Towards Quantum Mathematics Part II: Manifold Notions

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Vienna, Preprint ESI 556 (1998) 
May 27, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
Towards Quantum Mathematics Part II: Manifold Notions

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Abstract

Here we use the language of quantum set theory, developed in Part I of this work, to explore quantized (i.e. categorified) manifold notions. We first deal with the differentiable structure in the sense of an infinitesimal patching of tangent spaces and arrive at the (finite-dimensional) representations of (higher) groupoids this way. The relation to TQFT and to previous work of the author on a quantization of the category of topological spaces and continuous injections is pointed out. In a second approach, we deal with the topological level, first discretized by a triangulation and then in an easy to grasp continuous analog of this. Here, we are lead to non-abelian cohomology with $n$-th cohomology taking values in an $(n+1)$-Hilbert space (which is a weak $n$-category). In three dimensions, some of these cohomology classes give rise to spin networks labeled by representations of a quantum group. In higher dimensions, we expect higher spin complexes labeled by corresponding (higher) categorified structures. This points to a link between the conceptions of quantum geometry in quantum set theory and in recent work on quantum gravity.
1 Introduction

We start by quickly reviewing the content of the first part of this paper. We developed the old idea of von Neumann of a set theory with an internal quantum logic (see [1] and the literature cited therein) in a modern categorical guise, i.e., we took the objects of the category $H$ of (Pre-) Hilbert spaces and linear maps as the sets of the basic level (where level is to be understood in the sense of a quantum analog of the von Neumann hierarchy of classical set theory). We discussed that $H$ in several respects resembles a topos though there are decisive differences to one, the main one being that we do not get a universal pullback structure but a superposition structure of pullbacks. We interpreted this as indicating that quantum set theory describes the observation of a quantum system by a quantum system and the iteration of this process. Next, we investigated number systems in quantum set theory. We found that the natural numbers can either be understood as given by the universal infinite-dimensional Hilbert space $\mathcal{N}$ (which we called the internal natural numbers since they are really an object in $H$) or as the category $\text{Hilb}$ of finite-dimensional Hilbert spaces (which we called the external natural numbers). Using the internal natural numbers, the complex numbers are given as the algebra of linear operators on $\mathcal{N}$ and the reals as the self-adjoint operators. Modules over the natural numbers (treated in the external version now) turned out to be module categories over $\text{Hilb}$. In this way, we could clarify the relationship between categorification and quantization: Categorification can be understood as viewing a structure in the sense of quantum set theory. The higher order categorifications turned out to correspond to sets of a higher level, i.e., the tower of categorifications can be seen as the quantum analog of the von Neumann hierarchy of classical set theory. The different concepts of number system in quantum set theory also allowed a glimpse at the relationship between the approach to quantization via categorification and the one pursued by noncommutative geometry: In the sense of quantum set theory, the former one is in spirit a discretized version of the latter. But on higher levels of categorification both approaches mix and more work is necessary to deepen the understanding of the relationship. Here, higher categorification is to be understood in the spherical sense of (weak-) n-categories. Actually, there are two different set concepts present in quantum set theory: Sets as conceived of as extensional realizations of logic ($\text{quantum logic sets}$) and sets as carriers of algebraic structures ($\text{quantum algebraic sets}$). The von Neumann hierarchy of the latter ones gives the tower of the (more general)
weak n-tuple categories (for the general notions of higher category theory see [14]).

Finally, we made a suggestion how to understand all the different levels of quantum set theory (or the different categorifications) as different realizations of one and the same abstract structure, namely by viewing the structure of quantum mechanics as abstractly given in terms of arrow language, i.e. it can be realized in different categories $\mathcal{C}$. Here $\mathcal{C} = \text{Set}$ gives the usual Hilbert space structure while $\mathcal{C} = \text{Cat}$ leads to the level of module categories, and so on. We called this the universality postulate for quantum mechanics since it states that quantum mechanics is universal in the categorical sense.

In this paper we want to apply the language developed in the first part especially to the question of developing an analog of the manifold notion in quantum set theory. Using the approach of noncommutative geometry to quantization, there is a manifold concept developed in the spirit of algebraic geometry (see [2]). We will focus on the approach by categorification here. As in Part I, we should mention that this series of papers is intended to explore the general landscape of quantum set theory to get an idea of the most interesting regions for future more detailed research. Therefore not all details are developed with complete rigour, yet.

