On the Essential Spectrum of Two Dimensional Periodic Magnetic Schrödinger Operators

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Abstract

For two dimensional Schrödinger operators with a nonzero constant magnetic field perturbed by an infinite number of periodically disposed, long range magnetic and electric wells, it is proven that when the inter-well distance $(R)$ grows to infinity, the essential spectrum near the eigenvalues of the “one well Hamiltonian” is located in mini-bands whose width shrink faster than any exponential with $R$. This should be compared with our previous result [5], which stated that in the case of compactly supported wells, the mini-bands shrink Gaussian like with $R$.

1 Introduction

In this paper we continue the study (begun in [4,5]) of the spectral properties of two dimensional magnetic Schrödinger operators. In [4] we considered the “one well problem” i.e.

$$H = (p - a_0 - a)^2 + V,$$

where $a_0$ corresponds to a nonzero constant magnetic field, $B_0$, the magnetic perturbation $B'(x) = \text{curl } a(x)$ is long range but bounded in the sense that:

$$b \equiv max \{ ||D^\alpha B'||_\infty, |\alpha| \leq 1 \} < \infty$$

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and the scalar perturbation $V = V_1 + V_2$ obeys:

$$V_1 \in L^2(\mathbb{R}^2), \quad V_2 \in L^\infty(\mathbb{R}^2)$$  \hspace{1cm} (1.3)

It is known that if both the magnetic and the scalar perturbations are vanishing at infinity, then (see [6,10]):

$$\sigma_{ess}(H) = \sigma_L(B_0) = \{(2n + 1)B_0 \mid n = 0,1,\ldots\}$$  \hspace{1cm} (1.4)

While in [4] we studied the exponential decay at infinity of the eigenfunctions of $H$, in [5] we dealt with the multiple well case. The reasoning was that the Landau spectrum suffers a radical change and one is expecting to find essential spectrum and gaps in between the Landau levels. The fact that the “one well” perturbations were taken compactly supported allowed us to conclude that the Hamiltonian with an infinite number of wells (not necessarily disposed in a periodic lattice) has (for a sufficiently large interwell distance) essential spectrum near the discrete eigenvalues of $H$, and is located in mini-bands whose width presents a Gaussian decay with the interwell distance.

In this paper we intend to study the case in which the “one well” perturbations are no longer compactly supported, but periodic. We will show that (see Theorem 2.1 for the precise statement) the essential spectrum which appears near a discrete eigenvalue of $H$ looks like a mini-band too, and its width is shrinking (at least) faster than any exponential. This together with an improved resolvent estimate for the “one well” Hamiltonian (see Lemma 2.1) is what we add new to the already known results. Nevertheless, we want to stress that the magnetic perturbation is considered to be the magnetic field which is not always a very regular perturbation and one has to be a little bit more careful with it.

Other results very close in spirit to ours were obtained in [1] (the so called “geometric perturbation theory”) and [11]. For other aspects of these problems, see for example [2,7,8,9,12] and references therein.

2 Preliminaries and the results

As already said, we shall consider only the two dimensional case (i.e. the particle is confined in the plane $x_3 = 0$ and the magnetic field is orthogonal
to that plane). Let $B(x) \in C^1(\mathbb{R}^2)$. We shall use the following family of vector potentials corresponding to $B(x)$:

$$a(x, x') = \int_0^1 ds \, s \, B(x' + s(x - x')) \wedge (x - x') \quad (2.1)$$

For $x' = 0$, this is nothing but the usual transversal gauge (see e.g. [15]):

$$a(x, 0) = a(x) = \int_0^1 ds \, s \, B(s \, x) \wedge x \quad (2.2)$$

If we define (see [5] for details):

$$\varphi(x, x') = -\left(\int_0^1 dt \int_0^1 ds \, s \, B(s \, t \, (x - x') + s \, x') \right) \cdot (x \wedge x'), \quad (2.3)$$

then one can see that:

$$\nabla_x \varphi(x, x') = a(x) - a(x, x') \quad (2.4)$$

If $B(x) = B_0$ is constant, then

$$\varphi_0(x, x') = -\frac{1}{2}B_0 \left(x_1 \, x_2' - x_1' \, x_2\right)$$

$$a_0(x, x') = \frac{1}{2}B_0 \wedge (x - x') \quad (2.5)$$

The Hamiltonian of a particle in the presence of a magnetic field and a scalar potential $V$ is (in the transversal gauge):

$$H = (p - a(x))^2 + V(x)$$

$$p = \left(-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}\right)$$

$$a(x) = \left(-x_2 \int_0^1 ds \, s \, B(s \, x), x_1 \int_0^1 ds \, s \, B(s \, x)\right) \quad (2.6)$$

Our first result (used in the estimation on the width of the mini-bands which appear in the multiple well case, but also interesting in itself) states the following:
Lemma 2.1 Let $B_0 > 0$, $B'(x) \in C^1(\mathbb{R}^2)$ and $B(x) = B_0 + B'(x)$. Let $V_1 \in L^2(\mathbb{R}^2)$, $V_2 \in L^\infty(\mathbb{R}^2)$ and $V = V_1 + V_2$.

