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Boltzmanngasse 9
A-1090 Wien, Austria

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Vienna, Preprint ESI 744 (1999) August 30, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

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Abstract. We review the localization formula due to Berline-Vergne and Atiyah-Bott, with applications to the exact stationary phase phenomenon discovered by Duistermaat-Heckman. We explain the Weil model of equivariant cohomology and recall its relation to BRST. We show how to quantize the Weil model, and obtain new localization formulas which, in particular, apply to Hamiltonian spaces with group valued moment maps.

1 Introduction

The purpose of these lecture notes is to present the *localization formulas* for equivariant cocycles. The localization phenomenon was first discovered by Duistermaat and Heckman in [DH], and then explained in the works of Berline-Vergne [BV] and Atiyah-Bott [AB]. The main idea of the localization formulas is similar to the residue formula: a multi-dimensional integral is evaluated exactly by summing up a number of the *fixed point contributions*.

In Section 2 we review the localization formula of [BV] and [AB]. We use an elementary example of the sphere S^2 as an illustration. Then, we outline the relation between the localization formulas and Hamiltonian Mechanics, and recover the Duistermaat-Heckman formula [DH].

In Section 3 we discuss the relations between the localization formulas and the group actions. In the case of the Duistermaat-Heckman formula, localization is intimately related to the symmetry group of the underlying Hamiltonian system. In particular, we compare the equivariant differential to the BRST differential.

In Section 4 we explain how to *quantize* the equivariant cohomology. This Section is based on the papers [AMM], [AM], [AMW1] and [AMW2]. We end up by presenting the new localization formula which is derived in [AMW2]. Some simple applications of this new formula can be found in [P]. Section 4 is based on the joint works with A.Malkin, E.Meinrenken and C.Woodward.

These notes do not touch upon various applications of localization formulas in Physics. Usually, one proceeds by extrapolating the localization phenomenon to path integrals. Some of the most exciting examples of this approach can be found in [MNP], [W2], [G], [BT], [MNS]. In fact, [W2] was the original motivation for the formulas of Section 3.

I am grateful to the organizers and participants of the 38th Schladming Winter School for the inspiring atmosphere!

2 Localization Formulas

In this Section we review the *localization formula* due to Berline-Vergne [BV] and Atiyah-Bott [AB]. It is then used to derive the *exact stationary phase* formula due to Duistermaat and Heckman [DH]. The presentation is illustrated at the elementary example of sphere S^2.

2.1 Stationary phase method

In this section we recall the stationary phase method. It applies when one is interested in the asymptotic behavior at large s of the integral

$$I(s) = \int_{-\infty}^{\infty} \mathrm{d}x e^{isf(x)} g(x). \tag{1}$$

Here we assume that functions $f(x)$ and $g(x)$ are real, and sufficiently smooth.

At large $s > 0$ the leading contribution into the integral (1) is given by the neighborhood of the critical points of $f(x)$, where its derivative in x vanishes. Let x_0 be such a critical point. Then, one can approximate $f(x)$ near x_0 by the first two terms of the Taylor series,

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots,$$

where ... stand for the higher order terms.

The leading contribution of the critical point x_0 into the integral $I(s)$ is given by a simpler integral

$$I_0(s) = g(x_0) e^{isf(x_0)} \int_{-\infty}^{\infty} \mathrm{d}x e^{\frac{i}{2} s f''(x_0)(x-x_0)^2}.$$

This integral is Gaussian, and can be computed explicitly,

$$I_0(s) = g(x_0) e^{i(sf(x_0) + \varepsilon \frac{\pi}{4})} \left(\frac{2\pi}{s|f''(x_0)|} \right)^{\frac{1}{2}}$$

Here ε is the sign on the second derivative $f''(x_0)$.

A similar formula holds for multi-dimensional integrals,

$$I(s) = \int \mathrm{d}^n x g(x) e^{isf(x)}. \tag{2}$$

Again, the leading contrinution into the asymptotics at large s is given by the critical points of $f(x)$, where its gradient vanishes $\nabla f = 0$. At the critical point x_0 one can expand $f(x)$ into the Taylor series,

$$f(x) = f(x_0) + \frac{1}{2} \sum_{i,j=1}^{n} f''_{ij}(x_0)(x - x_0)_i (x - x_0)_j + \ldots, \tag{3}$$

where

$$f''_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

We assume that the critical point is non-degenerate, that is, the matrix f''_{ij} is invertible. Then, the leading contribution of x_0 into the integral $I(s)$ is of the form,

$$I_0(s) = g(x_0) e^{isf(x_0)} \left(\frac{2\pi}{s}\right)^{\frac{n}{2}} \frac{e^{i\sigma\frac{\pi}{4}}}{|\det(f''(x_0))|^{\frac{1}{2}}}. \tag{4}$$

Here $\sigma = \sigma_+ - \sigma_-$ is the signature of the matrix f''_{ij}, σ_+ and σ_- are numbers of positive and negative eigenvalues, respectively.

In general, one can have several critical points. Then, on can add the leading contributions (4) to obtain the approximate answer for $I(s)$,

$$I(s) \approx \left(\frac{2\pi}{s}\right)^{\frac{n}{2}} \sum_i g(x_i) e^{isf(x_i)} \frac{e^{i\sigma_i\frac{\pi}{4}}}{|\det(f''(x_i))|^{\frac{1}{2}}} \tag{5}$$

Of course, there is no reason for the right hand side to be the exact answer for $I(s)$. But sometimes this is the case! Such a situation is called *exact stationary phase*, and will be studied in these notes.

Example: sphere S^2 The simplest example of the exact stationary phase phenomenon is the computation of the following integral. Consider the unit sphere S^2 defined by equation $x^2 + y^2 + z^2 = 1$. We choose $g(x, y, z) = 1$ and $f(x, y, z) = z$. Then, the integral $I(s)$ is of the form,

$$I(s) = \int_{S^2} \mathrm{d}A\; e^{isz}, \tag{6}$$

where $\mathrm{d}A$ is the area element normalized in the standard way, $\int_{S^2} \mathrm{d}A = 4\pi$.

The critical points of the function $f(x, y, z) = z$ are the North and the South poles of the sphere. At both points one can use x and y as local coordinates to obtain,

$$z \approx 1 - \frac{1}{2}(x^2 + y^2)$$

near the North pole, and

$$z \approx -1 + \frac{1}{2}(x^2 + y^2)$$

near the South pole. Thus, for the stationary phase approximation one obtains,

$$I(s) \approx \frac{2\pi}{s}\,(-ie^{is} + ie^{-is}) = 4\pi\,\frac{\sin(s)}{s}. \tag{7}$$

Here we have used that at both North and South poles $\det(f_{ij}) = 1$, and that $\sigma_N = -2$ and $\sigma_S = 2$.

