Semi-group Domination and Eigenvalue Estimates

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ABSTRACT. For a class of integral operators it is shown that if the integral kernel of one operator majorates the kernel of the other one then certain forms of eigenvalue estimates are inherited. The operators must be related to a positivity preserving semigroup and, respectively, a positively dominated semigroup. It follows, in particular, that any, sufficiently regular, eigenvalue estimate for the Schrödinger operator is carried over automatically to the magnetic Schrödinger operator, regardless of the method of obtaining the estimate.

1. There are two natural notions of positivity for operators acting in $L_2(X)$ where $X$ is a space with a $\sigma$-finite measure. On the one hand, it is the positivity in the operator sense: $K$ is non-negative if $(Ku,u) \geq 0$ for any $u$ in the domain $\mathcal{D}(K)$ of the operator $K$. Another notion of positivity is related to the lattice structure of the $L_2$ space: $K$ is positivity preserving (P.P.) if $(Ku)(x) \geq 0$ almost everywhere (a.e) for any a.e. positive function $u \in \mathcal{D}(K)$. For particular operators, interaction of these notions proves to be a rather effective tool in the analysis, see, e.g. [RSi, v.2,4] and [CFKirSi].

Having a positivity preserving operator $K$, we say that the operator $L$ is dominated by $K$ ($L \preceq K$) if $|(Lu)(x)| \leq (K|u|)(x)$, $u \in L_2$, for almost all $x$ (the term 'majorated' is also used in the literature). For integral operators, with kernels $K(x,y), L(x,y)$ domination is equivalent to the inequality $|L(x,y)| \leq K(x,y)$ for almost all $(x,y)$. A natural question is which properties of the operator $K$ are inherited by $L$. It is obvious that boundedness is inherited. It was shown in [P] that compactness is inherited as well. Now, the 'quality' of a compact operator can be described by the rate of decay of its $s$-numbers (or eigenvalues, if operators are self-adjoint). But the simple two-dimensional example $L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $L \preceq K$ shows that while the naturally expected inequality for the largest eigenvalues of $L$ and $K$ holds, it may be violated for other eigenvalues. Thus one must
not expect that, generally, \( L \preceq K \) implies the inequality \( n(\lambda, L) \leq n(\lambda, K) \) for the distribution functions for the \( s \)-numbers of \( L \) and \( K \).

The other way for measuring compactness is by membership to the Shatten ideals of compact operators. It is easy to show (see, e.g., [Si1]) that if the operator \( K \) belongs to the Hilbert-Schmidt ideal \( \mathcal{E}_2 \) then the same holds for \( L \). This statement is correct also for the classes \( \mathcal{E}_p \) consisting of compact operators for which the sequence of \( s \)-numbers belongs to \( l_p \), provided \( p \) is an even integer ([Si1]). However, the conjecture made by B. Simon in [Si1], that this remains true for all \( \mathcal{E}_p, p \geq 2 \), was refuted by V. Peller in [Pe]: for any \( p \) which is not an even integer, there exist operators \( L \preceq K \) such that \( K \in \mathcal{E}_p \) but \( L \notin \mathcal{E}_p \). It is still unclear, what is the situation here for other ideals of compact operators, in particular for the classes \( \Sigma_p \) consisting of operators with \( n(\lambda) = O(\lambda^{-1/p}) \).

The study of operator domination was partially motivated by the spectral analysis of the magnetic Schrödinger operator. The diamagnetic inequality implies that the resolvent (or, which is equivalent, the heat semi-group) of the Schrödinger operator

\[
H_{a,V} = -\left(\nabla + ia\right)^2 - V
\]

with electric potential \(-V\) and vector magnetic potential \(a\) is dominated by that of \(H_{0,V}\). It might be very tempting to derive eigenvalue estimates for the magnetic operator from the non-magnetic ones, using the diamagnetic inequality only. This would, in particular, lead automatically to estimates not depending on the magnetic field, which agrees with the physical intuition. However the example constructed in [AHSi] shows that even in this concrete situation switching on the magnetic field may move some eigenvalues in the 'wrong' direction, therefore domination does not generally imply inequalities for eigenvalues. Thus obtaining eigenvalue estimates for the magnetic Schrödinger operator became a separate problem, and many methods used in the non-magnetic situation did not work in the magnetic case.

In the present paper we address the general question on comparison of eigenvalues of operators. In Section 2 we show that if the operators \( K \) and \( L \) are related, in a certain way, to positivity preserving and positively dominated semi-groups, then any, sufficiently regular, eigenvalue estimate for \( K \) implies the same estimate for \( L \), with, probably, somewhat worse constant. The \( \mathcal{E}_p \)- and \( \Sigma_p \)-domination questions get here an affirmative answer as well. Further on, we apply general results to estimating negative eigenvalues of abstract Schrödinger-like operators. In this context, one may associate two semi-groups to such operators. The first one was used by E. Lieb [L1] in path integral form and in [RoSol1] in an abstract formulation. Another semi-group was considered in the paper by P. Li and S.-T. Yau [LiY], later in [MeRo] and in the most abstract form in [LeSol]. We use a combination of these approaches. In Section 3 we consider the most simple situation with a very regular 'potential'. Then, in Section 4, the general situation, with, possibly, fairly singular potentials, is treated.

In Sect.5 we describe certain general constructions providing one with pairs of operators to which the above abstract results can be applied. Some of then in-
volve introducing 'magnetic field' in some way. Others deal with automorphic operators. The abstract eigenvalue comparison results of the paper are, actually, 'meta-theorems': although producing no new estimates themselves, they carry over the existing (and all future) estimates to dominated operators, regardless of the method used. Not for all problems described in Sect.5, the eigenvalue estimates for the 'dominator' are presently known. As soon as such estimates are obtained, a similar bound follows automatically for the 'dominated' one. However in the present, there are still a number of concrete problems where application of our results proves to be useful. They are related, chiefly, to the magnetic Schrödinger operators. Here, the situation was mostly settled in dimensions greater than 2 in [BSol]. However, in certain singular situations (say, for potentials being measures) and for two dimensions, there still was a gap between non-magnetic and magnetic results, and this gap is now closed by applying our abstract theorem. In Sect.6 we describe these and some other applications.

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2. We start with a general eigenvalue comparison theorem for semi-group generators. Let $X$ be a space with a $\sigma$-finite measure. In the Hilbert space $L_2(X)$ we consider two positive self-adjoint operators $S$ and $T$. These operators generate contraction semi-groups $P(t) = \exp(-tS)$ and $Q(t) = \exp(-tT)$, $t \geq 0$ in $L_2(X)$. We suppose that the semi-group $Q(t)$ is positivity preserving and $P(t) \preceq Q(t), t \geq 0$. Denote by $N(\lambda, T)$ the distribution function for the eigenvalues of the operator $T$, i.e. the number of the points of the spectrum of $T$ in $(0, \lambda)$; as usual, this quantity is set to infinity if there are points of the essential spectrum of $T$ below $\lambda$. For a function $\varphi(\lambda), \lambda > 0$, growing subexponentially, we denote by $\hat{\varphi}(t)$ the Laplace transform of $\varphi$. Such function $\varphi$ is called regular if

$$t \hat{\varphi}(t) \leq C_{\varphi} \varphi(t^{-1}), t > 0,$$

for a certain constant $C_{\varphi}$. There are quite a plenty of regular functions, which, obviously, form a convex cone in the space of functions on the semi-axis. In particular, power functions, these times logarithm to some power, etc. are regular.

For a compact operator $K$ with $s$-numbers $s_j(K)$ we will denote by $\|K\|_{\Sigma_p}, 0 < p < \infty$, the expression $\sup\{s_j(K)^{1/p}\}$ and by $\|K\|_{\mathfrak{C}_p}, 0 < p < \infty$, the quantity $(\sum s_j(K)^p)^{1/p}$. It is well known that the latter formula for $p \geq 1$ defines a norm in $\mathfrak{C}_p$; otherwise these expressions give quasi-norms in corresponding classes.

