The Extension Theorem

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THE EXTENSION THEOREM

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To Ludwig Danzer on the occasion of his 70th birthday

Abstract

Given a compact convex polyhedron, can it tile space in a transitive (or in a regular) way? We discuss in the paper the so-called extension theorem giving conditions under which there is unique extension of a finite polyhedral complex, which consists of replicas of the given polyhedron, to a global isohedral tiling. The extension theorem gives a way to get all possible regular tilings with the given polyhedron. The well-known results on fundamental domains in the case of a translational group or of a Coxeter group generated by mirrors follow from the extension theorem too. The extension theorem gives a method of describing which finite point sets can admit extension to a regular point orbits with respect to crystallographic groups.

1. Introduction

A basic problem we are going to discuss in this paper is to describe conditions to help determine whether a fixed polyhedron admits an isohedral tiling (or in another terminology a regular tiling or tile-1-transitive tiling). This problem is very close to a question which goes back to Poincaré’s investigations on Fuchian groups in ([1]). Poincaré initiated the powerful method of studying and describing discrete groups by means of their fundamental domains. This method has since been strongly developed (see for instance [9], [17], [16], [19]). To my knowledge, the most complete development of this idea for space of arbitrary dimension can be found in [7]. After getting acquainted with Venkov’s work [6] on parallelohedra Alexandrov realized that an abstract complex built up with finitely many different shapes of convex polyhedra can be embedded (i.e. mapped in a one-to-one way) into space if and only if the tiling property holds around each face of dimension \(d - 2\).

Basically all these papers are concerned with the question whether a polyhedron represents a fundamental domain of some group of isometries. This is a good place to recall the part B of Hilbert’s eighteenth problem [2]:

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Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by mean of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.

Question B asks (in part) “whether there exists a tile such that no symmetry group acts transitively on any tiling in which it appears” ([21], p. 22).

An example of such a three-dimensional tile was found by Reinhardt in [3] and in the euclidean plane by Heesch [5]. These tiles admit no isohedral tiling.

However there are also polyhedra which admit isohedral tilings but are not fundamental domains for any group. It is unclear how many such tiles there are but they do exist. This fact motivated us to present a criterion to test whether a convex polyhedron will tile space in a face-to-face and isohedral way independent of whether it is a fundamental domain or not.

We do not care about whether a tile of the isohedral tiling is a fundamental domain of the symmetry group of the tiling or even whether it has a proper subgroup which still operates transitively on the tiling.

In this context we will be interested in the following questions concerning a given polyhedron $P$:

1. does the polyhedron $P$ admit at least one isohedral (not necessarily fundamental) tiling?

2. how can one get all isohedral tilings by replicas of $P$?

Since a polyhedron may admit several different tilings, the following question arises. Does there exist and how can one describe such a compound of replicas of $P$ (i.e. a finite, possibly small, monohedral complex) that guarantees existence of a unique isohedral tiling containing this compound as a finite subcomplex?

This question has been answered in terms of coronas. The term of a corona was introduced by P. Engel though the concept of a corona under the name of a neighborhood had been in active use after the discovery in 1974 of the local approach both for regular tilings and point sets [11], [26].

The local theorem for Delone sets (for tilings) states that a Delone set (a tiling) all of whose points (tiles) have the same neighborhoods (coronas) of some radius is regular. However, as was understood very soon, the local theorems have the following serious defect: they says nothing about the neighborhood itself. The first explicit intrinsic, i.e. depending only on a neighborhood itself, description of the properties of such a neighborhood in case of point sets was given in [12]. In this paper it was also said that the question whether a finite point set can be extended to a regular point set can be reduced to a similar question for some finite polyhedral complexes, i.e. to the extension theorem for polyhedra. Though the extension theorem and some its applications have
been presented several times (see for instance [14], [20], [28]) this paper aims to present for the first time a more detail exposition of this theorem.

The corona approach may seem inconvenient but the method of Poincaré is rather complicated even for the fundamental polyhedron case and we do not see at all how it can be modified for more the general case. Moreover, as will be shown, a few well-known descriptions of fundamental polyhedra can be easily deduced by the corona approach.

2. Definitions and notations

Given a tiling, a corona about a tile $P$ is, roughly speaking, a special kind of a finite complex of the tiles surrounding the tile $P$. In a $d$-dimensional tiling there can be considered exactly $d$ different kinds of coronas depending on the choice of the dimension of faces at which they are constructed. Let us fix dimension, say $d - 1$, of faces in the tiling, then the *corona of radius 1 about the polyhedron $P$ at $(d - 1)$-dimensional faces* is a complex consisting of the tile $P$ and all tiles of the tiling which share a $(d - 1)$-face with $P$. To get the *corona of radius 1 at $(d - 2)$-faces* we need to add to the corona at $(d - 1)$-faces of $P$ all tiles which cover all $(d - 2)$-faces of the tile $P$ as well. The corona at $i$-dimensional faces (about $P$ of radius 1) consists of all tiles adjacent with $P$ at all faces of dimensions equal or greater than $i$. A corona of arbitrary (integer positive) radius is determined in recurrent way. It is clear that the coronas at $(d - 1)$-faces of a fixed radius are the least numerous complexes among the coronas of the same radius but at faces of lower dimensions. Nevertheless, it turns out that in the Local Theorems for regular and multiregular tilings ([18], [26]) it suffices to consider only the coronas at faces of dimension $d - 1$.

