\sim}.$$

Since a representation $\rho^*$ of $\text{GL}(J, \mathbb{C})$ with $\text{supp} \lambda_0$ finite extends in a canonical way to a holomorphic representation of the group $\text{GL}(L^2(J))$, the proof is complete.

**Remark III.16.** (a) It is interesting to observe that, even though there exist unitary highest weight representations of covering groups of $K = L_\mathbb{C}$ which do not factor to the group $K$ (this happens if the numbers $M$ or $m$ are not integral which is possible in case (1) and (2)), the corresponding representation of $L$ is almost always well defined in the sense that it factors through the inclusion $L \to K$. The only exception is the case $\mathbb{K} = \mathbb{C}$ with $M + m \neq 0$.

(b) Suppose that $\mathbb{K} = \mathbb{C}$. Then $2\epsilon = m + M$ may be any real number. Then $g \mapsto \det((gg^*)^\sim)^\sim$ is not defined as a representation of $L$, and we have to consider its universal covering group $L \cong \text{GL}(J, \mathbb{C})$ instead. Only for $\epsilon \in \mathbb{Z}$ this representation factors over the group $L$.

**IV. The Representation on the $L^2$-spaces**

In this section we eventually come to the description of the representation of the group $U_1 \rtimes L_1$, resp. $U_2 \rtimes L_2$, on the vector-valued $L^2$-spaces on $U^*_1$, resp. $U^*_2$, that we assign to an admissible representation of $L$. First we explain that the realization on $U^*_1$ leads in a quite straightforward manner to a representation of $U_1 \rtimes L_1$. The interesting behavior shows up if one wants to make the projective representation of the bigger group $U_2 \rtimes L_2$ on the $L^2$-space explicit. To assign a sense to the cocycles for this group, one encounters for the realization on $U^*_1$, we first have to pass to a realization on $U^*_2$. The existence of the measures on $U_2^*$ is obtained by extending a modified representation $\rho^*$ from the domain $\Sigma_1$ to the bigger domain $\Sigma_2$. This is done by replacing the determinant factor in the representation (cf. the Factorization Theorem) by the regularized Hilbert-Schmidt determinant. After working out the cocycles for the action of the group $U_2 \rtimes L_2$, we show that the unitary multiplier representation of $U_2 \rtimes L_2$ obtained that way leads to a non-trivial central extension of the group, hence cannot be modified to a unitary representation of the group itself.

The following proposition generalizes the discussion in Example II.5 to the case of higher-dimensional representations of $L$.

**Proposition IV.1.** If $\rho : L \to \text{GL}(V)$ is an admissible representation, then it uniquely extends to a continuous representation $\rho$ of $L_1$ and there exists a unique $\text{Herm}^+(V)$-valued measure $\mu_1$ on $U^*_1$ with

$$\mathcal{L}(\mu_1)(x) = \rho(1 + x) \quad \text{for} \ x \in \Omega_1 - 1 \subseteq U_1.$$
The prescription
\[(\pi_1(u,g),f)(\alpha) := e^{i\alpha(u)} e^{\frac{i\alpha}{g} (1-gg^\top)} \rho(g^{-\top}), f(g^{-1},\alpha)\]
defines a unitary representation of the group \(U_1 \times L_1\) on the \(V\)-valued \(L^2\) space \(L^2(U_1^*,\mu_1)\). The corresponding representation on the reproducing kernel Hilbert space \(H_K \subseteq \text{Hol}(\Sigma_1,V)\), where \(K(z,w) = \rho\left(\frac{z}{w}\right)\) and the isometry \(L^2(U_1^*,\mu_1) \to H_K\) is given by \(f \mapsto \hat{f}\) with
\[
\langle \hat{f}(z),v \rangle = \langle f, e_{-\frac{z}{2iu}} v \rangle, \quad z \in \Sigma_1
\]
is given by
\[
\langle \pi_1(u,g),f\rangle(z) = \rho(g), \hat{f}(g^{-1}.(z-2iu)) \quad \text{for} \quad g \in L_1, u \in U_1, z \in \Sigma_1.
\]

**Proof.** First we use Theorem III.9 to see that \(\rho\) extends to a continuous representation of the Banach Lie group \(L_1\). In view of the continuity of \(\rho\), the function
\[\Omega_1 - 1 \to B(V), \quad x \mapsto \rho(1+x)\]
is positive definite as a function on the convex open domain \(\Omega_1\). Now Theorem I.7 implies the existence of a unique \(V^+\)-valued measure \(\mu_1\) on \(U_1^*\) with the required properties.

One easily checks that (4.1) defines a representation on the space \(V^U_1^*\) of all \(V\)-valued functions on \(U_1^*\). We claim that the measure \(\mu_1\) transforms under \(L_1\) according to
\[
d\mu_1(g^{-1}.\alpha) = \rho(g^{-1}) \cdot \left(e^{\alpha(1-gg^\top)} d\mu_1(\alpha)\right) \cdot \rho(g^{-\top}).
\]
This will be done by showing that both measures have the same Laplace transforms. For the left hand side we get
\[
\int_{U_1^*} e^{-\alpha(x)} d\mu_1(g^{-1}.\alpha) = \int_{U_1^*} e^{-(\alpha(g.\alpha)(x))} d\mu_1(\alpha) = \int_{U_1^*} e^{-\alpha(g^{-\top}.x)} d\mu_1(\alpha) = \rho(1+g^{-1}.x)
\]
and for the right hand side
\[
\rho(g^{-1}) \left( \int_{U_1^*} e^{-\alpha(x)} e^{\alpha(1-gg^\top)} d\mu_1(\alpha) \right) \rho(g^{-\top}) = \rho(g^{-1}) \left( \int_{U_1^*} e^{-\alpha(x-1+gg^\top)} d\mu_1(\alpha) \right) \rho(g^{-\top})
\]
\[
= \rho(g^{-1})\rho(g^{-1}xg^{-\top}+1) = \rho(1+g^{-1}.x).
\]
That the action of \(L_1\) on \(V^U_1^*\) induces a unitary action on the space \(L^2(U_1^*,\mu_1)\) can now be seen as follows. If \(f \in L^2(U_1^*,\mu_1)\), then
\[
\|\pi_1(g)f\|_2^2 = \int_{U_1^*} P_{(g.f)(\alpha)} d\mu_1(\alpha) = \int_{U_1^*} e^{\alpha(1-gg^\top)} \rho(g^{-\top}) P_{(g^{-1}.\alpha)} \rho(g^{-1}) d\mu_1(\alpha)
\]
\[
= \int_{U_1^*} P_{(g^{-1}.\alpha)} \left(e^{\alpha(1-gg^\top)} \rho(g^{-1}) d\mu_1(\alpha)\right) \rho(g^{-\top}) \int_{U_1^*} P_{(g^{-1}.\alpha)} d\mu_1(g^{-1}.\alpha) = \|f\|_2^2.
\]
To obtain the formula for the action on \(H_K\), we first observe that for each \(v \in V\) we have
\[
\langle \epsilon_{iu} f \rangle(z),v) = \langle \epsilon_{iu} f, e_{-\frac{z}{2iu}} v \rangle = \langle f, e_{-\frac{z}{2iu}} e_{-\frac{z}{2iu}} v \rangle = \langle f, e_{-\frac{z}{2iu}} v \rangle = \langle \hat{f}(z-2iu),v \rangle.
\]
For \(g \in L_1\) and \(z \in \Sigma_1\) we obtain
\[
\langle (g.\epsilon_{-\frac{2}{iu}} v)\rangle(\alpha) = e^{\frac{i\alpha}{g} (1-gg^\top)} e_{-\frac{z}{2iu}} (g^{-1}.\alpha) \rho(g^{-\top}),v = e^{\frac{i\alpha}{g} (1-gg^\top)} e_{-\frac{z}{2iu}} (g^{-1}.\alpha) \rho(g^{-\top}),v.
\]
For \(g \in L_1\) and \(z \in \Sigma_1\) we thus get
\[
\langle (g.f)\rangle(z),v) = \langle g.f, e_{-\frac{z}{2iu}} v \rangle = \langle f, g^{-1}.e_{-\frac{z}{2iu}} v \rangle = \langle f, e_{-\frac{z}{2iu}} \rho(g^{-\top}).v \rangle
\]
\[
= \langle \hat{f}(g^{-1}.z),\rho(g^{-\top}).v \rangle = \langle \rho(g).\hat{f}(g^{-1}.z),v \rangle.
\]
Putting these formulas together, the assertion follows.