Define $a(x) = \int_0^1 ds \, B(sx) \wedge x$; then $H = (p - a)^2 + V$ is essentially self adjoint on $C_c^\infty(\mathbb{R}^2)$.

Define $\mathcal{H}_\mu \equiv L^2(\mathbb{R}^2, dx \exp(2\mu \cdot \cdot \cdot ))$, where $< \cdot , \cdot > = (1 + |\cdot|^2)^{\frac{\mu}{2}}$ and $\mu \geq 0$.

Let $z$ be a complex number such that dist\{z, $\sigma(H)$\} = $d > 0$ and let $\chi_n(x)$, $n \in \mathbb{N}$ be the characteristic function of \{x|x| \geq n\}.

Define $b_n = \max(||\chi_n D^a B'||_\infty, \alpha \leq 1)$ and $v_n = ||\chi_n V_1||_2 + ||\chi_n V_2||_\infty$, obeying the condition $\lim_{n \to \infty} (b_n + v_n) = 0$. Take $n \in \mathbb{R}^2$, $|n| = 1$; then for every positive number $\mu$ one has that the following operators:

$$A_1(z) = (H - z)^{-1}$$

and

$$A_2(z) = \left[ n \cdot (p - a)(H - z)^{-1} \right]$$

belong to $B(\mathcal{H}_\mu)$ and their norms are uniform in $0 < d_0 \leq d \leq d_1 < \infty$.

From now on, by a single well we will understand a magnetic field $B(x) = B_0 + B'(x)$ and a scalar potential $V$, obeying the following condition (in order to insures the existence of (2.8)):

$$|D^a B'(\cdot)| + |V(\cdot)| < < \cdot , \cdot >^{\beta}, \quad \beta > 2, \quad |\alpha| \leq 1. \quad (2.7)$$

If $a$ gives $B = B_0 + B'$, then the one well Hamiltonian will be $H = (p - a)^2 + V$.

Denote with $\Gamma = \mathbb{Z}^2$ and define a dilated, rectangular lattice $\Gamma_R = \{R\gamma | \gamma \in \Gamma, \quad R > 0\}$. The periodic potentials will be:

$$V_{\Gamma_R}(\cdot) = \sum_{\gamma \in \Gamma} V(\cdot - R\gamma), \quad B_{\Gamma_R}(\cdot) = B_0 + \sum_{\gamma \in \Gamma} B'(\cdot - R\gamma). \quad (2.8)$$

If $a_{\Gamma_R}$ stands for the (transversal) gauge which gives $B_{\Gamma_R}$, then our Hamiltonian will be $H_{\Gamma_R} = (p - a_{\Gamma_R})^2 + V_{\Gamma_R}$.

Let $c < d$ such that $[c, d] \cap \sigma_L(B_0) = \emptyset$ (see (1.4)) and let $\sigma(H) \cap (c, d) = \{E_1\}$, where $E_1$ is a discrete eigenvalue of $H$ (for simplicity we’re considering it not degenerate; see Remark 4 at the end of the proofs for a short discussion on the degenerate case).

Then the existence of highly localised essential spectrum near $E_1$ is stated in the following
Theorem 2.1  Fix $\mu > 0$. Then there exists a real function $E_1(R)$ obeying $\lim_{R \to \infty} E_1(R) = E_1$, and an $R(\mu) > 0$ such that for every $R > R(\mu)$ one has:

i) $\sigma(H_{\Gamma_R}) \cap [c, d] \subset [E_1(R) - 2\exp(-\mu R), E_1(R) + 2\exp(-\mu R)]$

and

ii) $\dim \text{Ran} P_{\Gamma_R}([c, d]) = \infty$ (i.e. the above intersection is not the empty set).

3 The proofs

3.1 An improved resolvent estimate

Before starting the proof of Lemma 2.1, let’s remark the following

Proposition 3.1 Let $K$ be a compact in $\mathbb{C}$ and $z \in K$. If $A_1(z)$ is bounded, then $A_2(z)$ is bounded and there exists a constant $C(K)$ such that

$$||A_2(z)|| \leq C(K) (1 + ||A_1(z)||)$$

Proof:

In the argument we’re presenting here appear the same basic ideas as in the proof of Theorem B.6.5, from [14] (which treats atomic Hamiltonians without magnetic fields and with polynomial weights). Nevertheless, we decided to give a rather detailed proof in order to have a better control on the bounds (see the reasoning for this in Remark 2 below the proof of Lemma 2.1).

