Einstein–Weyl Structures from Hyper–Kähler Metrics with Conformal Killing Vectors

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Abstract

We consider four (real or complex) dimensional hyper-Kähler metrics with a conformal symmetry $K$. The three-dimensional space of orbits of $K$ is shown to have an Einstein–Weyl structure which admits a shear-free geodesics congruence for which the twist is a constant multiple of the divergence. In this case the Einstein–Weyl equations reduce down to a single second order PDE for one function. The Lax representation, Lie point symmetries, hidden symmetries and the recursion operator associated with this PDE are found, and some group invariant solutions are considered.

1 Three-dimensional Einstein–Weyl spaces

Three-dimensional Einstein–Weyl (EW) geometries were first considered by Cartan [3] and then rediscovered by Hitchin [8] in the context of twistor theory. They constitute an interesting generalisation of (the otherwise locally trivial) Einstein condition in three dimensions.

In this paper we shall consider four-dimensional anti-self-dual (ASD) vacuum (or complexified hyper-Kähler) spaces with a conformal symmetry. By a general construction [9] such spaces will give rise to Einstein–Weyl structures on the space of trajectories of the given conformal symmetry $K$. The cases where $K$ is a pure Killing vector or a tri-holomorphic homothety have been extensively studied [1, 15, 4, 10]. Therefore we shall consider the most general case of $K$ being a conformal, non-triholomorphic Killing vector. We begin by collecting various definitions and formulae concerning three-dimensional Einstein–Weyl spaces (see [11] for a fuller account). In the next section we shall give the canonical form of an allowed conformal Killing vector in a natural coordinate system associated with the Kähler potential. Then we shall look at solutions to a non-linear Monge–Ampère equation (the so called ‘first heavenly equation’ [12]) (2.7) for the Kähler potential which admit the symmetry $K$. This will give rise to a new integrable system in three dimensions and to the corresponding EW geometries. In Section 3 we shall give the Lax representation of the reduced equations. When Euclidean reality conditions are imposed (Section 5) we shall recover some known results [1, 15] as limiting cases of our construction. In Section 6 we shall find and classify the Lie point symmetries of the field equations in three dimensions (and so the Killing vectors of the associated Weyl structure), and consider some group invariant solutions. In Section 7 we shall study hidden symmetries and the recursion operator associated to the three-dimensional system. In Section 8 we shall show that the EW structures studied in this paper admit a shear free geodesic congruence for which twist and divergence are linearly dependent.

Let $W$ be an $n$-dimensional complex manifold, with a torsion-free connection $D$ and a conformal metric $[h]$. We shall call $W$ a Weyl space if the null geodesics of $[h]$ are also geodesics for $D$. This condition is equivalent to

$$D_i h_{j k} = \nu_i h_{j k}$$

(1.1)

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Let $x$ and $\nu$ be a complexified hyper-Kähler (i.e. ASD vacuum) metric on a complex four-manifold $\mathcal{M}$ and $\mathbb{R}^{4,4} = (u, z, \bar{u}, \bar{z})$ be a null coordinate system on $\mathcal{M}$. Locally $g$ can be put in the form

$$ds^2 = \Omega_{uv} du dv + \Omega_{uz} du d\bar{z} + \Omega_{\bar{u}z} d\bar{u} d\bar{z} + \Omega_{\bar{u}\bar{z}} d\bar{u} d\bar{z}$$

(subscripts denote partial differentiation) where $\Omega = \Omega(u, z, \bar{u}, \bar{z})$ is solution of the first heavenly equation [12]

$$\Omega_{uz} \Omega_{\bar{u}\bar{z}} - \Omega_{u\bar{z}} \Omega_{\bar{u}z} = 1.$$  

(2.7)
where symmetric spinors $\phi_{A'B'}$ and $\psi_{AB}$ are respectively self-dual and anti-self-dual parts of the covariant derivative of $K$. The well known formula $\nabla_a K_b = R_{abcd} K^d$ relating the second covariant derivative of $K$ to the Riemannian curvature implies that in vacuum

$$\nabla_{A'}\phi_{B'C'} = 2C_{A'B'C'D'} K^{D'} - 2\varepsilon_{A'B'} \nabla_{C'} [A'], \nabla_{A'A'} \phi_{B'B'} = 0, C_{A'B'C'D'} \nabla_{A'D'} = 0.$$

Here $C_{A'B'C'D'}$ and $C_{A'B'C'}$ are respectively the ASD and SD Killing spinors. In particular in an ASD vacuum $\phi_{A'B'} = \text{const}$ and $\eta = \text{const}$ (or the space time is of type N). In this paper we shall analyse a situation where $K$ is not hyper-surface orthogonal and $\det(\phi_{A'B'}) \neq 0, \eta \neq 0$.

**Lemma 2.1** In an ASD vacuum the most general conformal Killing vector with $\det(\phi_{A'B'}) \neq 0$ can be transformed to the form

$$K = \eta (z \partial_z + \bar{z} \partial_{\bar{z}}) + \rho (\partial_z - \bar{z} \partial_{\bar{z}}).$$

**Proof.** In the adopted coordinate system a basis of SD two-forms is

$$\Sigma^{0'1'0'} = d\bar{u} \wedge dz, \quad \Sigma^{0'1'1'} = dw \wedge dz,$$

$$\Sigma^{1'0'1'} = \Omega_{u\bar{u}} dw \wedge d\bar{u} + \Omega_{u\bar{u}} dz \wedge d\bar{u} + \Omega_{\bar{u}\bar{u}} dz \wedge dz.$$  