In the following it will be important to go one step further and to realize the measure on the space \(U_1^*\) instead of \(U_1^*\). To obtain this realization, we need the generalized determinant.
**Definition IV.2.** Let $H$ be a Hilbert space and $X \in B_2(H)$. Then $(1 + X) e^{-X} - 1 \in B_1(H)$ follows from $1 + X - e^X = X^2(\cdots)$. Hence the generalized determinant

$$
\det_2(1 + X) := \det((1 + X)e^{-X})
$$

makes sense for $X \in B_2(H)$ (cf. [Mi89, Prop. 6.2.3]). This means that for $g \in \text{GL}_2(H)$ we have

$$
\det_2(g) = \det(ge^{1-g}).
$$

For $g \in \text{GL}_1(H)$ this simplifies to $\det_2(g) = \det(g)e^{\text{tr}(1-g)}$. □

**Lemma IV.3.** The generalized determinant satisfies for $x, y \in 1 + B_2(H)$ the relation

$$
\det_2(xy) = \det_2(yx) = \det_2(x)\det_2(y)e^{-\text{tr}(1)}(y^{-1}|y^{-1}| - 1).
$$

**Proof.** (cf. [Mi89, p.138]) First we observe that both sides are continuous functions on $(1 + B_2(H)) \times (1 + B_2(H))$. Hence it suffices to prove the equality for $(x, y)$ in the dense subset $(1 + B_1(H)) \times (1 + B_1(H))$, where we have

$$
\begin{align*}
\det_2(xy) &= \det(x)\det(y)e^{1-xy} = \det(x)e^{1-x}\det(y)e^{1-y}e^{\text{tr}(1+y^{-1}+1-1-xy)} \\
&= \det_2(x)\det_2(y) e^{\text{tr}(1+y^{-1})} = \det_2(x)\det_2(y) e^{\text{tr}(1)}(y^{-1} - 1).
\end{align*}
$$

□

Let $\rho$ be an admissible representation of $L_1$, and for $\mathbb{K} = \mathbb{C}$ we assume that $\varepsilon = 0$ in the factorization from Theorem III.9:

$$
\rho(g) = \det_2(g^{-1}g^{-1})^\frac{1}{2}\rho_0(g) \quad \text{for all } g \in L_1
$$

with $k \in \mathbb{N}_0$. It is clear that this expression does not make sense for $g \in L_2$. Since $\rho_0$ extends in a canonical way to $L_2$, we define a continuous map

$$
\rho_2: L_2 \to \text{GL}(V), \quad g \mapsto \det_2(g^{-1}g^{-1})^\frac{1}{2}\rho_0(g)
$$

and note that for $g \in L_1$ we have $\rho_2(g) = e^{\frac{1}{2}\text{tr}(1-g^{-1}g^{-1})}\rho(g)$ (cf. Definition II.6). Thus $\rho_2|_{L_1}$ is a multiplier representation which defines the same projective representation as $\rho_0$. The following lemma displays the corresponding cocycle.

**Lemma IV.4.** For $x, y \in L_2$ we have

$$
\rho_2(xy) = \rho_2(x)\rho_2(y)e^{-\varepsilon\text{tr}(1)}(y^{-1}x^{-1} - 1)(y^{-1}x^{-1} - 1)) \quad \text{and} \quad \rho_2(x^\top) = \rho_2(x)^*.
$$

**Proof.** The first formula is an immediate consequence of Lemma IV.3, and the second one follows from $\rho_2(g^\top) = \det_2(g^{-1}g^{-1})^\frac{1}{2}\rho_0(g^\top) = \det_2(g^{-1}g^{-1})^\frac{1}{2}\rho_0(g)^* = \rho_2(g)^*$. □

Even though $\text{tr}$ does not make sense as a linear functional on $U_2$, we want to assign a sense to the functional $(g, \text{tr} - \text{tr})$ for $g \in L_2$. Here one has to be careful because naively one would consider

$$
(g^{-1}, \text{tr} - \text{tr})(x) = \text{tr}(gxg^\top - x)
$$

which in general does not make sense for $g \in L_2$ because there exist orthogonal operators $u$ and Hilbert-Schmidt operators $x$ for which $uxu^{-1} - x$ is not of trace class. Nevertheless, the approach described in the following lemma works.
Lemma IV.5. For $g \in L_2$ and $x \in U_2$ we define $c(g) \in U_2^*$ by
\[ c(g)(x) := \frac{k_d}{2} \text{tr} \left( (g^{-\top} g^{-1} - 1)x \right). \]

(i) We have the cocycle identity $c(g_1 g_2) = g_1 c(g_2) + c(g_1)$ for $g_1, g_2 \in L$.
(ii) $a.g := g^{-1} (a - c(g))$ defines an affine right action of $L_2$ on $U_2^*$.