However, in the case where we start with a finite complex and have no tiling apriori, the consideration of the coronas at faces of codimension 1 becomes unsufficient. First of all we need to choose the minimal dimension of faces at which we should best consider coronas for our goal. Certainly, one could take this dimension to be possibly small, for instance, equal to 0. However, the choice of the fullest coronas would make checking whether such a corona exists a very hard task. In order to make our theorem more easily applicable this minimal dimension must be as large as possible. The treatment of the problem shows that the dimension of faces at which we consider coronas is $d - 2$. So, we will consider coronas at faces of dimension $d - 2$.

Before the rigorous definitions which follow, we need to make one more important remark. Since we are interested in coronas which are extended to *tilings* of the whole space the coronas seem to have to be *embedded* in space. But, indeed, by the following definition of a corona, the polyhedra entering the corona are allowed to overlap (for reasons see Remark 3.7 in Section 3). In order to describe correctly a self-intersecting corona we need first to introduce the concept of an abstract corona. Only after that
we will consider a realization of the abstract corona in space by means of a special immersion of it into space. From now on, by corona we will mean such a map of an abstract corona into space.

Given a convex closed euclidean (hyperbolic or spherical) \( d \)-dimensional polyhedron \( P \), consider an abstract locally finite complex \( C \) which besides standard requirements for a complex also satisfies the following:

(i) the complex \( C \) consists of replicas of \( P \) (the monohedrality condition);

(ii) given polyhedra \( P' \) and \( P'' \in C \), there exists a path \( P_0 (= P') , \ldots , P_m (= P'') \) in that any any two successive polyhedra \( P_i \) and \( P_{i+1} \) share \( (d-2) \)-face (the connectedness condition).

To define coronas at faces of dimension \( d-2 \) we introduce the following definition of distance in the complex \( C \).

**Definition 2.1.** The distance \( d(Q, Q') \) between polyhedra \( Q \) and \( Q' \in C \) is the length \( l \) of the shortest path \( P_0(= Q)P_1 \ldots P_{i-1}P_i(= Q') \) in which any two successive polyhedra \( P_{i-1} \) and \( P_i \) have a \( (d-2) \)-face in common.

Now we can describe what an abstract corona is.

**Definition 2.2.** Given a positive integer \( k \), a complex \( C \) satisfying (i) and (ii) is called an abstract corona (at faces of dimension \( d-2 \)) of radius \( k \) about the polyhedron \( P \in C \) if the following hold:

(AC1) the distance \( d(P, Q) \leq k \) for all \( Q \in C \);

(AC2) if \( d(P, Q) \leq k-1 \) then for each \( (d-2) \)-face \( F^{d-2} \) of \( Q \) in \( C \) there are \( d \)-polyhedra \( Q_i, i = 1, 2, \ldots , m \), that form a cyclic sequence around the \( (d-2) \)-face \( F^{d-2} \), i.e. such that \( Q_1 = Q_m = Q \) and \( Q_i \cap Q_{i+1} = F_{i}^{d-1} \cap F^{d-2} \) for any \( i = 1, \ldots , m \).
A corona of radius 2 around a polyhedron $P$; each $(d-2)$-face of a polyhedron $Q$ in the 1-corona of $P$ is enclosed by polyhedra $Q_i$ of the 2-corona.

Denote an abstract corona of radius $k$ about $P$ by $C_k(P)$ and realize it in space by means of a map (if it exists)

$$f : C_k(P) \to X^d, \quad \text{where } X^d \text{ is euclidean, hyperbolic or spherical space},$$

such that the following holds:

(M1) $f$ is an isometry when restricted to each $d$-polyhedron $Q \in C_k(P)$;

(M2) if polyhedra $Q$ and $Q' \in C_k(P)$ share a $(d-2)$-face then their images $Q$ and $Q'$ do not overlap in space.

If the polyhedra $Q$ and $Q'$ meet at a $(d-1)$-face $F$ they automatically then share all $(d-2)$-dimensional faces bounding the $(d-1)$-face $F$. By the condition (M2), for two polyhedra $Q$ and $Q \in C_k(P)$ sharing a $(d-1)$-face $F$ their images $Q : = f(Q)$
and $Q': = f(Q')$ are on opposite sides of the $(d - 1)$-hyperplane supporting the image $f(F)$.

**Definition 2.3.** The map $f(C_k(P))$ is called a *corona* about the polyhedron $P(= f(P))$ of radius $k$ and will be denoted $C_k(P)$.

The corona $C_k(P)$ centered at a polyhedron $P$ consists of pairwise congruent convex polyhedra. Since we allow the corona’s polyhedra to overlap it is not a complex embedded in space. However, the map $f$ is an immersion of the complex $C_k(P)$ at any its $(d-2)$-dimensional faces, i.e. $f$ is a homeomorphic mapping in some sufficiently small neighborhood of any relatively interior point of each $(d-2)$-face.

From now on we will say that a polyhedron $P$ admits a corona of radius $k$ if there exist a monohedral abstract corona $C_k(P)$ and a map $f$ fulfilling conditions (M1) and (M2).

**Remark 2.4.** Thus, the existence of a corona does not imply an *embedding* of the corona. But the corona is definitely required to be immersed at any of its $(d-2)$-faces. In other words, the map $f$ is *not* a homeomorphism between the abstract corona and its realization in space $X^d$.

3. The Extension Theorem

Recall that in an isohedral tiling the symmetry group operates transitively on the tiles. In particular, for every tile $P$ and its neighbour $P'$ at a common $(d-1)$-dimensional face there is an isometry $g$ from the symmetry group of the tiling that sends $P$ into $P'$ and the corona $C_k(P)$ into the corona $C_k(P')$. The coronas of these neighboring polyhedra overlap. The image $C_k(P)g$ agrees with the corona $C_k(P)$ in the sense that the symmetry either takes polyhedron from $C_k(P)$ to some polyhedron of $C_k(P)$ or moves it out the corona.