Let $K = K^A \partial_A + \tilde{K}^A \partial_{\tilde{A}}$, where $w^A = (u, z)$ and $\tilde{w}^A = (\tilde{u}, \tilde{z})$. The action of $K$ on self-dual two forms is determined by

$$\mathcal{L}_K \Sigma^{0'0'} = m \Sigma^{0'0'} + n \Sigma^{1'1'},$$

$$\mathcal{L}_K \Sigma^{1'1'} = \tilde{m} \Sigma^{0'0'} + \tilde{n} \Sigma^{1'1'},$$

$$\mathcal{L}_K \Sigma^{0'1'} = \eta \Sigma^{0'0'}. $$

for some constants $m, \tilde{m}, n, \tilde{n}$. This is because for non-degenerate $\phi_{A'B'}$ the Kähler structure can be identified with $dK = \phi_{A'B'} \Sigma^{A'B'}$. It follows that $n = \tilde{n} = 0$, and $K^A = mw^A, \tilde{K}^A = \tilde{m} \tilde{w}^A$. From $2\Sigma^{0'0'} \wedge \Sigma^{1'1'} = \Sigma^{0'1'} \wedge \Sigma^{0'1'}$ we find that $\eta := (m + \tilde{m})/2$. Define $\rho := (m - \tilde{m})/2$. We have the freedom to transform $w^A \rightarrow W^A(w^B)$ and $\tilde{w}^A \rightarrow \tilde{W}^A(\tilde{w}^B)$ in a way which preserves $\Sigma^{0'0'}$ and $\Sigma^{1'1'}$. Put $Z = z^2/2, W = w/z, \tilde{Z} = \tilde{z}^2/2, \tilde{W} = \tilde{u}/\tilde{z}$. This yields (coming back to $(w^A, \tilde{w}^A)$) (2.8). Now

$$\nabla_{A'} K_{A'B'} = \left( \begin{array}{cc} 0 & \rho + \eta \\ \rho - \eta & 0 \end{array} \right).$$

\[ \square \]

The real form of the Killing vector (2.8) also appears in the list of Lie point symmetries of (2.7) given in [2].

### 2.1 Symmetry reduction

In this section we shall look at the heavenly equation (2.7) with the additional constraint $\mathcal{L}_K g = \eta g$. This will lead to a new integrable equation describing a class of three-dimensional Einstein–Weyl geometries.

**Proposition 2.2** Every ASD vacuum metric with conformal symmetry is locally given by

$$ds^2 = \epsilon^{n'1'}(V^{-1}h + V(dt + \omega)^2),$$

where

$$h = -\epsilon^{2\rho \mu} du d\bar{u} - \frac{1}{16}(\eta^2 F du + \eta (F_w dw - F_{\bar{u}} d\bar{u}))^2,$$

$$\omega = \frac{\eta F_w - F_{u\bar{u}}}{\eta^2 F - F_{uu}} du + \eta F_{\bar{u}} d\bar{u},$$

$$V = \frac{1}{4}(\eta^2 F - F_{uu}).$$

\[ 3 \]
for constants \( \eta, \rho \in \mathbb{C} \).

**Corollary 2.3** The metric \( h \) is defined on the space \( W \) of trajectories of \( K \) in \( M \). From Proposition 1.1 it follows that \( h \) is the most general EW metric which arises as a reduction of ASD vacuum solutions by a conformal Killing vector. Equation (2.12) is therefore equivalent to the Einstein–Weyl equations (1.2).

**Proof.** The general ASDV metric can locally be given by (2.6). From Lemma 2.1 it follows that we can take \( K \) as in (2.8). Perform the coordinate transformation \((z, \bar{z}) \to (t, u)\) given by

\[
2t := \ln(1^{1/m}z^{1/m}), \quad 2u := \ln(1^{1/m}\bar{z}^{-1/m}).
\]

In these coordinates \( K = \partial_t \) and so \( \Omega(t, u, w, \bar{u}) = e^{nt}F(u, w, \bar{u}) \). The first heavenly equation is equivalent to (2.12). Rewriting the metric (2.6) in the new coordinate system yields (2.9) and \( \det(h) = -(1/4)\bar{V}^2e^{4nt} \).

The dual to \( K \) is \( K = e^{nt}V(dt + \omega) \). From Proposition 1.1 we find the EW one-form to be

\[
\nu = 2\ast_g \frac{K \wedge dK}{|K|^2} = 2e^{nt}V \ast_g ((dt + \omega) \wedge dw)
\]

\[
= 4pdu + \frac{(2\eta + 4\rho)(\eta F_w - F_{uu})du + (2\eta + 4\rho)(\eta F_{\bar{w}} + F_{u\bar{w}})d\bar{u}}{\eta^2 F - F_{uu}}
\]

where \( \ast_g \) is the Hodge operator determined by \( g \).

\( \square \)

## 3 Lax representation

In this section we shall represent equation (2.12) as the integrability condition for a linear system of equations. We shall interpret the Lax pair as a (minitwistor) distribution on a reduced projective spin bundle. The Lax pair for the first heavenly equation

\[
L_0 : = \Omega_{\bar{w}} \partial_{\bar{z}} - \Omega_{w} \partial_{z} - \lambda \partial_{\bar{w}},
\]

\[
L_1 : = \Omega_{z} \partial_{\bar{z}} - \Omega_{\bar{z}} \partial_{z} - \lambda \partial_{w},
\]

(3.13)

is defined on the five complex dimensional correspondence space \( \mathcal{F} = M \times \mathbb{CP}^1 \). Here \( \lambda \in \mathbb{CP}^1 \) parametrises null self-dual surfaces passing through a point in \( M \). Equations \( L_0 \Psi = L_1 \Psi = 0 \) have solutions in \( \mathcal{F} \) provided that \( \Omega \) satisfies the first heavenly equation (2.7). The formulation (3.13) is crucial to the twistor construction, as the projective twistor space on \( M \) arises as a factor space of \( \mathcal{F} \) by the distribution \( \{L_0, L_1\} \).

