On the Discrete Spectrum of Many-Particle Neutral Systems in Magnetic Field with Fixed Pseudomomentum

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We study the discrete spectrum of many-particle hamiltonians $H_0$ of the neutral systems in a homogeneous magnetic field with fixed pseudomomentum. The general theorem is proved, which describes (under some conditions) the discrete spectrum of the operator $H_0$ in the properties of the discrete spectrum of some effective one-dimensional differential operators with the known structure of the discrete spectrum. On this base the conditions of the finiteness and infinity and the spectral asymptotics are obtained for the operator $H_0$. The results of the paper can be applied to hamiltonians of arbitrary atom. This paper continues the investigation, which was started in [1].

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In this paper we prolong the study of the spectrum of many-particle hamiltonians for the neutral systems in a homogeneous magnetic field after fixation of pseudomomentum and separation the center-of-mass motion in the direction of the field, which was began in [1]. We start here from two results of [1]: the form of the operator $H_0$ (see (1.1)) and the theorem on the localization of the essential spectrum. The main aim of this investigation is the obtaining of the information on the discrete spectrum of the operator $H_0$, since the discrete spectrum was studied before only for the systems of the hydrogen atom type [1].

Similar to the most number of the papers on the discrete spectrum we consider here the case, when the bound of the discrete (and essential) spectrum of the operator $H_0$ is determined only by the decompositions $Z_2 = \{C_1, C_2\}$ of the initial system 2) into two stable subsystems with fixed pseudomomentum, that is when the lower bound of the energy operator $H_{03}(Z_2)$ (see (1.4)) of such decomposed system $Z_2 = \{C_1, C_2\}$ is the point of the discrete spectrum of this operator. Under this condition at the first time we obtained here the sufficient conditions of the finiteness and infinity of the discrete spectrum (Theorems 1.2, 1.3) and (in the case of infinity of the discrete spectrum) the spectral asymptotics (Theorem 1.4).

To determine the field of the application of these results we undertook the study of the spectral properties of the mentioned operators $H_{03}(Z_2)$, for which $\inf H_{03}(Z_2)$ is the beginning of the essential spectrum of the considered operator $H_0$. It is shown that if the clusters $C_1, C_2$ in $Z_2$ are not neutral, then the bound of the essential spectrum of the operator $H_{03}(Z_2)$ is determined by some decompositions $Z'_3 = \{C'_1, C'_2, C'_3\}$ of the compound system $Z_2$ by means of the partition some of subsystems $C_1, C_2$ (Theorem 1.5). Let us denote by $H_{03}(Z'_3)$ the energy operator of the system $Z'_3$. It follows from this result that the condition of the applicability of the Theorems 1.2-1.4 is the inequality

$$\inf H_{03}(Z_2) < \inf H_{03}(Z'_3).$$

(0.1)

To verify it in the case of $n$-electron atoms we establish here some sufficient condition, guaranteeing the validity of the inequality (0.1) (Theorem 1.6). Using it we were succeed in the proof (0.1) for $n = 1, 2$. But when this paper was in preparing for print it was proved [1.2, that (0.1) holds for arbitrary $n$. Thus our results (Theorems 1.2, 1.4) can be applied for any atoms.

In the conclusion let us note that the principal results of the paper — the Theorems 1.2, 1.5, 1.6 — are established with the further development of the geometric methods in the direction of the using more complicated (than in [1–3]) types of the decompositions of the configuration space and that all assertions of the paper take into account the permutation symmetry of the system.

2) We say that the system is decomposed into subsystems $C_1, C_2$, if its hamiltonian does not contain the potentials of the interaction between the particles from different subsystems. At the same time the differences of coordinates $x_i - x_j, y_i - y_j, i \in C_1, j \in C_2$ can be present in kinetic part of the hamiltonian.
§ 1. Definitions and the principal results

1.1 We consider the neutral $n$-particle quantum system $Z_1 = \{1, 2, \ldots, n\}$ in a homogeneous magnetic field with the direction of $z$-axes. Let $m_i, e_i, r_i = (x_i, y_i, z_i)$ be the mass, the charge and the radius-vector of the particle number $i$, $M = \sum_{j=1}^{n} m_j$, $Q = \sum_{j=1}^{n} e_j = 0$. We denote the relative coordinates of the particle number $j$ by $q_j$:

$$q_j = (q_{j1}, q_{j2}, q_{j3}) = (q_{j1}, q_{j2}, q_{j3}) = r_j - \sum_{i=1}^{n} m_i r_i \cdot M^{11}.$$  

Then the energy operator of the system $Z_1$ after the separation of the center-mass-motion in the direction of the third axes and after the restriction to the subspace of the functions with the fixed values $\nu_1, \nu_2$ of the components of pseudomomentum can be written in the following form [1]

$$H_0 = T_1^0 + T_3^0 + F + V(q),$$  

(1.1)

where

$$T_1^0 = \sum_{i=1}^{n} m_i \sum_{p=1}^{2} \left( \frac{1}{t} \nabla_{t_{ip}}^0 - D_{t_{ip}} \right)^2, \quad T_3^0 = - \sum_{i=1}^{n} m_i (\nabla_{t_{i3}}^0)^2,$$

$$\nabla_{t_{ip}}^0 = \frac{1}{m_i} \frac{\partial}{\partial q_{ip}} - \frac{1}{M} \sum_{j=1}^{n} \frac{\partial}{\partial q_{jp}}, \quad p = 1, 2, 3$$  