In order to work with bounded weights only, we’ll consider that $f_m(x)$ is a sequence of some smooth and bounded functions such that $f_m(x) = <x>$ if $|x| \leq m/2$ and with all their derivatives (up to the second order) uniformly bounded in $m$.

Because $b_0 + v_0 < \infty$, there exists a $\lambda < 0$ such that

$$\lambda < -1 - (\inf \sigma(H))$$

and therefore $R(\lambda) = (H - \lambda)^{-1} > 0$.

Denote with $R^0(\lambda) = [(p-a)^2 - \lambda]^{-1}$; then for $-\lambda$ large enough, one has:

$$\left\| \left(1 + (R^0(\lambda))^2 V(R^0(\lambda))^2 \right)^{-1} \right\|_{B(\mathcal{H}_0)} < 1/2$$

(3.2)
and
\[ R(\lambda) = R^0(\lambda)^\frac{1}{R^0(\lambda)} \left( 1 + (R^0(\lambda))^\frac{1}{2} V(R^0(\lambda))^\frac{1}{2} \right)^{-1} R^0(\lambda)^\frac{1}{2} \] (3.3)

Moreover, provided \(-\lambda\) large enough, one can be sure that \(R(\lambda) \in \mathcal{B}(\mathcal{H}_\mu)\) and
\[ \| \exp(\mu < \cdot >)R(\lambda)\exp(-\mu < \cdot >) \|_{\mathcal{B}(\mathcal{H}_0)} < \infty. \] (3.4)

Then \(A_2(z)\) can be rewritten (as an operator in \(\mathcal{B}(\mathcal{H}_0)\)):
\[ \exp(\mu < \cdot >) [n \cdot (p - a)R(\lambda) + (z - \lambda)R(\lambda)R(z)] \exp(-\mu < \cdot >) \] (3.5)

so it would be sufficient proving that \(A_2(\lambda)\) is bounded.

Let \(\phi, \psi \in C_0^\infty(\mathbb{R}^2)\) with their support included in \(|x| < M < \infty.\) If one manages to prove that:
\[
\begin{align*}
|\langle \exp(\mu < \cdot >)\phi, [n \cdot (p - a)] R(\lambda) \psi > | & \leq \\
& \leq \text{const} (1 + \|A_1(\lambda)\|) \|\phi\|_{\mathcal{H}_0} \|\psi\|_{\mathcal{H}_\mu}
\end{align*}
\] (3.6)

then via a density argument, the proof would be concluded.

Let’s compute \((M < m / 2)\):
\[
\begin{align*}
\langle \exp(\mu < \cdot >)\phi, [n \cdot (p - a)] R(\lambda) \psi >_{\mathcal{H}_0} &= \\
&= \langle [n \cdot (p - a)] \exp(\mu f_m(\cdot))\phi, R(\lambda) \psi >_{\mathcal{H}_0} = \\
&= \langle [n \cdot (p - a)]\phi, \exp(\mu f_m(\cdot)) R(\lambda) \psi > + \\
&+ i \mu \langle [n \cdot (\nabla f_m(\cdot))] \phi, \exp(\mu f_m(\cdot)) R(\lambda) \psi >
\end{align*}
\] (3.7)

The second term of the last equality is easy to control; we will treat here only the first one.

Commuting \(\exp(\mu f_m(\cdot))\) with \(R(\lambda)\), one obtains:
\[
\exp(\mu f_m(\cdot)) R(\lambda) \psi = \\
= R(\lambda) \exp(\mu < \cdot >) \psi + R(\lambda) [H, \exp(\mu f_m(\cdot))] R(\lambda) \psi = \\
= R(\lambda) \exp(\mu < \cdot >) \psi + R(\lambda) (\mu^2 (\nabla f_m(\cdot))^2 + \mu \Delta f_m(\cdot)) \times \\
\times \exp(\mu f_m(\cdot)) R(\lambda) \psi - 2 i \mu R(\lambda) [(p - a) \cdot (\nabla f_m(\cdot))] \times \\
\times \exp(\mu f_m(\cdot)) R(\lambda) \psi.
\] (3.8)