Proof. (i) For each $x \in U_2$ we have the relation
\[
\begin{align*}
  c(g_1 g_2)(x) &= \frac{k_d}{2} \text{tr} \left( (g_1^{-\top} g_2^{-\top} g_2^{-1} g_1^{-1} - 1)x \right) \\
  &= \frac{k_d}{2} \text{tr} \left( (g_1^{-\top} g_2^{-\top} g_2^{-1} g_1^{-1} - g_1^{-\top} g_1^{-1})x \right) + \frac{k_d}{2} \text{tr} \left( (g_1^{-\top} g_1^{-1} - 1)x \right) \\
  &= \frac{k_d}{2} \text{tr} \left( (g_2^{-\top} g_2^{-1} - 1)g_1^{-1} x g_1^{-\top} \right) + c(g_1)(x) \\
  &= (g_1 c(g_2))(x) + c(g_1)(x).
\end{align*}
\]

(ii) This is an immediate consequence of (i).

Proposition IV.6. If $\rho$ is an admissible representation of $L_1$, $\rho_2 : L_2 \to \text{GL}(V)$ is defined by
\[
\rho_2(g) = \text{det}_{\mathbb{R}}(g^{-\top} g^{-1})^{\frac{k_d}{2}} \rho_0(g),
\]
and $\varphi : \Omega_2 \to B(V)$ by $\varphi(x) := \text{det}_{\mathbb{R}}(x^{-\top} x)^{-\frac{k_d}{2}} \rho_0(x)$, then there exists a unique $\text{Herm}^+(V)$-valued measure $\mu_2$ on $U_2^*$ with
\[
\mathcal{L}(\mu_2)(x) = \varphi(1 + x) \quad \text{for} \quad 1 + x \in \Omega_2.
\]
Moreover
\[
(\sigma_2(g) f)(\alpha) = e^{\frac{k_d}{2} \text{tr} x} (1 - g g^{-\top}) \rho_2(g^{-\top}) f (g^{-1} (a - c(g))) = \text{det}_{\mathbb{R}}(g g^{-\top})^{\frac{k_d}{2}} e^{\frac{k_d}{2} \text{tr} x} \rho_2(g^{-\top}) f (g^{-1} (a - c(g)))
\]
defines a unitary representation of $L_2$ on the space $L^2(U_2^*, \mu_2)$.

Proof. For $a \in U_1^*$ we write $\delta_a$ for the corresponding point measure and note that the convolution $\delta_a * \mu_1$ is a measure on $U_1^*$ with
\[
\mathcal{L}(\delta_a * \mu_1)(x) = e^{-\frac{k_d}{2} \text{tr} x} \rho_1(1 + x).
\]
The function $\varphi$ is positive definite because it is continuous, and for $1 + x \in \Omega_1$ we have
\[
\varphi(1 + x) = e^{\frac{k_d}{2} \text{tr} x} \rho_1(1 + x) = e^{\frac{k_d}{2} \text{tr} x} \mathcal{L}(\mu_1)(x) = \mathcal{L}(\delta_{1 + x} * \mu_1)(x).
\]
Using Theorem I.7(iii), we therefore obtain a unique $\text{Herm}^+(V)$-valued measure $\mu_2$ on $U_2^*$ with $\mathcal{L}(\mu_2)(x) = \varphi(1 + x)$ for $1 + x \in \Omega_2$.

To see that the action (4.2) induces a unitary representation of $L_2$ on the space $L^2(U_2^*, \mu_2)$, we first show that
\[
d \mu_2(g^{-1} a) = \rho_2(g^{-1}) \left( e^{\frac{k_d}{2} \text{tr} x} (1 - g g^{-\top}) d \mu_2(a + c(g)) \right) \rho_2(g^{-\top}).
\]
This will be done by showing that both have the same Laplace transforms. First we observe that
\[
\begin{align*}
  \text{det}_{\mathbb{R}}(1 + g^{-1} x g^{-\top})^{-\frac{k_d}{2}} &= \text{det}_{\mathbb{R}}(1 + g^{-\top} g^{-1} x)^{-\frac{k_d}{2}} \\
  &= \text{det}_{\mathbb{R}}(g g^{-\top})^{\frac{k_d}{2}} \text{det}_{\mathbb{R}}(g g^{-\top} + x)^{-\frac{k_d}{2}} e^{-\frac{k_d}{2} \text{tr} ((g g^{-\top} - 1)x)} \\
  &= \text{det}_{\mathbb{R}}(g g^{-\top})^{\frac{k_d}{2}} \text{det}_{\mathbb{R}}(g g^{-\top} + x)^{-\frac{k_d}{2}} e^{-\frac{k_d}{2} \text{tr} (1 - g^{-\top} g^{-1})x} \\
  &= \text{det}_{\mathbb{R}}(g g^{-\top})^{\frac{k_d}{2}} e^{c(g)(x)} \text{det}_{\mathbb{R}}(g g^{-\top} + x)^{-\frac{k_d}{2}}.
\end{align*}
\]
For the Laplace transform of the left hand side of (4.3) we now get
\[
\int_{\mathbb{U}_2} e^{-\alpha(x)} d\mu_2(g^{-1}, \alpha) = \det_{\mathbb{H}_2}(1 + g^{-1}x) - E \rho_0(1 + g^{-1}x)
\]
\[
= \det_{\mathbb{H}_2}(1 + g^{-1}xg^{-T}) - E \rho_0(1 + g^{-1}xg^{-T})
\]
\[
(4.4) \det_{\mathbb{H}_2}(gg^T) E e^{c(g)(x)} \det_{\mathbb{H}_2}(gg^T + x) - E \rho_0(g^{-1}) \rho_0(g^{-1})(gg^T + x) \rho_0(g^{-T})
\]
\[
= \det_{\mathbb{H}_2}(gg^T) E \det_{\mathbb{H}_2}(g^{-T}g) E e^{c(g)(x)} \rho_0(g^{-1}) \phi(gg^T + x) \rho_0(g^{-T})
\]
\[
= e^{c(g)(x)} \rho_2(g^{-1}) \phi(gg^T + x) \rho_2(g^{-T})
\]
and for the Laplace transform of the right hand side of (4.3) we obtain
\[
\rho_2(g^{-1}) \int_{\mathbb{U}_2} e^{-\alpha(x)} e^{(\alpha + c(g))(1 - gg^T)} d\mu_2(\alpha + c(g)) \rho_2(g^{-T})
\]
\[
= \rho_2(g^{-1}) \int_{\mathbb{U}_2} e^{-\alpha(1 - gg^T)} d\mu_2(\alpha) \rho_2(g^{-T})
\]
\[
= \rho_2(g^{-1}) e^{c(g)(x)} \int_{\mathbb{U}_2} e^{-\alpha(x - 1) + 2g^T} d\mu_2(\alpha) \rho_2(g^{-T})
\]
\[
= \rho_2(g^{-1}) e^{c(g)(x)} \phi(x + gg^T) \rho_2(g^{-T}).
\]