Now we can compare the 'approximate' result (7) with the exact calculation. It is convenient to use polar angles $0 < \theta < \pi, 0 < \phi < 2\pi$. Then, the coordinate function $z = \cos(\theta)$, and the area element is $\mathrm{d}A = \mathrm{d}\cos(\theta)\mathrm{d}\phi$. The simple calcuation gives,

$$I(s) = \int \mathrm{d}\cos(\theta)\mathrm{d}\phi e^{is\cos(\theta)} = 2\pi\, \frac{e^{is} - e^{-is}}{is} = 4\pi\, \frac{\sin(s)}{s}. \tag{8}$$

This expression coincides with the stationary phase result (7).

In the following sections we shall see that the equality of the exact and approximate results (7) amd (8) is not a coincidence. In fact, this is the symplect example of *equivariant localization*.

2.2 Equivariant cohomology

Stokes's theorem and residue formula The main tool in proving the localization formula will be the generalization of the Stokes's integration formula. The latter states that given an exact differential form α, $\alpha = \mathrm{d}\beta$, its integral over the domain D can be expressed as an integral of β over the boundary of D,

$$\int_D \mathrm{d}\beta = \int_{\partial D} \beta. \tag{9}$$

As a warm up excercise we prove the standard residue formula using the Stokes's formula (9). Given a function $f(z)$ analytic on the complex plane with the exception of some finite number of poles, its integral over a closed contour C is given by the sum of residues at the poles inside C,

$$\frac{1}{2\pi i} \int_C f(z)\mathrm{d}z = \sum_i \mathrm{res_{z_i} f}. \tag{10}$$

We naturally choose β in the form,

$$\beta = \frac{1}{2\pi i}\, f(z)\mathrm{d}z,$$

the domain D is the interior of C, and its boundary is C. The form α is given by equation,

$$\alpha = \mathrm{d}\beta = \frac{1}{2\pi i}\mathrm{d}(f(z)\mathrm{d}z) = \frac{1}{2\pi i}(\bar{\partial} f)\mathrm{d}\bar{z}\wedge \mathrm{d}z,$$

where $\bar{\partial} f$ is the partical derivative in $\bar{z}$. The function $f(z)$ is analytic. Hence, α vanishes everywhere except for the poles. We conclude, that α is a distribution supported at some number of points. Such a distribution is a sum of δ-functions and its derivatives. The only terms which contribute into the integral of α over D are δ-functions at the poles. They give rise to the residues,

$$\bar{\partial}\, \frac{\mathrm{res_{z_i} f}}{2\pi i(z - z_i)} = (\mathrm{res_{z_i} f})\delta(\mathrm{z} - \mathrm{z_i}).$$

Now we use the Stokes's formula,

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z = \int_D \sum_i (\mathrm{res_{z_i} f})\delta(\mathrm{z}-\mathrm{z_i}) = \sum_{\mathrm{i}} \mathrm{res_{z_i} f},$$

and recover the residue formula.

S^1-action and fixed points The derivation of the localization formula requires more structure on the integration domain. In particular, the notion of symmetry plays an important role. We assume that our symmetry is continuous. In particular, this may be the action of the circle group S^1, which is our main example in this Section. For instance, in the case of S^2 there is an action of S^1 by rotations around the z-axis.

We always choose the integration domain to be a compact manifold M without boundary (as in the case of S^2). The S^1-action defines a vector field $v = \partial/\partial\phi$ on M. Zeroes of v correspond to fixed points of the circle action. For simplicity, we assume that all the fixed points are isolated. This is only possible if the dimension of the manifold is even, $n = 2m$. Given such a fixed point x_0 one can write the action near this point as

$$x^i(\phi) = x_0^i + R^i_j(\phi)(x-x_0)^j + \ldots,$$

where $\ldots$ stand for higher order terms in $x - x_0$. It is easy to see that one can lenearize the action (drop higher order terms). The matrix R gives a linear representation of S^1,

$$R(\phi_1)R(\phi_2) = R(\phi_1+\phi_2),$$

and satisfies condition $R(2\pi) = \mathrm{id}$. By the appropriate choice of the basis such a matrix can always be represented as a direct sum of 2×2 blocks, each block of the form

$$\begin{pmatrix} \cos(\nu\phi) & \sin(\nu\phi) \\ -\sin(\nu\phi) & \cos(\nu\phi) \end{pmatrix},$$

where ν is an integer. We can denote the corresponding local coordinates by x_i, y_i where $i = 1\ldots m$. In these local coordinates the vector field v has the form,

$$v = \sum_{i=1}^m \nu_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right).$$

The integers λ_i are called indices of the vector field v at the point x_0.

In fact the indices ν_i are defined up to a sign: the flip of coordinates x_i and y_i changes the sign of the corresponding index. In what follows we shall need a product of all indices corresponding to the given fixed point, $\nu_1 \ldots \nu_m$. It is well defined is the tangent space at the fixed point is oriented: one should choose the coordinate system $(x_1, y_1, \ldots, x_m, y_m)$ with positive orinetation. This condition determines the product of indices in a unique way. In particular, if the manifold is oriented, the products of indices at fixed points are well defined.

In the following we shall use that one can always choose an S^1-invariant metric on M. Indeed, given any metric, one can always average it over the S^1-action. In particular, this metric at the fixed point x_0 can always be chosen in the form,

$$g=\sum_{i=1}^{m}(\mathrm{d}x_i^2+\mathrm{d}y_i^2). \tag{11}$$

Equivariant differential Now we are ready to define the equivariant differential, and equivariant cohomology. We define the space of *equivariant forms* on M as S^1 invariant differential fomrs with values in functions of one variable, which we denote by ξ. Typically, such a differential form is a polynomial,

$$\alpha(\xi)=\sum_{s=0}^{N}\alpha^s\xi^s,$$

where α^s are S^1-invariant differential forms. We shall also need equivariant forms with more complicated ξ-dependence.

Sometimes it is convenient to decompose equivariant forms according to the degree,

$$\alpha(\xi)=\sum_{j=0}^{n}\alpha_j(\xi),$$

where $\alpha_j(\xi)$ is a form of degree j which takes values in functions of ξ.

The differential on the space of equivariant forms is defined by formula,

$$\mathrm{d}_{S^1}=\mathrm{d}+i\xi\iota_v, \tag{12}$$

where ι_v is the contraction with respect to the vector field v. One can assign to parameter ξ degree 2 in order to make the equivariant differential homogeneous. Unfortunately, this arrangement is only meaningful for equivariant differential forms polynomial in ξ.

It is the basic property of the differential (12) that it squares to zero on the space of equivariant forms. Indeed,

$$\mathrm{d}_{S^1}^2=(\mathrm{d}+i\xi\iota_v)^2=i\xi(\mathrm{d}\iota_v+\iota_v\mathrm{d})=i\xi L_v,$$

where we have used Cartan's formula for L_v. The Lie derivative L_v vanishes on equivariant forms, and so does $\mathrm{d}_{S^1}^2$.