Let \( \pi_{A^1} = (\pi_0, \pi_{1}) \) be coordinates on the fibers of a bundle \( S_{A^1} \) of primed spinors. The space \( \mathcal{F} \) can be regarded as the projectivised version of \( S_{A^1} \) in a sense that \( \lambda = \pi_0/\pi_1 \). Define the Lie lift of a Killing vector \( K \) to \( \mathcal{F} \) by

\[
\bar{K} := K + Q\partial_{\lambda}, \quad \text{where} \quad Q := \pi_{A^1} \pi_{B^1} \delta^{A^1 B^1}/(\pi_{1})^2.
\]

(3.14)

The flow of \( \bar{K} \) in \( \mathcal{F} \) determines the behaviour of \( \alpha \)-planes under the action of \( K \) in \( M \). The linear system \( L_A \) for equation (2.7) is given by (3.13). The vector fields \( \{L_0, L_1, \bar{K}\} \) span an integrable distribution. This can be seen as follows:

\[
[K, L_A] = -\pi^{A^1}(\phi_{A^1 B^1} \epsilon_A B + \psi_{A^1} B^1 + (1/2)\bar{\eta}_A B^1)\nabla_{BB^1} = -\pi^{A^1}j_B (\nabla_{A A^1} + (\psi_{A} B^1 + \eta A B )L_B).
\]

(3.15)
\( \tilde{K} = K + \pi_A \phi^{AB} \frac{\partial}{\partial \pi^B} + \frac{1}{2} \eta \pi^A \frac{\partial}{\partial \pi^A}, \) \tag{3.15}

so that \( [\tilde{K}, L_A] = 0 \) modulo \( L_A \).

The projection of \( \tilde{K} \) to \( \mathcal{F} \) is given by (3.14), where the factor \( \pi^3_1 \) is used to dehomogenise a section of \( \mathcal{O}(2) \). If \( K \) is given by (2.8) then \( \tilde{K} = K + \rho \lambda \delta x \). Introduce an invariant spectral parameter \( \tilde{\lambda} \) (which is constant along \( \tilde{K} \)) by \( (\lambda, t) \rightarrow (\tilde{\lambda} := \lambda e^{-\delta}, t := t) \). In the new coordinates

\[
\partial_t = \partial_t - \rho \lambda \delta x, \quad \partial_x = e^{\delta t} \partial_x, \quad \text{so that} \quad \tilde{K} = \partial_t.
\]

The linear system for the reduced equation is obtained from (3.13) by rewriting it in \((u, \tilde{w}, u, \tilde{t}, \tilde{\lambda})\) coordinates and ignoring \( \partial_t \). This yields (after rescaling)

\[
\begin{align*}
L_{\hat{w}} &= m e^{\hat{w} u} \left( F_{\bar{w} w} \left( \frac{\partial}{\partial w} + \rho \tilde{\lambda} \frac{\partial}{\partial \lambda} \right) + (\eta F_{vw} - F_{w uv}) \frac{\partial}{\partial \tilde{w}} \right) + 2\lambda \frac{\partial}{\partial \hat{w}} \cdot (3.16)
\end{align*}
\]

The mini-twistor space corresponding to solutions of (2.12) is the quotient of \( \mathcal{F} \) by the integrable distribution \((L_{\hat{w}}, L_{\hat{t}}, \tilde{K})\).

The existence of a mini-twistor distribution follows from Hitchin’s construction [8]; the basic mini-twistor correspondence states that points in \( \mathcal{W} \) correspond in \( Z \) to rational curves with normal bundle \( \mathcal{O}(2) \). Let \( \gamma \) be the line in \( Z \) that corresponds to \( x \in \mathcal{W} \). The normal bundle to \( \gamma \) consists of tangent vectors at \( x \) (horizontally lifted to \( T_{(\gamma, \lambda)} \mathcal{F}_W \)) modulo the twistor distribution. Therefore we have a sequence of sheaves over \( \mathbb{C}P^1 \)

\[
0 \longrightarrow D_W \longrightarrow \mathbb{C}^3 \longrightarrow \mathcal{O}(2) \longrightarrow 0.
\]

We shall identify \( T^* \mathcal{W} \approx S^{(A^t \otimes S^B)} \). The map \( \mathbb{C}^3 \longrightarrow \mathcal{O}(2) \) is given by \( V^{A'B'} \rightarrow V^{A'B'} \pi_A \pi_B \). Its kernel consists of vectors of the form \( \pi^{(A'B')} \) with \( \pi^{B'} \) varying. The twistor distribution is therefore \( D_W = O(-1) \otimes S^{A^t} \) and so \( L_{A^t} \) is the global section of \( \Gamma(D_W \otimes \mathcal{O}(1) \otimes S_{A^t}) \). Let \( Z \) be a totally geodesic two-plane corresponding to a point \( Z \) of a mini-twistor space. This two plane is spanned by vectors of the form \( V^e = \pi^{(A'B')} \) with \( \pi^{A^t} \) fixed. Let \( W = \pi^{(A'B')} \) be another vector tangent to \( Z \). The Frobenius theorem implies that the Lie bracket \([V, W]\) must be tangent to some geodesic in \( Z \), i.e. \([V, W] = a \pi + b W \) for some \( a, b \). The last equation determines the mini-twistor distribution \( L_{A^t} \) to be a horizontal lift of \( \pi^{B'} D_{A'B'} \) to the weighted spin bundle by demanding \( L_{A^t} \pi_{e^t} = 0 \). The integrability conditions imply \( [L_{A^t}, L_{B^t}] = 0 \), (mod \( L_{A^t} \)). In fact if one picks two independent solutions of a ‘neutrino’ equation on the EW background, say \( \rho^{A^t} \) and \( \lambda^{A^t} \), then \( \tilde{L}_{\hat{w}} := \rho^{A^t} L_{A^t}, \) and \( \tilde{L}_{\hat{t}} := \lambda^{A^t} L_{A^t} \) commute exactly: \( [\tilde{L}_{\hat{w}}, \tilde{L}_{\hat{t}}] = 0 \).