That the action of \( L_2 \) on \( V_{\mathbb{U}_2} \) induces a unitary action on the space \( L^2(U_2^*, \mu_2) \) now follows for \( f \in L^2(U_2^*, \mu_2) \) from
\[
\|g.f\|^2 = \int_{\mathbb{U}_2} P_{(g.f)(\alpha)} d\mu_2(\alpha) = \int_{\mathbb{U}_2} e^{a(1 - gg^T)} \rho_2(g^{-T}) P_{f(g^{-1}, (\alpha + c(g)) \rho_2(1 - gg^T)} d\mu_2(\alpha)
\]
\[
= \int_{\mathbb{U}_2} e^{a(1 - gg^T)} \rho_2(g^{-T}) P_{f(g^{-1}, \alpha) \rho_2(1 - gg^T)} d\mu_2(\alpha + c(g))
\]
\[
= \int_{\mathbb{U}_2} P_{f(g^{-1}, \alpha)} e^{a(1 - gg^T)} (\rho_2(g^{-1}) d\mu_2(\alpha + c(g)) \rho_2(g^{-T}))
\]
\[
= \int_{\mathbb{U}_2} P_{f(g^{-1}, \alpha)} d\mu_2(\alpha + c(g)) = \|f\|^2.
\]

It is instructive to see how the cocycle representation \( \pi_2 \) of \( U_2 \rtimes L_2 \) on \( L^2(U_2^*, \mu_2) \) looks like in the realization of the space \( L^2(U_2^*, \mu_2) \) as a reproducing kernel Hilbert space \( \mathcal{H}_K \subseteq \mathcal{H}_1(\Sigma_2, V) \), where the kernel is given by \( K(z, w) = \rho_2(gz, \mu_2) \), and the isometry is given by \( f \mapsto \hat{f} \) with
\[
\langle \hat{f}(z), v \rangle = \langle f, e^{-1} v \rangle, \quad z \in \Sigma_2
\]
(cf. Theorem 1.7).

**Proposition IV.7.** For \( g \in L_2 \) and \( z \in \Sigma_2 \) we have
\[
(\pi_2(g)f)(z) = e^{\frac{1}{2} g(z)(1 - 1)} \rho_2(g) \hat{f}(g^{-1}, z).
\]

**Proof.** For \( g \in L_2 \) and \( z \in \Sigma_2 \) we have, in view of Proposition IV.6,
\[
(g(e^{-1}v))(\alpha) = e^{\frac{1}{2} a(1 - gg^T)} e^{-a(1 - gg^T)(\alpha - c(g))} \rho_2(g^{-T}). v
\]
\[
= e^{\frac{1}{2} g(z)(1 - 1)} e^{-a(1 - gg^T)(\alpha - c(g))} \rho_2(g^{-T}). v
\]
\[
= e^{\frac{1}{2} g(z)(1 - 1)} e^{-a(1 - gg^T)(\alpha)} \rho_2(g^{-T}). v
\]
\[
= e^{\frac{1}{2} g(z)(1 - 1)} e^{-a(1 - gg^T)(\alpha)} \rho_2(g^{-T}). v
\]
\[
= e^{\frac{1}{2} g(z)(1 - 1)} e^{-a(1 - gg^T)(\alpha)} \rho_2(g^{-T}). v
\]
In view of
\[
c(g)(g(z - 1)) = \frac{k}{2} \text{tr}((g^{-T} g^{-1} - 1)g(z - 1)g^T) = \frac{k}{2} \text{tr}(g^{-T} g^{-1} - 1)g(z - 1)) = \frac{k}{2} \text{tr}((1 - g^T g)(z - 1)) = -c(g^{-1})(z - 1),
\]
this simplifies to
\[
\langle (g, (e_{-\frac{k}{2}d} v))(\alpha) = e^{-\frac{k}{2}(g^{-1})z} e_{-\frac{k}{2}d} \rho_2 (g^{-T})v.
\]
For \( g \in L_2 \) and \( z \in \Sigma_2 \) we thus obtain with Lemma IV.4:
\[
\langle (g, f)(z), v) = \langle g, f, e_{-\frac{k}{2}d} v) = \langle f, g^{-1}, e_{-\frac{k}{2}d} v) = \langle f, e^{-\frac{k}{2}(g^{-1})} e_{-\frac{k}{2}d} \rho_2 (g^{-T})v.
\]

One motivation to look for a realization on the bigger space \( U_2^* \) is that this realization fits well with the action of the larger group \( L_3 \) suggested by the discussion in Example I.8 on the spaces \( L^2(U_2^*, \nu_2^1) \), resp. \( L^2(U_1^*, \nu_2^1) \). The natural isomorphism between these two spaces is given by
\[
\eta: L^2(U_2^*, \nu_2^1) \rightarrow L^2(U_1^*, \nu_2^1), \quad \eta(f)(\alpha) = f(\frac{k}{2} \text{tr} + \alpha |v_i|).
\]

**Remark IV.8.** (a) Let \( \mu_1 \) be the measure on \( U_2^* \) with \( \mathcal{L}(\mu_1)(x) = \rho(1 + x) \) for \( 1 + x \in \Omega_1 \). Since the right hand side is a differentiable function on the open domain \( \Omega_1 - 1 \subseteq U_1 \), we can calculate its derivative as follows:
\[
d\rho(1)(y) = d\mathcal{L}(\mu_1)(0)(y) = - \int_{v_1} a(y) \, d\mu_1(\alpha).
\]
Hence we can view the linear function \( -d\rho(1): U_1 \rightarrow \text{Herm}(V) \) as the expectation value \( E(\mu_1) \) of the measure \( \mu_1 \) on \( U_1^* \). For the special case \( \rho(\gamma) = \det_\gamma g^{-\frac{k}{2}} \) we obtain in particular
\[
E(\mu_2^1) = \frac{k}{2} \text{tr}_\mathbb{R} \quad \text{and} \quad E(\delta_{-\frac{k}{2}d} \mu_1) = E(\nu_2^1) - \frac{k}{2} \text{tr}_\mathbb{R} = 0.
\]
In general
\[
E(\delta_{-\frac{k}{2}d} \mu_1) = E(\mu_1) - \frac{k}{2} \text{tr}_\mathbb{R} = -d\rho(1)
\]
extends to a continuous linear function on \( U_2^* \). Since the expectation value of the shifted measure is a linear function \( U_1 \rightarrow \text{Herm}(V) \) that extends continuously to the bigger space \( U_2^* \), this shift is the most natural one, to obtain a related measure on the space \( U_2^* \). For the scalar case we have in particular \( E(\mu_2) = 0 \).