(1.2)

$$D_{t_{i1}} = \nu_1 \left( \frac{1}{M} - \frac{1}{nm_1} \right) + B \sum_{j=1}^{n} (n_{i} - q_{i2}) \left( e_i - e_j \right) \left( \frac{e_i - e_j}{nm_1} + \frac{2e_j}{M} \right),$$

$$D_{t_{i2}} = \nu_2 \left( \frac{1}{M} - \frac{1}{nm_1} \right) - B \sum_{j=1}^{n} (q_{i1} - q_{i1}) \left( e_i - e_j \right) \left( \frac{e_i - e_j}{nm_1} + \frac{2e_j}{M} \right),$$

$$F = (\nu_1 + \mathcal{E}_2)^2 + (\nu_2 - \mathcal{E}_1)^2, \quad \mathcal{E}_j = 2B \sum_{j=1}^{n} q_{ij} e_i, \quad V(q) = \sum_{s < t} V_{st}(\{|q_s - q_t|\}).$$

Concerning the potentials $V_{st}(\{|q_i|\})$ we assume that

$$V_{st}(\{|q_i|\}) \in \mathcal{L}_{2, loc}(R^3), \quad V_{st}(\{|q_i|\}) \in C^2 \quad \text{if} \quad |q_i| \neq 0,$$

$$V_{st}(\{|q_i|\}) = e_i |q_i|^\gamma \quad \text{if} \quad |q_i| > a,$$

where $\gamma > 0, \ a > 0$ are some constants.

1.2 Let

$$R_0 = \{q | q = (q_1, \ldots, q_n), \ q_i = r_i - \sum_{j=1}^{n} m_j r_j \cdot M^{11} \ i = 1, \ldots, n\}.$$  

$R_0$ is the configuration space of the relative motion of the system $Z_1$; it is evident that if $q \in R_0$

$$\sum_{j=1}^{n} m_j q_j = 0.$$  

The operator $H_0$ is considered in the space $\mathcal{L}_2(R_0)$. Let us extend $H_0$ from the domain $\mathcal{D}(H_0) = C_0^\infty(R_0)$ to self-adjoint operator, conserving for the obtained operator and its domain the notations $H_0$ and $\mathcal{D}(H_0)$. Let $S$ and $A$ be respectively the group of the permutations of the identical particles of the system $Z_1$ and arbitrary type of the irreducible representation of the
Let \( Z_s = \{C_1, \ldots, C_s\} \) be the arbitrary decomposition of the system \( Z_1 \) into \( s \) nonintersecting nonempty clusters \( C_i \),

\[
M[C_i] = \sum_{i \in C_i} m_i, \quad Q[C_i] = \sum_{i \in C_i} \epsilon_i,
\]

\[
R_{03}(Z_s) = \{q | q \in R_0, \quad \sum_{k \in C_i} m_k q_{k3} = 0 \quad t = 1, 2, \ldots, s\},
\]

\[
R_{c3}(Z_s) = \{q | q \in R_0, \quad q_j = (0, 0, q_{j3}) \quad j = 1, \ldots, n, \quad (q, q')_1 = 0 \quad \forall q' \in R_{03}(Z_s)\},
\]

where for any \( q = (q_1, \ldots, q_n) \), \( q' = (q'_1, \ldots, q'_n) \) from \( R_0 \)

\[
(q, q')_1 = \sum_{j=1}^n m_j (q_j - q'_j)_{R^3}.
\]

Evidently,

\[
R_0 = R_{03}(Z_s) \oplus R_{c3}(Z_s).
\]

\( R_{03}(Z_s) \) and \( R_{c3}(Z_s) \) are the configuration space of the relative motion and the space of the motion of the centers-of-masses for the clusters \( C_1 \ldots C_s \) in the direction of \( z \)-axes.

The energy operator of the compound system \( Z_s \) (which consists of the noninteracting one with other clusters \( C_i \) after the separation of the motion of center-of-mass every cluster in the direction of \( z \)-axes), can be written in the form

\[
H_{03}(Z_s) = T^0_\perp + T^0_\parallel(Z_s) + F + V_{Z_s}(q), \tag{1.4}
\]

where

\[
T^0_\parallel(Z_s) = -\sum_{i=1}^s \sum_{i \in C_i} \frac{1}{m_i} \left( \frac{\partial}{\partial q_{i3}(Z_s)} - \frac{m_i}{M[C_i]} \sum_{p \in C_i} \frac{\partial}{\partial q_{p3}} \right)^2,
\]

\[
V_{Z_s}(q) = \frac{1}{2} \sum_{i=1}^s \sum_{i,j \in C_i, i \neq j} V_{ij}(q_i - q_j),
\]

\[
q_i - q_j = (q_{i1} - q_{j1}, q_{i2} - q_{j2}, q_{i3}(Z_s) - q_{j3}(Z_s)),
\]

\[
q_{i3}(Z_s) = q_{i3} - \frac{1}{M[C_i]} \sum_{k \in C_i} m_k q_{k3} \quad \text{if} \quad i \in C_i.
\]

The operator \( H_{03}(Z_s) \) will be considered in the space \( L_2(R_{03}(Z_s)) \).