**Theorem 1.** Suppose that for the operator $T$ the estimate

$$N(\lambda, T) \leq \varphi(\lambda)$$

holds for all $\lambda > 0$, with a certain non-decreasing function $\varphi(\lambda)$ of subexponential growth. If the semi-group $P(t) = \exp(-tS)$ is dominated by $Q(t) = \exp(-tT)$,
$t \geq 0$, then

(i) for the eigenvalue distribution function for $S$ one has

$$N(\lambda, S) \leq e^{-\lambda} \hat{\phi}(\lambda^{-1}), 0 < \lambda < \infty,$$

in particular, for a regular function $\varphi$,

$$N(\lambda, S) \leq e C \varphi(\lambda), 0 < \lambda < \infty;$$

(ii) if the operator $T^{-1}$ belongs to the class $\Sigma_p, 0 < p < \infty$, then $S^{-1} \in \Sigma_p$ and

$$\|S^{-1}\|_{\Sigma_p} \leq C(p)\|T^{-1}\|_{\Sigma_p};$$

(iii) if the operator $T^{-1}$ belongs to the class $\mathcal{S}_p, 0 < p < \infty$, then $S^{-1} \in \mathcal{S}_p$ and

$$\|S^{-1}\|_{\mathcal{S}_p} \leq \|T^{-1}\|_{\mathcal{S}_p}.$$

**Proof.** The trace of the positive operator $Q(t)$ can be expressed as

$$t \int_0^\infty n(\lambda, T) \exp(-t\lambda) d\lambda \leq t\hat{\phi}(t), t > 0.$$

Since the function $\varphi$ grows subexponentially, this integral converges, and thus $Q(t)$ belongs to the trace class $\mathcal{E}_1$ and, therefore, to the Hilbert-Schmidt class $\mathcal{E}_2$ for any positive $t$. This implies that the operators $P(t)$ dominated by $Q(t)$ also belong to the Hilbert-Schmidt class for all positive $t$, with domination of Hilbert-Schmidt norms. Thus

$$\text{Tr } P(t) = \|P(t)\|_{\mathcal{E}_1} = \|P(t/2)\|^2_{\mathcal{E}_2} \leq \|Q(t/2)\|^2_{\mathcal{E}_2} = \text{Tr } Q(t) \leq t\hat{\phi}(t).$$

Being expressed in the terms of eigenvalues $\lambda_j$ of the operator $S$, this gives

$$\text{Tr } P(t) = \sum_{j=1}^\infty e^{-\lambda_j} \leq t\hat{\phi}(t).$$

Now, for a given positive $\lambda$, we set $t = \lambda^{-1}$ in (5). Since the first $n(\lambda, S)$ terms in the sum in (5) are greater than $e^{-1}$, we arrive at (3). The regularity condition (1) gives (4). Since the power function is regular, we obtain part (ii) of the Theorem, with $C(p) = e\Gamma(1 + \frac{1}{p})$. Finally, $\|S^{-1}\|_{\mathcal{E}_p}^p = \Gamma(\frac{1}{p})^{-1} \int_0^\infty t^{\frac{1}{p} - 1} \text{Tr} \exp(-tS) dt$, and, similarly, for the operator $T^{-1}$; in this way (iii) follows from the inequality $\text{Tr} \exp(-tS) \leq \text{Tr} \exp(-tT)$.

In applications below, we will encounter semi-groups acting in Hilbert spaces which, unlike $L_2$, do not necessarily have intrinsic Hilbert lattice structure, but rather are subspaces in $L_2$. To treat this situation, we require a generalisation of Theorem 1 and a procedure of *extension* of semi-groups. Suppose that $\mathcal{H}_T$ is
a closed subspace in \( L_2(X) \) and let \( Q_T(t) \) be a semi-group in \( \mathcal{H}_T \) generated by a positive self-adjoint operator \( T \). We associate to it a family \( Q(t) \) of operators in \( L_2(X) \) setting \( Q(t) = \mathcal{P}_T^* Q_T(t) \mathcal{P}_T \) where \( \mathcal{P}_T : L_2(X) \to \mathcal{H}_T \) is the orthogonal projection. Since \( \mathcal{P}_T \mathcal{P}^*_T = \mathbf{1}_{\mathcal{H}_T} \), the family \( Q(t) \) possesses the semi-group property, but \( Q(0) = \mathcal{P}_T \mathcal{P}_T \), which, generally, does not coincide with \( \mathbf{1}_{L_2(X)} \). Having another semi-group \( P_S(t) \) acting in a subspace \( \mathcal{H}_S \), we construct the extended semigroup \( P(t) = \mathcal{P}_S^* P_S(t) \mathcal{P}_T \) in \( L_2(X) \) in the same way. Similar to the the semi-group situation, domination of extended semi-groups \( P(t) \leq Q(t) \) is equivalent to the domination of extended resolvents \( \mathcal{P}_S^* (S + t)^{-1} \mathcal{P}_S \leq \mathcal{P}_T^* (T + t)^{-1} \mathcal{P}_T \) in \( L_2(X) \).

**Theorem 2.** Let \( T \) and \( S \) be positive self-adjoint operators in subspaces \( \mathcal{H}_T, \mathcal{H}_S \) and the families \( Q(t) \) and \( P(t) \) be constructed as above. Suppose that \( Q(t) \) is positivity preserving and \( P(t) \leq Q(t), t > 0 \) Then all assertions of Theorem 1 hold true.

**Proof.** Everything goes exactly in the same way as in the proof of Theorem 1, as soon as we show that \( \text{Tr } Q(t) = \sum \exp(t \lambda_j(T)) \) and similarly for the operator \( S \). However this is quite clear: just take the orthonormal basis \( \{u_j\} \) in \( \mathcal{H}_T \) consisting of eigenvectors of \( T \) and complement \( \{P_T u_j\} \) by an orthonormal basis \( \{v_i\} \) in \( L_2(X) \oplus \mathcal{H}_T \), so that \( \mathcal{P}_T v_i = 0 \). Then we have \( \text{Tr } Q(t) = \sum_j (\mathcal{P}_S^* Q_T(t) \mathcal{P}_T P_T u_j, \mathcal{P}_T u_j) + \sum_i (\mathcal{P}_T^* Q_T(t) \mathcal{P}_T v_i, v_i) = \sum_j (Q_T(t) u_j, u_j) \).

3. We consider Schrödinger-like operators now. We start with the most regular case; here it is possible to avoid troublesome technicalities present in the general situation. Let \( A \) and \( B \) be positive definite operators in \( L_2(X) \), with the semi-group \( \exp(-tB) \) positivity preserving and \( \exp(-tA) \) dominated by \( \exp(-tB) \), \( t > 0 \). For a non-negative measurable function \( V(x) \), with \( V \) and \( V^{-1} \) both bounded, we define the operators \( B - qV, A - qV, q \geq 0 \). Denote by \( N_-(B - qV), N_-(A - qV) \) the number of negative eigenvalues of these operators, with the usual agreement on the negative essential spectrum.