## 4 Reality conditions

To obtain real Einstein–Weyl metrics we have to impose reality conditions on the coordinates \((u, \tilde{w}, u, \tilde{z})\):

- The reduction from the Euclidean slice \((\pi = 2, \bar{w} = -\tilde{w})\) yields positive definite EW metrics with \( u := iv \in \mathbb{R} \). Without loss of generality we can impose the condition \( \sinh u = 1 \), so \( \eta = \cos \alpha \) and \( \rho = i \sin \alpha \). The Euclidean version of (2.12) is then

\[
(F \cos^2 \alpha + F_{\bar{w}}) F_{\bar{w} \bar{w}} - (F_{\bar{w}} \cos \alpha - i F_{\bar{w} \bar{w}})(F_{\bar{w}} \cos \alpha + i F_{\bar{w} \bar{w}}) = 4 e^{-2 \sin \alpha}.
\]  \tag{4.17}

To obtain another form introduce \( G \) by \( G = e^{i \sin \alpha} F \). The transformed equation, the metric (rescaled by \( e^{2 \sin \alpha} \)) and the EW one-form are:

\[
(G + G_{\bar{w}} - 2 G_{\sin \alpha}) G_{\bar{w} \bar{w}} - (e^{i \alpha} G_{\bar{w} \bar{w}} - i G_{\bar{w} \bar{w}})(e^{-i \alpha} G_{\bar{w} \bar{w}} + i G_{\bar{w} \bar{w}}) = 4,
\]  \tag{4.18}
\[ \nu = -2 \sin \omega \omega + \frac{(2 + 2 \sin^2 \alpha)(G_{\omega \omega} - G_{\omega \omega}) + i \sin 2\alpha (G_{\omega \omega} - G_{\omega \omega})}{G + G_{\omega \omega} - 2G_{\omega \omega} \sin \alpha} \]  
\[ \psi = \frac{2(\cos \alpha + 2i \sin \alpha)G_{\omega \omega} + 2(\cos \alpha - 2i \sin \alpha)G_{\omega \omega}}{G + G_{\omega \omega} - 2G_{\omega \omega} \sin \alpha} \]  

- On an ultra-hyperbolic slice we have \( \tau = \tilde{\tau}, \omega = \tilde{\omega} \) which again implies \( u = iv \). The metric (2.9) has signature \((++-+)\). Another possibility is to take all coordinates as real. This gives a different real metric of signature \((++-+)\). The function \( F \) is real and \( \mu = \sin \alpha, \rho = \cosh \alpha \).

The analogous reality conditions are imposed on the linear system (3.16). From now on we shall be mostly concerned with the positive definite case. The correspondence space is now viewed as a real six-dimensional manifold. The real lift of a Killing vector is \( \bar{K} = i \partial \bar{v} + i \sin \alpha (\lambda \partial \alpha - \bar{\nabla} \bar{v}) \).

5 Special cases

Solutions to (4.17) describe the most general EW metrics which arise as reductions of hyper-Kähler structures. In this section we look at limiting cases and recover hyper-CR EW spaces [4], and LeBrun-Ward EW spaces which come from the \( SU(\infty) \) Toda equation. The real form of the Killing vector (2.8) is a linear combination of a rotation and a dilation;

\[ K = K_D \cos \alpha + K_R \sin \alpha, \quad \alpha \in [-\pi/2, 0], \quad K_D := z \partial z + \bar{\nabla} \bar{\tau}, \quad K_R := i(z \partial z - \bar{\nabla} \bar{\tau}). \]

5.1 LeBrun-Ward spaces

Take \( \alpha = -\pi/2 \). Then \( K \) is a pure Killing vector which does not preserve the complex structures on \( \mathcal{M} \). This case was studied in [1, 15, 16]. Put \( F = j, F = \bar{\phi} \) and rewrite equation (4.17) as

\[ \begin{align*}
d\bar{\phi} \wedge d\bar{j} \wedge d\bar{w} &= 4 e^{iz} d\omega \wedge d\bar{\omega} \wedge d\nu \\
d\bar{j} \wedge d\omega \wedge d\nu &= d\bar{\phi} \wedge d\bar{\omega} \wedge d\bar{w}.
\end{align*} \]

Use \( (j, w, \bar{\omega}) \) as coordinates and eliminate \( \bar{\phi} \) to obtain

\[ v_w \bar{\omega} + 2(e^{iz})_{jj} = 0 \]

which is the \( SU(\infty) \) Toda (or Boyer-Finley) equation [1]. The metric (2.9) reduces to

\[ h = e^{iz} d\omega d\bar{\omega} + \frac{1}{16} f^2, \quad \nu = 4v_j d\bar{j}. \]

This class of EW spaces is characterised by the existence of a twist-free, shear-free geodesic congruence [13].

Let us come back to complex coordinates and put \( w = e^{i\theta}, \bar{w} = e^{i\theta} \) and \( M = 2u + 2s \). In the \((s, u, \theta)\) coordinates equation (5.22) and the metric become

\[ M_{ss} - M_{\theta\theta} - 8(e^M)_{jj} = 0, \quad h = -e^M(ds^2 + d\theta^2) - \frac{1}{16} d\bar{j}^2. \]

Imposing a symmetry in \( \theta = \ln(\sqrt{u}/\bar{u}) \) direction we arrive at

\[ M_{ss} - 8(e^M)_{jj} = 0, \]

which was solved by Ward [15] who transformed it to a linear equation. The conclusion is that LeBrun-Ward EW metrics with \( \omega \partial \omega - \bar{\omega} \partial \bar{\omega} \) symmetry are solved by the same ansatz as those with \( \omega \partial \omega - \partial \bar{\omega} \) symmetry. In Subsection 6.1 it will be shown that imposing \( \omega \partial \omega - \partial \bar{\omega} \) symmetry leads to a linear equation even if \( \alpha \) is arbitrary.
Put $a = 0$. Then $K$ is a triholomorphic conformal symmetry. The corresponding EW metrics were in [7] called ‘special’, and then referred to as hyper CR (since each complex structure on $\mathcal{M}$ defines a CR structure on $\mathcal{W}$). They are characterised by the existence of a sphere of shear-free and divergence-free geodesic congruences. The equation (4.17) reduces to

$$F_{uv}(F + F_{uv}) - (F_{uw} + iF_{uw})(F_{uw} - iF_{uw}) = 4,$$

which is the form given in [14]. The corresponding Lax pair is

$$L_{01} = e^{i\lambda} \left( iF_{uw} \frac{\partial}{\partial v} - (F_{uw} + iF_{uw}) \frac{\partial}{\partial w} \right) + 2\lambda \frac{\partial}{\partial w},$$

$$L_{12} = e^{i\lambda} \left( (F_{uw} + iF_{uw}) \frac{\partial}{\partial v} - (F_{uw} + iF_{uw}) \frac{\partial}{\partial w} \right) - 2i\lambda \frac{\partial}{\partial v}.$$

