On the Non-Existence of Local Smoothing for Oscillatory Integrals with Concave Phase

Björn Gabriel Walther

Vienna, Preprint ESI 746 (1999)  
September 3, 1999

Supported by Federal Ministry of Science and Transport, Austria  
Available via http://www.esi.ac.at
ON THE NON-EXISTENCE OF LOCAL SMOOTHING
FOR OSCILLATORY INTEGRALS WITH CONCAVE PHASE

BJÖRN GABRIEL WALther

ABSTRACT. In this note we prove the non-existence of local smoothing in the sense of \( L^2 \)
for oscillatory integrals with concave phase. We use integration by parts to control remainder terms
and use the oscillation at infinity of Bessel functions to reduce the higher dimensional problem
to the one-dimensional.

1. Introduction and Notation

1.1. Definition. For \( x \) and \( \xi \) in \( \mathbb{R}^n \) we let \( x \xi = x_1 \xi_1 + \ldots + x_n \xi_n \). If \( f \) is in the Schwartz class \( S(\mathbb{R}^n) \)
we define
\[
(S^a f)[x](t) = \int_{\mathbb{R}^n} e^{i(x \xi + t |\xi|^a)} \hat{f}(\xi) \, d\xi.
\]
Here \( \hat{f} \) is the Fourier transform of \( f \),
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \xi} f(x) \, dx.
\]
The Fourier transformation extends to an isomorphism on \( S'(\mathbb{R}^n) \), the space of tempered distributions on \( \mathbb{R}^n \).

1.2. Purpose of this note. For \( a = 2 \) the expression \( (S^a f)[x](t) \) is a solution to the time-dependent
Schrödinger equation \( (\Delta_x - i \partial_t) u = 0 \). Smoothing properties for solutions to this and related equations
have been studied e.g. in Ben-Artzi, Klainerman [1], Craig, Kappeler, Strauss [2], Kato, Yajima [3],
Sjölin [5] and [11] where it is has been shown that one can gain \( 1/2 \) of a derivative in the sense
of \( L^2 \) by integrating with respect to \( x \) and \( t \) controlling the contribution given by the low frequencies
of \( f \). For \( a > 1 \) the gain of derivatives is at least \((a - 1)/2\). See e.g. Sjölin [5] and [10].
Similar results using homogeneous Sobolev norms can be found e.g. in Kenig, Ponce, Vega [4].
Interesting discussions on (local) smoothing can be found in Sogge [7] and in the references cited there.

In Sjögren, Sjölin [6] it was shown that one can gain at most 1/2 derivatives in the case \( a = 2 \).
In [12] the case \( a > 1 \) for this sharpness problem is treated and the result there is that one can gain
at most \((a - 1)/2\) derivatives.

For \( a = 1 \) the expression \( (S^a f)[x](t) \) is a solution to the classical wave-equation. For such solutions
it is known (see e.g. §5.1 below) that there is no smoothing in the \( L^2 \)-sense. The purpose of this note
is to show that this non-existence of smoothing holds also in the case \( 0 < a < 1 \). The counterexamples
we provide are radial. As in [12] we analyze the high frequencies of \( f \) in the case \( n = 1 \) and transfer
the result to higher dimensions using the oscillation at infinity of one single Bessel function.
1.3. **Notation.** We introduce fractional Sobolev spaces

\[ H^s \left( \mathbb{R}^n \right) = \left\{ f \in \mathcal{S}' \left( \mathbb{R}^n \right) : \|f\|_{H^s \left( \mathbb{R}^n \right)}^2 = \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^s |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}. \]

By \( B^n \) we denote the open unit ball in \( \mathbb{R}^n \). \( B^1 \) will be denoted by \( B \).

Numbers denoted by \( C \) (sometimes endowed with subscripts) may be different at each occurrence even within the same chain of equalities or inequalities. In \( \S\S 3.4 \) and 4.2.3 the notation \( C_{m,n} \) will be used where \( m \) and \( n \) are positive integers. In this notation \( m,n \) refers to inequality \((m,n)\) from which the existence of \( C_{m,n} \) follows.