We need to reflect this in the extension theorem in some appropriate way. However, we have only one corona, not a tiling and, furthermore, polyhedra in our corona a priori are allowed to overlap. The notion of which coronas agree gets more complicated.

**Definition 3.1.** Consider the abstract corona $C_k(P)$ and a polyhedron $Q \in C_k(P)$. The following subcomplex

$$U_k(Q) : \{ Q' \in C_k(P) \mid d(Q, Q') \leq k \}$$

is called a *$k$-neighbourhood* $U_k(Q)$ of the polyhedron $Q$ in the abstract corona $C_k(P)$. We call the image $U_k(Q) := f(U_k)$ the *$k$-neighbourhood* of the polyhedron $Q$ in the corona $C_k(P)$.

**Definition 3.2.** Given a $d$-polyhedron $P$ along with a (monohedral) corona $C_k(P)$ of radius $k$, let $P' \in C_k(P)$ share a $(d-1)$-face with $P$, let $U_k(P')$ be its $k$-neighbourhood in $C_k(P)$, and $\partial$ an isometry such that $P\partial = P'$. We say that the
image $C_k(P)\partial$ agrees with $C_k(P)$ if the neighbourhood $U_k(P')$ belongs to $C_k(P)\partial$ as well.

**Remark 3.3.** The last definition expresses what happens in isohedral tilings. Indeed, if in an isohedral tiling one corona around $P$ moves under an appropriate symmetry $\partial$ into a corona around $P'$ with $d(P, P') = 1$, then $C_k(P') = C_k(P)\partial$ overlap, and furthermore

$$C_k(P') \cap C_k(P)\partial = U_k(P').$$

This relationship corresponds to the definition of which coronas agree. Note that it is unclear whether the last relationship is true for $P$ and $P'$ with $d(P, P') > 1$.

Denote by $S_k(P)$ the group of isometries of space which leave invariant the center $P$ and send each polyhedron $Q$ of the corona $C_k(P)$ to some polyhedron $Q' \in C_k(P)$.

It is obvious that $S_k(P) \subseteq S_{k-1}(P)$. Note also that the group may be not the full symmetry group of the corona because it is possible that the corona may also admit automorphisms which do not leave $P$ invariant ( if $C_k(P)$ has also another center).

**Theorem 3.4.** Let a convex polyhedron $P \subset X^d$, ($X^d$ is either euclidean $E^d$, or hyperbolic $H^d$, or spherical space $S^d$ ) admit a (monohedral) corona $C_k(P)$ of some radius $k$, fulfilling the following:

(i) $S_{k-1}(P) = S_k(P)$, and

(ii) for each polyhedron $P\nu$ which meets the polyhedron $P$ at a $(d - 1)$-face $F\nu \subset P$, $\nu = 1, \ldots, n$, there is an isometry $\partial\nu$ such that $P \partial\nu = P\nu$ and the image $C_k(P)\partial\nu$ agrees with $C_k(P)$.

Then

(1) the corona $C_k(P)$ is an embedded complex and admits an extension to an isohedral tiling $T$;

(2) the isohedral tiling $T$ is uniquely determined by the corona $C_k(P)$;

(3) the (full) symmetry group $S(T)$ of the tiling $T$ contains a group $\langle \partial\nu \rangle$, generated by the isometries $\partial\nu$, and the group $S_k(P)$: $S(T) \supseteq \langle \partial\nu \rangle$, $S(T) \supset S_k(P)$.

(4) the group $\langle \partial\nu, \nu = 1, \ldots, m \rangle$ operates on $T$ transitively;

(5) the point group $S_k(P)$ is the stabilizer of the tile $P$ in the symmetry group $S(T)$.

**Remark 3.5.** The conditions (i) and (ii) obviously hold for every isohedral tiling for some value of $k$. Thus these conditions not only are sufficient but necessary too. This is important because it shows that the theorem gives a finite procedure for finding
all isohedral tilings of space by the given polyhedron $P$. By constructing and testing all possible coronas of bounded radius one can determine all possible isohedral tilings with replicas of $P$. Only the coronas satisfying the hypothesis of the extension theorem can extend to an isohedral tiling.

**Remark 3.6.** We note that both conditions (i) and (ii) are essential. Consider Böröczky’s tiling of the hyperbolic plane by congruent pentagonal shapes ([10], [22], [23]). It is well-known that this shape admits no isohedral tiling (in fact, even no crystallographic). Meanwhile this shape $P$ has only one corona $C_1(P)$ of radius 1. It fulfills the agreement condition (ii) and does not satisfy condition (i) on $S_{k-1}(P) = S_k(P)$. This corona has two different extensions to coronas of radius 2. Each of them fulfills condition (i) for $k = 2$ but condition (ii) does not hold for both of them. Exactly the same can be said about coronas $C_k(P)$ for any $k > 2$. So there is no monohedral corona about this shape which would fulfill both conditions.

**Remark 3.7.** As shown in [27], in contrast to $\mathbb{H}^2$ in the euclidean plane the identity of the first coronas in a tiling implies its regularity although not all these corona fulfill condition (i). The matter is that in the 2-dimensional euclidean case the identity of the first coronas about a given shape causes the uniqueness of the second corona which turns out to fulfill both conditions (i) and (ii).

**Remark 3.8.** The size of coronas to be tested is bounded, depending on the order of the symmetry group of the polyhedron $P$. However, in contrast to the hyperbolic and spherical cases, the total number of $(d-1)$-faces in a $d$-dimensional euclidean face-to-face space-filler is bounded by some constant $c(d)$ ([8], see also [24]). From this one can get a corresponding upper bound for the radius of monohedral coronas to be tested.