Using the differential (12) one can define equivariantly closed forms, $\mathrm{d}_{S^1}\alpha=0$, and equivariantly exact forms $\alpha=\mathrm{d}_{S^1}\beta$. Because $\mathrm{d}_{S^1}^2=0$, equivariantly exct forms are automatically equivariantly closed, and one can define equivariant cohomology $H_{S^1}(M)$ as the quotient of the space of (equivariantly) closed forms by the space of (equivariantly) exact forms. If $\alpha(\xi)$ is an equivariant cocycle, it satisfies the closedness condition,

$$(\mathrm{d}+i\xi\iota_v)\alpha(\xi)=0,$$

which implies a number of equations for the forms $\alpha_k(\xi)$,

$$d\alpha_{k-2}(\xi) + i\xi\iota_v\alpha_k(\xi) = 0. \tag{13}$$

Note that this recurrence relation has step 2. Hence, odd and even degree parts of an equivarient cocycle are also equivariant cocycles. If the manifold M is even dimensional, the closedness condition relates the top degree component $\alpha_n(\xi)$ and the function $\alpha_0(\xi)$. We shall see that exactly this relation is used in the localization formula.

The Stokes' integration formula generalizes to equivariantly exact forms. Indeed,

$$\int_D (d + i\xi\iota_v)\beta = \int_{\partial D} \beta,$$

where $\int_D \iota_v\beta = 0$ because the integrand has a vanishing top degree component (at least one degree is eaten up by ι_v). In particular, if the integration domain has no boundary, the integral of any equivariantly exact form vanishes, and the integration map descends to equivariant cohomology. That is, given a class $[\alpha] \in H_{S^1}(M)$ one can choose any representative α in integrate it over M. The result is a function of ξ which is independent of the representative: the representatives differ by an exact form, and the integral of an exact form vanishes.

The localization formula is a tool of computing integral of equivariant cocycles in terms of fixed points. This formula was discovered by Belrine-Vergne [BV] and by Atiyah-Bott [AB]. For an equivariant cocycle $\alpha(\xi)$, its integral over M is given by,

$$\int_M \alpha(\xi) = \left(\frac{2\pi}{i\xi}\right)^{\frac{n}{2}} \sum_p \frac{(\alpha_0(\xi))(x_p)}{\nu_1^p \dots \nu_m^p}, \tag{14}$$

where the index p labels fixed points of the circle action (we assume that all of them are isolated), and $\nu_1^p, \dots, \nu_m^p$ are indices of the p's fixed point.

Note that the integral on the left hand side of (14) depends only on the top degree component $\alpha_n(\xi)$ of the cocycle $\alpha(\xi)$. At the same time the right hand side is expressed in terms of the zero degree component $\alpha_0(\xi)$. This is possible because $\alpha_n(\xi)$ and $\alpha_0(\xi)$ are related by the recurrence relations (13) expressing closedness of $\alpha(\xi)$.

Also note that even if the equivariant form $\alpha(\xi)$ is smooth at $\xi = 0$, the right hand side of (14) contains the divergent factor $\xi^{-n/2}$. The singulariy at $\xi = 0$ is canceled by the sum of contributions of fixed points, which has a zero of degree $n/2$ at $\xi = 0$. This idea leads to *reside formulas* [JK].

2.3 Proof of localization formula

In this section we give a proof of the localization formula (14). This proof was suggested by Witten in [W1]. The idea is as follows. Choose an S^1-invariant metric g on M, with behavior at fixed points given by (11), and define a 1-form

$$\psi = g(v, \cdot),$$

such that $\iota_u \psi = g(v,u)$ for any vector field u on M. The form ψ is S^1-invariant (because the metric is S^1-invariant). Define an equivariantly exact form

$$\beta(\xi) = \mathrm{d}_{S^1}\psi = \mathrm{d}\psi + i\xi\iota_v\psi = \mathrm{d}\psi + i\xi g(v,v).$$

Note that near a fixed point ψ is of the form

$$\psi \approx -\frac{1}{2}\sum_{k=1}^{m} \nu_k (x_k \mathrm{d}y_k - y_k \mathrm{d}x_k),$$

and the form $\beta(\xi)$ is given by

$$\beta(\xi) \approx -\sum_{k=1}^{m} \nu_k \mathrm{d}x_k \wedge \mathrm{d}y_k + \frac{i\xi}{2}\sum_{k=1}^{m} \nu_k^2 (x_k^2 + y_k^2).$$

Let us consider the equivariant form

$$e^{is\beta(\xi)} - 1 = \sum_{k=1}^{\infty} \frac{(is)^k}{k!}\,(\mathrm{d}_{S^1}\psi)^k = \mathrm{d}_{S^1}\left(\sum_{k=1}^{\infty} \frac{(is)^k}{k!}\;\psi(\mathrm{d}_{S^1}\psi)^{k-1}\right).$$

It is equivariantly exact, and hence,

$$\int_M \alpha(\xi) = \int_M \alpha(\xi) e^{is\beta(\xi)} \tag{15}$$

for any value of the parameter s. In particular, if $\xi > 0$, one van consider the limit $s \to +\infty$. On one hand, the asymptotics of the integral (15) can be computed using the stationary phase method. But on the other hand, the answer does not depend on s. Hence, it is sufficient to extract the term in the asymptotics which does not depend on s, and this will be the exact answer for the integral!

At large s the form

$$e^{is\beta(\xi)} = e^{is\mathrm{d}\psi - s\xi g(v,v)}$$

is exponentially small everywhere except for the small neighborhoods of the fixed points where $v = 0$ and $g(v,v) = 0$. So, the fixed points are at the same time the critical points which give contributions into the stationary phase asymptotics. The leading contribution in s of the critical point x_p is given by

$$(\alpha_0(\xi))(x_p) \prod_{k=1}^{m} \left(is\nu_k^p \int \mathrm{d}x_k \mathrm{d}y_k\, e^{-\frac{s\xi}{2}\sum (\nu_k^i)^2 (x_k^2 + y_k^2)} \right) \tag{16}$$

Here we have used the fact that the 2-form $is\mathrm{d}\psi$ which enters the integrand is proportional to s, and, hence, it gives the leading contribution into the integration measure. The integral in (16) is Gaussian. It yields

$$\left(\frac{2\pi}{i\xi}\right)^m \frac{(\alpha_0(\xi))(x_p)}{\nu_1^p \ldots \nu_m^p}$$

which is indeed independent of s. Summing up these contributions for all fixed points x_p we obtain the localization formula (14).

2.4 Duistermaat-Heckman formula

Perhaps, the most well-known application of the localization formulas is the Duistermaat-Heckman formula in symplectic geometry [DH]. In fact, it was discovered before the general localization principle was formulated.

The framework is as follows: we have a closed 2-form ω,

$$\mathrm{d}\omega = 0,$$

which satisfies the nodegeneracy condition. That is,

$$\iota_u \omega = 0$$

for some vector field u on M implies $u = 0$. The pair (M, ω) is called a symplectic manifold. A standard example is R^{2m} with coordinates p_i and q_i and the 2-form

$$\omega = \sum_{i=1}^{m} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

A vector field v is called Hamiltonian if there exists a function H such that

$$\iota_v \omega + \mathrm{d}H = 0.$$

The function H is called the Hamiltonian of v. A symplectic manifold is always even dimensional (otherwise ω is necessarily degenerate), and has the volume form

$$\mathcal{L} = \frac{\omega^m}{m!} = [e^{\omega}]_{top}$$

called the Liouville form. The volume form $\mathcal{L}$ is invariant with respect to all Hamiltonian vector fields.