**Theorem 3.** Suppose that

\[
(7) \quad N_-(B - qV) \leq \varphi(q)
\]

for all \( q > 0 \) for some non-decreasing function \( \varphi(q) \) of subexponential growth. Then

\[
N_-(A - qV) \leq eq^{-1} \varphi(q^{-1}), q > 0;
\]

\[
(8) \quad \int_0^\infty N_-(A - qV)q^{p-1}dq \leq \int_0^\infty N_-(A - qV)q^{p-1}dq, 0 < p < \infty,
\]

as long as the integral in the right-hand side is finite; and if the function \( \varphi \) in (7) is regular then

\[
(9) \quad N_-(A - qV) \leq c_C \varphi(q),
\]
Proof. We will construct semi-group generators related to our Schrödinger-like operators, and apply Theorem 1. According to the Birman-Schwinger principle [B],

\[ N_-(B - qV) = n(\tau, K_B), N_-(A - qV) = n(\tau, K_A), \tau = q^{-1/2}, \]

where

\[ (10) \quad K_A = V^{1/2} A^{-1/2}, K_B = V^{1/2} B^{-1/2}, \]

and \( n(\tau, K) \) denotes the distribution function for \( s \)-numbers \( s_j(K) \) of \( K : n(\tau, K) = \# \{ j : s_j(K) > \tau \} \) for a compact operator \( K \) in \( L_2(X) \). Under our conditions, these operators are bounded, they are products of bounded operators, and \( K_B \) is compact. Moreover, they have trivial kernels and therefore the self-adjoint compact operators \( K_B K_B^* = V^{1/2} B^{-1} V^{1/2} \) and \( K_A K_A^* = V^{1/2} A^{-1} V^{1/2} \) have self-adjoint positive-definite inverses, respectively, \( T_B \) and \( T_A \). The estimate (7) can therefore be written as \( N(\lambda, T_B) \leq \varphi(\lambda), \lambda > 0 \). So, Theorem 1, with \( T = T_B \) and \( S = T_A \), will give us the required estimate as soon as we show that the semi-group \( \exp(-tT_B) \) dominates \( \exp(-tT_A) \). This is equivalent to the resolvent domination

\[ (T_A + t)^{-1} \lesssim (T_B + t)^{-1}, t > 0. \]

We can write \((T_A + t)^{-1} = (V^{-1/2} A V^{-1/2} + t)^{-1} = V^{1/2} (A + tV)^{-1} V^{1/2} \), and a similar identity for \( T_B \). Now it remains to notice that the Trotter formula implies \((A + tV)^{-1} \approx (B + tV)^{-1}\) and to use positivity of \( V \).

Remark The relation (8) can be expressed also as

\[ \|K_A\|_{\Psi_p} \leq \|K_B\|_{\Psi_p}, 0 \leq p \leq \infty, \]

at the same time, for the regular function \( \varphi(q) = q^p \), (9) implies

\[ \|K_A\|_{\Sigma_p} \leq C(p) \|K_B\|_{\Sigma_p}, 0 \leq p \leq \infty, \]

4. We move now on to the general situation. Suppose that operators \( A, B \) in \( L_2(X) \) are positive and \( \exp(-tA) \approx \exp(-tB) \), \( t \geq 0 \). Let an operator \( J \) be defined on the linear sum \( D(B^{1/2}) + D(A^{1/2}) \) and act into \( L_2(Y) \) where \( Y \) is a space with a \( \sigma \)-finite measure \( dv \). We suppose that the operator \( J : D(B^{1/2}) \to L_2(Y) \) is bounded. The operator \( J \) has also to possess certain positivity properties. First, due to the fact that \( B \) generates a P.P. semi-group, for any function \( u \in D(B^{1/2}) \), its absolute value \( |u| \) belongs to \( D(B^{1/2}) \). We suppose that for such \( u \), \( |Ju| = J|u| \); this, in particular, implies that \( J \) is P.P. Further on, if \( u \in D(A^{1/2}) \), then \( |u| \in D(B^{1/2}) \), so \( J|u| \) is defined for these \( u \) as well, and we suppose again that \( |Ju| = J|u| \) a.e., \( u \in D(A^{1/2}) \). The semi-group domination implies that \( \|A^{1/2}u\| \geq \|B^{1/2}u\|, u \in D(A^{1/2}) \), therefore, our operator \( J \) is bounded on \( D(A^{1/2}) \). We assume that \( \text{Re} (J\text{sign} u, Ju) = (Ju, Ju) \) for \( u \in D(A^{1/2}), 0 \leq v \in D(B^{1/2}) \).
Theorem 4. Let the form \( \|Ju\|_{L^2(Y)}^2 \) be finitely-simally form-bounded with respect to \( B \). Denote by \( A-qV, B-qV, q \geq 0 \), the operators in \( L_2(X) \) defined by the the quadratic forms \( \|A^{1/2}u\|_{L^2(X)}^2-q\|Ju\|_{L^2(Y)}^2, \|B^{1/2}u\|_{L^2(X)}^2-q\|Ju\|_{L^2(Y)}^2 \) considered initially on \( D(A^{1/2}), D(B^{1/2}) \), respectively. Then the statements of Theorem 3 hold true.

Proof. We suppose that \( A, B \) are positive definite – this assumption is, as usual, disposed of by considering the operators \( A+\varepsilon, B+\varepsilon, \varepsilon > 0 \), instead of \( A, B \) and then setting \( \varepsilon \to 0 \). Consider operators \( K_A = JA^{-1/2}, K_B = JB^{-1/2} \) which are bounded from \( L_2(X) \) to \( L_2(Y) \). According to the Birman-Schwinger principle, we have \( N_-(B-qV) = n(q^{-1}/2), K_B = n(q^{-1}, K_B K_B) = n(q^{-1}, K_B K_B) \), and, similarly, for the operator \( A-qV \). Denote by \( \mathcal{H}_B \) the closure of the range of \( K_B \) in \( L_2(Y) \) and by \( \mathcal{P}_B : L_2(Y) \to \mathcal{H}_B \) the orthogonal projection. We set \( K_B = \mathcal{P}_B K_B \) and consider it as operator from \( D(B^{1/2}) \) to \( \mathcal{H}_B \). Of course, \( K_B K_B^* \) and \( K_B^* K_B \) have same nonzero eigenvalues, therefore one may replace \( K_B K_B^* \) by \( K_B K_B^* \) in the above Birman-Schwinger formulas. Now, since the range of \( K_B \) is dense in \( \mathcal{H}_B \), the operator \( K_B^* : \mathcal{H}_B \to D(B^{1/2}) \) has trivial null space, and the same is true for the operator \( K_B K_B^* \) in \( \mathcal{H}_B \). Thus the range of the latter operator is dense in \( \mathcal{H}_B \) and therefore it has a positive self-adjoint inverse \( T_B = (K_B K_B^*)^{-1} \). We are going to prove that the extended semi-group \( Q(t) = \mathcal{P}_B \exp(-tT_B) \mathcal{P}_B \) in \( L_2(Y) \) is positivity preserving and that this family dominates a similar extended semi-group \( \tilde{P}(t) \) constructed for the operator \( A \). After that, it remains to apply Theorem 2.