In section 2, we develop a first version of a categorification of the manifold concept. It corresponds to a categorification of discretized manifolds in the sense of manifold like objects with the tangent space structure given by a module over the natural numbers (this discretized manifold notion can be understood as a functor from a groupoid to the category of modules over the natural numbers). We will discuss the relationship to TQFTs and our work in [12].

In section 3 then, we investigate a second approach to the categorification of the manifold concept. While in section 2 we consider the differentiable structure in the sense of patching of tangent spaces, in section 3 we directly deal with the topological level. We consider first a discretized version by using a triangulation of the manifold but then get an easy translation to the continuous case. As the result, we find non-abelian cohomology classes (simplicial, respectively sheaf cohomology) with $n$-th cohomology taking values in an $(n+1)$-Hilbert space (which is a weak $n$-category). In the case of sheaf cohomology, we find that the classes correspond to bundles on the classical manifold with fibers isomorphic to some $n$-Hilbert space. In the three-dimensional case, some of these bundles give rise to spin networks labeled by the representations of a quantum group. For four dimensions, we
expect spin foams labeled by representations of a Hopf category. In this way, the abstract quantum set theoretic approach suggested by von Neumann is close to the view on quantum geometry now emerging from quantum gravity (see [27], [28] and the literature cited therein).

Section 4 contains some concluding remarks and an outlook on future work.

2 The naive approach

Remember that a manifold can be defined in terms of two families \( (U_i)_{i \in I} \) and \( (U_{ij})_{(i,j) \in I \times I} \) of open subsets of some topological vector space (with \( I \) an index set) and a family of differentiable functions (called transition functions)

\[ \varphi_{ji} : U_{ij} \to U_{ji} \]

subject to the requirements

\[ U_{ii} = U_i \]

for the open sets and

\[ \varphi_{ii} = id_{u_i} \]

and

\[ \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \]

(whenever the left hand side is defined) for the transition functions (the latter two are called compatibility conditions for the transition functions). We could use a differentiated version of this (describing the infinitesimal patching of the tangent spaces) with transition functions \( d\varphi_{ji} \) being operator valued functions, i.e. at each point \( x \in U_{ij} \) we get a linear operator \( d\varphi_{ji}(x) \) on a vector space isomorphic to the one, we started with.

Now, suppose a module category \((\mathcal{M}, \oplus, \otimes)\) over \( \text{Hilb} \) to be given. In addition, suppose we have some other category \( \mathcal{B} \). We imagine the class of objects of \( \mathcal{B} \) as the set of points to be included in our manifold, i.e. as the analog of the disjoint union of all the \( U_i \). We should then have a relation telling us which points have to be identified, i.e. have to be regarded as one and the same point of the manifold, only given in different coordinatizations in the different \( U_i \). We require that this be encoded in the structure of \( \mathcal{B} \) in the sense that there is an arrow between two objects of \( \mathcal{B} \) precisely if they have
to be identified. Since this should be a symmetric relation, we immediately conclude that $B$ is a groupoid. The structure which is to become the analog of the tangent bundle is then given by a functor

$$\mathcal{F} : B \to 2V$$

to the weak 2-category of module categories over $\text{Hilb}$. We require that

$$\mathcal{F}(B) \cong M$$

for every object $B$ of $B$. Then by the fact that $\mathcal{F}$ is a functor, we immediately have a kind of differential transition functions between tangent spaces which have to be identified. Besides this, by the nature of $B$ and the functoriality property of $\mathcal{F}$, we have an analog of the compatibility conditions for transition functions. In this sense, every functor $\mathcal{F}$ of the kind above can be regarded as a categorification of a manifold with tangent space given by $M$.

There is one problem in the above construction which we turn to now, namely we considered $2V$ as a category only, forgetting about its 2-morphism level of structure. According to the philosophy behind the universality pos-tulate for quantum mechanics, we presented in the first part, considering representations of an object in $2V$ means that the object has to be a 2-category itself. So, actually we should turn $B$ into a 2-groupoid. Introducing invertible 2-morphisms in $B$ means that we introduce an additional relation saying when two identifications of points are equivalent. This makes sense since in contrast to the situation for classical manifolds, we allowed for the possibility that there are different inequivalent ways to identify two points (we did not require that $B$ should only have empty or one element homomorphism classes because in the categorical scheme this restriction does not seem to be needed). If we allow $B$ even to become a weak 2-groupoid, we use the 2-level to formulate a kind of weak compatibility conditions at the first level. So, to have objects of the same categorical type as domain and codomain of $\mathcal{F}$, we henceforth allow $B$ to be a weak 2-groupoid and $\mathcal{F}$ a weak 2-functor. So, in a first approach to the manifold notion, we draw the following conclusion:

**Conclusion 1:** A categorification of a manifold (or a manifold at the first level of quantum set theory) with tangent space a module category $M$ is a weak 2-functor from a weak 2-groupoid $B$ to $2V$ satisfying

$$\mathcal{F}(B) \cong M$$
for every object $B$ of $\mathcal{B}$.

This immediately generalizes to the higher levels of quantum set theory. A manifold on the $n$-th level is then a weak $(n+1)$-functor from a weak $(n+1)$-groupoid to the weak $(n+1)$-category $(n+1)V$ of weak module $n$-categories over $n\text{Vect}$ (the finite-dimensional analog of $n\text{V}$, including Hilbert space structures we should actually take $n\text{Hilb}$, the category of finite-dimensional $n$-Hilbert spaces instead, see [6] for an approach to $n$-Hilbert spaces), again satisfying

$$\forall B \in \text{Obj}(\mathcal{B}) \quad \mathcal{F}(B) \cong \mathcal{M}$$

On the $\omega$-level we then have representations of weak $\omega$-groupoids.

**Remark 1** In Part I of this work we have seen that we get the cubic versions of higher categories instead of the spherical ones if we use quantum algebraic instead of quantum logic sets. In this case we would get representations of weak $n$-tuple-groupoids and on the $\omega$-level groupoidic versions of Batanin's monoidal globular categories (MGCs, see [3] and [4]).

In conclusion, we see that the theory of manifolds in quantum set theory is just the (finite-dimensional, and when considering the full Hilbert space structure, we should assume unitary) representation theory of higher groupoids. There is an old conjecture of Grothendieck (see [9]) saying that the category $\text{Top}$ of topological spaces and continuous maps is as a weak $\omega$-category equivalent to the weak $\omega$-category of weak $\omega$-groupoids. Here $\text{Top}$ becomes a weak $\omega$-category in the following way: Objects and 1-morphisms are clear, as 2-morphisms take the homotopies between them. Now, homotopies are themselves maps, so, we can consider homotopies between them which we take as the 3-morphisms. In this way we proceed. Taking the discussion of the above paragraphs and the relationship between categorification and quantization from Part I together, we reach the following conclusion:

**Conclusion 2:** Under the assumption that the conjecture of Grothendieck mentioned above is true, an infinite quantization (in the sense of categorification) of the manifold concept results in the representation theory of topology (as considered up to arbitrary homotopies).
Remark 2 The idea of infinite quantization of the differential geometric structure of general relativity has been uttered at least twice (see [10] and [11]) in the Euclidean path integral approach to quantum gravity, so, it is interesting that we get an idea of the structures emerging on the completion of this process. But one has to be careful in trying to establish a connection between the two approaches because, as we mentioned in Part I, quantization by categorification is related to a discretization in the sense that the manifolds in quantum set theory defined above are actually the quantizations of manifold like structures with modules over the natural numbers as tangent spaces. Besides this, one has to be careful about the way the patching of the tangent structures is defined (see below). In spite of these warnings, the conclusion above suggests that quantum theory may have a power for understanding the notion of topology far beyond the calculation of topological invariants (see also the first part of [12]).

Since manifolds of the first level in quantum set theory already correspond to weak 2-groupoids, one can formulate a notion of manifold of the zeroth level, i.e. a patching of pure sets of the basic level (this is possible because in quantum set theory pure sets carry already a linear structure). A manifold of the zeroth level is then a (finite-dimensional, unitary) representation of a groupoid.