Now assume that M is compact, and carries a circle action. In addition, let the corresponding vector field v be Hamiltonian, with Hamltonian H. Then, one can define the following integral,

$$I(\xi) = \int_M \mathcal{L}\, e^{i\xi H}, \tag{17}$$

which is called the Duistermaat-Heckman integral, and can be evaluated using localization theorm.

First, we define the equivariant extension of the symplectic form ω,

$$\omega(\xi) = \omega + i\xi H.$$

It is an equivariantly closed form,

$$\mathrm{d}_{S^1}\omega(\xi) = (\mathrm{d} + i\xi\iota_v)(\omega + i\xi H) = \mathrm{d}\omega + i\xi(\iota_v\omega + \mathrm{d}H) = 0.$$

Here we have used closedness of ω and the definition of the Hamiltonian vector field.

Next, we define an equivarinat Liouville form,

$$\mathcal{L}(\xi) = e^{\omega(\xi)} = e^{i\xi H} \sum_{k=1}^{m} \frac{\omega^k}{k!},$$

where the sum terminates because higher powers of ω vanish. The form $\mathcal{L}(\xi)$ is also equivariantly closed, and, moreover,

$$\int_M \mathcal{L}(\xi) = \int \mathcal{L}\ e^{i\xi H} = I(\xi).$$

Now we apply the localization formula (14) to the left hand side to obtain the Duistermaat-Heckman formula,

$$I(\xi) = \left(\frac{2\pi}{i\xi}\right)^m \sum_p \frac{e^{i\xi H(x_p)}}{\lambda_1^p \dots \lambda_m^p}. \tag{18}$$

Example: sphere S^2. Getting back to the example of the sphere S^2, we show that our observation on exact stationary phase is a particular case of the Duistermaat-Heckman formula.

Let us choose the area form on the sphere as the symplectic form. It is clearly closed, and non-degenerate. In the polar angles θ, ϕ the vector field generating rotations around z-axis is of the form $v = \partial/\partial\phi$. Then, the Hamiltonian of v is determined by equation,

$$\iota\left(\frac{\partial}{\partial\phi}\right) \mathrm{d}\cos(\theta)\mathrm{d}\phi + \mathrm{d}H = 0,$$

which imples $H = \cos(\theta) = z$ (up to a shift by a constant). Thus, the integral which we would like to compute,

$$I(\xi) = \int_{S^2} \mathrm{d}A\ e^{i\xi z}$$

is the Duistermaat-Heckman integral, and is given by the Duistermaat-Heckman formula (18).

There are two fixed points of the S^1-action on S^2, the North pole and the South pole. The values of the Hamiltonian are given by $z_N = 1, z_S = -1$, and the indices of the S^1-action are $\nu^N = 1$ and $\nu^S = -1$. Then, formula (18) yields

$$I(\xi) = \frac{2\pi}{i\xi}\ \left(\frac{e^{i\xi z_N}}{\nu^N} - \frac{e^{-i\xi z_S}}{\nu^S}\right) = 4\pi\ \frac{\sin(\xi)}{\xi}$$

confirming the results we obtained before.

3 Weil model of equivariant cohomology

In this Section we develop technical tools for dealing with equivariant cohomology for any compact group G. We begin by introducing the Weil algebra, and the Weil model of equivariant cohomology. Then we establish the equivalence to the Cartan model which we used in the case of $G = S^1$. Finally, we give an expression for the equivariant Liouville form in the Weil model, and introduce equivariant cohomology with generalized coefficients.

For another physicist-oriented review of the subject see [CMR].

3.1 Weil algebra and Weil differential

Group actions on manifolds. In general, we shall study the situation when the group acting on M is not necessarily a circle S^1. Let G be a compact connected Lie group, and $\mathcal{G}$ be its Lie algebra. In many situations it will be convenient to choose a basis $\{e_a\} \subset \mathcal{G}$, and the dual basis $\{e^a\}$ in the space $\mathcal{G}^*$. We denote by f_{ab}^c the structure constants in this basis,

$$[e_a, e_b] = f_{ab}^c e_c.$$

If the group G acts on the manifold M, one can associate to each element $e \in \mathcal{G}$ a fundamental vector field on M which we denote by e_M. For instance, the vector fields corresponding to the basis elements e_a are $(e_a)_M$. The Lie derivatives and contractions corresponding to these fundamental vector fields act on the space of differential forms $\Omega(M)$. We denote them by L_a and ι_a, respectively. They satisfy the following relations,

$$\begin{aligned} [L_a, \iota_b] &= f_{ab}^c \iota_c, \\ [L_a, L_b] &= f_{ab}^c L_c, \\ [d, \iota_a] &= L_a. \end{aligned} \tag{19}$$

Here d is the de Rham differential, and $[,]$ stands for the super-commutator. For instance, $[d, \iota_a] = d\iota_a + \iota_a d$.

In a more abstract setting we can say that equations (19) define a superalgebra $\hat{\mathcal{G}}$ with generators L_a, ι_a, d. If M is a G-manifold, the space of forms $\Omega(M)$ carries a representation of $\hat{\mathcal{G}}$, where L_a are represented by Lie derivatives, ι_a by contractions, and d by the de Rham differential.

Weil algebra. In this section we construct a special representation of the algebra $\hat{\mathcal{G}}$ called the Weil algebra. It was suggested by H.Cartan in C1 as an 'algebraic model' of the space of forms on the classifying space EG.

By definition, the Weil algebra W_G is the product of the symmetric and exterior algebras of the dual space to the Lie algebra of G,

$$W_G := S\mathcal{G}^* \otimes \wedge\mathcal{G}^*. \tag{20}$$

The algebra $S\mathcal{G}^*$ is the algebra of polynomials on $\mathcal{G}$. It is convenient to introduce generators v^a of $S\mathcal{G}^*$ corresponding to the basis elements $e^a \in \mathcal{G}^*$. The generators v^a correspond to linear functions on $\mathcal{G}$, and naturally commute with each other,

$$v^a v^b - v^b v^a = 0.$$

We denote the generators of the exterior algebra $\wedge\mathcal{G}^*$ by y^a. They satisfy the anti-commutation relations,

$$y^a y^b + y^b y^a = 0.$$

One can introduce a grading on W_G by assigning degree 2 to v^a and degree 1 to y^a,

$$W_G^l = \oplus_{j+2k=l} S^k\mathcal{G}^* \otimes \wedge^j \mathcal{G}^*.$$

Following H.Cartan, one can view W_G as a model of the space of forms on EG, such that each y^a corresponds to a 1-form, and each v^a corresponds to a 2-form.