Considering the resolvent of \( T_B \), we write it as

\[
(T_B + t)^{-1} = ((K_B K_B^*)^{-1} + t)^{-1} = K_B (1 + tK_B^* K_B)^{-1} K_B^* .
\]

One can obtain this identity from

\[
((K K^*)^{-1} + t)^{-1} = K (1 + tK^* K)^{-1} K^* , \quad t > 0 ,
\]

for any bounded operator \( K \). In its turn, for sufficiently small \( t \), (11) follows from the obvious formula \( ((K^* K)^{-1} + t)^{-1} = (1 + t(K^* K)^{-1}) K^* K \) and the Taylor expansion, and then it extends by analyticity to all positive \( t \). However, \( K_B K_B^* = K_B^* K_B \) as operators in \( L_2(X) \), \( \mathcal{P}_B^* K_B = K_B, K_B^* \mathcal{P}_B = K_B^* \), and therefore the extended resolvent of \( T_B \) in \( L_2(Y) \) has the form

\[
\mathcal{P}_B(T_B + t)^{-1} \mathcal{P}_B = K_B (1 + tK_B^* K_B)^{-1} K_B^* .
\]

Now we need some regularisation in (12). Let \( \varepsilon > 0 \). We have \( K_B \exp(-\varepsilon B) = J_B(\varepsilon) B^{-1/2} \) where \( J_B(\varepsilon) = \exp(-\varepsilon B) : L_2(X) \to L_2(Y) \) is a bounded operator. As \( \varepsilon \to +0 \), \( K_B \exp(-\varepsilon B) \to K_B \) strongly. Similarly, \( \exp(-\varepsilon B) K_B^* = K_B^* \exp(-\varepsilon B) = J_B(\varepsilon) B^{-1/2} K_B^* \).
$B^{-1/2}(J_B(\varepsilon))^*$ and \(\exp(-\varepsilon B)K^*_B \to K^*_B\) as \(\varepsilon \to +0\). Now, since the strong convergence of operators respects the P.P. property (we will use this recurrently), it is sufficient to prove that

\[
J_B(\varepsilon)[B^{-1/2}(1 + tK^*_B K_B)^{-1} B^{-1/2}]J_B(\varepsilon)^*
\]

is P.P. Due to our conditions imposed on \(J\) and \(B\), both operators \(J_B(\varepsilon)\) and \(J_B(\varepsilon)^*\) are P.P. As for the middle term in brackets in (13), it equals \((B + tJ^*J)^{-1}\) where the operator \(B + tJ^*J\) corresponds to the quadratic form \(\|B^{1/2}u\|_{L_2(X)}^2 + t\|Ju\|_{L_2(Y)}^2\) defined on \(\mathcal{D}(B^{1/2})\). To justify the latter statement, we take any element \(u \in L_2(X)\) and denote \(B^{-1/2}(1 + tK^*_B K_B)^{-1} B^{-1/2}u\) by \(v\) which, of course, belongs to \(\mathcal{D}(B^{1/2})\). Then we have \(B^{-1/2}u = (1 + tK^*_B K_B)B^{1/2}v\), and therefore \((u, v) = (B^{-1/2}u, B^{1/2}v) = ((1 + tK^*_B K_B)B^{1/2}v, B^{1/2}v) = \|B^{1/2}v\|_{L_2(X)}^2 + t\|K_B B^{1/2}v\|_{L_2(Y)}^2\). So it remains to notice that \(K_B B^{1/2}v = Jv, v \in \mathcal{D}(B^{1/2})\). Now the properties of \(J\) and \(B\) imply that Beurling-Deni conditions are satisfied for the form \(\|B^{1/2}u\|_{L_2(X)}^2 + \|Ju\|_{L_2(Y)}^2\) on \(\mathcal{D}(B^{1/2})\). Thus we have established that \(T_B\), after extension, generates a P.P. family in \(L_2(Y)\).

We can repeat the same procedure starting with the operator \(A\) instead of \(B\) and obtain a self-adjoint operator \(T_A = (K_A K_A^*)^{-1}\) in \(\mathcal{H}_A\) where \(K_A = JA^{-1/2} : L_2(X) \to L_2(Y)\) and \(\mathcal{H}_A\) is the closure of the range of \(JA^{-1/2}\). In order to use Theorem 1, we check that the extended semi-group generated by \(T_B\) dominates the one generated by \(T_A\). It suffices to establish this domination for the regularized extended resolvents of these operators:

\[
J_A(\varepsilon)(A + tJ^*J)^{-1}J_A(\varepsilon)^* \preceq J_B(\varepsilon)(B + tJ^*J)^{-1}J_B(\varepsilon)^*,
\]

where \(J_A(\varepsilon) = J \exp(-\varepsilon A) : L_2(X) \to L_2(Y)\) for \(\varepsilon > 0\). According to our conditions, \(J_A(\varepsilon) \preceq J_B(\varepsilon)\). The domination relation is respected by the operation of taking adjoints (this is shown, e.g. in [Pe], Lemma in Sect.8). Thus \(J_A(\varepsilon)^* \preceq J_B(\varepsilon)^*\) as well. Finally, consider the middle terms \((A + tJ^*J)^{-1}\) and \((B + tJ^*J)^{-1}\) in (14). To prove domination, using the criterion obtained in [Ou, Theorem 3.3], it is sufficient to check the Kato-type inequality

\[
\text{Re} \left( ((A+tJ^*J)^{1/2} \text{sign} g)(A+tJ^*J)^{1/2}g \right) \geq (B+tJ^*J)^{1/2}f, (B+tJ^*J)^{1/2}|g|
\]

for \(f \in \mathcal{D}((B+tJ^*J)^{1/2})\), \(f \geq 0\) and \(g \in \mathcal{D}((A+tJ^*J)^{1/2})\), so that \(f \leq |g|\), as well as the implication \(g \in \mathcal{D}((A+tJ^*J)^{1/2}) \Rightarrow |g| \in \mathcal{D}((B+tJ^*J)^{1/2})\). Under our conditions for \(J\), the latter implication follows from the same fact for \(t = 0\). To establish the inequality (15), we note again that it holds for \(t = 0\) since \(\exp(-sA) \preceq \exp(-sB)\). Thus, taking into account the explicit expression for the bilinear forms for operators \((A+tJ^*J)\) and \((B+tJ^*J)\), we have only to check that \(\text{Re} (Jf \text{sign} g, Jg) \geq (Jf, Jg)|g|\), but here, due to the properties of the operator \(J\), we actually even have the equality \(\bullet\)
5. In this section we give some examples of operators $A, B, J$ satisfying general conditions of Sect.4. We describe here three basic methods of establishing semi-group domination and, correspondingly, three groups of examples.

5.1. The study of semi-group domination in 70-s was motivated by the Schrödinger operator with magnetic field and its generalisations. In [Si2] domination was proved along the lines of the following reasoning. Let $B$ be (probably, locally) represented as the form sum of operators $B_j$, each of them being a generator of P.P. semi-group. The operator $A$ is defined as the form-sum of operators $A_j = \psi_j^{-1} B_j \psi_j$, where the functions $\psi_j$ have absolute value 1 and are regular enough - so that the form sum of $A_j$ makes sense. Then the relation $\exp(-tA) \lesssim \exp(-tB)$ follows from the appropriate version of the Trotter formula. This method, used by B. Simon for the magnetic Schrödinger operator, can be easily carried over to certain other situations. So, let $X$ be a complete Riemannian manifold and $B$ the Laplace-Beltrami operator on $X$. To specify $B$ as the Hilbert space operator, we determine it by means of the usual quadratic form defined initially on $C_0^\infty(X)$. The magnetic potential $a$ is a 1-form on $X$, belonging to $L_{2,\text{loc}}(X)$, the magnetic Schrödinger operator $A$ is defined by the similar quadratic form with $\nabla$ replaced by $\nabla + ia$ in the local representation, and the Trotter formula reasoning goes through. To generalise this, one can relax smoothness and ellipticity conditions (implied by the Riemannian picture). Say, let $Z_j$ be a system of real vector fields on $X$, with locally bounded coefficients. We set $B = \sum Z_j^* Z_j$, the operator corresponding to the quadratic form $\int_X \sum |Z_j u|^2 dx$ defined initially on $C_0^\infty(X)$. For a system of functions $\psi_j(x)$, $|\psi_j(x)| = 1$, the 'magnetic operator' is $A = \sum \psi_j^{-1} Z_j^* Z_j \psi_j$. It corresponds to the quadratic form $\int \sum Z_j(\psi_j u)^2 dx$, $u \in C_0^\infty(X)$, and is well defined provided $Z_j \psi_j \in L_{2,\text{loc}}$. Another modification dealing with the quadratic form $\sum \int_{\mathbb{R}^d} g_{jk} D_j u D_k u dx$, $X = \mathbb{R}^d$, for the operator $B$, with a different sort of degeneration and with coefficients having singularities, $g \in L_p$, $g^{-1} \in L_q$, $p^{-1} + q^{-1} = 2/d$, was considered in [MeRoz]. Generally, in such cases, one has to prove that the 'magnetic' quadratic form is closable, so that it defines a self-adjoint operator, but this is made in a standard way, see, e.g. the reasoning in [MeRoz].