2. **The Theorem**

2.1. **Theorem.** Let \( 0 < a \leq 1 \). Assume that there exists a number \( C \) independent of \( f \) such that

\[ \|S^a f\|_{L^2(B^{s+a})} \leq C \|f\|_{H^s \left( \mathbb{R}^n \right)} . \]  

Then \( s > 0 \).

2.2. **Remark.** Although this result is already known in the case \( a = 1 \) it is included here since our method for \( 0 < a < 1 \) can be extended without any change of the argument to the case \( a = 1 \).

3. **Some preparation**

3.1. **Definition.** For real numbers \( \lambda > -1/2 \) we define the Bessel function of order \( \lambda \) by

\[ J_\lambda (r) = \frac{r^\lambda}{2^\lambda \Gamma(\lambda + 1/2) \Gamma(1/2)} \int_B e^{irp} (1 - \rho^2)^{\lambda-1/2} \, d\rho. \]

3.2. **Theorem.** (Cf. Stein, Weiss [8, theorem 3.10 p. 158].) Let \( f(\xi) = f_0(|\xi|) |\xi|^{-(n/2)+1/2} \). Then

\[ \hat{f}(x) = (2\pi)^{n/2} |x|^{-n/2+1} \int_0^\infty f_0(\rho) J_{n/2-1}(\rho |x|) \rho^{1/2} \, d\rho. \]

3.3. **Theorem** (Asymptotics of the Bessel function. ([8, lemma 3.11 p. 158].)) There is a number \( C \) independent of \( \rho \) such that

\[ \left| J_{n/2-1}(\rho) - \left( \frac{2}{\pi} \right)^{1/2} \rho^{-1/2} \cos \left[ \rho - \frac{(n-1)\pi}{4} \right] \right| \leq C \rho^{-3/2}, \quad \rho \geq 1. \]

4. **Proof of the theorem**

4.1. **Proof of 2.1 in the case \( n = 1 \)**

4.1.1. Let \( I_N = N + N^{1/2}[0,3] \) where \( N \geq 1 \) and let \( I_N' \) be the middle third of \( I_N \). Then \( I_N \subseteq [N,4N] \). Choose a function \( g_N \in \mathcal{C}_0^\infty (\mathbb{R}) \) such that

\[ g_N(\mathbb{R} \setminus I_N) = 0, \quad g_N(\mathbb{R} \subseteq [0,1] \quad \text{and} \quad g_N(I_N') = 1 \]

and such that

\[ \|g_N\|_{L^\infty (\mathbb{R})} \leq CN^{-1/2} \]

where \( C \) is independent of \( N \). Set \( \Phi_\circ (\xi, x, t) = x_\xi + t \xi^2, \xi > 0 \). Then

\[ |\Phi_\circ (\xi, x, t)| \geq \frac{|x|}{2} \text{ and } |\Phi_\circ'' (\xi, x, t)| \leq 4a(1-a)N^{n-2} \]

assuming that \( x \notin 2B, t \in B \) and \( \xi \in I_N \).
4.1.2. Main term estimate. If \( \hat{f}_N(\xi) = g_N(\xi) \), then there is a number \( C > 0 \) independent of \( N \) such that
\[
\frac{1}{C} N^{2s+1/2} \leq \|f_N\|_{H^s(\mathbb{R})}^2 \leq C N^{2s+1/2}.
\]
Parseval’s formula gives
\[
\| (S^a f_N)(t) \|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \left| e^{iHr} \hat{f}_N(\xi) \right|^2 d\xi = \frac{1}{2\pi} \|f_N\|_{H^s(\mathbb{R})}^2.
\]
Integrating with respect to \( t \in B \) gives together with (4.3) that
\[
\|S^a f_N\|_{L^2(\mathbb{R}\times B)}^2 \geq C N^{1/2}, \quad C > 0
\]
for some number \( C \) independent of \( N \). This is the main term estimate.