### 6 Lie point symmetries

In order to find the Lie algebra of infinitesimal symmetries of (4.18) we shall convert it to system of differential forms. Introduce $Q$ and $J$ by $J := G_{\overline{w}}$, $Q := (e^{i\alpha}G - iG_{\nu})$

$$\omega_1 := iQ \wedge dJ \wedge d\overline{w} + e^{-i\alpha}(Q dJ - J dQ) \wedge d\overline{w} \wedge dv,$$

$$\omega_2 := dQ \wedge dv + e^{i\alpha}J dw \wedge d\overline{w} \wedge dv - idJ \wedge dw \wedge d\overline{w}.$$  

This system forms a closed differential ideal. Its integral manifold is a subspace of $\mathbb{R}^6$ on which $\omega_\mu = 0$. This integral manifold represents a solution to (4.18).

Let $X$ be a vector field on $\mathbb{R}^6$. The action of $X$ does not change the integral manifold if $\mathcal{L}_X \omega_\mu = \Lambda_\mu^\nu \omega_\nu$, where $\mu, \nu = 1, 2$ and $\Lambda_\mu^\nu$ is a matrix of differential forms. The general solution is

$$X = (A w + B) \frac{\partial}{\partial w} + (\overline{A} \overline{w} + \overline{B}) \frac{\partial}{\partial \overline{w}} + C \frac{\partial}{\partial v} + \frac{1}{2}(A + \overline{A})G \frac{\partial}{\partial G},$$

$$+ D_1 e^{i \sin \alpha \cos (v \cos \alpha)} \frac{\partial}{\partial G} + D_2 e^{i \sin \alpha} \sin (v \cos \alpha) \frac{\partial}{\partial G},$$

where $A, B \in \mathbb{C}$, and $C, D_1, D_2 \in \mathbb{R}$ are constants. Real generators are

$$X_1 = \partial_v + \partial_w, \quad X_2 = i(\partial_w - \partial_{\overline{w}}), \quad X_3 = i(w \partial_w - \overline{w} \partial_{\overline{w}}),$$

$$X_4 = \partial_{\overline{w}}, \quad X_5 = w \partial_w + \overline{w} \partial_{\overline{w}} + G \partial_G,$$

$$X_6 = e^{i \sin \alpha} \sin (v \cos \alpha) \partial_G, \quad X_7 = e^{i \sin \alpha} \cos (v \cos \alpha) \partial_G.$$  

The commutation relations between these vector fields are given by the following table, the entry in row $i$ and column $j$ representing $[X_i, X_j]$.

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<td>0</td>
<td>0</td>
<td>$-\sin \alpha X_7 + \cos \alpha X_6$</td>
<td>$X_7$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

1 Note that the corresponding algebra of Lie point symmetries for the heavenly equation (2.7) is infinite-dimensional [2]. In order to obtain a finite-dimensional algebra one needs to factorize it by the infinite-dimensional gauge algebra corresponding to the freedom in the definition of $\Omega$. In our case the gauge freedom in $\Omega$ was already used to find the canonical form of the Killing vector. There is no residual gauge freedom in $F$. 

---

7
6.1 Group invariant solutions

We can simplify equation (4.18) by looking at group invariant solutions. The finite transformation generated by $X_7$ does not change the metric. The one by $X_5$ rescales it by a constant factor. All transformations are conformal Killing vectors for $h$.

- $X_3 = i(u\partial_u - w\partial_w)$ and the corresponding solutions depend on $(t, R := \ln(w/u))$. This will lead to a new 2D integrable system (6.26). Multiplying (4.18) by $e^R$ yields

$$
(G + G_{uv} - 2\sin \alpha G_v)G_{RR} - (e^{i\alpha} G_R - iG_v R)(e^{-i\alpha} G_R + iG_v R) = 4e^R.
$$

The ideal (6.24) reduces to

$$
0 = i\partial_t Q \wedge \partial J + e^{-i\alpha} (J\partial_t Q \wedge \partial t - Q\partial_t J \wedge \partial t) - 4d(e^R) \wedge \partial t,
$$

where $J = G_R$, $Q = (e^{i\alpha} G - iG_v)$. Eliminate $Q$ and use $(J, t)$ as coordinates to obtain an equation for $R(J, t)$

$$
4(e^R)_{tJ} + R_{tt} + 2(JR_t)_t \sin \alpha + J(JR_t)_{tJ} = 0. \quad (6.27)
$$

Putting $R(J, t) = f(J) + g(t)$ yields (for constant $\alpha$)

$$
R(J, t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3 \tanh \sqrt{4J^2 + 1} + \alpha_3.
$$

A simple solution to (4.18) is

$$
G = e^{-\sin \alpha (w/u) + \frac{b}{1 + 3\sin^2 \alpha}}. \quad (6.28)
$$

It has

$$
\Omega(\omega, z, \overline{\omega}, \overline{z}) = (z \overline{z} [\cos^2 \alpha]/2 \left( \frac{\omega}{\overline{\omega}} \right)^{(i\sin \alpha \cos \alpha)/2} \frac{w}{b} + (z \overline{z})^{1 + (\cos \alpha)/2} \frac{4b}{1 + 3\sin^2 \alpha}. \quad (6.29)
$$

Calculation of curvature components shows it describes a flat metric on $\mathbb{R}^4$. Therefore the corresponding EW metric belongs to a class described in [11].

- $X_4 = \partial_u$. Equation (4.18) reduces to $GG_{\overline{w}w} - G_{\overline{w}}G_w = 4$. Define $\Psi(\omega, \overline{\omega})$ by $e^\Psi = G$. The EW structure is (after rescaling by $16e^{-2\Psi}$) given by

$$
\begin{align*}
\omega_h &= 16e^{-2\Psi} dw d\overline{\omega} - (dw - i e^{-i\alpha} \Psi_w dw + i e^{i\alpha} \Psi_{\overline{w}} d\overline{\omega})^2, \\
\nu_h &= -2\sin \alpha dv + (2\sin^2 \alpha + i \sin 2\alpha) \Psi_w dw + (2\sin^2 \alpha - i \sin 2\alpha) \Psi_{\overline{w}} d\overline{\omega},
\end{align*}
$$

where $\Psi_w = 4e^{-2\Psi}$ (Liouville equation).