The same approach works for discrete operators. Say, for a lattice $\mathbb{Z}^d$ we define the Laplacian $B = \sum \delta_k^* \delta_k$ by means of the quadratic form $\sum_k \sum_{x \in \mathbb{Z}^d} |\delta_k u(x)|^2$ where $\delta_k$ is the difference in direction $k$. The form is defined initially on functions with finite support. The magnetic operator $A$ corresponds to the quadratic form $\sum_k \sum_{x \in \mathbb{Z}^d} |\delta_k(\psi_k(x) u(x))|^2$ where $|\psi_k(x)| = 1$, $x \in \mathbb{Z}^d$, $k = 1, \ldots, d$. And again the Trotter formula gives domination. For more general graphs, it is more convenient to use other methods, see next section.

5.2. A different way of establishing domination, found in 70-s, is based on the Kato inequality (see, e.g., [Si3]). A development of this method, enabling one to consider quadratic forms rather than operators, was proposed by E-M. Ouhabaz in [Ou]. We have already used one of his results when proving Theorem 4. Here
we discuss application of another theorem from [Ou]. Suppose that \(a\) and \(b\) are two closed quadratic forms in \(L_2(X)\) with domains \(D(a), D(b)\), and they define self-adjoint operators \(A, B\). The set \(D(a)\) is called an ideal for \(D(b)\) if \(u \in D(b)\) for any \(u \in D(a)\) and for any \(u \in D(a), v \in D(b)\), such that \(|v| \leq |u|\), the function \(v\) sign \(u\) belongs to \(D(a)\). According to Corollary 3.5 in [Ou], if the forms \(a, b\) are restrictions of a closed form \(\mathfrak{h}\) and the operator corresponding to \(\mathfrak{h}\) generates a P.P. semi-group, the domination \(\exp(-tA) \trianglelefteq \exp(-tB)\) is equivalent to \(D(a)\) being an ideal of \(D(b)\). In particular, \(B\) generates a P.P. semi-group if \(D(b)\) is an ideal of itself.

We apply this criterium in the following situation. Let \(M\) be a non-compact complete Riemannian manifold and \(\Gamma\) a discrete group of isometries of \(M\). Consider a fundamental domain \(X = \Gamma \setminus M\) of the group \(\Gamma\). The Neumann Laplacian \(H\) in \(L_2(X)\), defined by means of the usual Dirichlet form \(\mathfrak{h}[u] = \int_X |\nabla u|^2 dx\) initially on \(C_0^\infty(X)\) generates a P.P. semi-group, as it follows from the usual Kato’s inequality. We consider two restrictions of \(\mathfrak{h}\). One of them, \(b\) is defined on the functions with periodic boundary conditions: \(u(\gamma x) = u(x), \ x, \gamma x \in \partial X\). This gives us the periodic Laplacian \(B\) on \(X\). The other operator is the automorphic one. If \(\chi\) is a character of the group \(\Gamma\), i.e. its one-dimensional unitary representation, then the automorphic operator \(A = A_{\chi}\) corresponds to the restriction of \(\mathfrak{h}\) to the functions with the condition \(u(x) = \chi(\gamma)u(\gamma x); x, \gamma x \in \partial X\). The criterium above easily implies domination \(\exp(-tA_{\chi}) \trianglelefteq \exp(-tB)\). In this construction the group structure is not that essential. More generally, let \(X\) be a manifold with piecewise smooth boundary \(\partial X\). Suppose that there exists a smooth section \(\gamma\) of the smooth part of \(\partial X\), such that the metrics at the boundary is periodic with respect to \(\gamma\). The Neumann and periodic Laplacians are defined as before. Now let \(\chi(x)\) be a measurable function on the boundary such that \(|\chi(x)| = 1\). Then the operator \(A_{\chi}\) corresponds to the Dirichlet form restricted to the functions with boundary conditions \(\chi(x)u(x) = \chi(\gamma x)u(\gamma x); x \in \partial X\), and domination follows.

The quadratic form criterium from [Ou, Theorem 3.3] works effectively for operators on graphs. Let the infinite graph \(G\) have \(V\) as the set of vertices and the number \(n(x) = \#\{y \in V : y \sim x\}\) be finite for every \(x\), where \(\sim\) denotes the relation of vertices \(a, b\) being connected by a wedge. We associate the weight \(n(x)\) to the vertex \(x\) and define the Laplacian \(-\Delta_G\) by means of the form \(\frac{1}{2} \sum_{x \sim y} \langle u(x) - u(y) \rangle^2\). The discrete version of the magnetic potential is a real function \(\psi(x, y) = -\psi(y, x)\) defined on the set of pairs \(x \sim y\) and the the discrete magnetic Laplacian is defined by means of the form \(1/2 \sum_{x \sim y} |u(x) - \exp(i \psi(x, y)u(y))|^2\). The domination conditions, similar to (15), are now checked by simple calculations.

5.3. One more method of establishing domination also works in the automorphic situation, but admits a different sort of generalisations. As before, let \(X\) be a fundamental domain of a group \(\Gamma\) of isometries of \(M\) and \(H\) be the Laplacian on \(M\), generating a P.P. semi-group \(R(t)\) in \(L_2(M)\). We can define the operator family
where $\mathcal{P}$ is the restriction operator from $L^2(M)$ to $L^2(X)$, $\mathcal{P}^* : L^2(X) \to L^2(M)$ its adjoint and $\gamma^*$ denotes the natural action of $G$ in $L^2(X)$. Supposing that the sequence of operator norms $\|\mathcal{P} \mathcal{R}(t) \gamma^* \mathcal{P}^*\|$ belongs to $l_1$, uniformly in $t$ (essentially, this means a certain rate of decay of the heat kernel away from the diagonal), it is easy to show that $Q(t)$ is a strongly continuous semi-group in $L^2(X)$. We take its generator as $B$. P.P. property follows directly from (16) since all terms in the sum are P.P. operators. At the same time, for a character $\chi$, we define the strongly continuous semi-group $P_\chi(t)$ in $L^2(X)$ by setting $(P_\chi(t)u)(x) = \sum_{\gamma \in \Gamma} \chi(\gamma)(\mathcal{P} \mathcal{R}(t) \gamma^* \mathcal{P}^* u)(x)$ and denote by $A_\chi$ its generator. Having these representations, the semi-group domination becomes obvious, and thus the pair $(B, A_\chi)$ fits in the picture of Sect.4.

More generally, one may drop the manifold structure and perform the same construction, supposing only that $M$ is a general space with a $\sigma$-finite measure, $\Gamma$ is a group of measure preserving bijections of $M$ (having only a measure-zero set of fixed points). Having a positive operator $H$ in $L^2(M)$ commuting with the action of $\Gamma$, a character $\chi$ and a measurable set $X$ intersecting over a single point with each orbit of $\Gamma$, we can define operators $A_\chi$ and $B$ by means of the above formulas, provided $H$ generates a P.P. semi-group. This works, in particular, in the discrete case with $M$ being an infinite graph, $\Gamma$ a group of its automorphisms without fixed points and $X$ a system of representatives of nodes from every orbit of $\Gamma$. The 'magnetic' and 'automorphic' constructions may be inter-related; see, e.g. [Su].