Why did we term this approach to a categorified manifold notion the naive one? Because it only reflects the algebraic part in the construction of a manifold, the relation saying which tangent spaces should be identified and the operators giving the isomorphisms for the identifications. It does not involve the topological part, saying that these operators should be given in terms of operator-valued continuous functions with domain isomorphic to some open subset of the module giving the structure of the tangent spaces (which is itself supposed to be topological). Nevertheless, this makes sense in our context because what we categorified is the notion of a manifold like structure with tangent space isomorphic to a module over the natural numbers, i.e. a kind of discretization of a classical manifold. In this discrete setting the topological aspects disappear, what remains is a relation giving the identification of points and a linear isomorphism for each pair of such points (it does no longer make sense to collect these operators together into functions $d\varphi_{ji}$ since there would anyway be no continuity requirement on them).
Let us conclude this section by shortly considering the relationship of our manifold notion in quantum set theory to general weak $n$-functors from a weak $n$-category to the weak $n$-category of weak $(n-1)$-module categories over the corresponding higher categorification of $\text{Hilb}$. Dropping the groupoid requirement and the requirement of isomorphy of tangent spaces means that the transformations (i.e. the analogs of the $d\varphi_{ji}(x)$) no longer have to be invertible, i.e. we can transform the information from one tangent space to another but possibly not vice versa. This is a kind of generalization of the manifold concept to singular cases, e.g. variable dimension of the tangent space structures.

**Remark 3** Benabou in [13] mentioned already the close conceptual relation between functorially indexed category structures and differential geometry.

Examples of weak $n$-functors of this kind with domain not a higher groupoid are extended TQFTs. Though there is in general no invertibility of the arrows in a higher cobordism category, there are duality notions present by taking reverse cobordisms (but the composite of a cobordism and its reverse is in general not equivalent to the identity). So, extended TQFTs appear as singular quantized manifolds.

Another example are the structures emerging in [12]. There we considered a quantization of the category $\text{Topi}$ of topological spaces and continuous injections. There was a natural candidate for inclusion even of the underlying set theory into the process of quantization then, and categories in this set theory lead to functorially indexed double categories with module structure. We remarked there already that these can be seen as a kind of categorification of the manifold concept (which was interesting because they appeared as state spaces of topology in the non-classical set theory there). We now understand clearly in which sense these structures relate to categorifications of manifolds: They are singular quantum manifolds again but this time related to the manifold notion based on quantum algebraic sets. This is an example where the more general cubic higher category notion arises in a physical context. Besides this, we see that a non-classical set theory of the kind emerging in [12] can be gained via representation theory in quantum set theory. In Part I we remarked that, following the universality requirement of set theory, all higher mathematical structures emerging should again be considered as
sets. In this sense, a set theory suitable for quantum topology appears as a special sector of general quantum set theory.

In conclusion, we have seen the following in this section: The representation theory of (higher) category structures in universal quantum mechanics leads to (singular) manifolds in quantum set theory. This connection between representation theory and manifold notion is new and not known from classical mathematics.

3 The simplicial approach

In this section, we investigate a second approach to the question of a categorification of the manifold notion. While we considered the differential structure of a manifold (patching of tangent spaces) in the last section, here we deal with the topological side. We directly start from the observation that categorifications are actually quantizations of discretized versions of the structures of classical mathematics, e.g. a finite-dimensional module category over \( \text{Hilb} \) (or a 2-vector space in the terminology of [15] and [16]) is not really a complex vector space in quantum set theory but the analog of a module over the natural numbers. So, if we try to categorify the manifold notion, it is natural to start with the triangulation of a classical manifold (and we will see that a natural candidate for the continuous case emerges then, too).

Suppose a finite triangulation (we restrict to the finite case for simplicity here) of a classical manifold \( M \) is given, i.e. a homeomorphism to a finite simplicial polyeder. This defines an abstract simplicial complex \( S \) which reflects the topological properties of the manifold. A simplicial polyeder is a representation (subject to certain conditions) of a simplicial complex in Euclidean space. Now, a topological space can always be thought of as a category by taking its points as objects and the continuous paths as morphisms. In this way the representation can be understood as a functor. We can therefore ask for representations of an abstract simplicial complex in other categories (where we allow for representations by weak functors). This is in the spirit of the universality postulate for quantum mechanics, we suggested in Part I. Here we view the simplicial structure of the triangulation of a classical manifold as universal in the same sense, i.e. we view it as an abstract concept which can have representations in different categories \( C \). We view these
representations then as manifolds in \( C \).