1.4 Let \( \hat{S} = \hat{S}(Z_s) \) be the group of the permutational symmetry of the system \( Z_s \), generated by the groups \( S[C_i] \quad t = 1, 2, \ldots, s \) of the permutations of the identical particles in the clusters \( C_1, \ldots, C_s \) and by the permutations of the identical clusters \( C_i \) from \( Z_s \), if such clusters are in

\( 3 \) for the physical applicability of the results one have to take only such types \( \alpha \), which are permitted for the system \( Z_1 \) by Pauli principle; in this paper \( \alpha \) is arbitrary type of the representation of \( S \).
Let us denote the types of the representations of the group $\hat{S}$ in $L_2(R_0)$, $L_2(R_{\alpha}(Z_s))$ and in $L_2(R_{\alpha}(Z_s))$ respectively by $\hat{\alpha}, \hat{\alpha_0}$ and $\hat{\alpha_c}$; their matrices are denoted by $D_{\hat{\alpha}}, D_{\hat{\alpha_0}}$, and $D_{\hat{\alpha_c}}$, their dimensions are denoted by $|\hat{\alpha}|, |\hat{\alpha_0}|$ and $|\hat{\alpha_c}|$. Let us write (following to scheme of [2]), that

$$(\hat{\alpha_0}, \hat{\alpha_c}) \prec \alpha,$$ if at least one irreducible component of the tensor product of the representations $D_{\hat{\alpha}}(\hat{\alpha_0}) \otimes D_{\hat{\alpha_c}}(\hat{\alpha_c})$ of the group $\hat{S}(Z_s)$ is contained in the irreducible representation of the type $\alpha$ of the group $S$ after restriction it from $S$ to $\hat{S}$,

$$\hat{\alpha_0} \prec \alpha,$$ if such type $\hat{\alpha}$ exists, that $(\hat{\alpha_0}, \hat{\alpha_c}) \prec \alpha$.

Let us denote by $P^{(\hat{\alpha_0})}(Z_s)$ the projector in the space $L_2(R_{\alpha}(Z_s))$ onto the subspace of functions, which are transformed according to the representation of the type $\hat{\alpha_0}$, and put

$$H_{03}(\hat{\alpha}) (Z_s) = H_{03}(Z_s) P^{(\hat{\alpha_0})},$$

$$H_{03}(\alpha; Z_s) = \sum_{\hat{\alpha_0} \prec \alpha} H_{03}^{(\hat{\alpha_0})} (Z_s) = \sum_{\hat{\alpha_0} \prec \alpha} H_{03}(Z_s) P^{(\hat{\alpha_0})},$$

$$\mu^{(\alpha)} = \min_{Z_s, \alpha \geq 2} \inf H_{03}(\alpha; Z_s).$$

If the decomposition $Z'_k = \{C'_1, \ldots, C'_k\}$ is obtained from $Z_s$ by a partition at least one cluster from $Z_s$, then we shall write $Z'_k < Z_s$. Since

$$\inf H_{03}(\alpha; Z'_k) \leq \inf H_{03}(\alpha; Z'_k)$$

when $Z'_k < Z_s,$

then

$$\mu^{(\alpha)} = \min_{Z_s} \inf H_{03}(\alpha; Z_s). \quad (1.5)$$

Further for the arbitrary operator $A$ we denote by $\sigma_{ess}(A)$, $\sigma_{d}(A)$, $\sigma_p(A)$ and $\sigma_{pp}(A)$ its essential, discrete, point and pure point spectrum respectively.

1.5 Theorem 1.1 For arbitrary $\alpha$

$$\sigma_{ess}(H_0^\alpha) = [\mu^\alpha; +\infty). \quad (1.6)$$

The proof of the Theorem 1.1 was obtained in [1]. We present here the formulation of this theorem, since we shall use it many times and since we should like formulate more simple definition for the relation $\hat{\alpha_0}(Z_s) \prec \alpha$, which is used in the definition of the operator $H_{03}(\alpha; Z_s)$ and consequently in the expression (1.5) for the number $\mu^{(\alpha)}$.

Let $N_0^\alpha (Z_s)$ be the multiplicity of the representation of the type $\hat{\alpha}$ of the group $\hat{S}(Z_s)$ in the representation of the type $\alpha$ after the restriction of the last one from $S$ to $\hat{S}(Z_s)$. If the clusters $C_1, C_2$ are not identical ($C_1 \neq C_2$), then the group $\hat{S}(Z_s)$ in $L_3(R_{\alpha}(Z_s))$ has only identical representation, since the permutations from $\hat{S}(Z_s)$ do not change the coordinates of the vectors from $R_{\alpha}(Z_s)$. That is why at this case $D_{\hat{\alpha}} \otimes D_{\hat{\alpha_c}} = D_{\hat{\alpha}}$ and the relation $\hat{\alpha_0}(Z_s) \prec \alpha$ means that $N_0^\alpha \geq 1$. If the clusters $C_1$ and $C_2$ are identical ($C_1 \sim C_2$), then the group $\hat{S}(Z_s)$ in the space $R_{\alpha}(Z_s)$ two one-dimensional irreducible representations; their types we denote by $\hat{\alpha_c}^\pm$. For them the number 1 corresponds to the permutations of the particles inside $C_j$ $j = 1, 2$ in the both representations and to the permutation of the clusters $C_1 \leftrightarrow C_2$ in the representation of the type $\hat{\alpha_c}^+$; the number $-1$ corresponds to this transposition of the clusters in the representation