**The projective representation of the affine group**

In Proposition IV.6 above we have constructed a natural representation of the group \( L_2 \) on the space \( L^2(U_2^*, \mu) \) given by
\[
(\pi_2(g, f)(\alpha) = e^{\frac{k}{2} \alpha(1 - gg^T)} \rho_2(g^{-T})f(g^{-1})(\alpha - c(g))).
\]
On the other hand we have seen in Proposition IV.1 how \( L_1 \) acts on the space \( L^2(U_1^*, \mu_1) \). On the group \( L_1 \) both representations are compatible in the sense that the unitary map
\[
\eta: L^2(U_1^*, \mu_1) \rightarrow L^2(U_2^*, \mu_2), \quad \eta(f)(\alpha) = f(\frac{k}{2} \text{tr} + \alpha |v_i|)
\]
is $L_1$-equivariant.

If we consider the representation of $U_1$ on $L^2(U^*_2, \mu_2)$ obtained by the requirement that $\eta$ is equivariant, we find the formula

$$\langle \pi_1(u), f \rangle (a) = e^{i(2k + \alpha)(u)} f(u).$$

Since the functional $\text{tr}$ cannot be extended in a canonical way to $U_2$, this formula cannot be used to obtain a representation of $U_2$. Thus we simply put

$$\langle \pi_2(u), f \rangle (a) = e^{ia(u)} f(a), \quad u \in U_2, a \in U^*_2,$$

and note that for $u \in U_1$ we have $\pi_2(u) = e^{-ikd(u)} \pi_1(u)$, so that the corresponding projective representations are the same. For $(u, g) \in U_2 \times L_2$ we define

$$\pi_2(u, g) := \pi_2(u) \pi_2(g)$$

and claim that

$$\pi_2(g) \pi_2(u) \pi_2(g^{-1}) = e^{-ic(g)[g, u]} \pi_2(g, u).$$

In fact, for $f: U^*_2 \to V$ and $h: U^*_2 \to C$ we have

$$\langle (\pi_2(g), h f) \rangle (a) = h(g^{-1}, (a - c(g))) \cdot \langle \pi_2(g), f \rangle (a)$$

and therefore

$$\langle \pi_2(g) \pi_2(u) \pi_2(g^{-1}), f \rangle (a) = e^{i(a-c)(g-u)} f(a) = e^{-ic[g][g, u]} \langle \pi_2(g, u), f \rangle (a).$$

More explicitly, we have

$$c(g)(g, u) = \frac{k d}{2} \text{tr} \left( (g^{-1} g - 1) g u g \right) = \frac{k d}{2} \text{tr} \left( g^{-1} g - 1 \right) g u = \frac{k d}{2} \text{tr} \left( (1 - g^{-1} g) u \right).$$

On the group $G := U_2 \times L_2$ we now consider the multiplier given by

$$m \left( (u_1, g_1), (u_2, g_2) \right) = e^{-ic(g_1)[g_1, u_2]}.$$

This multiplier defines a central extension $\hat{G} \to G$ with $\hat{G} = G \times \mathbb{T}$ and the multiplication

$$(u_1, g_1, z_1) \cdot (u_2, g_2, z_2) = (u_1 + g_1 \cdot u_2, g_1 g_2, z_1 z_2 e^{-ic(g_1)[g_1, u_2]}),$$

and we obtain a unitary representation of $\hat{G}$ on $L^2(U^*_2, \mu_2)$ by $\hat{\pi}_2(u, g, z) := z \pi_2(u, g)$.

Since the multiplier $m: G \times G \to \mathbb{T}$ is smooth, it is clear that $\hat{G}$ is a Banach Lie group.

**Proposition 4.9.** The central extension $\hat{G} \to G$ defined by the multiplier $m$ is non-trivial as a central extension of Banach Lie groups.

**Proof.** To see that the central extension $\hat{G}$ of $G$ is non-trivial, it suffices to show that it is non-trivial on the Lie algebra level. According to [Va85, Th. 7.37], the Lie algebra cocycle defined by $m$ is given by

$$c(X, Y) = \frac{\partial^2}{\partial t \partial s} \bigg|_{t, s = 0} m(\exp tx, \exp sy) - \frac{\partial^2}{\partial t \partial s} \bigg|_{t, s = 0} m(\exp sy, \exp tx).$$

It follows immediately that $c(X, Y) = 0$ for $X, Y \in u_0 \cong U_2$ or $X, Y \in iL_2$. For $u \in U_2$ and $X \in L_2$ we have $m(\exp u, \exp sX) = 1$ and

$$m(\exp sX, tu) = e^{-ic(\exp sX)} \exp sX u \cdot e^{-i s \text{tr} \left( (1 - \exp (sX) T) \exp sX T u \right)}.$$

Hence

$$c(X, u) = \frac{d}{ds} \bigg|_{s = 0} - \frac{k}{2} \text{tr} \left( (1 - \exp (sX) T) \exp sX u \right) = i \frac{k}{2} \text{tr} \left( (X + X^T) u \right).$$

For $X = X^T$ we obtain in particular $c(X, u) = ik \text{tr}(X u)$. On the other hand $[X, u] = X u + u X^T = X u + u X$. If $c$ is a trivial cocycle, then there exists a continuous linear functional $f$ on $g = U_2 \times L_2$ with $\lambda([X, Y]) = c(X, Y)$ for $X, Y \in g$ and in particular

$$\lambda(X u + u X) = ik \text{tr}(X u) = i \frac{k}{2} \text{tr}(X u + u X).$$

This implies that $\lambda = i \frac{k}{2} \text{tr}$ on the space $U_1$ of all trace class operators in $U_2$, contradicting the continuity on $U_2$. ∎
Problem IV. Describe the space \( H^1_\varepsilon(L_2, U^*_2) = Z^1_\varepsilon(L_2, U^*_2)/U^*_2 \) of equivalence classes of continuous 1-cocycles of \( L_2 \) with values in \( U^*_2 \). The same calculation as in the proof of Lemma IV.5 shows that for each \( A \in U_5 \) the prescription \( e_A(g)(x) := \text{tr}(g^{-1} A g^{-1} - A)|x| \) defines an element in \( Z^1_\varepsilon(L_2, U^*_2) \), and this cocycle is trivial if and only if \( A \in U_2 \). We thus obtain an inclusion \( U_5/U_2 \hookrightarrow H^1_\varepsilon(L_2, U^*_2) \). Is this map surjective? \[\Box\]