4.1.3. Remainder estimate. We shall now proceed by proving that there is a number \( C \) independent of \( N \) such that
\[
\|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)} \leq C
\]
using integration by parts. We have
\[
\|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)} \leq \int_{\mathbb{R}} \left\| e^{iHr} \hat{f}_N(\xi, x, t) \right\| d\xi \leq \|g_N\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{d\xi}{\Phi(\xi, x, t)} + \int_{\mathbb{R}} \frac{\Phi(\xi, x, t)}{\Phi(\xi, x, t)} d\xi.
\]
Taking the \( L^2(\mathbb{R}\setminus2B) \)-norm for fixed \( t \) gives
\[
\|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)} \leq C \left( N^{-1/2} N^{1/2} + N^{1/2} N^{1/2} \right) \leq C
\]
where we have used the estimates in (4.1) and (4.2). The numbers \( C \) are independent of \( N \). Hence
\[
\|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)} \leq C
\]
where \( C \) is independent of \( N \).

4.1.4. Conclusion. Now suppose that
\[
\|S^a f\|_{L^2(B^2)} \leq C \|f\|_{H^s(\mathbb{R})}
\]
where \( C \) is independent of \( f \). By rescaling it follows that there is another number \( C \) independent of \( f \) such that
\[
\|S^a f\|_{L^2(2B^2)} \leq C \|f\|_{H^s(\mathbb{R})}.
\]
By (4.3)-(4.7) there are numbers \( C_{4.3}, C_{4.4}, C_{4.5} \) and \( C_{4.7} \) independent of \( N \) such that
\[
0 < C_{4.4} N^{1/2} \leq \|S^a f_N\|_{L^2(\mathbb{R}\times B)}^2 = \|S^a f_N\|_{L^2(2B^2)}^2 + \|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)}^2 \leq C_{4.7} \|f_N\|_{H^s(\mathbb{R})}^2 + \|S^a f_N\|_{L^2((\mathbb{R}\setminus2B)\times B)}^2 \leq C_{4.7} C_{4.3} N^{2s+1/2} + C_{4.5}.
\]
It follows that
\[
0 < C_{4.4} \leq C_{4.7} C_{4.3} N^{2s} + C_{4.5} N^{-1/2}.
\]
Since this inequality shall hold for all \( N \) we have to choose \( s \geq 0 \).

4.2. Proof of 2.1 in the case \( n > 1 \).
4.2.1. Main term estimate. If \( \hat{f}_N(\xi) = g_N(|\xi|)|\xi|^{-n/2+1/2} \), then there is a number \( C > 0 \) independent of \( N \) such that
\[
\frac{1}{C} N^{2s+1/2} \leq \|f_N\|_{H^s(\mathbb{R}^n)}^2 \leq C N^{2s+1/2},
\]
where \( g_N \) is defined as in \( \S 4.1.1 \).

Parseval’s formula gives
\[
\| (S^a f_N)(t) \|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| e^{it\xi} \hat{f}_N(\xi) \right|^2 d\xi = \frac{1}{(2\pi)^n} \|f_N\|_{H^s(\mathbb{R}^n)}^2.
\]

Integrating with respect to \( t \in B \) gives together with (4.8) that
\[
\| S^a f_N \|_{L^2(\mathbb{R}^n \times B)}^2 \geq C N^{1/2}, \quad C > 0
\]
for some number \( C \) independent of \( N \). This is the main term estimate.

4.2.2. Remainder estimate. It follows from the definition of Bessel functions (\( \S 3.1 \)) and from theorem 3.2 and 3.3 that
\[
(S^a f_N)[x](t) = (2\pi)^{n/2} |x|^{-n/2+1/2} \left\{ \left( \overline{S^a g_N} \right) [x](t) + (K g_N)[|x|](t) \right\}
\]
where
\[
\left( \overline{S^a g_N} \right)[r](t) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty \cos \left( r \rho - \frac{(n-1)\pi}{4} \right) e^{it\rho} g_N(\rho) d\rho
\]
and
\[
\| (K g_N)[r] \|_{L^2(B)} \leq C \int_0^\infty \frac{|g_N(\rho)|}{\sqrt{1+r\rho}} d\rho
\]
with \( C \) independent of \( N \). Also
\[
\left( \overline{S^a g_N} \right)[r] = \frac{1}{\sqrt{2\pi}} \left\{ e^{-i(n-1)\pi/4} \left( S^a \overline{g_N} \right)[r] + e^{i(n-1)\pi/4} \left( S^a \overline{g_N} \right)[-r] \right\}
\]
where \( \overline{g} \) is the inverse Fourier transform of \( g \). We have
\[
\| K g_N \|_{L^2(\mathbb{R}^\infty \times B)}^2 \leq \int_2^\infty \left( \int_0^\infty \frac{|g_N(\rho)|}{\sqrt{1+r\rho}} d\rho \right)^2 dr \leq \int_0^\infty \frac{N dr}{(1+Nr)^2} dr = 1.
\]