The general solution to the Liouville equation is

$$
e^\Psi = \frac{i(P - \overline{P})}{4\sqrt{P_w P_{\overline{w}}}}.
$$

---

\*With the definition $\xi := \ln J, M := M(v, \xi) = R - 2\xi$ we have

$$
M_{ss} + 2M_s \sin \alpha + M_{tt} + 4e^M (M_{\xi\xi} + M_{\overline{\xi}}^2 + 3M_{\xi}^2 + 2) = 0. \quad (6.26)
$$
Define new coordinates $(\phi, \theta, \psi)$ by
\[ u = 2b \tan(\theta/2) e^{i\phi}, \quad dv = \cos a (d\psi - d\phi) + (\sin a) \tan(\theta/2) d\theta \]
to obtain
\[ h = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 a (d\psi - \cos \theta d\phi)^2, \quad \nu = -2a (d\psi - \cos \theta d\phi) \]
which is the EW structure on the Berger sphere. Calculating the curvature components shows that the corresponding hyper-Kähler metric is flat. The transformation of solution (6.29) corresponding to Lie point symmetries
\[ \psi = \nu \tan(\theta/2) \]
gives a new solution. In particular (6.28) can be obtained in this way. Therefore the metric corresponding to (6.28) also describes a Berger sphere. If $\alpha = 0$ then (6.29) and (6.28) coincide and give the standard metric on $S^3$.

- $X_2 = i(\partial_u - \partial_\nu)$ (or $X_1$). This reduction leads to a linear equation. Put $w + \overline{\nu} = f$ to obtain
\[ (G + G_{e, v} - 2\sin a G_v) G_{f f} - (e^{i\alpha} G_R - i G_{e, f}) (e^{-i\alpha} G_f + i G_{e, f}) = 4. \]

With the definition $J := G_f$, $Q := (e^{i\alpha} G - i G_v)$ this yields
\[ 0 = i dQ \wedge dJ + e^{-i\alpha} (J dQ \wedge dv - Q dJ \wedge dv) - 4 d\nu \wedge dv, \]
\[ 0 = dQ \wedge dv - e^{i\alpha} J dJ \wedge dv - i dJ \wedge dv. \]

Now eliminate $Q$ and use $(v, \xi = \ln J)$ as coordinates to obtain a linear equation for $f(\xi, v)$
\[ 4 e^{-2\xi} (f_{\xi \xi} - f_\xi) + f_{vv} + 2 \sin a f_{\xi v} + f_{\xi v} = 0. \]

7 Hidden symmetries

In this section we shall find a recursion procedure for generating ‘hidden symmetries’ of (4.17). We start with discussing the general conformally invariant wave equation in Einstein–Weyl background.

A tensor object $T$ which transforms as
\[ T \rightarrow \phi^m T \quad \text{when} \quad h_{ij} \rightarrow \phi^3 h_{ij} \]
is said to be conformally invariant of weight $m$. Let $\beta$ be a $p$-form of weight $m$. The covariant derivative
\[ D \beta := d\beta - \frac{m}{2} \nu \wedge \beta \]
is a well defined $p + 1$ form of weight $m$. Its Hodge dual, $*_h D \beta$, is a $(2 - p)$-form of weight $m + 1 - p$. Therefore we can write the weighted Weyl wave operator which takes $p$-forms of weight $m$ to $(3 - p)$-forms of weight $m + 1 - p$
\[ D *_h D = \left( d - \frac{m + 1 - p}{2} \nu \wedge \right) *_h \left( d - \frac{m}{2} \nu \wedge \right). \]
Another possibility is to set \( k \) and let \( g \) gauge freedom. In \([4]\) it was assumed that \( k = 0, m = -1 \) and \( \nabla_i \nu^i = 0 \) (the Gauduchon gauge) which led to the derivative of the generalised monopole equation (1.4):

\[
\nabla^i \nabla_i \phi - \left( m + \frac{1}{2} \right) \nu^i \nabla_i \phi + \frac{1}{4} \left( m(m+1) \nu^i \nu_i - 2m \nabla^i \nu_i \right) \phi = k \left( R + 2 \nabla^i \nu_i - \frac{1}{2} \nu^i \nu^i \right) \phi.
\]

At this stage one can make some choices concerning the values of \( m \) and \( k \). One can also fix the gauge freedom. In \([4]\) it was assumed that \( k = 0, m = -1 \) and \( \nabla_i \nu^i = 0 \) (the Gauduchon gauge) which led to the derivative of the generalised monopole equation (1.4):

\[
\nabla^i \nabla_i \phi + \frac{1}{2} \nu^i \nabla_i \phi = 0.
\]

Another possibility is to set \( m = -(1/2), k = 1/8 \). With this choice equation (7.32) simplifies to

\[
\nabla^i \nabla_i \phi = \frac{1}{8} R \phi,
\]

which is the well known conformally invariant wave equation in the 3D Riemannian geometry. Note that the gauge freedom was not fixed to derive the last equation. All we did was to get rid of the 'non-Riemannian' data.