V. Ergodicity of the scalar measures on \( U^* \)

As before, let \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \) and consider the special case where the set \( J \) is countable, so that we may w.l.o.g. assume that \( J = \mathbb{N} \). Then \( U = \text{Herm}(\mathbb{N}, \mathbb{K}) \) the Jordan algebra of finite \( \mathbb{N} \times \mathbb{N} \)-matrices and the algebraic dual space \( U^* \) is the space of all infinite hermitian matrices. As a subgroup of \( \mathbb{L} \), the connected group \( L_K := U(\mathbb{N}, \mathbb{K}) = \lim U(n, \mathbb{K}) \) acts in a natural way on \( U^* \). In this subsection we explain how a recent result of O. Ol’shanskii and A. Vershik on \( L_K \)-ergodic measures on these spaces is related to the measures considered in Section II.

Theorem V.1. (Ol’shanskii–Vershik) Let \( \mu \) be an \( L_K \)-invariant probability measure on \( \text{Herm}(\mathbb{N}, \mathbb{K})^* \) and \( \hat{\mu} : \text{Herm}(\mathbb{N}, \mathbb{K}) \to \mathbb{C} \) its Fourier transform. Then \( \mu \) is ergodic if and only if \( \mu \) is multiplicative in the sense that there exists a function \( F : \mathbb{R} \to \mathbb{C} \) such that if \( A = \text{diag}(\lambda_1, \ldots, \lambda_r) \in \text{Herm}(r, \mathbb{K}) \subseteq \text{Herm}(\mathbb{N}, \mathbb{K}) \), then

\[
\hat{\mu}(A) = \prod_{j=1}^r F(\lambda_j).
\]

In particular the measures \( \mu \) on \( \text{Herm}(\mathbb{N}, \mathbb{K})^* \) whose Fourier transform is given by the following formula are ergodic:

\[
\hat{\mu}(A) = e^{i\beta_1 \text{tr}A - \beta_2 \sum_{k=1}^\infty \text{det}
\quad \left(e^{-i\beta_1 A} (1 - i\beta_k A)^{-1}\right),
\]

where \( \text{det} \) denotes the holomorphic extension of the real determinant function of the Jordan algebra \( \text{Herm}(\mathbb{N}, \mathbb{K}) \) to \( \text{Herm}(\mathbb{N}, \mathbb{K})^* \), \( \beta_1 \in \mathbb{R}, \beta_2 \geq 0 \), and \( \sum_{k=1}^\infty x_k < \infty \).

Note that the factor \( e^{i\beta_1 \text{tr}A} \) corresponds to the point measure in \( \beta_1 \text{tr} \), and that \( e^{-\beta_2 \sum_{k=1}^\infty \text{tr}} \) corresponds to a Gaussian measure \( \gamma_U \) with respect to the real scalar product \( (A, B) = \beta_2 \text{tr}B(AB) \). The corresponding measures are supported by the positive cone if and only if \( \beta_2 = 0 \), \( x_j > 0 \) for all \( j \) and \( \sum_{k=1}^\infty x_k \leq \beta_1 \). ([OV96, Rem. 2.11]) The paper [OV96] also contains the result that for \( \mathbb{K} = \mathbb{C} \) (the argument also works for \( \mathbb{K} = \mathbb{R}, \mathbb{H} \) ) these are all the ergodic \( L_K \)-invariant probability measures.

Proof. (cf. [OV96]) The first assertion (cf. [OV96, Th. 2.1]) is proved in [Ol90] for spherical functions of the pairs \( (\text{GL}(\mathbb{N}, \mathbb{K}), U(\mathbb{N}, \mathbb{K})_0) \). Here we need the analog result for the corresponding motion groups

\[
G = \text{Herm}(\mathbb{N}, \mathbb{K}) \times U(\mathbb{N}, \mathbb{K})_0.
\]

Nevertheless, [Ol90, Th. 23.6] also covers this case, and a closer inspection of the proof of [Ol90, Th. 23.8] shows that it carries over from \( \text{GL}(\mathbb{N}, \mathbb{K}) \) to the motion groups \( G \).

The second assertion is also discussed extensively in [OV96]. To explain why formula (3.1) is the Fourier transform of a probability measure, we explain how such measures can be constructed if \( x_n = 0 \) for \( n > k \).

Let \( z \in \mathbb{R} \) and \( E \in \text{Herm}(k, \mathbb{K}) \). We consider the map

\[
\Phi_{z; E} : (\mathbb{K}^d)^k \cong \mathbb{M}(k; \mathbb{K}) \to U^*, \quad v = (v_1, \ldots, v_k) \mapsto z1 + \frac{1}{2} z Ev^*.
\]

where \( Ev^* \) should be read as a matrix product. We may w.l.o.g. assume that

\[
E = \text{diag}(x_1, \ldots, x_k).
\]
Then $\Phi_{z,E}$ is $L_K$-equivariant and the image of the Gaussian measure on $(\mathbb{K}^2)^k$ is an analog of a Wishart distribution (cf. Definition II.2). Let $\gamma$ denote the Gaussian measure on $(\mathbb{K}^2)^k$ and put $\nu_{z,E} := \Phi_{z,E}^* \gamma$. If $E = E_1 + E_2$ and $z = z_1 + z_2$, then we pointwise have $\Phi_{z,E} = \Phi_{z_1,E_1} + \Phi_{z_2,E_2}$. Therefore

$$\mu_{z,E} = \mu_{z_1,E_1} * \mu_{z_2,E_2}. $$

To calculate the Fourier transforms it therefore suffices to do that for the special case where $E$ is a rank-1-matrix, say $E = x_1 E_{11}$ and $z = 0$. Then

$$\Phi_{z,E}(v) = \frac{x_1}{2} v_1 E_{11} v^* = \frac{x_1}{2} v_1 v_1^*; $$

so that, up to the factor $x_1$, the measure $\nu_{0,E}$ is a Wishart distribution, and the same calculation as in Example II.5 yields

$$\hat{\mu}_{0,E}(A) = \det(1 - ix_1 A)^{-1}. $$

This shows that in general we have the formula

$$\hat{\mu}_{z,E}(A) = \e^{iz \cdot \text{tr} A} \prod_{j=1}^k \det(1 - i x_j A)^{-1} = \e^{i(z + \sum_{\ell=1}^k x_\ell) \cdot \text{tr} A} \prod_{j=1}^k \det (\e^{-ix_j A} (1 - i x_j A)^{-1}). $$

The general formula (3.1) is obtained for the infinite convolution product

$$\mu = \delta_{\delta_1} \delta_1 * \gamma U * \mu_{-x_1,E_{11}} * \mu_{-x_2,E_{22}} * \cdots. $$

**Corollary V.2.** The measures $\nu_k$ on the positive cone in $\text{Herm}(\mathbb{N}, \mathbb{K})^*$ are ergodic for the action of the group $L_K = U(\mathbb{N}, \mathbb{K})$. The representation of the semidirect product group $U \rtimes L_K$ on $L^2(U^*, \nu_k)$ is irreducible.