There is an action of $\hat{\mathcal{G}}$ on W_G defined as follows. Operators L_a are defined on generators,

$$L_a(v^c) = -f^c_{ab} v^b, \ L_a(y^b) = -f^c_{ab} y^c,$$

and extended by the Leibniz rule. In a similar fashion, one defines contractions ι_a,

$$\iota_a(v^b) = 0, \ \iota_a(y^b) = \delta^b_a.$$

Finally, the *Weil differential* d is defined by

$$d(y^a) = v^a - \frac{1}{2} f^a_{bc} y^b y^c, \ d(v^a) = -f^a_{bc} y^b v^c.$$

These formulas have a simple geometric meaning: if one interprets y^a as components of a connection on a principal G-bundle, then the first formula,

$$v^a = dy^a + \frac{1}{2} f^a_{bc} y^b y^c,$$

is the standard definition of the curvature. The second formula,

$$dv^a + f^a_{bc} y^b v^c = 0$$

gives the Bianchi identity.

Relation to BRST. The differential on W_G can be written in the form,

$$d = y^a(L_a \otimes 1) + (v^a - \frac{1}{2} f^a_{bc} y^b y^c)\iota_a,$$

where $(L_a \otimes 1)$ is the Lie derivative acting only on the elements of $S\mathcal{G}^*$.

It is often compared to the *BRST differential* which is defined as follows. Let V be a representation of the group G. Then, the BRST differential acts on the space $V \otimes \wedge\mathcal{G}^*$, and is given by formula,

$$d_{BRST} = y^a(L_a \otimes 1) - \frac{1}{2} f^a_{bc} y^b y^c \iota_a.$$

In the physical interpretation, y^a are called ghosts, and denoted by c^a. The dual contractions ι_a are called anti-ghosts, and denoted by b_a. The ghosts and anti-ghosts (generators of $\wedge\mathcal{G}^*$ and contractions) satisfy the anti-commutation relation,

$$c^k b_l + b_l c^k = \delta^k_l.$$

If we introduce a special notation for generators of the G-action on V, $T_a := (L_a \otimes 1)$, we get the standard formula for the BRST differential,

$$d_{BRST} = c^a T_a - \frac{1}{2} f^a_{bc} c^b c^c b_a.$$

The main difference between the Weil differential (and the equivariant differential) and the BRST differential is the extra term $v^a \iota_a$ in the Weil differential. One can interpret it as a BRST differential for the abelian Lie algebra $\mathcal{G}^*$ with generators v^a and ghosts $b_a := \iota_a$. One can say that the Weil differential is a sum of two BRST differentials,

$$d = d^{\mathcal{G}}_{BRST} + d^{\mathcal{G}^*}_{BRST}.$$

3.2 Weil model of equivariant cohomology

In this section we define the Weil model of equivariant cohomology, and prove that it is equivalent to the Cartan model introduced before. Then, we extend our consideration to equivariant cohomology with generalized coefficients.

Definition of the Weil model. Let M be a G-manifold. It is our goal to define the space of equivariant forms in the Weil model, and the equivariant differential on this space.

Consider the product $\Omega(M) \otimes W_G$ of the space of differential forms on M and of the Weil algebra W_G. If one interprets W_G as the space of differential forms on EG, the product is naturally interpreted as $\Omega(M \times EG)$. Both $\Omega(M)$ and W_G carry representations of $\hat{\mathcal{G}}$. Hence, one can define the diagonal action on the tensor product. That is, L_a, ι_a and d are defined as operators on $(\Omega(M) \otimes S\mathcal{G}^*)$ by formulas,

$$L_a = L_a \otimes 1 + 1 \otimes L_a,$$
$$\iota_a = \iota_a \otimes 1 + 1 \otimes \iota_a,$$
$$d = d \otimes 1 + 1 \otimes d.$$

We define the space of equivariant forms on M as the *basic part* of $\Omega(M) \otimes W_G$,

$$\Omega_G(M) := \{\alpha \in \Omega(M) \otimes W_G | \, L_a\alpha = 0, \ \iota_a\alpha = 0\}.$$

In more geometric terms we are looking at the principal G-bundle

$$M \times EG \to (M \times EG)/G,$$

and define $\Omega_G(M)$ as the space of basic forms. These are forms which can be obtained as pull-backs of differential forms on the quotient space $(M \times EG)/G$.

The space of equivariant forms $\Omega_G(M)$ carries the action of the combined differential $(d \otimes 1 + 1 \otimes d)$. One defines the equivariant cohomology of M as

$$H_G(M) := H(\Omega_G(M), d \otimes 1 + 1 \otimes d).$$

In the next section we show that this definition is equivalent to the definition in the Cartan model which we used in the case of $G = S^1$.

Equivalence to the Cartan model. In Section 2 we used a simpler model for S^1-equivariant cohomology. This model does not use anti-commuting variables y^a, and is called the Cartan model [C2]. A simple transformation which establishes the relation between Weil and Cartan models was suggested by Kalkman [K].

Let us define an operator Φ on the space $\Omega(M) \otimes W_G$ by formula,

$$\Phi := \exp(-\iota_a \otimes y^a).$$

The key property of Φ is

$$\Phi(\iota_a \otimes 1 + 1 \otimes \iota_a)\Phi^{-1} = 1 \otimes \iota_a. \tag{21}$$

In order to prove this equality we use the formula,

$$\Phi X \Phi^{-1} = \sum_{j=0}^{\infty} \frac{1}{j!} \, \mathrm{ad}^{\mathrm{j}}(-\iota_{\mathrm{a}} \otimes \mathrm{y}^{\mathrm{a}})\mathrm{X}. \tag{22}$$

The calculation gives,

$$\mathrm{ad}(-\iota_{\mathrm{a}} \otimes \mathrm{y}^{\mathrm{a}})(\iota_{\mathrm{a}} \otimes 1 + 1 \otimes \iota_{\mathrm{a}}) = -\iota_{\mathrm{a}} \otimes 1,$$
$$\frac{1}{2!} \, \mathrm{ad}^2(-\iota_{\mathrm{a}} \otimes \mathrm{y}^{\mathrm{a}})(\iota_{\mathrm{a}} \otimes 1 + 1 \otimes \iota_{\mathrm{a}}) = 0.$$

After substitution to (22) we obtain equation (21).

The action of Φ maps the forms annihilated by $(\iota_a \otimes 1 + 1 \otimes \iota_a)$ to the forms annihilated by $1 \otimes \iota_a$. That is, it is mapped to $\Omega(M) \otimes S\mathcal{G}^*$. Taking into account that Φ commutes with the diagonal action of L_a, we conclude that the space of equivariant forms $\Omega_G(M)$ is mapped to

$$\Phi : \Omega_G(M) \to (\Omega(M) \otimes S\mathcal{G}^*)^G.$$

This is the new model of the space of equivariant forms called Cartan model. We already worked with it in the case of $G = S^1$.