4) when we consider the representation of the group $S$ or of its subgroups in any subspace from $L_2(R_0)$, we always consider the representation by the operators $T_\alpha$ (see §1.2).
of the type $\alpha_e^1$. That is why the representation $D_g^\alpha := \hat{D}_g^{\alpha_0} \otimes D_g^{\alpha_c}$ is irreducible if $\hat{\alpha}_c = \hat{\alpha}_c^\pm$ and $\hat{\alpha} = \hat{\alpha}_0$ if $\hat{\alpha}_c = \hat{\alpha}_c^\pm$; when $\hat{\alpha}_c = \hat{\alpha}_c^\perp$ we denote its type by $-\hat{\alpha}_0$. In this case the relation $\hat{\alpha}_0 \prec \alpha$ means that at least one value $N_{\alpha_0}^\alpha(Z_2)$, $N_{\perp \alpha_0}^\alpha(Z_2)$ is not equal to zero. Let
\[
d_{Z_2}(\alpha, \hat{\alpha}_0) = N_{\alpha_0}^\alpha(Z_2) \quad \text{if } C_1 \neq C_2,
\]
\[
d_{Z_2}(\alpha, \hat{\alpha}_0) = N_{\alpha_0}^\alpha(Z_2) + N_{\perp \alpha_0}^\alpha(Z_2) \quad \text{if } C_1 \sim C_2.
\]
Then the condition $\hat{\alpha}_0 \prec \alpha$ is equivalent to the condition $d_{Z_2}(\alpha, \hat{\alpha}_0) \geq 1$.

1.6 Let
\[
O(\alpha) = \{Z_2 | \inf \{H_{\alpha_3}(\alpha; Z_2) = \mu(\alpha)\} \},
\]
\[
O_0(\alpha) = \{Z_2 | Z_2 \in O(\alpha), \mu(\alpha) \in \sigma_2(H_{\alpha_3}(\alpha; Z_2))\}.
\]

$O_0(\alpha)$ is the set of such stable two-cluster systems $Z_2$, for which the lower bound of the energy is the lower bound of the essential spectrum of the operator $H_{\alpha_3}(\alpha)$. We shall study the discrete spectrum of the operator $H_{\alpha_3}(\alpha)$ under condition that

\[
O(\alpha) = O_0(\alpha).
\]

Of course this condition holds not for any systems, but it is usual condition for the investigation of the discrete spectrum (see for example [2-4]). Moreover let us note, that when magnetic field is absent, the equality (1.7) holds (in particular) for any atoms, their positive and negative (with one extra electron) ions [4]. For $Z_2 \in O_0(\alpha)$ we denote by $W(\alpha; Z_2)$ the eigenspace of the operator $H_{\alpha_3}(\alpha; Z_2)$, corresponding to its eigenvalue $\mu(\alpha)$.

In addition to the equality (1.7) we assume also that:

A1) All decompositions $Z_2' \in O(\alpha)$ can be obtained from one decomposition $Z_2 \in O(\alpha)$ by the permutations of the identical particles;

A2) The representation of the group $\hat{S}(Z_2)$ in the space $W(\alpha; Z_2)$ is irreducible, i.e. there is such type $\hat{\alpha}_0$, that
\[
P(\hat{\alpha}_0)W(\alpha; Z_2) = W(\alpha; Z_2) \quad \text{and} \quad \dim W(\alpha; Z_2) = |\hat{\alpha}_0|.
\]

The conditions A1), A2) have pure technical nature and they are used for the simplification of the formulations and the proofs. We shall show the needed changing in the formulations, if A1) or A2) does not hold.

1.7 The condition (1.7) suggests that the income of the decomposition $Z_2 = \{C_1, C_2\}$ in the discrete spectrum of the operator $H_{\alpha_3}(\alpha)$ have to depend on the interaction between clusters $C_1, C_2$ from $Z_2$, when the system $Z_2$ is in the ground state. If we estimate this interaction from above and from below for the large values of $|\zeta_3| = \left[\sum_{\gamma \in C_1} m_{\gamma}^{C_1} - \sum_{\gamma \in C_2} m_{\gamma}^{C_2}\right]$, then we can get one-dimensional effective potentials
\[
V_{Z_2}^\pm(\zeta_3) = Q(Z_2)|\zeta_3|^{1-\gamma} \pm c_3^0|\zeta_3|^{1-\gamma_0}, \quad \text{if } |\zeta_3| \geq b_0, \quad \text{if } |\zeta_3| < b_0,
\]
and further introduce the effective one-dimensional operators in $L_2(R^1)$:
\[
h_{Z_2}^\pm = -M(Z_2)^{1+1} \frac{d^2}{d\zeta_3^2} + V_{Z_2}^\pm(\zeta_3),
\]
where $b_0$ and $c_0$ are some constants, $M(Z_2)^{1+1} = M[C_1]^{1+1} + M[C_2]^{1+1}$, $Q(Z_2) = Q[C_1] \cdot Q[C_2]$, $\gamma_0 = \min(\gamma + d, 3)$ and $d = 2$, if the condition $A_2$ or more general condition $A_3$ holds:
A3) All functions from $W(\alpha; Z_2)$, corresponding to the same type of the permutational symmetry have the same parity with respect to the inversion $q_{i3} \to -q_{i3}$;  
\[ d = 1 \] in the other cases.