Because
\[ \|n \cdot (p - a)(R^0(\lambda))^\frac{1}{2} \|_{\mathcal{B}(\mathcal{H}_0)} < \infty \] (3.9)
and
\[ \sup_m \left| \langle R^\alpha(\lambda) \rangle \hat{\gamma}(\mathbf{p} - \mathbf{a}) \cdot (\nabla f_m) \right|_{L^2(\mathcal{H}_0)} < \infty, \]  
then using (3.3) and taking the supremum over \( m \), one obtains (3.6).\( \square \)

In the proof of Lemma 2.1, a central role will be played by the following Lemma (proved in [4] in a slightly different form as Corollary 3.1), which controls the “asymptotic Hamiltonians”:

**Lemma 3.1** Let \( z \) be a complex number, with \( \text{dist}(z, \sigma_L(B_0)) = d > 0 \).

Define:
\[ B_n(x) = B_0 + B'_n(x), \quad V_n = V_{1,n} + V_{2,n} \]
and
\[ b_n = \max \left( \sup_{\mathbf{x} \in \mathbb{R}^2} |D^\alpha B'_n(\mathbf{x})|, |\alpha| \leq 1 \right); \quad v_n = ||V_{1,n}||_2 + ||V_{2,n}||_\infty, \]
with
\[ \lim_{n \to \infty} (b_n + v_n) = 0. \]

Denote
\[ H_n = (\mathbf{p} - \mathbf{a}_n)^2 + V_n. \]

Fix \( \mu > 0 \). Then there exists \( N(\mu, d) > 0 \) large enough such that for any \( n \geq N(\mu, d) \) one has that \( z \notin \sigma(H_n) \) and
\[ ||\exp(\mu < \cdot >)(H_n - z)^{-1} \exp(-\mu < \cdot >)||_{L^2(\mathcal{H}_0)} = C(\mu, d). \]  

**Proof of Lemma 2.1:**

The main idea of the proof consists in constructing an “almost inverse” operator for \( H - z \), using the fact that after “gauging” away some magnetic phases and going sufficiently far from the origin, the particle “feels” only smaller and smaller magnetic and scalar perturbations. In that region, its dynamics should be very close to that of a particle which is placed in a globally small perturbation, for which we have the informations given in Lemma 3.1. The “almost inverse” operator will be constructed via a “cut and paste” technique which takes into account that for long range magnetic fields the gauge transformations we need can be performed only locally.

In what follows, all the cut-off functions are of the following type:
\[ g \in C^\infty(\mathbb{R}^2), \quad 0 \leq g(\mathbf{x}) \leq 1 \]  

(3.12)
with convex support. Define:

\[
g_i(\mathbf{x}) = \begin{cases} 
1 & \text{if } |x_1| \leq \frac{1}{2}, \; i \in \{1, 2\} \\
0 & \text{if } |x_1| \geq \frac{3}{2} \text{ or } |x_2| \geq \frac{3}{2}
\end{cases}
\]  \hspace{1cm} (3.13)

\[
g_1(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_1 \geq \frac{3}{2} \text{ and } |x_2| \leq \frac{1}{2} \\
0 & \text{if } x_1 \leq \frac{1}{2} \text{ or } |x_2| \geq \frac{3}{2}
\end{cases}, \quad \tilde{g}_1(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_1 \geq \frac{1}{4} \\
0 & \text{if } x_1 \leq \frac{1}{8}
\end{cases},
\]  \hspace{1cm} (3.14)

\[
g_2(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_2 \geq \frac{3}{2}, \; \tilde{g}_2(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_2 \geq \frac{1}{4} \\
0 & \text{if } x_2 \leq \frac{1}{8}
\end{cases},
\]  \hspace{1cm} (3.15)

\[
g_3(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_1 \leq -\frac{3}{2} \text{ and } |x_2| \leq \frac{1}{2} \\
0 & \text{if } x_1 \geq -\frac{1}{2} \text{ or } |x_2| \geq \frac{3}{2}
\end{cases}, \quad \tilde{g}_3(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_1 \leq -\frac{1}{4} \\
0 & \text{if } x_1 \geq -\frac{1}{8}
\end{cases},
\]  \hspace{1cm} (3.16)

\[
g_4(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_2 \leq -\frac{3}{2}, \; \tilde{g}_4(\mathbf{x}) = \begin{cases} 
1 & \text{if } x_2 \leq -\frac{1}{4} \\
0 & \text{if } x_2 \geq -\frac{1}{8}
\end{cases},
\]  \hspace{1cm} (3.17)

In addition, we want that the functions defined above to form a quadratic partition of unity:

\[
\sum_{j=0}^{4} g_j^2(\mathbf{x}) = 1
\]  \hspace{1cm} (3.18)