5.4. Having created a certain stock of such pairs of operators, one can construct more by considering functions of operators and their combinations. Thus, if for given $A$ and $B$, the semi-group domination takes place, the same holds for $F(A)$ and $F(B)$ if $F(s)$ is such a positive function on the semi-axis $s > 0$ that $(-1)^k F^{(k)}(s) \leq 0$, $k > 0$ (see [BrKiRob]). The most important examples here are $F(s) = s^\alpha$, $0 < \alpha < 1$ and $F(s) = (s + 1)^\alpha - 1$, $0 < \alpha < 1$ appearing in some relativistic quantum models. One gets more examples by taking convex combinations of admissible pairs, obtained, possibly, by different methods. Even more new pairs one obtains by taking tensor products: if $\exp(-tA_j) \preceq \exp(-tB_j), j = 1, 2$, then $\exp(-tA) \preceq \exp(-tB)$ where $A = A_1 \otimes 1 + 1 \otimes A_2$ and $B = B_1 \otimes 1 + 1 \otimes B_2$. This latter construction may be localized.

5.5. To construct examples of the operator $J$, we take a measurable subset $Y$ in $X$, a $\sigma$-finite measure $\nu'$ on $Y$ and a $\nu'$-measurable, non-negative function $V$ on $Y$. We set $dv = V \nu'/\nu$. Suppose that on a certain common form core $\mathcal{M}$ for the operators $B$ and $A$, the restriction on $Y$ of functions $u \in \mathcal{M}$ makes sense and
the linear restriction operator \( J : \mathcal{M} \to L_2(Y, dv) \) is bounded in \( \mathcal{D}(B^{1/2}) \)-norm. Then it is bounded in \( \mathcal{D}(A^{1/2}) \)-norm as well and can be extended by continuity to \( \mathcal{D}(B^{1/2}) + \mathcal{D}(A^{1/2}) \). The conditions imposed on \( J \) above are satisfied provided \( J \) maps non-negative functions to non-negative. This all takes place in the examples of \( A, B \) above related to the magnetic field: the space \( C_0^\infty \) acts as this common form core. The conditions on \( Y, \nu', V \) granting boundedness depend on the operators \( A, B \). In particular, for operators related with the Laplacian, they may be expressed in the terms of capacity and can be found in [M]. Note that we might include \( V \) into the measure \( \nu \) from the very beginning, but the separate treatment makes particular examples more accessible – see next section. This construction may involve measures having both nontrivial absolutely continuous and singular parts with respect to \( dx \).

For the situations of Sect. 5.2, 5.3, there may be no obvious common form core for \( A, B \). Thus certain additional restrictions have to be imposed on \( (Y, dv) \). Say, if the measure \( dv \) is absolutely continuous with respect to \( dx \), then the natural operator of restriction to \( Y \) is well defined on \( L_2(X) \), so only boundedness conditions have to be checked. In the presence of a singular component, the operator \( J \) is well defined if, for example, in the manifold case, the support of the singular part of the measure \( dv \) lies in the interior of \( X \).

6. Now we discuss some examples where, in the situations described above, there already exist eigenvalue estimates for the dominator \( B \) or they can be easily obtained by standard methods, so that application of our theorems produces new results. Note that quite often, as soon as the correct order estimate for eigenvalues is established, the asymptotic formula for eigenvalues follows in a routine way. We will not discuss such possibilities here. we restrict ourselves to second order operators; lower order ones, such as generalisation of the relativistic Schrödinger operator, can also be treated.

6.1. We start with the well studied case of the Schrödinger operator \( B - qV \) in \( \mathbb{R}^d \), \( B = -\Delta \), with \( 0 \leq V \in L_{d/2,\text{loc}} \). The fundamental result here is the CLR estimate \( N_-(B - qV) \leq C(d)q^{d/2} \int V^{d/2} dx \) for \( d \geq 3 \), which is valid as long as the integral converges. For the magnetic Schrödinger operator \( A - qV, A = -\Delta_a = -(\nabla + ia)^2 \) with magnetic potential \( a \) in \( L_{2,\text{loc}} \) we take the support of the function \( V \) as \( Y, dv = V dx \) and consider the identity operator \( J : C_0^\infty(\mathbb{R}^d) \to L_2(Y) \). Since \( C_0^\infty(\mathbb{R}^d) \) is a common form core for \( A \) and \( B \) and the function \( q^{d/2} \) is regular, we can apply Theorem 4 and obtain the magnetic CLR estimate \( N_-(A - qV) \leq C'(d)q^{d/2} \int V^{d/2} dx \), as long as \( V \in L_{d/2} \). This result was proved earlier by other, more specific methods, in [Si4] and later in [MeRo] and [RoSp1]. Note, however, that in [Si4] and [RoSp1] the constant in the estimate does not worsen compared with the nonmagnetic case. If \( V \) is still in \( L_{d/2,\text{loc}} \) but decays at infinity more slowly, so that it is not in \( L_{d/2} \) and the CLR-bound is useless, a number of estimates for \( N_-(B - qV) \) were obtained in
There, it is shown also that if a certain eigenvalue estimate is established for the Schrödinger operator by the particular method of that paper, then the same estimate holds for the magnetic case. Our Theorem 4 may be applied in this situation as well, but it produces no new results. Note that in [BSol] only domination was used for carrying-over the estimates.

If the function \( U \geq 0 \) tends to infinity as \( x \to \infty \), so that the operator \(-\Delta + U\) has discrete spectrum, the usual quadratic forms manipulations derive the estimate \( N(\lambda, -\Delta + U) \leq C \int (\lambda - U)^{d/2} \, dx, \lambda \geq 0 \), from the magnetic CLR-estimate. However, if the potential \( U \) tends to infinity not in all directions or not uniformly, so that the integral in the last formula diverges, the spectrum still may be discrete, but a different sort of eigenvalue estimates takes place. So, in [Sol1], the case of a homogeneous potential \( U \) was considered, \( U(x) = |x|^\kappa f(\omega), \omega \in S^{d-1}, \kappa \geq 0 \). If \( f^{-1} \) does not belong to \( L_\theta(S^{d-1}), \theta = d\kappa^{-1}(1 + \kappa/2) \) then the integral in the CLR-estimate diverges. Suppose that the function \( f \) is continuous and vanishes only on a Lipschitz subset of dimension \( d_0 - 1 \) in \( S^{d-1} \), moreover the order of vanishing is not higher than \( \kappa - \kappa_0 \) at any point, \( 0 < \kappa_0 < \kappa \) then \( N(\lambda, -\Delta + U) \leq C(U)\lambda^{d_0\kappa_0^{-1}(1+\kappa/2)}, \) provided \( d_0 \kappa_0^{-1} > d\kappa^{-1} \). Since the semi-group generated by \(-\Delta + U\) dominates the one generated by \(-\Delta_0 + U\), Theorem 1 implies the same order of estimate for the magnetic Schrödinger operator. In the same way, in the case when \( d_0 \kappa_0^{-1} = d\kappa^{-1} \), one gets estimates with a logarithm term. So, for the potential \( U(x_1, x_2) = x_1^2 x_2^2 \) in \( \mathbb{R}^2 \), the estimate \( N(\lambda, -\Delta + U) \leq C\lambda \log \lambda \) holds.

6.2. In the two-dimensional case, the problem of eigenvalue estimates is considerably more delicate. The situation becomes somewhat more simple when a regularizing positive term is added to the Laplacian, so that the virtual level at zero is removed.