For any $\lambda < 0$ we denote by $N(\lambda; h^\pm_{Z_2})$ and by $N(\mu^{(\alpha)} + \lambda; H_0^{(\alpha)})$ the dimensions of linear spans of the eigenfunctions of the operators $h^\pm_{Z_2}$ and $H_0^{(\alpha)}$, corresponding to the eigenvalues of these operators, which do not exceed respectively $\lambda$ and $\mu^{(\alpha)} + \lambda$.

1.8 The principal result of this paper is Theorem 1.2.

**Theorem 1.2** Let the relation (1.7) holds and $Q(Z_2) < 0 \forall Z_2 \in O(\alpha)$. Then for any type $\alpha$ of the irreducible representation of the group $S$, for all $\lambda < 0$ and for some constant $\varepsilon$

\[ -\varepsilon + d_{Z_2}(\alpha, \hat{\alpha}_0)N(\lambda; h^\pm_{Z_2}) \leq N(\mu^{(\alpha)} + \lambda; H_0^{(\alpha)}) \leq \varepsilon + 
\]

\[ + d_{Z_2}(\alpha, \hat{\alpha}_0)N(\lambda; h^\pm_{Z_2}), \]  \hspace{1cm} (1.10)

**Remarks** 1. Effective operators $h^\pm_{Z_2}$ do not depend explicitly on the fixed value of pseudomomentum $\nu = (\nu_1, \nu_2)$, but the constant $\varepsilon$ in (1.10) can depend on $\nu$. Moreover (and it is especially important) the lower bound of the essential spectrum $\mu^{(\alpha)}$, the set $O(\alpha)$, and consequently the effective potentials can depend on $\nu$. But now we are not ready to consider some examples.

2. If $Q(Z_2) = 0$, then the proof of the Theorem 1.2 can be applied also, but to get the informative results it is necessary to have more precise expressions for effective potentials $V^{\pm}_{Z_2}(\xi)$.

We do not obtain them here not to complicate the paper but mainly since if $Q(Z_2) \equiv Q[C_1 \cdot Q[C_2] = 0$ then $Q[C_1] = -Q[C_2] = 0$, and for this case we have no approaches to verify the condition (1.7) (see Theorem 1.5) and we can not show no one example when this condition holds.

3. When the condition $A_1$ is not valid we decompose the set $O(\alpha)$ into such nonintersecting subsets $K_i$, that all $Z_2'$ from $K_i$ are obtained from some $Z_2i \in K_i$ by the permutations of the identical particles and no one $Z_2 \in K_j$ for $j \neq i$ does not have this property. Then instead of the relation (1.10) we shall obtain the relation

\[ -\varepsilon + \sum_i d_{Z_2i}(\alpha; \hat{\alpha}_{0i})N(\lambda; h^\pm_{Z_2i}) \leq N(\mu^{(\alpha)} + \lambda; H_0^{(\alpha)}) \leq \varepsilon + 
\]

\[ + \sum_i d_{Z_2i}(\alpha, \hat{\alpha}_{0i})N(\lambda; h^\pm_{Z_2i}), \]  \hspace{1cm} (1.11)

where $\hat{\alpha}_{0i}$ are determined for the decomposition $Z_2i$ in the same way as $\hat{\alpha}_0$ for $Z_2$. Moreover we assume, that for every $Z_2i$, the condition $A_3$ holds, if we put there $Z_2 = Z_2i, \hat{\alpha}_0 = \hat{\alpha}_{0i}$.

4. If the condition $A_2$ does not hold for a decomposition $Z_2$ or for some decomposition $Z_{2i}$ (see the remark 3), then in the inequalities (1.10), (1.11) instead of the coefficients $d_{Z_2}(\alpha, \hat{\alpha}_0)$ and $d_{Z_2i}(\alpha, \hat{\alpha}_{0i})$ we have to write respectively $d_{Z_2}(\alpha) = \sum_{s=1}^{\infty} k_s^{(\alpha)} d_{Z_2}(\alpha, \hat{\alpha}_0^{(s)})$ and $d_{Z_2i}(\alpha) = \sum_{t=1}^{f_i} k_t^{(i)} d_{Z_2i}(\alpha, \hat{\alpha}_{0i}^{(t)})$, where $\hat{\alpha}_0^{(s)}$ and $\hat{\alpha}_{0i}^{(t)}$ are all types of such irreducible representations.

---

\(^5\)The condition $A_2$ is sufficient for the validity of the condition $A_3$, since if $A_2$ does not hold for some type $\hat{\alpha}_0$ of the irreducible representation of the group $S(Z_2)$ in $W(\alpha; Z_2)$, then the subspace $W(\alpha; Z_2)$ contains at least two basises of the representation of the type $\hat{\alpha}_0$, and one of them consists of the even functions, the other consists of the odd functions with respect to the inversion $q_{i3} \to -q_{i3}$, but it contradicts to the condition $\dim W(\alpha; Z_2) = |\hat{\alpha}_0|$, contained in $A_2$. 