**Remark 3.9.** Though we assume the corona to be not embedded, just immersed at $(d-2)$-faces, from the extension theorem it follows that indeed it has to be and it is an embedded complex. Certainly, it is clear that a corona which is immersed but not embedded cannot extend to a tiling of space. It would seem natural to require it in advance. As a practical matter, for testing whether a polyhedron admits the starting corona, it is certainly easier to check only matching conditions at $(d-2)$-faces then to check why a set of adjacent polyhedra forms a self-nonoverlapping complex named a corona. For instance, later we will consider the so-called Coxeter polyhedra. It seems obvious that these polyhedra can form coronas fulfilling matching conditions at $(d-2)$-faces. But it is not easy to check whether the Coxeter polyhedra entering these coronas do not overlap. Note that if we have found out in some way that some polyhedra entering the corona do overlap, certainly, we may remove this corona from further considerations.

**Remark 3.10.** The extension theorem ceases to be true in non-simply connected
spaces, although the local theorem remains there true as well. It makes sense to quote here the statement of the local theorem for tilings ([26]):

A tiling $T$ of $X^d$ is isohedral if and only if there exists an integer $k > 0$ for which conditions a) and b) hold:

a) The $k$-th coronas $C_k(P)$ around all $P$ in $T$ are pairwise congruent,

b) $S_{k-1}(P) = S_k(P)$, $P \in T$.

Moreover, the group $S_k(P)$ is the stabilizer of $P$ in $S(T)$.

Though this statement means by $X^d$ one of three simply-connected spaces of constant curvature, in fact the simply-connectedness is not required. Suppose that there is a monohedral tiling of a space of constant curvature such that all stable coronas in it are pairwise identical then the tiling is isohedral. By identical coronas we mean ones that can be superposed by some isometry of the space. The proof of this more general theorem repeats the proof for the simply-connected space word by word.

On the other hand, given a locally euclidean two-dimensional surface of the cylinder of revolution it is easy to present on this nonsimply-connected surface examples of monohedral coronas that fulfill all the hypothesis of the extension theorem but admit no extension to global tilings of the cylinder.

4. The outline of the proof

The key idea of the proof is as follows. First, under the hypothesis of the theorem, we construct a new topological space $\bar{X}^d$ which is a polyhedral complex made of pasted together congruent copies of a convex polyhedron $P$. This polyhedral complex can be mapped in $X^d$ by a mapping which is an isometry on each $d$-cell of the complex and a homeomorphism around each $(d - 2)$-dimensional face of the complex $\bar{X}^d$. By Alexandrov’s theorem [7] this mapping is a homeomorphic mapping of $\bar{X}^d$ onto $X^d$. In other words, the polyhedral complex $\bar{X}^d$ is a tiling of $X^d$. Now one needs to note that this tiling has pairwise identical coronas $\bar{C}_k(P)$, $P \in T$, with $S_{k-1}(P) = S_k(P)$. By the local theorem for tilings the tiling is isohedral.

The realization of this idea starts with the following lemmata. Each of them is easy to prove. The proofs of the lemmas can be found in [29].

**Lemma 4.1.** If an isometry $\partial$ is such that $P \partial = P$, and $C_k(P) \partial$ agrees with $C_k(P)$ then $\partial \in S_k(P) \partial^\nu$.

**Lemma 4.2.** For every pair of isometries $\partial^\nu$ and $s$, where $s \in S_k$, there is a pair of isometries $\partial^\nu$ and $s'$, $s' \in S_k$ such that

$$(s \partial^\nu)^{-1} = s' \partial^\nu.$$
Lemma 4.3. For each pair of isometries \( \partial_v, s \), where \( s \in S_k(P) \), one has such a pair of isometries \( \partial_{v''}, s'' \in S_k(P) \), such that
\[
\partial_v s = s'' \partial_{v''}.
\]

Lemma 4.4. Let \( F^{d-2} \) be \((d-2)\)-face of the polyhedron \( P \) and \( Q_0 (= P), Q_1, \ldots, Q_m (= P) \) a circuit of all polyhedra in the corona \( C_k(P) \) that share the face \( F^{d-2} \). Then there is an appropriate sequence of facet generators \( \partial_{v_1}, \partial_{v_2}, \ldots, \partial_{v_m} \) such that
1. \( P \partial_{v_1} = Q_1, P \partial_{v_2} \partial_{v_1} = P_2, \ldots, P \partial_{v_m} \ldots \partial_{v_2} \partial_{v_1} = Q_m (= P) \);
2. \( C_k(P) \partial_{v_1} \ldots \partial_{v_2} \partial_{v_1} \) agrees with \( C_k(P) \) for any \( i = 1, \ldots, m \);
3. \( C_k(P) \partial_{v_1} \ldots \partial_{v_2} \partial_{v_1} \) contains \( C_{k-1}(P), i = 1, \ldots, m \).

Let
\[
\Gamma := \{ \partial_v, \nu = 1, \ldots, m; S_k(P) \},
\]
i.e. the group \( \Gamma \) is generated by the \( \partial_v \) and the group \( S_k(P) \).

From Lemmas 4.3 and 4.4, we have:

**Lemma 4.5.** Each element \( \gamma \in \Gamma \) can be presented in the form
\[
\gamma = s \partial_{v_m} \ldots \partial_{v_1}, \quad s \in S_k(P).
\]

Denote by \( G := \{ S_k \gamma \} \) the set of all cosets of \( S_k \) in \( \Gamma \) and denote an individual coset by \( g \).