**Proof.** For $z = 0$ and $E_k := E_{11} + \ldots + E_{kk}$ we obtain in particular

$$\hat{\mu}_{0,E_k}(A) = \prod_{j=1}^k \det(1 - i A)^{-1} = \det(1 - i A)^{-k}, $$

i.e., $\nu_k = \mu_{0,E_k}$ is the measure from Example II.5. Now the ergodicity follows from Theorem V.1.

To see that the natural representation of the group $G := U \rtimes L_K$ on $H := L^2(U^*, \nu_k)$ is irreducible, we first observe that the von Neumann algebra of the multiplication operators coming from $L^\infty(U^*, \nu_k)$ is a maximal abelian subalgebra of $B(H)$ which is generated by the image of the representation of $U$. Hence the commutant of the representation of $G$ consists of the $L_K$-invariant elements in $L^\infty(U^*, \nu_k)$, which, in view of the ergodicity of the measure, form a one-dimensional space. Now the irreducibility of the representation is a consequence of Schur’s Lemma.

Note that Corollary V.2 is false in the finite-dimensional case. In this sense the irreducibility of the action of $U \rtimes L_K$ on $L^2(U^*, \nu_k)$ is a new phenomenon in the infinite-dimensional setting.

**Problem V.** Can anything comparable to Corollary V.2 be said about operator-valued measures?

---

**VI. Additional Information on the Measures**

Let $\rho$ be an admissible representation of $L$ and $k$ the corresponding number which can be computed as explained in Theorem III.4 from the highest weight $\lambda$ of the corresponding representation of $L_0$. We further write $\mu$ for the measure on $U^*$ with $L(\mu)(x) = \rho(1 + x)$ for $x \in \Omega$. For a trace class operator $S \in \text{Herm}_1(V)$ we write $\mu^S$ for the scalar measure obtained by $\mu^S(E) = \text{tr}(\mu(E)S)$. 

Lemma VI.1. The measure $\mu^S$ is absolutely continuous with respect to the measure $\nu_k$ on $U^*$. 

Proof. We have to show that $\nu_k(M) = 0$ implies that $\mu^S(M) = 0$ holds for a measurable subset $M \subseteq U^*$. Since each $S$ can be approximated by a positive linear combination of rank one operators, we may w.l.o.g. assume that $S = P_v$ for some $v \in V$. We put $\mu^v := \mu_{P_v}$. Then for each $M \in \mathcal{S}$ the mapping $V \to \mathbb{R}$, $v \mapsto \mu^v(M) = (\mu(M), v, v)$ is continuous, so that it suffices to prove $\mu^v(M) = 0$ for each $v$ in a dense subspace $V_0 \subseteq V$.

To see how we can obtain a suitable subspace, we recall that the representation of $L_1$ on $V$ is a direct limit of representations of finite-dimensional subgroups $L_E \subseteq L$ corresponding to finite-dimensional Jordan algebras $\text{Herm}(E, \mathbb{K})$, where $E \subseteq J$ is a finite subset. The corresponding subspace $V_0 := \bigcup V_E$ of $V$ is dense.

If $\rho_E$ is the corresponding representation of $L_E$ on $V_E$, then the corresponding positive definite function on $\Omega_E$ is given by $\varphi_E(x) = P_E\varphi(x + e - e_E)P_E^*$, where $P_E: V \to V$ is the orthogonal projection onto $V_E$. Let $\mu_E$ denote the corresponding $\text{Herm}^+(V_E)$-valued measure on $\Omega_E$. Then

$$L(\mu_E)(x) = \varphi_E(x + e_E) = P_E\varphi(x + e)P_E^* = P_EL(\mu)(x)P_E^* = L(\mu_{P_E})(x)$$

for $x \in \Omega_E$ implies that the restriction map $\gamma^*_E: U^* \to U^*_E$, which is the adjoint of the inclusion map $\gamma_E: U_E \to U$, satisfies $(\gamma^*_E)^* \mu_{P_E} = \mu_E$.

According to [HN97, Th. V.17] (or [Cl95]), the measure $\mu_E$ has a density $\Phi_E$ with respect to the measure $\nu_{E,k} := (\gamma^*_E)^*\nu_k$, i.e., $\mu_E = \Phi_E \cdot \nu_{E,k}$. For $U_E \subseteq U_F$ we likewise have

$$(\gamma^*_F)^* \mu_{F,P_F} = \mu_E$$

which implies that the density $\Phi_F$ satisfies $\Phi_E \circ \gamma^*_E = P_E\Phi_F P_E^*$ (cf. [Ne98a, Lemma I.18]). We further conclude that

$$\Phi_E \circ \gamma^*_E = \Phi_E \circ \gamma^*_E \circ \gamma^*_F = P_E(\Phi_F \circ \gamma^*_F)P_E^*.$$

This shows that the function $\Phi := \Phi_E \circ \gamma^*_E: U^* \to \text{Herm}(V_E)$ is measurable with

$$(\gamma^*_F)^*(\Phi \nu_k) = (P_E\Phi_F P_E^*)\nu_{F,k} = \mu_{F,P_F}$$

whenever $U_E \subseteq U_F$. This proves that $\Phi \nu_k = \mu_{P_F}$, so that $\mu_{P_F}$ has a density for each $E$. This completes the proof of the lemma. $
$

Theorem VI.2. If $\mu$ is the unique measure on $U^*$ with $L(\mu)(x) = \rho(1 + x)$ for $x \in \Omega$, and $k \in \mathbb{N}_0$ as in Theorem III.4, then the set $\Omega^*_k$ of elements of rank $\leq k$ in the cone $\Omega^* \subseteq U^*$ is thick with respect to the measure $\mu$. If, in addition, $J$ is countable, then $\Omega^*_k$ is a measurable subset of $U^*$. 