---
of the groups $\hat{S}(Z_2)$ and $\hat{S}(Z_{2i})$, for which respectively

$P^{(0)}(\alpha; Z_2) \neq 0 \quad s = 1, \ldots, s_0,$

$P^{(s)}(\alpha; Z_{2i}) \neq 0 \quad t = 1, 2, \ldots, t_0,$

$k_0^{(s)}$ and $k_i^{(t)}$ are the multiplicities of the representations of the types $\hat{\alpha}_0^{(s)}$ and $\hat{\alpha}_i^{(t)}$ in the spaces $W(\alpha; Z_2)$ and $W(\alpha; Z_{2i})$. At this case the exponent $\gamma_0$ in the second term of the effective potential (1.8) is equal to $\min(\gamma + 2, 3)$ if the condition $A_3$ (§1.7) holds and it is equal to $\min(\gamma + 1, 3)$ if this condition does not hold.

5. The effective operators $h_{Z_2}^\pm$ and the formulation of the Theorem 1.2 are practically the same as the effective operators and Theorem 1 in [3], but in fact the results of this paper and [3] describe the spectral properties of the different operators. In [3] we studied the restrictions of the operators of many-particle systems to the subspaces of the functions with a fixed type $m$ of $SO(2)$ symmetry, while here we consider the restrictions of the many-particle hamiltonians to the subspaces of the functions with fixed value of pseudomomentum. But the main difference consists in the fact that the results [3] can not be applied to the neutral systems (which are considered here) on principle, since for such systems the using of $SO(2)$ symmetry does not make possible to find the bottom of the essential spectrum (there is no HVZ-theorem) and consequently we can not study the discrete spectrum.

1.9 Using the Theorem 1.2 and the known properties of the operators $h_{Z_2}^\pm$ [5,6], one can establish the spectral properties of the operator $H_{0}^{(\alpha)}$. Further we assume that the type $\alpha$ is arbitrary, the condition (1.7) holds and $Q(Z_2) < 0$ for $\forall Z_2 \in O(\alpha)$.

**Theorem 1.3** The discrete spectrum of the operator $H_{0}^{(\alpha)}$

is finite, if $\gamma > 2$ or if $\gamma = 2$ and $Q(Z_2) \geq -4^{\pm 1}M(Z_2)^{\pm 1}$

is infinite, if $\gamma = 2$ and $Q(Z_2) < -4^{\pm 1}M(Z_2)^{\pm 1}$ or if $\gamma < 2$ and $Q(Z_2) < 0$

**Theorem 1.4**

1. At $\lambda \rightarrow -0, \quad \gamma < 2$ and $Q(Z_2) < 0$

$$N(\mu^{(\alpha)} + \lambda; H_{0}^{(\alpha)}) = |\lambda|^{(\gamma + 2)/2} d(\alpha, \hat{\alpha}_0) M(Z_2)^{1/2} |Q(Z_2)|^{1/\gamma} J(\gamma) + R,$$

where $J(\gamma) = \gamma^{\pm 1} \int \frac{1}{1 + (u - 1)^{1/2} u^{(\gamma + 1)/\gamma}} du, \quad R = O(|ln |\lambda||).$

2. If $\lambda \rightarrow -0, \quad \gamma = 2$ and $Q(Z_2) < -4^{\pm 1}M(Z_2)^{\pm 1}$

$$N(\mu^{(\alpha)} + \lambda; H_{0}^{(\alpha)}) = \frac{1}{2\pi} |ln |\lambda|| |d(\alpha, \hat{\alpha}_0) (4 Q(Z_2)|M(Z_2) - 1|)^{1/2} +$$

$$+ o(|ln |\lambda||).$$

**Remarks.**

1. The Theorems 1.3, 1.4 were formulated for simplicity under the assumptions $A_1, A_2)$. To obtain the formulations in a general case one have to use the remarks 3,4 to Theorem 1.2, and to take into account that if the assumption $A_2)$ does not hold and at least for one $Z_{2i}$ the assumption $A_3)$ does not hold also, then if $\gamma < 1$ we have to take $R = O(\lambda|^{(\gamma + 1)/2\gamma}).$

2. For $\gamma = 1$ (Coulomb potentials) $J(\gamma) = J(1) = \pi/2$.

1.10 Let us apply the obtained results to hamiltonian of helium atom with fixed pseudomomentum. Let $Z_1 = (1, 2, 3)$, where 1 is nuclear number, 2 and 3 are the numbers of electrons,
\( \frac{e_2 - e_3}{2} = m_2 = m_3, \quad v_{ij}(r_{ij}) = e_i e_j r_{ij}^{-1}. \) The possible types of the symmetry \( \alpha \) of the system \( Z_1 \) correspond to symmetrical (identical) and to antisymmetrical representations of the group of the permutations of two electrons. For both types \( \alpha \)

\[
O(\alpha) = \{Z_2(2), Z_2(3)\}, \quad \text{where } Z_2(j) = \{C_{1j}, C_{2j}\},
\]

\[
C_{1j} = Z_1 \setminus (j), \quad C_{2j} = (j), \quad j = 2, 3.
\]