Define:

\[
g_{j,n}(\mathbf{x}) = g_j\left(\frac{\mathbf{x}}{n}\right); \quad \tilde{g}_{j,n}(\mathbf{x}) = \tilde{g}_j\left(\frac{\mathbf{x}}{n}\right); \quad n \in \{1, 2, \ldots\}.
\]  \hspace{1cm} (3.19)

From these definitions one can see that all the derivatives of \( g_{j,n} \) converge uniformly to zero when \( n \) goes to infinity.
Take
\[ \{x^j\}_{j \in \{1, \ldots, 4\}}, \quad x^j \in \text{supp}\ g_{j,n}. \]
Define
\[
B_{j,n} = B_0 + \hat{g}_{j,n} B', \quad V_{j,n} = \hat{g}_{j,n} V, \quad \alpha_{j,n}(x) = \int_0^1 ds \, s B_{j,n}(sx) \wedge x, \quad j \in \{1, \ldots, 4\}. \tag{3.20}
\]
But (see (2.4) and (2.3)):
\[
\alpha(x) = \alpha(x, x^j) + \nabla_x \varphi(x, x^j) \quad \text{and} \quad \alpha_{j,n}(x) = \alpha_{j,n}(x, x^j) + \nabla_x \varphi_{j,n}(x, x^j), \quad j \in \{1, \ldots, 4\}, \tag{3.21}
\]
and for \( x \in \text{supp}\ g_{j,n} \), one can easily show that:
\[
\alpha(x, x^j) = \alpha_{j,n}(x, x^j). \]
Denote
\[
\delta \varphi_{j,n}(x, x^j) = \varphi(x, x^j) - \varphi_{j,n}(x, x^j); \]
Then
\[
(p - a) \exp(\iota \delta \varphi_{j,n}(\cdot, x^j))g_{j,n} = \exp(\iota \delta \varphi_{j,n}(\cdot, x^j))(p - \alpha_{j,n})g_{j,n} \tag{3.22}
\]
At this point we are able to define the “asymptotic Hamiltonians” (they have \textit{globally} small perturbations, uniformly in \( n \); see the conditions imposed on \( B' \) and \( V \)):
\[
H_{j,n} = (p - \alpha_{j,n})^2 + V_{j,n}, \quad j \in \{1, \ldots, 4\}. \]
Take \( z \) as stated. For \( n_0 \) large enough and \( n \geq n_0 \), Lemma 3.1 implies that
\[
\text{dist}(z, \sigma(H_{j,n})) \geq \frac{d}{2}. \tag{3.23}
\]
Define
\[
S(z) = g_{\alpha,n}(H - z)^{-1} g_{\alpha,n} + \sum_{j=1}^4 \exp(\iota \delta \varphi_{j,n}(\cdot, x^j))g_{j,n}((H_{j,n} - z)^{-1} g_{j,n} \exp(-\iota \delta \varphi_{j,n}(\cdot, x^j))). \tag{3.24}
\]
Then
\[(H - z)S(z) = 1 + T_0 + T_1,\]
where
\[T_0 = [H, g_{0,n}](H - z)^{-1}g_{0,n},\]
and
\[T_1 = \sum_{j=1}^{4} \exp(i \delta \varphi_{j,n}) [H_{j,n}, g_{j,n}] ((H_{j,n} - z)^{-1}g_{j,n} \exp(-i \delta \varphi_{j,n}).\]

where \([H_{j,n}, g_{j,n}] = -2i(\nabla g_{j,n}) \cdot (p - a_{j,n}) - (\Delta g_{j,n})\]

Fix \(\mu > 0\). Proposition 3.1, Lemma 3.1, the compact support of \(g_{0,n}\) and the fact that the derivatives of \(g_{j,n}\) behave at least like \(1/n\), insure the existence of an \(n(\mu, d)\) such that for \(n = n(\mu, d) + 1\) one has:
\[
\| \exp(\mu < \cdot >)T_1 \exp(-\mu < \cdot >) \| + \| T_0 \| \leq \frac{1}{4},
\]
\[
\| \exp(\mu < \cdot >)T_0 \| < \infty,
\]
\[
\| \exp(\mu < \cdot >)S(z) \exp(-\mu < \cdot >) \| \leq \text{const}(\mu, d).
\]

Then
\[
(1 + T_1 + T_0)^{-1} = (1 + T_1)^{-1} \left[ 1 + T_0(1 + T_1)^{-1} \right]^{-1} =
\]
\[
= (1 + T_1)^{-1} \left\{ 1 - T_0(1 + T_1)^{-1} \left[ 1 + T_0(1 + T_1)^{-1} \right]^{-1} \right\}
\]
and
\[
(H - z)^{-1} = S(z)(1 + T_1 + T_0)^{-1} =
\]
\[
= S(z)(1 + T_1)^{-1} -
\]
\[
- S(z)(1 + T_1)^{-1}T_0(1 + T_1)^{-1} \left[ 1 + T_0(1 + T_1)^{-1} \right]^{-1}. \]