The operator \( H(1, q) = -\Delta + 1 - qV, \ V \geq 0 \) in \( L_2(\mathbb{R}^2) \) was considered in a number of papers. Here, the latest results are obtained in [Sol3]. In particular, the conditions are found for the estimate

\[
N_-(H(1, q)) = O(q^\kappa), \kappa \geq 1,
\] (17)

to hold. Thus, if the function \( V \) belongs to the Orlicz space \( L \log(1 + L) \) locally and the sequence of its Orlicz norms over a grid of equal squares covering the plane belongs to the weak class \( l_{\kappa,w} \) for \( \kappa > 1 \) and to \( l_1 \) for \( \kappa = 1 \), (17) takes place. Since the power function \( q^\kappa \) is regular, Theorem 4 applies and it gives the estimate of the form (17) for the magnetic two-dimensional Schrödinger operator with a constant not depending on the magnetic field. A weaker result, with local \( L_r \)-norms belonging to \( l_1 \) is obtained in [RoSol2].

Another form of Orlicz type estimates was obtained by T.Weidl [W]. If we denote by \( V_0 \) the function \((qV - 1/2)_+\), then

\[
N_-(H(1, q)) \leq C(||V_0 \ln_+(V_0||V_0||^{-1})||_1 + ||V_0||_1(1 + \ln_+ ||V_0||_1)).
\] (18)
The function on the right-hand side of (18) is not necessarily regular in the sense of Sect. 2. However, if for a particular potential \( V \) it is, then (18) is valid for the magnetic operator as well. Anyway, the other statements of Theorems 3, 4 still apply without any assumptions on \( V \).

6.3. Another sort of regularizing terms were considered in [BLa], [Sol2] and [LaN]. Set \( H(r^{-2}, q) = -\Delta + r^{-2} - qV, r = |x| \). Following [Sol2], suppose that \( V \in L \log(1 + L) \) locally and let \( \{\zeta_j\}, j \geq 0 \) be the sequence of averaged Orlicz norms of \( V \) over the annuli \( r \in (2^{j-1}, 2^j), j > 0 \) and over the unit disk for \( j = 0 \). Then, according to [Sol2], \( N_- (H(r^{-2}, q)) \leq C q \|\{\zeta_j\}\|_1 \). Since the function in the right-hand side is regular, this estimate is carried over to the magnetic case. In the same way, all estimates obtained in [BLa] and [LaN] hold for the magnetic operator. We state here only the magnetic version of the result from [LaN]: if the potential \( V \) is radially symmetric then

\[
N_- (-\left(\nabla + ia \right)^2 + r^{-2} - qV) \leq C q \int_{\mathbb{R}^2} V(x)dx.
\]

As usual, the constant in (19) does not depend on the magnetic field (we will not mention this any more).

6.4. We consider now the two-dimensional Schrödinger operator without regularizing terms. As it is shown in [Sol2, 3] and [BLa], for the operator \(-\Delta - qV\) in \( L_2(\mathbb{R}^2) \), one has to consider two problems. One of them is the one-dimensional operator on the axis \(-\infty < \tau < \infty\):

\[
Z_q v(\tau) = -v''(\tau) - qF(\tau)v(\tau),
\]

where \( F(\tau) = (2\pi)^{-1} e^{2\tau} \int V(e^\tau, \omega) d\omega \). The other one is the same Schrödinger operator but with a Hardy type term added, as in Sect 6.3:

\[
H(r^{-2}, q) u = -\Delta u + r^{-2}u - qVu, r = |x|.
\]

Due to the splitting \( L_2 \) into the direct sum of the spaces of rotation invariant functions and of functions orthogonal to those, and the corresponding decomposition of the Birman-Schwinnger operator,

\[
N_- (-\Delta - qV) \leq N_- (Z_q) + N_- (H(r^{-2}, q)).
\]

Estimating the terms to the right in (22), one obtains bounds for \( N_- (-\Delta - qV) \). In [BLa], it is shown that the order of the terms in (22) may be different, and each of them may be leading. In the presence of the magnetic field, there’s no similar decomposition of the Birman-Schwinnger operator. However the domination leads,
via Theorem 4, to the following conditional result. Suppose that for a given $V$ one has an estimate $N_-(Z_q) + N_-(W_q) \leq \varphi(q)$ for a regular function $\varphi(q)$. Then

$$N_-(-(\nabla + ia)^2 - qV) \leq e\varphi(q).$$

Estimates for the first term in (22), having a power order, were obtained in [BLa] and [Sol2]. The second term was discussed above, in Sect. 6.3. Being put together, these results produce concrete eigenvalue estimates for the two-dimensional magnetic Schrödinger operator. In particular, the estimates hold for the case when one of the terms in (22) admits a regular estimate (as in Sect. 6.3) and the other one has a lower order, having thus been absorbed by the leading one, or the same order. Such situation is discussed in [BLa], Sect. 5. In particular, in the Orlicz space terms, like in Sect. 6.3, the estimate

$$N_-(\Delta - qV) \leq 1 + C q(\|\zeta_j\| + \int V(x) \log |x| dx)$$

obtained in [Sol2] carries over to the magnetic case.

6.5. As it is shown in [BLa, Sect. 7.2], the multi-dimensional Schrödinger operator

$$H(-\kappa_d r^{-2}, q) = -\Delta - \kappa_d r^{-2} - qV(x), \kappa_d = \frac{4}{(d-2)^2}, \ r = |x|, \text{in } \mathbb{R}^d, d \geq 3$$

has properties similar to the ones of the two-dimensional problem and can be treated in the similar way. In fact, the presence of the negative Hardy type term produces a virtual bound state at zero. Again, due to splitting, (22) holds, with the above expression for $F(\tau)$ replaced by its natural $d$-dimensional version $F(\tau) = \mathbb{S}^{d-1} \int_{\mathbb{S}^{d-1}} e^{2\tau V} (e^\tau, \omega) d\omega$. And thus our usual application to the magnetic operator follows: in the $d$-dimensional case estimate (23) holds, provided $N_-(Z_q) + N_-(W_q) \leq \varphi(q)$ for a regular function $\varphi(q)$. For the first term in (22) one can use the same estimates as before, see [BLa]; for the second term, the usual CLR-estimate applies.

6.6. Certain quantum models involve potentials, having support on zero-measure sets in $\mathbb{R}^d$, for example, on surfaces. To illustrate application of our Theorem 4 to this situation, we consider a model problem. Let $V(y) \geq 0$ be a function on $\mathbb{R}^{d-1}, d \geq 3$. The Schrödinger operator in $L^2(\mathbb{R}^d)$ with potential $-q\delta \otimes V$ is defined by means of the quadratic form $\int_{\mathbb{R}^d} |u|^2 dx - q \int_{\mathbb{R}^{d-1}} V(y) |u(0, y)|^2 dy$ defined initially on $C_0^\infty(\mathbb{R}^d)$. Negative spectrum estimates for this problem can be obtained by many ways; the following is, probably, the shortest. From the standard embedding theorem, the inequality $\|(\Delta_{d-1})^{1/4} I u\|_{L_2(\mathbb{R}^{d-1})} \leq C \|(\Delta_d)^{1/2} u\|_{L_2(\mathbb{R}^d)}$ follows for the trace operator $I : C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^{d-1})$ (here, for the sake of clarity, we denote by $\Delta_k$ the $k$-dimensional Laplacian. Therefore, the Birman-Schwinger
operator $K = V^{1/2} I (−Δ_d)^{−1/2}$ can be factorized as $K = ((V^{1/2} (−Δ_{d−1})^{−1/4}) × ((−Δ_{d−1})^{1/4} I (−Δ_d)^{−1/2})$. To the first factor we can apply the Cwikel estimate, and the second factor is bounded, which gives $N_− (D − qδ ⊗ V) ≤ C q^{d−1} \int_{R^{d−1}} V(y)^{d−1} dy$.

Now, applying Theorem 4, with $Y = R^{d−1}$ and the operator $J = V^{1/2} I$, we arrive at the above estimate for the magnetic Schrödinger operator. Of course, one can consider more general surfaces instead of $R^{d−1}$ and more general operators instead of the Laplacian.