Next, we compute the equivariant differential in the Cartan model. We apply formula (22) to the equivariant differential,

$$\mathrm{ad}(-\iota_\mathrm{a} \otimes \mathrm{y}^\mathrm{a})(\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}) = \mathrm{L}_\mathrm{a} \otimes \mathrm{y}^\mathrm{a} - \iota_\mathrm{a} \otimes (\mathrm{v}^\mathrm{a} - \frac{1}{2}\mathrm{f}^\mathrm{a}_{\mathrm{bc}}\mathrm{y}^\mathrm{b}\mathrm{y}^\mathrm{c}),$$
$$\frac{1}{2!}\mathrm{ad}^2(-\iota_\mathrm{a} \otimes \mathrm{y}^\mathrm{a})(\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}) = -\frac{1}{2}\mathrm{f}^\mathrm{c}_{\mathrm{ab}}\iota_\mathrm{c} \otimes \mathrm{y}^\mathrm{a}\mathrm{y}^\mathrm{b},$$
$$\frac{1}{3!}\mathrm{ad}^2(-\iota_\mathrm{a} \otimes \mathrm{y}^\mathrm{a})(\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}) = 0.$$

We add all the terms and take into account that $(L_a \otimes 1 + 1 \otimes L_a)$ and $1 \otimes \iota_a$ annihilate the image of the space of equivariant forms. The final result for the differential in the Cartan model is quite simple,

$$d_G := d \otimes 1 - \iota_a \otimes v^a.$$

In the case of $G = S^1$ one should put $v = -i\xi$ to recover the expression (12) for d_{S^1}.

Equivariant Liouville form. As before, examples of equivariant classes are provided by symplectic geometry. Let (M, ω) be a symplectic manifold, and assume that the G-action on M is Hamiltonian. That is, there is a set of Hamiltonians, H_a such that [1]

$$\iota_a \omega = dH_a, \tag{23}$$

and

$$L_a H_b = f^c_{ab} H_c. \tag{24}$$

The collection of functions H_a can be viewed as the *moment map* $H : M \to \mathcal{G}^*$ with $H_a = \langle H, e_a \rangle$. The property (24) expresses equivariance of the map H with respect to the G-action on M and the co-adjoint action on $\mathcal{G}^*$.

Again, one can define the equivariant extension of the Liouville form in Cartan model,

$$\omega(v) := \omega + H_a v^a.$$

This form is equivariantly closed,

$$d_G \omega(v) = (d - v^a \iota_a)(\omega + v^b H_b) = d\omega + v^a(dH_a - \iota_a \omega) = 0.$$

[1] Note that we have changed the sign convention in comparison to the previous Section.

The same form in the Weil model is expressed by

$$\omega_W := \Phi^{-1}\omega(v) = \omega - y^a dH_a + H_a(v^a - \frac{1}{2} f^a_{bc} y^b y^c),$$

where we have used the property $\iota_a dH_b = f^c_{ab} H_c$ implied by (24).

Finally, we introduce the equivariant Liouville forms in Cartan and Weil models,

$$\mathcal{L}(v) := e^{\omega(v)} = e^{\omega} \; e^{v^a H_a},$$

and

$$\mathcal{L}_W := e^{\omega_W} = \exp(\omega) \; \exp(-y^a dH_a) \; \exp\left(-\frac{1}{2} H_a f^a_{bc} y^b y^c\right) \; \exp(v^a H_a). \quad (25)$$

In the next Section we shall use the expression for $\mathcal{L}_W$ to intoduce the theory of 'group-valued' Hamltonians.

Generalized coefficients. The last technical ingredient needed in the next Section is the notion of equivariant cohomology with generalized coefficients. It is sometimes convenient to make a Fourier-Laplace transform in the variables v^a such that they become distributions on the space $\mathcal{G}^*$ supported at the origin,

$$v^a = -\frac{\partial}{\partial \mu_a} \; \delta_0,$$

where μ_a are linear coordinates on $\mathcal{G}^*$, and δ_0 is the δ-function supported at the origin.

Then, it is natural to replace the space of polynomials $S\mathcal{G}^*$ in the definition of W_G by the space of all compactly supported distributions $\mathcal{E}'(\mathcal{G}^*)$. We denote the extended Weil algebra by

$$\hat{W}_G := \mathcal{E}'(\mathcal{G}^*) \otimes \wedge \mathcal{G}^*,$$

and the corresponding equivariant cohomology by $\hat{H}_G(M)$.

For instance, in the new notations the equivariant Liouville form contains the δ-function supported at the value H of the moment map,

$$\mathcal{L}_W = \exp(\omega) \; \exp(-y^a dH_a) \; \exp\left(-\frac{1}{2} H_a f^a_{bc} y^b y^c\right) \; \delta_H.$$

The distribution part of $\mathcal{L}_W$ is not supported at the origin, and defines a class in $\hat{H}_G(M)$.

4 Group-valued equivariant localization

In this Section we explain how to *quantize* the Weil algebra [AM], and define the *group valued* equivariant cohomology. This leads to the new localization theorem [AMW2], and new moment map theory [AMM].

In contrast to the previous sections we only sketch the results. At this stage the proofs are too involved for these notes. So, refer the reader to the original papers.

4.1 Non-commutative Weil algebra

In this section we introduce the non-commutative counterpart of the Weil lagebra. In the exposition we follow [AM].

Invariant inner product on $\mathcal{G}$. As before, we assume that G is a compact connected Lie group. In addition, we choose an invariant innder product on the Lie algebra $\mathcal{G}$, and suppose that G is a direct product of a compact simply-connected Lie group and a torus.

We denote the innder product on $\mathcal{G}$ by $(\cdot,\cdot)$. It induces a number of new structures. First, one can identify $\mathcal{G}$ with its dual space $\mathcal{G}^*$. The basis $\{e_a\}$ can be chosen orthonormal, $(e_a, e_b) = \delta_{ab}$. The corresponding structure constants $[e_a, e_b] = f_{abc}e_c$ are anti-symmetric with respect to the permutation of any two indices. Thus, one can define an element $\phi \in (\wedge^3\mathcal{G})^G$ by formula,

$$\phi := \frac{1}{6} f_{abc} e_a \otimes e_b \otimes e_c.$$

We define the left- and right-invariant vector fields e^L_a and e^R_a on the group G, and the dual left- and right-invariant 1-forms, θ^L_a and θ^R_a. They satisfy the Maurer-Cartan structure equations,

$$d\theta^L_a = -\frac{1}{2} f_{abc}\theta^L_b\theta^L_c\,,\ d\theta^R_a = \frac{1}{2} f_{abc}\theta^R_b\theta^R_c.$$

Using the identification $\wedge^3\mathcal{G} \cong \wedge^3\mathcal{G}^*$, one can define a bi-invariant 3-form on G,

$$\eta := \frac{1}{12} f_{abc}\theta^L_a\theta^L_b\theta^L_c = \frac{1}{12} f_{abc}\theta^R_a\theta^R_b\theta^R_c.$$

Finally, we introduce the distributions on G with support at the group unit corresponding to the vector fields e^L_a and e^R_a,

$$u^L_a := -e^L_a\delta_e\,,\ u^R_a := -e^R_a\delta_e.$$

Here δ_e is the δ-function supported at the group unit.