### 7.1 The recursion procedure

Let \( \delta F \) be a linearised solution to (2.12) (i.e. \( F + \delta F \) satisfies (2.12) up to the linear terms in \( \delta F \)). Then

\[
\left( \left[ (\nu F_{uw} - F_{uw}) \frac{\partial^2}{\partial \rho \partial \tilde{w}} - (\nu F_{u\tilde{w}} + F_{u\tilde{w}}) \right] \frac{\partial^2}{\partial \rho \partial u} - (\nu^2 F_{uu} - F_{uu}) \right) \frac{\partial^2}{\partial \rho \partial u} + F_{uw} \frac{\partial^2}{\partial u \partial \tilde{w}}
\]

\[+ \eta \left[ (\nu F_{uw} - F_{uw}) \frac{\partial}{\partial \tilde{w}} + (\nu F_{u\tilde{w}} + F_{u\tilde{w}}) \right] \frac{\partial}{\partial u} \right) \delta F = F_{uw} \delta F.
\]

This equation can be viewed more geometrically: let \( \square_\Omega \) denote the wave operator on an ASDV curved background given by \( \Omega \), let \( \partial \Omega \) be the linearised solution to the first heavenly equation and let \( \mathcal{W}_\Omega \) be the kernel of \( \square_\Omega \). It is straightforward to check \([6]\) that \( \delta \Omega \in \mathcal{W}_\Omega \). Indeed, put \( \partial := d \tau \circ \partial \rho + d z \circ \partial \tilde{z} \), \( \tilde{\partial} := d \tilde{w} \circ \partial \rho + d \tilde{z} \circ \partial \tilde{z} \) and rewrite (2.7) as \( (\tilde{\partial} \tilde{\partial} (\Omega + \delta \Omega))^2 = \nu \). For the linearised solution we have

\[
0 = (\partial \delta \Omega \wedge \tilde{\partial} \delta \Omega) = d \delta \Omega \wedge (\partial - \tilde{\partial}) \delta \Omega = d s_g \ d \delta \Omega = \square_\Omega \delta \Omega.
\]

Now impose the additional constrain \( \mathcal{L}_R \delta \Omega = \eta \delta \Omega \). This implies \( \delta \Omega = e^{\nu t} \delta F \). This yields

\[
0 = d s_g d (e^{\nu t} \delta F) = e^{\nu t} \left( \eta^2 (dt \wedge s_g dt) \delta F + \eta d t \wedge s_g d \delta F + \eta d F \wedge s_g dt + d s_g d \delta F \right).
\]

But \( d s_g dt = \square_\Omega t = 0 \) and \( dt \wedge s_g dt = |dt|^2 \nu_g \), therefore (7.33) is equivalent to

\[
\square_\Omega \delta F + \eta^2 |dt|^2 \delta F = 0.
\]

There should exist a choice of \( m \) and \( k \) which, in the appropriate gauge, reduces equation (7.32) down to (7.33).

Let \( \mathcal{W}_F \) be the space of solutions to (7.33) around a given solution \( F \). We shall construct a map \( R: \mathcal{W}_F \longrightarrow \mathcal{W}_F \). Let us start from the recursion operator for the heavenly equation \([6]\). Let \( \phi \in \mathcal{W}_\Omega \). Define a recursion operator \( R: \mathcal{W}_\Omega \longrightarrow \mathcal{W}_\Omega \) by

\[
\phi^A\nabla_A \nabla_A R \phi = \phi^A\nabla_A \nabla_A \phi, \quad \phi^A = (1, 0), \quad \phi^A = (0, 1), \quad A = 0, 1,
\]

\[
(7.34)
\]
Suppose that we introduce an invariant spin frame \( T \) sequence of parameters in which freely only on a two dimensional surface. Where \( \gamma \) in coordinates \( (\gamma, \kappa, \kappa) \).

**Alternative formulations**

To construct a reduced recursion operator we should be able to Lie derive (7.34) along \( K \). In order to do so we introduce an invariant spin frame

\[
\hat{A}^i := e^{-(1/3)\eta^i A^i}, \quad i^A_i := e^{(1/3)\eta^i A^i},
\]

in which \( \hat{\lambda} = (\pi_A^i \hat{A}^i) / (\pi_A^i A^i) \). Note that now \( \Gamma_{A/B} \neq 0 \). Recursion relations are

\[
e^{-\rho^i} \nabla_A B^i (e^\eta R\delta F) = \nabla A^i e^\eta \delta F.
\]

This yields the following result.

**Proposition 7.1** The map \( R : \mathcal{W}_F \to \mathcal{W}_F \) defined by

\[
\begin{align*}
me^{\eta u} (F_{w\bar{u}} (\eta - \partial_u) - (\eta F_w - F_{w\bar{u}}) \partial_{\bar{u}}) R\delta F &= 2 \partial_w \delta F \\
\tilde{m}e^{\eta u} ((\eta F_{\bar{w}} + F_{\bar{w}w})(\eta - \partial_w) - (\eta^2 F - F_{w\bar{u}}) \partial_{\bar{u}}) R\delta F &= 2 (\eta + \partial_w) \delta F.
\end{align*}
\]

generates new elements of \( \mathcal{W}_F \) from the old ones.

By cross differentiating we verify that two equations in (7.35) are consistent as a consequence of (2.12).

We start the recursion from two solutions \((e^{-\eta u}, \frac{2m}{m + \eta} \epsilon^{mu})\) to (7.33). Equations (7.35) yield

\[
e^{-\eta u} \longrightarrow -\frac{\eta F_{w\bar{u}}}{2m} \longrightarrow ..., \quad \frac{2m}{m + \eta} \epsilon^{mu} \longrightarrow F_{w} \longrightarrow ...
\]

Suppose that \( F = F(u, w, \bar{u}, T) \) depends on three local coordinates on a complex EW space and a sequence of parameters \( T = (T_2, T_3, \ldots) \). Put

\[
\frac{\delta F}{\delta T_n} := R \left( \frac{2m}{m + \eta} \epsilon^{mu} \right),
\]

so that \( T_1 = w \). The recursion relations \( R(\partial_{T_n} F) = \partial_{T_{n+1}} F \) form an over-determined system of equations which involve arbitrarily many independent variables, but initial data can be specified freely only on a two dimensional surface.