Evidently, \( Q(Z_2(j)) < 0. \) The operator \( H_{03}(Z_2(j)) \) has no any permutational symmetry, so \( H_{03}(\alpha; Z_2(j)) = H_{03}(Z_2(j)) \). It will be shown in \( \S \) 1.14, that \( \sigma_d(H_{03}(Z_2(j))) \neq \emptyset \) and consequently (1.7) holds. Using Theorems 1.2, 1.4 and the remark 4 to Theorem 1.2, we obtain that

\[
N(\mu^* + \lambda; H_0^a) = |\lambda|^{1/2} \left( \frac{m_2(m_1 + m_2)}{M} \right)^{1/2} d(\alpha) \cdot \frac{\pi}{2} + O(|\ln|\lambda||),
\]

where \( d(\alpha) = d|\text{dim} W(\alpha; Z_2(j)) \). We think that \( d(\alpha) = 1 \), but it is not prove yet.

**1.11** The applicability of Theorems 1.2-1.4 to the actual systems depends on the validity for them the relation (1.7). To testify (1.7) further we establish the location of the set \( \sigma_{ess}(H_{03}(\alpha; Z_2)) \) (Theorem 1.5) and on this basis for the systems of atom type we obtain the simple sufficient condition of non emptiness the set \( \sigma_d(H_{03}(\alpha; Z_2)) \) (Theorem 1.6). Using these results we proved here the relation (1.7) for 3-particle systems which are like to helium atom.

But while this paper was in preparing for the print, the relation (1.7) was proved on the basis of the theorems of the type of Theorems 1.5, 1.6 for arbitrary atoms [12]. Thus our main results — Theorems 1.2-1.4 — are valid for hamiltonians of arbitrary atoms.

Let

\[
\mu(\alpha; Z_2) = \min_{Z' \subset Z_2} \inf_{Z' \subset Z_2} \text{inf } H_{03}(\alpha; Z'_2).
\]

(1.12)

**Theorem 1.5** Let \( Q(Z_2) < 0. \) Then

\[
\sigma_{ess}(H_{03}(\alpha; Z_2)) = [\mu(\alpha; Z_2), +\infty).
\]

(1.13)

**Remark.** Since every decomposition of the system \( Z_1 \) into \( s \) subsystems for \( s > 3 \) is at the same time a decomposition into 3 subsystems, then we may take in (1.12) only \( Z'_2 = Z'_3 \).

**1.12** To understand the sense of Theorem 1.5 let us compare the situations with magnetic field and without it. We denote by \( \hat{H}_0(Z'_2) \) the energy operator of the compound system \( Z'_2 = \{C'_1, \ldots, C'_s\} \) in the absence of magnetic field after the separation of center-of-mass motion of the clusters \( C'_1 \ldots C'_s \subseteq R^3. \) Then the operator \( \hat{H}_0(Z_2) \) corresponds to the operator \( H_{03}(Z_2) \) when magnetic field is absent. According to [7] (see also [8])

\[
\inf \{\lambda | \lambda \in \sigma_{ess}(\hat{H}_0(Z'_2))\} = \inf_{\{\psi_p\}} \lim_{\psi_p} (\hat{H}_0(Z_2)\psi_p, \psi_p),
\]

(1.14)

where \( \{\psi_p\} \) are the sequences of Weyl, corresponding to all possible decompositions \( Z'_2 = \{C'_1, C'_2\} \) of the initial system \( Z_2 = \{C_1, C_2\}. \) If \( \psi_p \) describes the decomposition \( Z'_2 = Z_2 \), that is \( \psi_p \) corresponds to the moving the particles of cluster \( C_1 \{C_2\} \) from cluster \( C_2 \{C_1\} \), then the motion to infinity of the subsystem \( C_j \) has to accompany with some decomposition of this subsystem, since the position of its center-of-mass is fixed. If \( \psi_p \) describes the decomposition \( Z'_2 = \{C'_1, C'_2\} \neq Z_2 \), then (since the interaction between the clusters \( C_1 \) and \( C_2 \) is absent) \( Z'_2 \) corresponds to the decomposition into clusters \( C_1 \cap C'_1, C_1 \cap C'_2, C_2 \cap C'_1, C_2 \cap C'_2 \), and at least
three of them are not empty. So when magnetic field is absent, each Weyl sequence $\psi_p$ for the operator $\mathcal{H}_0(Z_2)$ describes some decomposition $Z_2^s$ $s \geq 3$ and consequently,
\[ \inf\{\lambda|\lambda \in \sigma_{ess}(\mathcal{H}_0(Z_2))\} \geq \min_{Z_2^s < Z_2} \inf \mathcal{H}_0(Z_2^s). \] (1.15)