Definition of non-commutative Weil algebra. We recall that the Weil algebra W_G is a tensor product of symmetric and exterior algebras of the space $\mathcal{G}^*$. The non-commutative Weil algebra is a tensor product of the non-commutative counterparts of these algebras. The symmetric algebra is replaced by the *universal enveloping algebra* $U(\mathcal{G})$ with generators u_a and relations

$$u_a u_b - u_b u_a = f_{abc}u_c.$$

The exterior algebra is replaced by the Clifford algebra $\mathrm{Cl}(\mathcal{G})$ with generators x_a and relations

$$x_a x_b + x_b x_a = \delta_{ab}.$$

Here we use the fact that the basis $\{e_a\}$ is orthonormal. We denote the non-commutative Weil algebra by $\mathcal{W}_G$,

$$\mathcal{W}_G := U(\mathcal{G}) \otimes \mathrm{Cl}(\mathcal{G}).$$

Similar to W_G the algebra $\mathcal{W}_G$ carries a representation of $\hat{\mathcal{G}}$. The action of the LIe derivatives L_a is defined on generators,

$$L_a(u_b) = f_{abc}u_c, \ L_a(x_b) = f_{abc}x_c.$$

The contractions ι_a are given by formulas,

$$\iota_a(u_b) = 0, \ \iota_a(x_b) = \delta_{ab}.$$

Finally, the Weil differential has its analog on $\mathcal{W}_G$,

$$d(x_a) = u_a - \frac{1}{2} f_{abc}x_b x_c, \ d(u_a) = -f_{abc}x_b u_c.$$

These formulas are very simlar to those which define the $\hat{\mathcal{G}}$-action on W_G. The important difference is that in the non-commutative algebra $\mathcal{W}_G$, the operators L_a, ι_a, d are inner derivations,

$$L_a = \mathrm{ad}(\mathrm{u_a} - \frac{1}{2}\mathrm{f_{abc}x_b x_c}), \ \iota_a = \mathrm{ad}(\mathrm{x_a}), \ \mathrm{d} = \mathrm{ad}(\mathrm{x_a u_a} - \frac{1}{6}\mathrm{f_{abc}x_a x_b x_c}).$$

As in the case of W_G, one can introduce the non-commutative Weil algebra with generalized coefficients,

$$\hat{\mathcal{W}}_G := \mathcal{E}'(G) \otimes \mathrm{Cl}(\mathcal{G}).$$

For any G-manifold M one can now define the space of ‘group-valued’ equivariant forms, $(\Omega(M) \otimes \mathcal{W}_G)_{basic}$ and $(\Omega(M) \otimes \hat{\mathcal{W}}_G)_{basic}$ and the ‘group-valued’ equivariant cohomology,

$$\begin{aligned} \mathcal{H}_G(M) &:= H((\Omega(M) \otimes \mathcal{W}_G)_{basic}, d \otimes 1 + 1 \otimes d), \\ \hat{\mathcal{H}}_G(M) &:= H((\Omega(M) \otimes \hat{\mathcal{W}}_G)_{basic}, d \otimes 1 + 1 \otimes d) \end{aligned}$$

It is our main goal to present the localization formulas for classes in $\mathcal{H}_G(M)$ and in $\hat{\mathcal{H}}_G(M)$.

4.2 Group-valued moment maps

In this section we give examples of equivariant cocycles which give rise to classes in $\hat{\mathcal{H}}_G(M)$. We follow the paper [AMW1]. The idea is to find the counterpart of formula (25) for the equivariant Liouville form. We shall see that it naturally leads to moment maps with values in the Lie group rather than in the dual of the Lie algebra.

The right hand side of (25) is a product of four factors,

$$\exp(\omega)\ \exp(-y^a dH_a)\ \exp\left(-\frac{1}{2}H_a f_{abc} y^a y^b\right)\ \delta_H \in \Omega(M)\otimes\mathcal{E}'(\mathcal{G}^*)\otimes\wedge\mathcal{G}^*$$

where ω is a 2-form on M and H is the moment map $H : M \to \mathcal{G}^*$. In the group-valued case we still need a 2-form ω, but the moment map should take values in the group G, $\Phi : M \to G$. The reason is that instead of the space of distrinutions on the dual of the Lie algebra $\mathcal{E}'(\mathcal{G}^*)$ we now have the space of distributions on the group $\mathcal{E}'(G)$. Then, the first and the last terms in (25) have their counterparts, $\exp(\omega)$ and δ_Φ.

The third term is related to the *spinor representation* of G. In more detail, choose $H = H_a e_a \in g$ and define a map $\tau : G \to \mathrm{Cl}(\mathcal{G})$ by formula,

$$\tau(e^H) = \exp\left(-\frac{1}{2}H_a f_{abc} x_b x_c\right).$$

If G is a product of a compact simply-connected Lie group and a torus, the map τ is well-defined, and defines the representation of G (see *e.g.* [BGV]),

$$\tau(g_1 g_2) = \tau(g_1)\tau(g_2).$$

There are two possible candidates for the role of the second term, $\exp(-x_a\Phi^*\theta^L_a)$ and $\exp(-x_a\Phi^*\theta^R_a)$. We notice that

$$\exp(-x_a\Phi^*\theta^R_a)\tau(\Phi) = \tau(\Phi)\exp(-x_a\Phi^*\theta^L_a).$$

Thus, we can choose either left- or right-invariant Maurer-Cartan forms, but we should position them on the different sides of $\tau(\Phi)$.

Finally, our candidate for an group-valued equivariant Liouville form is,

$$\mathcal{L}_W = \exp(\omega)\ \exp(-x_a\Phi^*\theta^R_a)\tau(\Phi)\ \delta_\Phi. \tag{26}$$

The question is: under what conditions on ω and Φ, the form $\mathcal{L}_W$ is an equivariantly closed? According to [AMW1], there are 2 conditions to be satisfied: first, the differential of the 2-form ω is a pull-back of the bi-invariant 3-form η on G,

$$d\omega = \Phi^*\eta. \tag{27}$$

Second, there is an analog of the moment map condition,

$$\iota_a\omega = \frac{1}{2}\Phi^*(\theta^L_a + \theta^R_a). \tag{28}$$

We call a triple (M, ω, Φ) which satisfies these conditions a *group-valued Hamiltonian space*.

Examples of group-valued Hamiltonian spaces Our first example of a group-valued Hamiltonian G-space is the torus $T^2 = S^1 \times S^1$. We parametrize by it two angles, (ϕ, ψ), choose the S^1 action

$$\theta : (\phi, \psi) \mapsto (\phi, \psi + \theta)$$

and the two form,

$$\omega = \mathrm{d}\psi \wedge \mathrm{d}\phi.$$

Then,

$$\iota\left(\frac{\partial}{\partial\psi}\right)\omega = \mathrm{d}\phi,$$

and one can define the moment map $\Phi : (\phi, \psi) \mapsto \phi$ with values in the group S^1. The 3-form η vanishes on S^1 for dimensional reasons, which is consistent with closedness of ω.