A topological manifold (of the first level of quantum set theory) is then given by a representation of a simplicial complex in a 2-vector space. Here, we consider a simplicial complex as a category with the objects the 0-faces and the morphisms the 1-faces. If the simplicial complex involves simplexes of dimension higher than one, it has actually the structure of a higher category by attaching \( n \)-morphisms to the \( n \)-faces. The way this can be done is given in [20]. Here, for technical reasons, an orientation of the faces is assumed but since this is not an inherent structure of the complex, we assume that the morphisms (of all levels) are invertible, thereby dispensing of it again. According to the universality postulate for quantum mechanics of Part I, the categorical nature of an object should determine the way it is represented in quantum set theory. In this case, this means that simplicial complexes involving simplexes of higher dimension should be represented in higher quantized vector spaces.

Actually, the topological information of a simplicial complex is encoded in its cohomology. Therefore, we have to consider not simply the representations in a higher quantized vector space but simplicial cohomology theory with values in a higher quantized vector space. Categorical cohomology theory is considered in ([19] and [20]) where \( n \)-th cohomology takes values in an \( n \)-category. This is just what appears here. Let us shortly explain the idea (for the details, see the above cited literature): In abelian first cohomology (i.e. values in an abelian group) we have the 1-cocycle condition stating that

\[
a - b + c = 0
\]

This can easily be generalized to the case of values in a monoid by shifting \( b \) to the right hand side and to values in a category as

\[
b = a \circ c
\]

i.e. commutativity of a triangle. In the same way, the 2-cocycle condition from abelian cohomology

\[
a - b + c - d = 0
\]

can be generalized to the case of values in a 2-category as

\[
a \bullet c = b \bullet d
\]
where $\circ$ indicates the composition of 2-morphisms. Since we assume that 2-morphisms correspond to triangles, this can be expressed as a commutative tetrahedron. In this way one proceeds to $n$-th cohomology. A representation of a simplicial complex defines a 1-cocycle if $b = a \circ c$ holds whenever $a, b, c$ are the images (under the representation) of the faces of a triangle, a 2-cocycle if $a \circ c = b \circ d$ holds whenever $a, b, c, d$ are the images of the faces of a tetrahedron, and so on. Two 1-cocycles are equivalent as cohomology classes if their defining functors are naturally equivalent (and a corresponding condition can be given for the higher cocycles, see [19]).

As we see from the above, to an $(n + 1)$-simplex there corresponds a representation in an $n$-category because the $(n + 1)$-th level of the simplex structure is needed for the cocycle condition. Therefore, we reach the following conclusion:

**Conclusion 3:** The quantizations of a triangulation of a classical manifold of dimension $n + 1$ are given by the $n$-cohomology classes where $n$-cohomology takes values in an $(n + 1)$-Hilbert space (which is an $n$-category).

**Example 1** Consider a triangulation $S$ of a two-dimensional manifold and a representation in a one-dimensional 2-vector space, i.e. we have a functor from $S$ to Hilb. This means to every 0-face we attach a finite-dimensional Hilbert space and to every 1-face a linear operator. The cocycle condition is just factorization of the linear operators belonging to the boundary of a 2-simplex. In consequence, we get a discretized version of a vector bundle as a quantization of $S$.

Knowing simplicial cohomology for the triangulations, it is easy to give a continuous version, i.e. quantum representations of the full (non-triangulated) topological manifold by translating the construction to sheaf cohomology. For simplicity assume for the moment that we are dealing with a Stein manifold $M$, i.e. we can write everything in terms of Čech cohomology classes of a suitable covering $(U_i)_{i \in I}$. Assume first that $M$ is two-dimensional (Hausdorff, with countable base). We attach to every $U_i$ a function with values finite-dimensional Hilbert spaces and to every

$$U_{ij} = U_i \cap U_j$$
with $U_{ij} \neq \emptyset$ a function with values invertible linear operators $A_{ij}(x)$. These have to satisfy $A_{ii} = 1_{U_i}$ and the cocycle condition

$$A_{ki} = A_{kj}A_{ji}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$ (where the equation is supposed to hold at all points $x$ of the triple intersection). Since we should in a topological context assume continuity of the functions involved and Hilbert spaces are classified by their dimension (i.e. in a discrete way), we get constant functions on the $U_i$ in the first case (without loss of generality we assume the $U_i$ to be connected). If $M$ is connected, the above conditions imply that all the Hilbert spaces attached to different $U_i$ are isomorphic (i.e. of the same dimension $n \in \mathbb{N}$) and taking into account the Hilbert space (besides the linear) structure, the $A_{ij}(x)$ should all be from $SU(n)$. The cocycle condition then tells us that we have an $SU(n)$-bundle over $M$. In the non-connected case, we get an $SU(n)$-bundle for each connected component (with $n$ possibly varying). Two bundles have to be identified if the corresponding families $(A_{ij}(x))$ and $(A'_{ij}(x))$ differ by a 0-cochain, i.e. if

$$A'_{ij} = B_i A_{ij} B_j^{-1}$$

with $B_i, B_j \in SU(n)$.