Using (4.5) and (4.10) the remainder is now estimated as follows:
\[
\| S^a f_N \|_{L^2(\mathbb{R}^\infty \times 2B^* \times B)} \leq \| \overline{S^a g_N} \|_{L^2(2\mathbb{R}^\infty \times B)} + \| K g_N \|_{L^2(2\mathbb{R}^\infty \times B)} \leq C \| \overline{S^a g_N} \|_{L^2(\mathbb{R}^\infty \times B)} + \| K g_N \|_{L^2(\mathbb{R}^\infty \times B)} \leq C.
\]

Here \( C \) is independent of \( N \).

4.2.3. Conclusion. Now suppose that
\[
\| S^a f \|_{L^2(\mathbb{R}^\infty \times B^*+)} \leq C \| f \|_{H^s(\mathbb{R}^n)}
\]
where \( C \) is independent of \( f \). By rescaling it follows that there is another number \( C \) independent of \( f \) such that
\[
\| S^a f \|_{L^2(2\mathbb{R}^\infty \times B)} \leq C \| f \|_{H^s(\mathbb{R}^n)}.
\]
0 < C_{4.9} N^{1/2} \leq \|S^0 f_N\|_{L^2(B^* \times B)}^2 = \|S^0 f_N\|_{L^2(B^* \times B)}^2 + \|S^0 f_N\|_{L^2(B^* \times B)}^2 \leq \|S^0 f_N\|_{H^s(R^n)}^2 + \|S^0 f_N\|_{L^2(B^* \times B)}^2 \leq C_{4.12} N^{2s-1/2} + C_{4.11}.

It follows that

0 < C_{4.9} \leq C_{4.12} C_{4.8} N^{2s} + C_{4.11} N^{-1/2}.

Since this inequality shall hold for all $N$ we have to choose $s \geq 0$.

5. Concluding remarks

5.1. To the following theorem (see e.g. [10, theorem 14.2 p. 216] and the references cited in that paper) a standard argument\(^{1}\) (see e.g. [9, §2.3 p. 487-488]) can be applied to derive the necessary condition on $s$ for the inequality (2.1) with $a = 1$ to hold with a number $C$ independent of $f$.

5.2. Theorem. Assume that there exists a number $C$ independent of $f$ such that the inequality

$$\|S^1 f\|_{L^1(B^*, L^\infty(B))} \leq C \|f\|_{H^s(R^n)}.$$  

Then $s > 1/2$.

5.3. If the standard argument is applied to [9, theorem 1.2 (b) p. 486] and to [13, theorem 4.1 chapter 4 p. 27], i.e. to the case $0 < a < 1$ we get the following proposition:

5.4. Proposition. Let $0 < a < 1$. Assume that there exists a number $C$ independent of $f$ such that

$$\|S^0 f\|_{L^2(B^* \times B)} \leq C \|f\|_{H^s(R^n)}.$$  

Then $s \geq -a/4$.

5.5. To derive the necessary condition in theorem 2.1 for the case $0 < a < 1$, $n = 1$, using the standard argument would require that $f \in H^s(R)$, $s > a/2$ is a necessary condition for the maximal function of $S^0 f$ to be in $L^2_{loc}(R)$. However, it is known [9, theorem 1.2 (a) p. 486] that $f \in H^{s+1/2}(R)$, $s > 0$ is sufficient for the maximal function of $S^0 f$ to be in $L^2_{loc}(R)$. In conclusion, using the standard argument to derive necessary conditions for local smoothing is quite imprecise unless $a$ is close to 0.

References


\(^{1}\) This argument is referred to here as the standard argument.


Royal Institute of Technology, SE - 100 44 STOCKHOLM, SWEDEN

Current address: Brown University, Providence, RI 02912-1917, USA

E-mail address: WALTHER@math.kth.se