Proof. If $E \subseteq U^*$ is a measurable subset not intersecting the set $\Omega^*_k$, then Lemma VI.1 shows that for each positive trace class operators $S$ on $V$ we have $\mu^S(E) = \text{tr}(\mu(E)S) = 0$ because the set $\Omega^*_k$ contains the range of the map $Q$ (cf. Example II.5), hence is thick with respect to the measure $\nu_k = Q^*\gamma_V$. We conclude that $\mu(E) = 0$, and this means that $\Omega^*_k$ is thick with respect to $\mu$.

If, in addition, $J$ is countable, then the weak-$*$-topology turns $U^*$ into a metrizable separable topological vector spaces. If from that it follows easily that all closed subsets of $U^*$ are measurable and hence that $\Omega^*_k$ is measurable for each $k \in \mathbb{N}_0$. $
$

Remark VI.3. (a) Theorem VI.2 implies in particular that if $\mathcal{S}$ denotes the $\sigma$-algebra on $\Omega^*_k$ that we obtain by intersecting measurable subsets of $U^*$ with $\Omega^*_k$, then we obtain a measure $\tilde{\mu}$ on $\mathcal{S}$ by $\tilde{\mu}(E \cap \Omega_k) := \mu(E)$. This is compatible with the observation that one could also apply Theorem I.3 directly to the projective system of the domains $\Omega^*_{k,F}$ corresponding to the finite-dimensional Jordan algebras $\text{Herm}(F, \mathbb{K})$, $F \subseteq J$ finite.
(b) Let \( r: U_1^* \to U^* \) denote the restriction map. Then \( r^* \mu_1 = \mu \) and therefore each measurable subset \( E \subseteq U^* \) with \( E \cap \Omega_k^* = \emptyset \) satisfies
\[
\mu_1(r^{-1}(E)) = \mu(E) = 0.
\]
But this does not necessarily imply that \( r^{-1}(\Omega_k^*) \) is thick in \( U_1^* \) with respect to \( \mu_1 \) because if \( E \subseteq U_1^* \) is measurable with \( E \cap r^{-1}(\Omega_k^*) = \emptyset \), then \( r(E) \) need not be measurable in \( U_1^* \).

(c) One could also try to see that the measure \( \nu_k^1 \) can also be realized on the cone of positive functionals in \( U_1^* \) which can be identified in a canonical way with the cone of all bounded positive operators on \( H \) (cf. [Ne98a]).

If we consider the Jordan algebra \( U_1 \) as the direct limit of the set of all those finite-dimensional subspaces \( U_F, F \subseteq J \) with \( U_F = U_P \cap U_1^* - U_F \cap U_1^* \), then the algebraic dual \( U_1^* \) is the projective limit of the dual spaces \( U_F^* \) and likewise the dual cone \( (U_1^*)^* \) can be viewed as the projective limit of the finite-dimensional cones \( (U_F^*)^* \), where \( U_F^* := U_F \cap U_1^* \). Unfortunately the corresponding restriction maps \( (U_F^*)^* \to (U_1^*)^* \) are not always surjective if \( U_F \subseteq U_1 \), so that the assumption (2) of Theorem I.3 is not satisfied in an obvious fashion. ■

**Example VI.4.** We have seen above that for each admissible representation \( \rho \) of \( L \) there exists a \( k \in \mathbb{N}_0 \) such that the set \( \Omega_k^* \) is thick for the corresponding operator-valued measure \( \mu \) on \( U^* \). Moreover, for each positive trace class operator \( S \in \text{Herm}_1^+(V) \) the positive measure \( \mu^S \) on \( U_1 \) has a density with respect to the measure \( \nu_k^1 \). To see how one can make this weak type of a density more explicit, we consider the particular case where \( K = \mathbb{R} \) and
\[
\rho(g) = \det(g)^{-\frac{1}{2}}g^{-T}
\]
which corresponds to the odd part of the metaplectic representation of the two-fold covering of \( \text{Sp}(J, \mathbb{R}) \). On \( \mathfrak{f} = \mathfrak{gl}(J, \mathbb{C}) \) the highest weight is given by
\[
\lambda = \frac{1}{2}\text{tr} - \varepsilon_{j_0},
\]
where \( j_0 \in J \) is maximal with respect to the ordering defining the positive system \( \Delta_1^+ \). This shows that \( M = -\frac{1}{2} \) and therefore that \( k = 1 \) (cf. Theorem III.4). The decomposition \( \rho = \det^{-\frac{1}{2}}\rho_0 \) is given by \( \rho_0(g) = g^{-T} \), and for the measure \( \nu_1^1 \) corresponding to \( \det^{-\frac{1}{2}} \) we have
\[
\mathcal{L}(\nu_1^1)(x) = \det(1 + x)^{-\frac{1}{2}}, \quad 1 + x \in \Omega_1.
\]
The determinant function \( \det: GL_1(H) \to \mathbb{C}^\times \) is a homomorphism of Lie groups, so that its differential is given by
\[
d\det(g)(g.X) = \det(g)d\det(1)(X) = \det g \cdot \text{tr} X.
\]
Thus
\[
d\mathcal{L}(\nu_1^1)(x)(S) = \frac{1}{2}\det(1 + x)^{-\frac{1}{2}}\text{tr}((1 + x)^{-1}S).
\]
This means that the continuous linear functional \( d\mathcal{L}(\nu_1^1)(x) \) on \( U_1 \) is represented by the bounded hermitian operator
\[
-\frac{1}{2}(1 + x)^{-\frac{1}{2}}(1 + x)^{-1} = -\frac{1}{2}\rho(1 + x).
\]
We conclude that
\[
\rho(1 + x) = -2d\mathcal{L}(\nu_1^1)(x)
\]
and therefore that for each \( S \in \text{Herm}_1^+(V) \) we have
\[
\mathcal{L}(\mu^S)(x) = \text{tr}(\rho(1 + x)S) = -2d\mathcal{L}(\nu_1^1)(x)(S) = 2\int_{U_1^*} \text{tr}(aS)e^{-a(x)} \, d\nu_1^1(a).
\]
This means that
\[
d\mu(a) = 2a \cdot d\nu_1^1(a)
\]
holds in the weak sense that for each \( S \in \text{Herm}_1^+(V) \) we have \( d\mu^S(a) = 2a(S) \cdot d\nu_1^1(a) \). ■

Similar methods can be used to make other operator-valued measures \( \mu \) more concrete in the same sense. If \( k \) is larger than 1, then one has to use higher derivatives.
References