In conclusion, for a two-dimensional (for simplicity: connected) manifold, the quantum representations are given by the $SU(n)$-bundles on it and the space of representations is described by a classification of these bundles (where $n \in \mathbb{N}$ is the first of the needed parameters).

**Remark 4** One can, of course, ask what the representations of a one-dimensional manifold look like. Since $\mathbb{N}$ is the 0th-level analog of $\text{Hilb}$, the 0-cocycle condition leads to constant positive integer valued functions as the representations, i.e. to $\mathbb{N}$ itself as the space of representations.

Observe that a two-dimensional classical manifold can be represented in a one-dimensional quantum module. Considering representations in higher dimensional module categories over $\text{Hilb}$ means considering module categories over the rig category of $SU(n)$-bundles over $M$.

Now, consider a three-dimensional manifold $M$ (again connected, Hausdorff, paracompact). In this case, we have to study representations in a 2-category, i.e. if we first consider representations in a one-dimensional (higher)
quantized module, we have to consider functors to $\mathcal{2Hilb}$, the 2-category of finite-dimensional 2-Hilbert spaces (see [6], [15]). Again, we get isomorphy of the 2-Hilbert spaces attached to the different $U_i$ as a consequence of connectedness, and the 1-cocycle condition implies that we deal with a bundle of 2-Hilbert spaces (where the transition functors should satisfy a unitarity condition) over $M$. Since the 1-cocycle condition now has to be satisfied only up to a natural isomorphism, we get a weak bundle and the natural transformations involved satisfy a tetrahedron relation (see above). Second cohomology with values in $\mathcal{2Hilb}$ means a classification of weak bundles of this type. These weak bundles form a weak rig 2-category and considering representations of $M$ in higher dimensional weak module 2-categories over $\mathcal{2Hilb}$ involves the consideration of weak module 2-categories over this.

We should remark at this point that a universally agreed on notion of 2-Hilbert space (and even more of $n$-Hilbert space for higher $n$) is still in development, we use the notion suggested by Baez in [6] here. With some additional monoidal structure on them (turning them into a categorification of a $H^*$-algebra) these categories appear to be categories of representations of certain quantum super Hopf algebraoids (Hopf algebraoids are to Hopf algebras as groupoids to groups, i.e. the indexed case). This is proved for the symmetric monoidal case in [6] (as a generalization of the Doplicher-Roberts theorem) where one has representations of the compact super groupoids, and for parts of the general case (see e.g. [21]). The assumption of an algebra structure beyond the simple Hilbert space structure is what one will need when generalizing our approach to a set theory for quantum field theory (this was already clearly perceived by von Neumann, see [1]). So, we see that in this case we can reduce our weak bundles fiber-wise to certain quantum super Hopf algebraoids. In special cases, we get fiber-wise a quantum group. Since the work in [23], [24] suggests that in the case of a triangulation of $M$ even the base space should be described by noncommutative geometry, we have another connection here between a quantum set theory and a noncommutative geometry approach.

It is in principle clear how to proceed to the four- and higher dimensional case then. For four dimensions it might again be possible to reduce the bundle-structures appearing fiber-wise by representation theory (using the trialgebraic structures suggested in [25]). The $\omega$-level is then reached for infinite-dimensional manifolds, modelled on a separable Hilbert space. Quantum representations of general infinite-dimensional manifolds, modelled on a non-separable space, then for the first time ask for an extension of quantum
set theory beyond the \( \omega \)-level of the von Neumann hierarchy.

**Remark 5** Observe that category-valued cohomology, as described here, involves in the same way a dimensional ladder (see [25], [26]) as does TQFT. This is no accident, it just points to the fact that in spirit the hierarchy of extended TQFTs is closely related to cohomology (based on bordisms instead of simplicial complexes or open coverings).