Use (3.27) in (3.29), and the proof is finished. □

\textbf{Remark 1:}

For \(\alpha \in \mathbb{R}\), define the following analytic family of operators:
\[H(\alpha) = \exp(i\alpha < \cdot >)H \exp(-i\alpha < \cdot >)\]
Lemma 2.1 implies in particular that if \( z \in \rho(H) \) then \((H(\alpha) - z)^{-1}\) admits an entire extension to the whole complex plane and \( \rho(H) \subset \rho(H(\alpha)) \), therefore the essential spectrum of \( H(\alpha) \) is not moving towards its discrete eigenvalues (from the usual analytic perturbation theory we know that its discrete spectrum is independent of \( \alpha \)). Moreover, the projector \( P(\alpha) \) (written with the Riesz formula) associated with a discrete eigenvalue of \( H(\alpha) \) admits an entire extension too, which (via O’Connor’s Lemma) implies that the discrete eigenfunctions of \( H \) decay faster than any exponential (this result was obtained in a simpler way in [4]).

Remark 2:

Suppose that we would have a family of scalar and magnetic perturbations \( V_k \) and \( B'_k \) such that (see the formulation of Lemma 2.1):

\[
\lim_{n \to \infty} \{ \sup_k (b_{n,k}) + \sup_k (v_{n,k}) \} = 0
\]

and \( \inf_k \{ \text{dist}(z, \sigma(H_k)) \} \geq d > 0 \). Then we claim that the choices for \( \lambda \) in \((3.1)\) and \((3.2)\), the bounds in \((3.4), (3.2), (3.3), (3.9) \) and \((3.10)\), the choice of \( N(\mu, d) \) and the bound in \((3.11)\) can be made uniform in \( k \), and so are the bounds on the norms which appear in Lemma 2.1.

3.2 Periodic wells in two dimensions

Proof of Theorem 2.1:

i). The strategy in finding the spectrum location of the “whole” Hamiltonian \( H_{\Gamma R} \) in the given interval \([c, d]\) consists in constructing its resolvent \((H_{\Gamma R} - z)^{-1}\) for all \( z \in [c, d] \) situated outside of an exponentially small strip centred in \( E_1(R) \), where \( E_1(R) \) will turn out to be an eigenvalue of an “almost one well” Hamiltonian \( H_R \). So one has to make the following two steps:

a) to guess \( H_R \) and \( E_1(R) \)

and

b) to construct a bounded operator \( S_R(z) \) such that, as soon as \( R \geq R(\mu) \) one can write:

\[
(H_{\Gamma R} - z)S_R(z) = 1 + T_R(z) \quad (3.30)
\]

\[
\|T_R(z)\| < \frac{\exp(-\mu R)}{|z - E_1(R)|} + 1/2 \quad (3.31)
\]
Let us introduce some cut-off functions which will help in writing down the needed operators. Let \( j \in \{1, 2, 3\} \), and let \( g_j \) be of the same kind as in (3.12) and with convex supports included in \( |x| \leq 7/8 \). Choose \( g_i(x) = 1 \) if \( |x| \leq 1/4 \) and equal to zero if \( |x| \geq 3/4 \). Take then \( g_2 \) and \( g_3 \) such that \( g_2 g_3 = g_2 \) and \( g_1 g_2 = g_1 \). If \( \text{supp}(\partial g_2) \) denotes the support of at least one derivative of \( g_2 \), then we ask that:

\[
\delta = \text{dist}(\text{supp}(\partial g_2), \text{supp} g_1) > 0 \quad (3.32)
\]

Define then \( g_{i, \gamma, R} = g_j ((\cdot - R\gamma)/R) \). The final requirement will be:

\[
\sum_{\gamma \in \Gamma} g_{1, \gamma, R}(x) = 1 \quad (3.33)
\]

**Remark 3:**

It is very important to see that for any \( R > 0 \) and any \( \gamma \), the function \( g_{2, \gamma, R} \) has at most 8 neighbours whose supports are not disjoint from \( \text{supp} g_{2, \gamma, R} \). Moreover,

\[
\text{dist}(\text{supp}(\partial g_{2, \gamma, R}), \text{supp} g_{1, \gamma, R}) = \delta R, \quad g_{2, \gamma, R} g_{1, \gamma, R} = g_{1, \gamma, R} \quad (3.34)
\]