Now to define a graph \( \mathcal{G} \) of the cosets of \( G \) we define the vertex set of \( \mathcal{G} \) to be the set \( G \) of cosets. Two vertices of the graph \( \mathcal{G} \) \( g := S_k \gamma \) and \( g' := S_k \gamma' \) are joined by an edge (or adjacent) if there is a generator \( \partial_{v_i} \) such that \( S_k \gamma = S_k \partial_{v_i} \gamma \).

**Lemma 4.6.** The definition of adjacent classes is symmetrical, i.e., if \( S_k \gamma = S_k \partial_{v_i} \gamma \) then there exists a generator \( \partial_{v_i} \) such that \( S_k \partial_{v_i} \gamma = S_k \gamma' \).

**Lemma 4.7.** \( \mathcal{G} \) is a connected graph.

Lemma 4.7 follows from Lemma 4.5.

An element \( \gamma \in \Gamma \) induces a mapping \( \gamma^* : G \rightarrow G \), where \( \gamma^* : S_k \gamma \rightarrow S_k \gamma_i \gamma \), Since the mapping also preserves edges of the graph the following lemma is true.

**Lemma 4.7.** The mapping \( \gamma^* \) is an automorphism of the graph \( \mathcal{G} \).
The graph $G$ serves as an auxiliary tool for constructing a new topological space denoted by $\mathbf{X}^d$. Conditions (i) and (ii) of the theorem imply the existence of a finite group $S_k(P)$ and several isometries $\partial_x$. Let $\Gamma$ be as defined above. By the hypothesis of the theorem we have $\partial_x S_k = S_k \partial_x$ and hence the group $S_k$ is a normal subgroup in $\Gamma$.

Now one can glue from copies of the given polyhedron $P$ a "new" space $\mathbf{X}^d$. First, take the direct product $P \times G \ni (P, g)$ and associate the polyhedron $P$ with each coset $g - (P, g) := P_g = P \cdot S_k \gamma$. The location of the polyhedra $P_g$ in $\mathbf{X}^d$ does not depend on the choice of an isometry in the coset $g$. At the same time we note that polyhedra associated with different cosets may overlap or even coincide. Nevertheless, $P_g$ and $P_{g'}$, if $g \neq g'$, are considered as different polyhedra in the space $\mathbf{X}$ being constructed.

Introduce in $P \times G$ the appropriate incidences. First, we arrange which associated polyhedra have common $(d-1)$-dimensional faces. In particular, the polyhedron $P$ along with its neighbours at $(d-1)$-faces $P_{\nu}$ enter $P \times G$. Two associated polyhedra $P_g$ and $P_{g'}$ are said to have a common $(d-1)$-face if and only if $g$ and $g'$ are adjacent cosets, i.e. $g = S \gamma$ and $g' = S \nu \partial \gamma$ for some $\nu$. In the particular case when $\gamma = e \in \Gamma$ this means that the polyhedron $P$ has a common $(d-1)$-face $F_{\nu}$ with the polyhedron $P_{\nu} = P \cdot \partial \gamma$ for any $\nu \in \{1, \ldots, m\}$. Since $\partial \nu S_k = S_k \partial \nu$ this order of identifying $(d-1)$-dimensional faces in $P \times G$ fits well throughout the $P \times G$.

After determining the incidences at faces of dimension $d-1$ one can determine the incidences in lower dimensions. Since the graph $G$ is connected every pair $P_g$ and $P_{g'}$ can be linked by a "path" in $P \times G$ such that any two successive polyhedra share a $(d-1)$-face (strong connectedness). Now, if a $(d-1)$-face $F$ be shared by $P_g$ and $P_{g'}$, then all lower dimensional faces of this common $(d-1)$-face are called common faces of $P_g$ and $P_{g'}$.

In general, two polyhedra $P_g$ and $P_{g'}$ are thought to have a face $F^j$, $j < d-1$ in common, if the face belongs to one of them, say to $P_g$, and, furthermore, in $P \times G$ there is a path of polyhedra $P_{g_i}, \ldots, P_{g_m}$ linking $P_g$ and $P_{g'}$ such that

(a) for every $i = 1, \ldots, m - 1$ $P_{g_i}$ and $P_{g_{i+1}}$ have a $(d-1)$-face in common and

(b) for each $i$ this common $(d-1)$-face contains the face $F^j$.

When they are determined these incidences turn the direct product $P \times G$ into a topological space $\mathbf{X}^d$. Each polyhedral constituent of $\mathbf{X}^d$ is a convex polyhedron embedded in the space $\mathbf{X}^d$. This allows us to transform the topological space $\mathbf{X}^d$ into a metric space: the distance $d(\mathbf{x}_1, \mathbf{x}_2)$ is the length of the shortest piecewise straight line linking $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbf{X}^d$.

Define a mapping $f : \mathbf{X}^d \to \mathbf{X}^d$ by the natural isometrical embedding $f(\mathbf{x}) = \mathbf{x}$ of each polyhedron $P_g$. Since this map does not enlarge the distance $(d(\mathbf{x}_1, \mathbf{x}_2) \geq ||f(\mathbf{x}_1) - f(\mathbf{x}_2)||)$, it is continuous.
Now, we consider the coronas $C_k(P_g)$ about the polyhedra $P_g$. They all are congruent to the corona $C_k(P)$ which fulfils the hypothesis of the theorem. Therefore the corona $C_k(P')$ about quite arbitrary $P'$ of $\mathbf{X}^d$ is also immersed in $(d - 2)$-faces.

Thus, we get the map $f : \mathbf{X}^d \rightarrow \mathbf{X}^d$ to fulfil the following conditions:

1. $f$ is a natural isometrical embedding of each polyhedron $P_g \subset \mathbf{X}^d$ in $\mathbf{X}^d$.
2. The map $f$ is continuous on the polyhedral complex $\mathbf{X}^d$.
3. If two polyhedra of $\mathbf{X}^d$ share a $(d - 1)$-face they are on the opposite sides of the supporting hyperplane of the face.
4. The map $f$ is an immersion on each $(d - 2)$-face of $\mathbf{X}^d$.