### 8 Alternative formulations

Here we shall give an alternative formulation of equation (2.12). Define functions \((V, S, \tilde{S})\) by

\[
4V := \eta^2 F - F_{ww}, \quad 2S := \eta F_w - F_{w\bar{u}}, \quad 2\tilde{S} := \eta F_{\bar{w}} + F_{u\bar{w}},
\]

so equation (2.12) takes the form

\[
V = \frac{(-e^{2\rho u} + \delta S \tilde{S})\eta}{S_{\bar{w}} + \tilde{S}_w}, \quad S_u + \eta S = 2V_w, \quad -\tilde{S}_u + \eta \tilde{S} = 2V_{\bar{w}}.
\]  

\[(8.36)\]
\[
\begin{align*}
\text{ds}^2 &= \epsilon^{\eta} (V(dt^2 - du^2) + V^{-1}(S\bar{S} - e^{2\mu u})dw d\bar{w} + S(dt - du)dw + \bar{S}(dt + du)d\bar{w}) \\
&= \epsilon^{\eta} (V^{-1}h + V(dt + \omega)^2)
\end{align*}
\]

where
\[
\begin{align*}
h := -e^{\mu u}dw d\bar{w} - \left(Vdu + \frac{Sdw - \bar{S}d\bar{w}}{2}\right)^2 \\
\omega := \frac{Sdw + \bar{S}d\bar{w}}{2V},
\end{align*}
\]

and the EW one-form corresponding to \( h \) is
\[
\nu = 4\rho du + \frac{(\eta + 2\rho)Sdw + (\eta - 2\rho)\bar{S}d\bar{w}}{V}.
\]

Euclidean reality conditions force \( \bar{S} = -S \) and \( V \) real. On the ++ -- slice we have \( \bar{S} = S \), or alternatively (on a different real slice) functions \( V, S, \bar{S} \) real and independent. The orthonormal frame on the Euclidean slice is
\[
\begin{align*}
e^1 &= \frac{1}{2}(e^{imv} dw + e^{-imv} d\bar{w}), \quad \nabla_1 = e^{-imv} \partial_w + e^{imv} \partial_{\bar{w}} + i \frac{Se^{-imv} - \bar{S}e^{-imv}}{2V} \partial_\theta \\
e^2 &= \frac{i}{2}(e^{-imv} dw - e^{imv} d\bar{w}), \quad \nabla_2 = i(e^{-imv} \partial_w - e^{imv} \partial_{\bar{w}}) - \frac{Se^{-imv} + \bar{S}e^{imv}}{2V} \partial_\theta \\
e^3 &= Vdw - \frac{Sdw - \bar{S}d\bar{w}}{2}, \quad \nabla_3 = \frac{1}{V} \partial_\theta.
\end{align*}
\]

The EW one form is
\[
\nu = 2\omega \cos \alpha - \frac{4}{V} \sin \alpha e^3 = \frac{\cos \alpha (Sdw + \bar{S}d\bar{w}) - 2i\sin \alpha (\bar{S}d\bar{w} - Sdw)}{V} - 4\sin \alpha \partial v.
\]

Equations (8.36) can be rewritten in a compact form
\[
d e^3 = \omega \wedge e^3 \cos \alpha + \frac{\cos \alpha}{V} e^1 \wedge e^2,
\]

(8.39)

\( d(e^1 + ie^2) = e^{i\alpha} \omega \wedge (e^1 + ie^2) + \frac{ie^{i\alpha}}{V} e^3 \wedge (e^1 + ie^2) \) (8.40)

(the last relation is an identity). In fact the converse is true:

**Proposition 8.1** Let \( (e^1, e^2, e^3) \) be real one-forms which satisfy (8.39,8.40) for some real one-form \( \omega \), function \( V \) and constant \( \alpha \). Then there exist local coordinates \( w \in \mathbb{C}, v \in \mathbb{R} \) and a complex function \( S(w, \bar{v}, \bar{v}) \) such that \( (e^1, e^2, e^3) \) are of the form (8.37) and the Euclidean version of (8.36) is satisfied.

**Proof.** Equation (8.40) and the Frobenius theorem imply that \( e^1 + ie^2 = e^{i\alpha} dw \) (where \( m = e^{i\alpha} \)) for some complex functions \( \chi \) and \( w \), which therefore satisfy
\[
d\chi = \omega + \frac{i}{V} e^3 - Sdw
\]

for some \( S \). Put \( \chi = v + iY \) (for \( v, Y \in \mathbb{R} \)) so that
\[
dv = \frac{1}{V} e^3 + \frac{i}{2}(Sdw - \bar{S}d\bar{w}), \quad dY = \frac{1}{2}(Sdw + \bar{S}d\bar{w}) - \omega.
\]

Now we use the conformal freedom of (8.39,8.40) and rescale
\[
e^1 + ie^2 = \Phi e^{i\theta}(e^1 + ie^2), \quad e^3 = \Phi e^3, \quad \hat{V} = \Phi V,
\]
so that we can put \( Y = 0 \), and (8.40) is solved. Now
\[
\omega = \frac{Sdw + \overline{S}d\overline{w}}{2V}, \quad e^{3} = VdT - iSdw - \overline{S}d\overline{w},
\]
and the equation (8.39) gives (8.36).
\[\square\]

Recall that a geodesic congruence \( \Gamma \) in a region \( U \subset \mathcal{W} \) is a set of geodesics, one through each point of \( U \). Let \( W^i \) be a generator of \( \Gamma \) (a vector field tangent to \( \Gamma \)). Then the geodesic condition is \( W^j D_j W^i \sim W^i \). The formula (8.39) implies that \( e^{3} \) generates a shear-free geodesic congruence, with twist and divergence given by:
\[
\text{twist} = s_b (e^{3} \wedge de^{3}) = \frac{\cos \alpha}{V}, \quad \text{divergence} = s_b d s_b e^{3} = \frac{6 \sin \alpha}{V}.
\]

They are both solutions of the generalized monopole equation (1.4). Conversely, it follows from [5] that if the twist and the divergence of a shear-free geodesic congruence on an EW space are proportional, then this EW space arises as a reduction of a hyper-Kähler metric\(^5\). Therefore solutions to (4.18) (or equivalently the Euclidean version of (8.36)) are completely characterized by the existence of a shear-free geodesic congruence of the above type. It should however be stressed that, given an EW structure, there is no a priori way of telling if this special shear-free geodesic congruence exists. It would be interesting to find a local obstruction to the existence of such congruence.

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References


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