**Remark 6** The same results, we gained by considering quantum representations of a triangulation of a manifold and the sheaf theoretic analog of these, would also have followed if we had asked for quantization (and higher quantization) of integer valued cohomology on a manifold (since \( \text{Hilb} \) is the quantized analog of \( \mathbb{N} \), \( 2\text{Hilb} \) its next higher quantization, and so on, in quantum set theory, see Part I of this work).

Let us for a short moment return to the three dimensional case. If a principal bundle is given over \( M \) and a triangulation of \( M \), it is known how to construct a spin-network, using the dual CW-complex of the triangulation, from this data (the construction is just using the principal bundle to give a lattice gauge theory, see [27], [28] and the literature cited therein). Now, a lattice gauge theory does not actually use the data of the transition functions in the principal bundle. So, in the case of our weak bundles, we can use the fiber-wise reduction implied by representation theory (see above) and get spin networks carrying representations of certain quantum super Hopf algebraoids. Especially, we get the case of spin networks carrying representations of a quantum group (spin networks and lattice gauge theories of this kind are known in the literature, see e.g. [29], [30]). Since spin networks are known to arise as quantum 3-geometries in quantum gravity ([27], [28]), this very clearly shows in which sense our weak bundles from above contain the information on quantum representations of \( M \). This construction might in principle be extended to (closed) spin foams for the four-dimensional case and to higher spin complexes for higher dimensions (see [28]). We want to point out two things here: First, we regard it as another confirmation of the views on quantum geometry emerging now that similar structures are also arrived at by the very abstract and general quantum set theory approach suggested by von Neumann. Second, we get in this way a concrete prescription for the labeling of the spin complexes: In general, we have to use a labeling
prescribed via representation theory from higher categorifications. E.g. for a closed spin foam, we expect a labeling with the Hopf categories introduced in [25].

Remark 7 In section 2, discussing quantization of the differentiable structure of a manifold, we encountered representation theory of (higher) groupoids. Here, at least on the lowest level, representation theory of compact groupoids appears in the fibers. Considering a topological groupoid means forcing the equivalence relations, we discussed in section 2, to live in the setting of classical topology. If we quantize the topological level itself, these structures appear only on the level of fibers.

Remark 8 Because here the dimension of $M$ determines the level of quantization of the representation, the conjecture of Grothendieck (mentioned in section 2) implies that quantum representations of separable Hilbert space manifolds give already structures with general topology (up to arbitrary homotopies) appearing on the level of fibers.

One should remark that the use of an $(n - 2)$-dimensional spin complex for the study of an $n$-dimensional manifold is not arbitrary but - once one has decided to use a lattice gauge theory like approach - can be given an abstract categorical justification by the fact that a lattice gauge theory is a category of functors from the free category on the lattice to a group like object (see [32]). Higher codomain categories afford higher complexes then. But the fact that we get bundles with fibers group like objects for dimension three is automatic in our considerations.

4 Conclusion

In this paper we developed manifold notions in quantum set theory. Starting from the differentiable structure, we arrived at representation theory of higher groupoids. Using the topological structure, we got non-abelian cohomology classes and a link to spin networks as presently considered in approaches to quantum geometry from quantum gravity. We described how higher spin complexes should arise this way for the higher dimensions. We
got a prescription for the labeling of the spin complexes depending on the dimension.

We consider this work as a general exploration of the territory of quantum mathematics. Many parts had to remain tentative, in order to proceed to a not too narrow sketch of the landscape at all. Much therefore remains to be done. Many parts of quantum set theory remain unexplored here, e.g. the extension of the quantum von Neumann hierarchy beyond the $\omega$-level (which should be important for quantizations of infinite-dimensional manifolds modelled on a non-separable space) or the question of the meta-level and model theory (which is so decisive for our modern view of set theory). Besides this, in the realm explored in the papers presented here, one now has to go into the technical details and investigate examples. Both directions are future work we intend to undertake.

Acknowledgements:

I want to thank Prof. Dr. H. Grosse, Prof. Dr. M. Reeken and Dr. F. Leitenberger for discussions on the subjects involved. This work has been done during a stay at the Institute of Theoretical Physics of the University of Vienna and the Erwin Schrödinger Institute for Mathematical Physics, Vienna. The hospitality of both institutions is gratefully acknowledged. This work was made possible by a research grant under the Gemeinsames Hochschulsonderprogramm III von Bund und Ländern from the DAAD (German Academic Exchange Service).

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