Now one can apply the theorem of A. Alexandrov [7] which states

Assume that a polyhedral complex $\mathbf{X}^d$ is pasted of finite number of different modulo congruence polyhedra. A mapping $f : \mathbf{X}^d \rightarrow \mathbf{X}^d$ that fulfils conditions (1) - (3) is one-to-one if and only if the mapping $f$ is one-to-one about each $(d - 2)$-face of $\mathbf{X}^{d-2}$, that is, the mapping fulfils also condition (4).

By this theorem the map $f$ is a homeomorphism of the $\mathbf{X}^d$ onto $\mathbf{X}^d$. Therefore the complex $\mathbf{X}^d$ is an embedded complex in the space $\mathbf{X}^d$, i.e. it is a tiling with pairwise congruent coronas fulfilling condition (i) of the extension theorem and consequently fulfils the the hypothesis of the local theorem. Now the basic conclusion (1) of the theorem easily follows from the local theorem for tilings.

The other conclusions of the theorem are easily derived from this.

In the next sections we discuss possible applications of the extension theorem.

5. The theorem on parallelohedra

Remember that a convex $d$-polyhedron $P \subset \mathbf{E}^d$ is called a parallelohedron if it can tile $d$-dimensional euclidean space $\mathbf{E}^d$ by translation. A particular example of a tiling of space by parallelohedra is the Voronoi tiling for a point lattice. This tiling is face-to-face and, since the symmetry group of the lattice operates on the tiles transitively, it is isohedral. Furthermore, it is obvious that a Voronoi parallelohedron and all its $(d - 1)$-faces are centrally symmetric.

On the other hand, it is also obvious that not every tiling of space by parallelohedra is face-to-face and isohedral. The following facts are known about a parallelohedron (Minkowski, Delone, Alexandrov):

(i) a parallelohedron $P$ is centrally symmetric;
(ii) each $(d - 1)$-face of $P$ is centrally symmetric;
(iii) a projection of \( P \) along each \((d-2)\)-face of \( P \) on the 2-dimensional complementary plane is a parallelogram or a centrally symmetrical hexagon.

Venkov [6], Alexandrov [7], and McMullen [13] proved:

**Theorem 5.1.** If a convex polyhedron \( P \in \mathbb{E}^d \) satisfies the three above mentioned conditions (i), (ii), (iii) then \( P \) is a parallelohedron.

Due to the following lemma, theorem 5.1 can be also derived from the extension theorem.

**Lemma 5.2.** Let a convex polyhedron \( P \) satisfy conditions (i), (ii), (iii). Then \( P \) admits a monohedral corona \( C_1(P) \) such that

1. \( S(P) = S_1(P) \),
2. for each \((d-1)\)-face \( F_\nu, \nu = 1, \ldots, n, \) of the polyhedron \( P \) there is a translation \( \partial_\nu \) such that \( \partial_\nu(P) = P_\nu \) and the image \( C_1(P) \partial_\nu \) agrees with \( C_1(P) \), where \( P_\nu \in C_1(P) \) is the polyhedron adjacent to \( F_\nu \).

**Remark 5.3.** By definition 3.2 the condition "\( C_1(P) \partial_\nu \) agrees with \( C_1(P) \)" means here that all polyhedra of the corona \( C_1(P) \) which share at least one \((d-2)\)-dimensional face of the face \( F_\nu \) must be contained in \( C_1(P) \partial_\nu \) as well.

Lemma 5.2 has a proof which is analogous to the proof of Lemma 6.1 in the next section. By this lemma, the extension theorem implies immediately that a polyhedron \( P \) with properties (i), (ii), (iii) admits an isohedral tiling \( T \) whose full symmetry group \( S(T) \) is \( \Gamma = \langle \partial_\nu, S(P), i = 1,2,\ldots,m \rangle \). Now, by the conclusion of the extension theorem, since the abelian subgroup \( \langle \partial_\nu \rangle \) generated by translations \( \partial_\nu \) operates on the tiling \( T \) transitively, \( P \) is a parallelohedron.

Note that, in general, parallelohedra may admit different coronas fulfilling the hypothesis of the extension theorem. This means that parallelohedra may also admit other tilings which are no longer translationally transitive.

**6. Coxeter polyhedra**

Recall that a compact convex polyhedron in \( \mathbb{X}^d \) is called a *Coxeter polyhedron* if all its interfacial angles are equal to \( \frac{\pi}{m_{ij}} \), where \( m_{ij} \) is a positive integer number. In a very nice paper [4] Coxeter classified all such polyhedra for euclidean and spherical spaces. In particular he showed that all spherical Coxeter polyhedra are simplices and in euclidean space they are either simplices of the full dimension or the direct product of Coxeter simplices of lower dimensions.
The Coxeter polyhedra are distinguished among all polyhedra in that they are fundamental domains for discrete groups generated by the reflections in \((d-1)\)-planes (see [19]).

This result also can be derived from the extension theorem. Indeed, the following lemma says that the angle conditions in Coxeter polyhedra guarantee the existence of at least one corona which fulfills all the hypothesis of the extension theorem. Since this corona is constructed by means of reflections in facets, it extends to such a isohedral tiling that a Coxeter group \(<\partial_1,\ldots, \partial_\nu>\) generated by reflections in \((d-1)\)-dimensional faces acts transitively on the tiling. Moreover, by the extension theorem the symmetry group of the tile is its stabilizer in the symmetry group of the tiling.