Let us define \( H_R \). If:

\[
B_R(x) = B_0 + \left[ \sum_{\gamma \in \Gamma} B'(x - R\gamma) \right] g_{3, 0, R}(x) = \quad (3.35)
\]

\[
= B(x) + (1 - g_{3, 0, R}(x)) B'(x) + \left[ \sum_{\gamma \neq 0} B'(x - R\gamma) \right] g_{3, 0, R}(x),
\]

\[
V_R(x) = V_{\Gamma, R}(x) g_{3, 0, R}(x) = \quad (3.36)
\]

\[
= V(x) + (1 - g_{3, 0, R}(x)) V(x) + \left[ \sum_{\gamma \neq 0} V(x - R\gamma) \right] g_{3, 0, R}(x)
\]

and \( a_R \) is the transversal gauge corresponding to \( B_R(x) \), then \( H_R = (p - a_R)^2 + V_R \). From the above definitions and using (2.7), one can see that:

\[
\lim_{R \to \infty} \left\{ \sup_{x \in \mathbb{R}^3} \left[ |D^n (B - B_R)(x)| + |V - V_R(x)| \right] \right\} = 0 \quad (3.37)
\]
The stability of spectrum for such Hamiltonians (see [13] for a more general discussion) implies that for $R \geq R_0$, in $[c, d]$ lives only one not degenerated eigenvalue of $H_R$ (denoted with $E_1(R)$) which converges to $E_1$ when $R$ goes to infinity.

At this point we have to establish the link between $H_R$ and $H_{\Gamma R}$. Denote with $t_{\gamma R}$ the usual translation $((t_{\gamma R} f)(x) = f(x - R \gamma))$; from (2.1) and (2.8) one obtains that $a_{\Gamma R}(x, \gamma R) = a_{\Gamma R}(x - \gamma R)$ and one can write down a gauge phase $\varphi$, such that:

$$H_{\Gamma R} g_{2,\gamma R} \exp (i \varphi) \gamma R = \exp (i \varphi) \gamma R H_R g_{2,0,R}$$ (3.38)

We are able now to write down the approximation for the resolvent:

$$S_R(z) = \sum_{\gamma \in \Gamma} g_{2,\gamma R} \exp (i \varphi) \gamma R (H_R - z)^{-1} \gamma R g_{1,0,R}$$ (3.39)

where the infinite sum should be understood as a strong limit. Using (3.34) (see Remark 3) and the fact that $\{g_{\gamma, R}\}$ is a locally finite partition of unity, one obtains:

$$||S_R(z)|| \leq \text{const} |E_1(R) - z|^{-1}$$ (3.40)

where the above constant does not depend on $R$. Then:

$$(H_{\Gamma R} - z) S_R(z) = 1 + T_R(z)$$ (3.41)

$$T_R(z) = \sum_{\gamma \in \Gamma} \exp (i \varphi) \gamma R [H_R, g_{2,0,R}] (H_R - z) g_{1,0,R} \gamma R \exp (-i \varphi)$$

The operator $T_R(z)$ is also bounded and one obtains:

$$||T_R(z)|| \leq \text{const} ||[H_R, g_{2,0,R}] (H_R - z)^{-1} g_{1,0,R}||$$ (3.42)

where $[H_R, g_{2,0,R}] = -2i(\nabla g_{2,0,R}) \cdot (p - a_R) - \Delta g_{2,0,R}$. Because $z$ is situated in the interval $[c, d]$ which contains only $E_1$, one can find a contour $C$ in the complex plane which surrounds $E_1$ such that $\text{dist}\{\sigma(H), C\} \geq d > 0$; due to the spectrum stability of $H_R$, for $R > R_0$ one has $\text{dist}\{\sigma(H), C\} \geq d/2$. Riesz formula gives at once that:

$$P_R = -\frac{1}{2\pi i} \int_C d\xi (H_R - \xi)^{-1},$$

$$1 - P_R (H_R - z)^{-1} = \frac{1}{2\pi i} \int_C d\xi \frac{1}{\xi - z} (H_R - \xi)^{-1}$$

$$(H_R - z)^{-1} = \frac{1}{E_1(R) - z} P_R + (1 - P_R) (H_R - z)^{-1}$$ (3.43)
Take (see (3.32) for the definition of $\delta$) $\tilde{\mu} > 2\mu/\delta$. From (3.34), Lemma 2.1, Remark 2 and (3.43) one can write:

$$||T_R(z)|| \leq \text{const} \left| \left| \left( H_{R}, g_{z,0,R} \right) \left( H_{R} - z \right)^{-1} g_{z,0,R} \right| \right| \leq \frac{\text{const}}{|E_1(R) - z|} \exp(-\tilde{\mu}\delta R) + 1/2$$