6.7. Another class of very singular potentials involves measures supported on fractal sets. In [NaSol], a class of self-similar measures was considered. So, let $S = \{S_l\}, l = 1, ..., m$ be a set of contracting similitudes of $R^d$ with contraction coefficients $h_l$, and let $p = \{p_l\}$ be a system of positive weights, $p_1 + ... + p_m = 1$. Then there exists a unique nonempty compact set $C \subset R^d$ such that $C = \cup S_l C$ and a unique probability Borel measure $μ$ supported on $C$ which satisfies the self-similarity property $μ = \sum p_l μ \circ S_l^{−1}$. Under certain conditions, it was found in [NSol] that for the Schrödinger operator with $μ$ acting as a potential, there is an estimate $N_− (D − qμ) ≤ C q^{δ}$, where $δ$ is the unique positive solution of $\sum (h_l^{2−d} p_l)^δ = 1$.

Now we apply Theorem 4, with $Y = C$, and $J$ being the restriction operator to $C$ (this operator satisfies conditions of Theorem 4 since $C_0^\infty$ is a common form core, see Sect. 5.5). This leads us to the same eigenvalue estimate for the magnetic case.

6.8. In the two-dimensional case, a number of eigenvalue estimates were obtained in [LaN] for general measure-potentials for the Schrödinger operator, with the regularizing term $|r|^{-2}$, as in Sect. 6.3. All these estimates, via Theorem 4, are carried over to the magnetic case. We formulate here only one of corresponding results. Suppose that the finite measure $μ$ on $R^2$ is rotation invariant. We define the operator $−Δ_a + r^{−2} − qμ$ by means of the quadratic forms, as before. Then $N_− (−Δ_a + r^{−2} − qμ) ≤ C qμ(R^2)$.

6.9. To give an example of the construction in Sect. 5.2, consider the strip $X = R^{d−1} × (0, 1)$. The usual Schrödinger operator $−Δ − qV$ in $L_2(X)$, with periodic boundary conditions has spectral properties similar to the ones of the operators in Sect. 6.3, 6.4. Just in the same way, after the natural splitting of the Birman-Schwinger operator, one has

\begin{equation}
N_− (−Δ_a − qV) ≤ N_− (−Δ_a + 1 − qV) + N_− (−Δ_{d−1} − qF),
\end{equation}

where $F(x') = \frac{1}{0} \int V(x', t) dt$. For the first term in (24), one can apply the usual CLR-estimate. The second term in (24), presents the same sort of problem, but in a lower dimension, so one can apply estimates discussed in Sect. 6.3 for $d−1 ≤ 2$ or the CLR estimate for $d−1 ≥ 3$. So, in the latter case, we have

\begin{equation}
N_− (−Δ − qV) ≤ C q^{d/2} \int_X V(x)^{d/2} dx + C q^{(d−1)/2} \int_{R^{d−1}} F(x')^{(d−1)/2} dx'.
\end{equation}
Now, let $\chi(x')$ be a (say, locally smooth) function on $\mathbb{R}^{d-1}$, $|\chi(x')| = 1$. We consider the automorphic Laplacian $-\Delta \chi$ in $L_2(X)$ defined by the quadratic form $\langle \nabla u, \nabla u \rangle$ on the domain $C_0^\infty(X)$ with boundary condition $u(x', 1) = \chi(x')u(x', 0)$, as in Sect. 5.2. Applying Theorem 4, we arrive at the estimate of the form (25), with $\Delta \chi$ instead of $\Delta$.

6.10. In [LeSol], eigenvalue estimates were obtained for the Schrödinger operator on the hyperbolic space $\mathbb{H}^d$ which is realised as the half-space $\{x = (x', z), x' \in \mathbb{R}^{d-1}, z > 0\}$ with measure $z^{-d} dx'dz$. The ‘non-magnetic’ Laplacian $-\Delta_{\mathbb{H}}$ is defined by means of the quadratic form $\langle \nabla u, \nabla u \rangle$; the lower bound for its spectrum in $E_d = (d - 1)^2/4$. So, if we define the magnetic Laplacian $-\Delta_{\mathbb{H}, a}$ by means of the form $\langle \nabla u + 2aiu, \nabla u \rangle$, the estimate from [LeSol] is carried over to the magnetic case and gives

$$N_-(\Delta_{\mathbb{H}, a} - E_d - qV) \leq C q^{d/2} \int_{\mathbb{H}^d} V^{d/2} z^{-d} dx'dz.$$ 

6.11. A number of results can be obtained for discrete operators. Here, we mention only the magnetic generalization of the estimate obtained in [LeSol], Sect. 3.3. If the graph $\mathcal{G}$ represents a group of polynomial growth $D \geq 3$ then $N_-(\Delta_{\mathcal{G}, \psi} - qV) \leq C q^{D/2} \sum x \, V(x)^{D/2} \mu(x)$, with the constant, as usual, not depending on the magnetic field.

6.12. Finally, we return to the usual Schrödinger operator, but consider the situation of a large potential perturbed by a smaller one with a big coupling constant (a flea on an elephant, according to B.Simon). Let $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^m$, $d \geq 3$, and $U(x')$ be a positive function on $\mathbb{R}^k$, so that the $k$-dimensional operator $H_U = -\Delta_k + U$ has a positive lowest eigenvalue $E$, and the rest of the spectrum is isolated from $E$. Consider the operator $-\Delta + U - E - qV$ where $V \geq 0$. The CLR-estimate for this operator is useless because the integral $\int (qV + E - U)^{d/2} dx$ diverges, even for very rapidly decaying $V$. So, to obtain an estimate for the spectrum of this operator, we split $L_2(\mathbb{R}^d)$ into direct sum of $\{\phi_0\} \otimes L_2(\mathbb{R}^m)$ and the space $\mathcal{H}$ orthogonal to it, where $\phi_0$ is the ground state of $H_U$, $\phi_0 > 0$. Due to corresponding splitting of the Birman-Schwinger operator, just as in [BLa],[Sol2,3], we have $N_-(\Delta_d + U - E - qV) \leq N_-(\Delta_d + U - E + \vartheta - qV) + N_-(\Delta_m - qV_0)$, where $\vartheta$ is the distance from $E$ to the rest of the spectrum of $H_U$ and the effective $m$-dimensional potential $V_0 = \int \varphi_0(x')V(x', x')dx'$. Denote $\min(1, \vartheta/E)$ by $\rho$. Since $-\Delta_d + U - E \geq 0$ in the operator sense, we have $-\Delta_d + U - E + \vartheta - qV \geq \rho(-\Delta_d + U - E) + \vartheta - qV \geq \rho(-\Delta + U) - qV$. To the latter operator, one can apply the CLR-estimate, and thus obtain

$$N_-(\Delta_d + U - E - qV) \leq C \int_{\mathbb{R}^d} (qV - \rho U)^{d/2} dx + N_-(\Delta_m - qV_0).$$
To the second term in (26) one can apply the $m$-dimensional CLR-estimate, if $m \geq 3$ or a suitable lower-dimension estimate, if $m = 1,2$. Thus, if the right-hand side in (26) is regular, this estimate is, according to Theorem 4, carried over to the magnetic case. In particular, the first term in (26) becomes regular if we drop $U$ from it (making it larger).

For illustration, consider $U(x') = |x'|^2$, so that $\phi_0 = c_0 \exp(-|x'|^2/2)$, $E = k$, $\vartheta = 1$ and $m \geq 3$. Then

$$
N_-(\Delta_a + |x'|^2 - k - qV) \leq C q^{\frac{m}{2}} \int V^\frac{m}{2} dx + c' q^{\frac{m}{2}} \int \left[ \int e^{-\frac{|x'|^2}{2}} V(x', x'') dx' \right]^m dx''.
$$

**References**


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