**Lemma 6.1.** Let \(P\) be a Coxeter polyhedron. Then \(P\) admits a monohedral corona \(C_1(P)\) which is constructed by means of reflections at \((d-1)\)-facets and is such that

1. \(S(P) = S_1(P)\),

2. for each \((d-1)\)-face \(F_\nu\), \(\nu = 1, \ldots, n\), of the polyhedron \(P\) the mirror \(\partial_\nu\) makes \(C_1(P)\) and the image \(C_1(P)\partial_\nu\) agree, where \(P_\nu \in C_1(P)\) is the polyhedron adjacent to \(F_\nu\).

**Proof.** Given a Coxeter polyhedron \(P\), one can paste from replicas of \(P\) an abstract monohedral corona at \((d-2)\)-faces about \(P\) of radius 1 by means of reflections at each \((d-1)\)-face \(F_i\) so that the cycle of polyhedra around each \((d-2)\)-face \(F_{ij}\) consists of an even number \(2m_{ij}\) of members. Remember that the dihedral angle in the face \(F_{ij}\) is equal to \(\frac{\pi}{m_{ij}}\). Hence, if in the abstract corona a cycle around each \((d-2)\)-face \(F_{ij}\) contains precisely \(2m_{ij}\) polyhedra then the abstract corona can be mapped in \(\mathbb{X}^d\) so that this mapping is immersion around each \((d-2)\)-face of \(P\).

The polyhedra which enter the corona \(C_1(P)\) and meet at the \((d-2)\)-face \(F_{ij}\) are as follows:

\[ P, P\partial_1, P \partial_2\partial_1, \ldots, P (\partial_2 \partial_3)^{m_{ij}-1}, P \partial_1(\partial_2 \partial_3)^{m_{ij}-1}, \]

where \(\partial_1\) and \(\partial_2\) are reflections in hyperplanes of \(F_i\) and \(F_j\) respectively.

Before we prove conclusions (1) and (2) we denote by \(\mathcal{F}_i\) the set of all \((d-1)\)-faces \(F_j\) of \(P\) which share a \((d-2)\)-face with a \((d-1)\)-face \(F_i\).

Let \(s \in S(P)\), i.e. \(P s = P\). It moves a pair of adjacent \((d-1)\)-faces \(F_i\) and \(F_j\) into a pair of also adjacent faces \(F_k\) and \(F_l\). From \(F_i s = F_k\) and \(F_j s = F_l\) it easily follows that

\[ \partial_1 s = s \partial_2, \quad \partial_2 s = s \partial_1. \]

The set of all polyhedra in the corona \(C_1(P)\) which share a \((d-2)\)-face with \(P\) consists of

\[ P, P\partial_1, P \partial_2\partial_1, \ldots, P (\partial_2 \partial_3)^{m_{ij}-1}, P \partial_1(\partial_2 \partial_3)^{m_{ij}-1}, \]

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where $F_j \in \mathcal{F}_i$. Under action of $s$ this set moves to

$$P s, P s\partial_h, P s\partial_i\partial_k, \ldots, P s(\partial_i\partial_k)^{m_{ij}-1}, P s\partial_h(\partial_i\partial_k)^{m_{ij}-1},$$

where $m_{kl} = m_{ij}$.

Since $P s = P$ the last set of polyhedra also enters the corona $C_1(P)$. This proves that $S(P) = S_1(P)$.

Now we are to prove that $C_1(P) \partial_i$ agrees with $C_1(P)$, i.e. that the neighbourhood $U_1(P_i)$ of an adjacent polyhedron $P_i \in C_1(P)$ belongs to $C_1(P)\partial_i$ as well. Let $\mathcal{F}_i^{d-2}$ be the set of all $(d - 2)$-faces of the face $F_i$ of $P$. It is obvious that $U_1(P_i)$ consists of all polyhedra from $C_1(P)$ that share some $(d - 2)$-face from $\mathcal{F}_i^{d-2}$ and only of them. Each of them can be presented as either $P (\partial_j\partial_i)^m$, or $P \partial_j(\partial_j\partial_i)^m$. The following relationships are easy to check:

\begin{equation}
(\partial_j\partial_i)^m = (\partial_i\partial_j)^{m_{ij}-m},
\end{equation}

\begin{equation}
\partial_i(\partial_j\partial_i)^m = (\partial_i\partial_j)^m,
\end{equation}

\begin{equation}
(\partial_j\partial_i)^m\partial_i = (\partial_j\partial_i)^{m_{ij}-1}\partial_j = (\partial_i\partial_j)^{m_{ij}-m+1}\partial_j =
(\partial_i\partial_j)^{m_{ij}-m} = (\partial_i\partial_j)^{m_{ij}-m}. \tag{6.3}
\end{equation}

By formulae (6.2) and (6.3) one gets that under the reflection in the hyperplane of $F_i$ an each polyhedron from $U_1(P_i)$ moves into a polyhedron from $U_1(P_i)$ again. So $U_1(P) \subset C_1(P)\partial_i$ what proves Lemma 6.1.

**Remark 6.2.** Generally speaking, the corona which is generated by reflections is not unique. In principle, there can be other coronas constructed of replicas of a Coxeter polyhedron which also fulfill the hypothesis of the extension theorem. Such coronas also extend to other different isohedral tilings. The symmetry groups of these tilings do not contain the Coxeter group generated by reflections in facets of the original Coxeter polyhedra.

Lemma 6.1 admits generalization for a family of polyhedra which are generalizations of the Coxeter polyhedra.

**Definition 6.3.** Call a polyhedron $P$ quasi-Coxeter polyhedron if

(i) all dihedral angles of $P$ are equal to $2\pi/m_{ij}$.