Our second example is a bit more complicated. We consider the group $G = SU(2)$, choose any element $f \in G$, and consider the corresponding conjugacy class,

$$C_f := \{gfg^{-1} |\; g \in G\}.$$

In other words, these are all unitary 2 by 2 matrices with the same eigenvalues as f. If f is e or $-e$, the corresponding conjugacy class f is a point. Otherwise, it is a 2-sphere. We define the moment map on C_f as its embedding into G. Then, the 2-form ω is uniquely determined by the conditions (27) and (28). Up to a scalar factor, ω coincides with the area form $\mathrm{d}A$ induced by the identification with the 2-sphere. If the eigenvalues of f are $\exp(i\lambda)$ and $\exp(-i\lambda)$, one gets [AMW1],

$$\omega = \sin(\lambda)\; \mathrm{d}A.$$

Our last example is the product of two copies of $SU(2)$, $D := SU(2) \times SU(2)$. One can view it as a nonabelian counterpart of the torus T^2. We view D as an $SU(2) \times SU(2)$-manifold, with the action,

$$(g, h) : (a, b) \mapsto (gah^{-1}, gbh^{-1}),$$

and the moment map,

$$\Phi : (a, b) \mapsto (ab, a^{-1}b^{-1}).$$

The corresponding 2-form which satisfies conditions (27) and (28) is [AMW1],

$$\omega = \frac{1}{2}(a^*\theta^L_a b^*\theta^R_a + a^*\theta^R_a b^*\theta^L_a).$$

4.3 Group-valued localization

In this section we explain how the localization principle works for the classes in $\hat{\mathcal{H}}_G(M)$. We begin by recalling some standard fact from the theory of Lie groups. We assume that G is a direct product of a compact semi-simple Lie group and a compact torus.

Some facts about compact Lie groups. Let G be a product of a compact semi-simple Lie group and a torus. Let T be a maximal torus in G and $\mathcal{T}$ be its Lie algebra. The Lie algebra $\mathcal{T}$ contains the integral lattice $\Lambda \subset \mathcal{T}$,

$$\Lambda = \{x \in \mathcal{T} \mid \exp(x) = e\}.$$

The dual space $\mathcal{T}^*$ contains the dual lattice Λ^*. By choosing the set of positive roots R_+ we define the positive Weyl chamber $\mathcal{T}^*_+ \subset \mathcal{T}^*$, and the set of dominant weights $\Lambda^* \cap \mathcal{T}^*_+$. A dominant weight λ defines a unique irreducible highest weight representation V_λ of G. The representation V_λ contains the highest weight vector v_λ which satisfies the following conditions,

$$e_\alpha \cdot v_\lambda = 0,$$

where e_α are the generators corresponding to positive roots, and

$$h_\alpha \cdot v_\lambda = (\alpha, \lambda) v_\lambda,$$

where h_α are the elements of $\mathcal{T}$ corresponding to the roots. All irreducible representations V_λ possess a Hermitian invariant scalar product. We normalize v_λ such that $(v_\lambda, v_\lambda) = 1$. Then, for each dominant weight λ one can define two functions on G, the character,

$$\chi_\lambda(g) = \mathrm{Tr}_{\mathrm{V}_\lambda} \mathrm{g},$$

and the 'spherical harmonics',

$$\Delta_\lambda(g) = (v_\lambda, g \cdot v_\lambda).$$

A special weight is given by the half-sum of positive roots,

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

For example, for $G = SU(2)$ the representations are parametrized by the spin $j = 0, 1/2, 1 \ldots$. The weight ρ corresponds to $j = 1/2$. The corresponding representation is two-dimentional, and we obtain,

$$\chi_{\frac{1}{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d, \;\; \Delta_{\frac{1}{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a.$$

Usually, we identify $\mathcal{T}$ and $\mathcal{T}^*$ using the scalar product. Then, one can view the dominant weights as the elements of $\mathcal{T}$. Because the weights belong to a special lattice, the corresponding one-parameter subgroups $T_\lambda = \{\exp(s\lambda)\}$ (except for $\lambda = 0$) are circle subgroups of T. Note that a typical one-paraemter subgroup of T is dense in T. So, the subgroups T_λ are very special, and this will play an important role in the localization theorem for $\hat{\mathcal{H}}_G(M)$.

Finally, we recall that the *classical r-matrix* is an element in $\wedge^2\mathcal{G}$ defined by formula,

$$r = \sum_{\alpha\in R_+} e_\alpha \wedge e_{-\alpha}.$$

It is convenient to represent r as $r = r_{ab} t_a \otimes t_b$. The r-matrix satisfies the *calssical Yang-Baxter* equation,

$$\mathrm{Cycl_{abc}(r_{as}f_{sbt}r_{tc})} = \frac{1}{4}\mathrm{f_{abc}},$$

where $\mathrm{Cycl_{abc}}$ stands for summation over the cyclic permutations of the indeces a, b, c.

The localization formula. Now we are ready to formulate the new localization formula. Let M be a compact G-manifold, and $\alpha \in (\Omega(M)\otimes \hat{\mathcal{W}}_G)_{basic}$ be an equivariant cocycle,

$$(d_M + d_{\mathcal{W}})\alpha = 0.$$

Then, one can define an integral of α over M,

$$\int_M \alpha \in (\hat{W}_G)_{basic}.$$

The elements of the space $(\hat{W}_G)_{basic}$ are annihilated by contractions, and, hence, belong to $\mathcal{E}'(G)\otimes 1$. By G-invariance, these distributions should be conjugation-invariant, $(\hat{W}_G)_{basic} \cong \mathcal{E}'(G)^G$. A conjugation-invariant distribution is completely characterized by its pairings with charatcers of irreducible representations of G,

$$\alpha_\lambda = \langle \int_M \alpha, \chi_\lambda\rangle.$$

It is easy to show that the numbers α_λ do not depend on the representative in the cohomology class.

The localization formula [AMW2] gives expressions for α_λ in terms of the fixed points of the action of $T_{\lambda+\rho}$ (note the shift by ρ). As usual, we assume that all these fixed points are isolated. Then, one obtains,

$$\alpha_\lambda = \left(\frac{2\pi}{i}\right)^m \dim V_\lambda \sum_p \frac{\langle exp(\frac{1}{2}\iota(r))\alpha, \Delta_\lambda\rangle(p)}{\nu_1^p \dots \nu_m^p}. \tag{29}$$

Here the dimension of M is $2m$, the dimension of the representation V_λ is $\dim V_\lambda$, $\iota(r)$ is defined as $r_{ab}\iota_a\iota_b$, and ν_i^p are the indices of the circle action of $T_{\lambda+\rho}$ at the point p.

Formula (29) simplifies if M is a Hamiltonian space with group valued moment map, and the cocycle is the equivariant Liouville form on M. In this case the localization formula reads [AMW2],

$$\mathcal{L}_\lambda = \left(\frac{2\pi}{i}\right)^m \dim V_\lambda \sum_p \frac{\Delta_{\lambda+\rho}(H(p))}{\nu_1^p \dots \nu_m^p}. \tag{30}$$

Note that if p is a fixed point for some $T_{\lambda+\rho}$, the value of the moment map $H(p)$ belongs to the maximal torus T. The spherical harmonics $\Delta_{\lambda+\rho}$ defines a character of T which generalizes the expression $\exp(i\xi H(p))$ in the Duistermaat-Heckman formula. (18).

Some simple application of the formula (30) can be found in [P]. In particular, certain integrals over the sphere S^4, and the space $SU(2) \times SU(2) \cong S^3 \times S^3$ can be computed using this technique. A more ambitious task is to show that formula (18) gives precise meaning to the path integrals of [W2].

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