(3.44)

where the above constant depends on everything but $R$; take now $R(\mu)$ such that $\exp(-\tilde{\mu}\delta R(\mu)/2) < 1$; then for $R \geq R(\mu)$ and $|E_1(R) - z| \geq 2\exp(-\tilde{\mu}R)$ one finally obtains $||T_R(z)|| < 1$ and the proof of i) is finished.

ii). First of all, because $H_{\Gamma_\gamma}$ commutes with the magnetic translations it can not have discrete spectrum, therefore the only thing we should check is that its spectral projection corresponding to $[c, d]$ is not zero. For $R \geq R(\mu)$, one has:

$$P_{\Gamma_\gamma}([c, d]) = \frac{1}{2\pi i} \int_{c} d\xi \left( H_{\Gamma_\gamma} - \xi \right)^{-1}$$

(3.45)

and using the expression of $S_R(z)$, one obtains (R sufficiently large):

$$||P_{\Gamma_\gamma}([c, d]) - \sum_{\gamma \in \Gamma} g_{z,\gamma,R} \exp(\pm \varphi_\gamma) t_{\gamma,R} P_R t_{\gamma,R} \exp(\mp \varphi_\gamma) g_{z,\gamma,R}|| < 1/2$$

(3.46)

where $P_R$ is the one dimensional projector of $H_R$ which corresponds to $E_1(R)$. Denote with $\tilde{P}(R)$ the sum which appears in (3.46) and with $\psi_1(R)$ the normalised eigenfunction of $H_R$. If $P(\psi_1)$ denotes the projector (eigenvector) of $H$ corresponding to $E_1$, then we know that

$$\lim_{R \to \infty} ||P_R - P|| = 0$$

(3.47)

From now on, $\epsilon$ will mean a generic, infinitesimally small positive quantity; for $R_0 > 0$, denote with $\chi_0$ the characteristic function of $\{|x| \geq R_0\}$; provided $R_0$ sufficiently large, one has:

$$\sup_{R \geq R_0} ||P_R - P|| \leq \epsilon$$

(3.48)

If $U(R)$ denotes the Nagy unitary which intertwines $P_R$ and $P$, then $\psi_1(R) = U(R)\psi_1$ and $\sup_{R \geq R_0} ||U(R) - 1|| \leq \epsilon$. Due to the exponential localisation of $\psi_1$ near the origin, one has $||\chi_0\psi_1|| \leq \epsilon$, therefore $< \psi_1(R), (1 - \chi_0)\psi_1(R) > \geq 1 - \epsilon$ and moreover, $(1 - \chi_0)\psi_1(R)$ is compactly supported. It follows that:

$$\psi_1(R) = (\langle \psi_1(R), (1 - \chi_0)\psi_1(R) \rangle)^{-1} P_R (1 - \chi_0)\psi_1(R)$$

(3.49)
therefore
\[
\sup_{R \geq R(\mu)} \| \exp(\mu \cdot \cdot \cdot) \psi_1(R) \| \leq C(\mu) \tag{3.50}
\]
which means that \( \psi_1(R) \) has an uniform exponential localisation in \( R \geq R(\mu) \). Then \( \|P(R)\psi_1(R)\| > 1/2 \) for sufficiently large \( R \). This and (3.46) finally imply that \( \|P_{1,\mu}([c, d])\psi_1(R)\| > 0 \) and the proof is concluded.\( \Box \)

Remark 4:
If \( E \) is degenerated of order \( n \), (say \( n = 2 \)), then the above argument can be repeated in order to obtain that for \( R \geq R(\mu) \) and \( k \in \{1, 2\} \):
\[
\sup_{R \geq R(\mu)} \| \exp(\mu \cdot \cdot \cdot) \psi_k(R) \| \leq C(\mu) \tag{3.51}
\]
Indeed, one has to express \( \psi_{1,2}(R) \) with the help of \( P_R(1 - \chi_0)\psi_{1,2}(R) \), using that these two vectors are almost orthonormal uniformly in \( R \geq R_0 \).

Then \( H_R \) has two eigenvalues \( E_1(R) \) and \( E_2(R) \) near \( E \); looking at (3.43), one realizes that the only thing that changes is:
\[
P_R(H_R - z)^{-1} = \frac{|\psi_1(R) \rangle \langle \psi_1(R)|}{E_1(R) - z} + \frac{|\psi_2(R) \rangle \langle \psi_2(R)|}{E_2(R) - z} \tag{3.52}
\]
Using (3.51), one concludes that the spectrum of \( H_{1,\mu} \) is situated in two mini-bands (of the same type as in i)) localised near \( E_1(R) \) and \( E_2(R) \).

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References