(ii) if for the dihedral angle between adjacent facets $F_i$ and $F_j$ the value of $m_{ij}$ is odd, then $P$ is symmetrical w.r.t. the bisecting hyperpane containing $d - 2$-face $F_{ij} = F_i \cap F_j$.

**Remark 6.4.** A quasi-Coxeter polyhedron $P$ is a Coxeter polyhedron if all values of $m_{ij}$ are even.
Lemma 6.5. Let $P$ be a quasi-Coxeter polyhedron. Then $P$ admits a monohedral corona $C_1(P)$ which is constructed by means of reflections at $(d-1)$-facets and such that

1. $S(P) = S_1(P)$,
2. for each $(d-1)$-face $F_\nu$, $\nu = 1, \ldots, n$, of the polyhedron $P$ the reflection $\partial_\nu$ makes $C_1(P)$ and the image $C_1(P)\partial_\nu$ agree, where $P_\nu \in C_1(P)$ is the polyhedron adjacent to $F_\nu$, $\nu = 1, \ldots, m$.

Remark 6.6. By Lemma 6.5 a quasi-Coxeter polyhedron admits the first corona which by the extension theorem extends to a isohedral tiling. Note that the Coxeter group $< \partial_1, \ldots, \partial_n >$ generated by the reflections in the $(d-1)$-dimensional faces of $P$ still operates transitively on the tiling but the tile $P$ is not a fundamental domain of this group.

7. Crystallographic clusters

Now we will say a few words about one more application that might be quite important. Crystallographers often construct complicated crystallographic structures starting with a relatively simple star of links (a set of segments that link one point to some of its neighbouring points). This original structure may not suggest any obvious crystallographic structure. Yet they want to have as much assurance as possible about whether this star of links can be extended to a crystallographic structure. The next remark is designed to help them to some degree.

First, we give a definition of a $\rho$-point (see [12] and [25]).

Definition 7.1. Given a discrete point set $Y \subset X^d$ and $\rho > 0$, a point $x \in Y$ is an $\rho$-point of $Y$ if every closed ball $B_x(\rho)$ of radius $\rho$ that contains $x$ on its boundary also contains at least one more point of $Y \setminus \{x\}$. The point set $Y$ is called a set enclosing the point $x$ while the point $x$ is called an enclosed point in the set $Y$.

Definition 7.2. A point $x'$ of a set $Y$ in $X^d$ is adjacent to a point $x$ if there is a ball $B$ that contains $x$ and $x'$ on its boundary $\partial B$ and contains no point of $Y \setminus \{x, x'\}$.

Definition 7.3. A set $Cl_x$ in $X^d$ is called a cluster about a point $x \in Cl_x$ if $x$ is an enclosed point of $Cl$ and all of points in $Cl \setminus \{x\}$ are adjacent to $x$.

Since the Voronoi domain is determined uniquely by a set of adjacency vectors we get the following Proposition.

Proposition 7.4. A cluster $Cl_x$ uniquely determines the Voronoi domain w.r.t. the point $x$ independently of the extension of the cluster, i.e. if two sets $X$ and $X'$ have the same cluster $Cl_x$ w.r.t. the point $x$, $x \in Cl_x \subset X \cap X'$, then the respective Voronoi domains

$$V_{Cl_x}(x) = V_X(x) = V_{X'}(x)$$
Definition 7.5. A cluster \( Cl_x \) is \textit{crystallographic} if there is a regular point set \( X \) (a crystallographic group orbit of some point) such that

1. \( Cl_x \subseteq X \);
2. the set \( Cl_x \) remains a cluster in the orbit \( X \) as well.

In this case \( X \) is called a \textit{crystallographic extension} of a cluster \( Cl_x \).

Remark 7.6. Given a cluster \( Cl_x \) there is a relatively easy procedure based on the extension theorem which checks all possible crystallographic extensions of \( Cl_x \).

Indeed, by lemma 7.4 for any crystallographic extension \( X \) of the cluster \( Cl_x \), if exists, its Voronoi tiling is an isohedral tiling of space by replicas of the Voronoi domain \( V_{Cl}(x) \). So if we already have \( Cl_x \) and along with it \( V_{Cl}(x) \) then, by the extension theorem, we can enlist all possible isohedral tilings with this polyhedron. The Voronoi tilings corresponding to the crystallographic extensions of \( Cl_x \) are possibly only some of them. To distinguish only Voronoi tilings we decorate the Voronoi domain \( V_{Cl}(x) \) by its action center \( x \) inside and denote it by \( P(x) \). After that we consider all possible coronas of some radius \( k \) comprised from replicas of the decorated polyhedron \( P(x) \) such that

- (i) for every two polyhedra \( P(x') \) and \( P(x'') \) adjacent at \((d-1)\)-faces and their action centers \([x']\) and \([x'']\) the segment \([x',x'']\) is orthogonal to the hyperplane of the common facet and bisected by it;
- (ii) \( S_{k-1}(P) = S_k(P) \);
- (iii) for each polyhedron \( P_v \) which meets the polyhedron \( P \) at a \((d-1)\)-face \( F_v \subseteq P \), \( v = 1, \ldots, n \), there is an isometry \( \partial_v \) such that \( P \partial_v = P_v \) and the image \( C_k(P) \partial_v \) agrees with \( C_k(P) \).

By the extension theorem such a corona admits only one isohedral tiling. But if in points (ii) and (iii) we require all isometries to preserve the action centers of corresponding polyhedra then the corona extends to an isohedral tiling which is a Voronoi tiling for the tiles’s action centers. The set of action center form one of crystallographic extensions of \( Cl_x \) (for details see [28]).

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References