Now let us consider the situation with magnetic field. Similar to (1.14) (see [7,8])
\[ \inf\{\lambda|\lambda \in \sigma_{ess}(\mathcal{H}_{03}(Z_2))\} = \inf_{\{\psi_p\}} \lim (\mathcal{H}_{03}(Z_2)\psi_p, \psi_p), \] (1.16)
where $\{\psi_p\}$ are Weyl sequences for the operator $\mathcal{H}_{03}(Z_2)$. The center-of-mass motion of every cluster $C_j \in Z_2$ was separated only in the direction of $z$-axes. That is why our considerations in the absence of magnetic field can be applied only to such Weyl sequences, which describe the decompositions $Z_2^s \neq Z_2$ or the decomposition $Z_2^s = Z_2$ - but in the direction of $z$-axes. In other words, if the sequence $\psi_p$ corresponds to the decomposition $Z_2^s = \{C_1, C_2\} \neq Z_2$ or to the decomposition $Z_2^s = Z_2$ as the result of unlimited increasing of the distance between the particles of the clusters $C_1, C_2$ in the direction of $z$-axes, then we can conclude (similar to above) that $\psi_p$ describes a decomposition $Z_2^s < Z_2$ $s \geq 3$. However if the sequence $\psi_p$ describes the decomposition $Z_2^s = Z_2$ as the result of unlimited increasing of the distance between clusters $C_1, C_2$ only in the plane $x, y$, then a priori one can not make such conclusion. It means that we can not except the origin of the essential spectrum of the operator $\mathcal{H}_{03}(Z_2)$ as a consequence of the infinite increasing of the distance between clusters $C_1, C_2$ in the plane $x, y$. But Theorem 1.5 shows that in the case $Q(Z_2) < 0$ such situation is impossible and the bound of the essential spectrum of the operator $\mathcal{H}_0(\alpha; Z_2)$ is determined by a decomposition at least one cluster $C_i \in Z_2$ (but not by the motion of non decomposed clusters $C_1, C_2$ in the plane $x, y$). The proof of Theorem 1.5 is given in §§3.1–3.8.

1.13 For the system of the atom type using Theorem 1.5 one can obtain the simple sufficient condition of the inclusion $\mu^{(\alpha)} \in \sigma_d(\mathcal{H}_{03}(\alpha, Z_2))$ when $Z_2 \in \Omega(\alpha)$. Further we consider the systems $Z_1 = (1, \ldots, n)$ consisting of $(n-1)$ identical particles with the numbers 2, 3, ..., $n$ and the charges $\epsilon_i = \epsilon \quad i = 2, \ldots, n$, and the particle number 1 with the charge $\epsilon_1 = -(n-1)\epsilon, \ n \geq 2$. Let the potentials of the interaction
\[ V_{ij}(|r_1|) = \epsilon_i \epsilon_j \cdot |r_1|^{1-\gamma} \quad 0 < \gamma < 1, \ 5. \]
It was proved in [9] without the account of the symmetry (when the symmetry is taken into account it follows from [1]) that the set $\Omega(\alpha)$ contains all two-cluster decompositions $Z_2(j) = \{C_1, C_2\}$, where $C_1 = Z_1 \setminus (j), \ C_2 = (j) \ j = 2, 3, \ldots, n$ so $\mu^{(\alpha)} = \inf \mathcal{H}_{03}(\alpha; Z_2(2))$. It follows from Theorem 1.5 that
\[ \mu(\alpha; Z_2) = \inf \mathcal{H}_{03}(\alpha; Z_3(2, 3)), \]
where $Z_3(2, 3) = \{Z_1 \setminus (2, 3); (2); (3)\}$

**Theorem 1.6** Let
\[ \mu(\alpha; Z_2) \in \sigma_p(\mathcal{H}_{03}(\alpha; Z_3(2, 3))). \] (1.17)
Then
\[ \mu^{(\alpha)} \in \sigma_d(\mathcal{H}_{03}(\alpha; Z_2(2))). \] (1.18)

**Remark.** Since the operators $\mathcal{H}_{03}(\alpha; Z_2(i))$ $i = 2, 3, \ldots, n$ and $\mathcal{H}_{03}(\alpha; Z_3(i, j))$ $j \neq i, \ i, j = 2, 3, \ldots, n$ are unitary equivalent respectively to the operators $\mathcal{H}_{03}(\alpha; Z_2(2))$ and $\mathcal{H}_{03}(\alpha; Z_3(2, 3))$, then in the formulation of Theorem 1.6 one can take instead of the decompositions $Z_2(2)$ and $Z_3(2, 3)$ the decompositions $Z_2(i)$ and $Z_3(i, j)$ for any $i, j \geq 2, \ j \neq i$. 10
1.14 The proof of Theorem 1.6 is given at the end of §3. Here we show that for 3-particle system \( Z_1 = \{1, 2, 3\} \) of the type of helium atom the condition (1.17) holds. For the considered case \( Z_2(2) = \{(1, 3), (2)\} \), \( Z_3 = Z_3(2, 3) = \{(1), (2), (3)\} \) and the operator \( H_{03}(Z_3) \) is the operator of 3 noninteracting particles with the fixed pseudomomentum (PM) and with the separated motion of every particle in the direction of \( z \)-axes. In other words \( H_{03}(Z_3) \) is the operator, obtained from the operator

\[
H_{03}(Z_3) = \sum_{j=1}^{3} m_j \left( \frac{1}{i} \nabla_{j\perp} - \epsilon_j A_j \right)^2
\]

afterfixation of PM. The operator \( H_{03}(Z_3) \) has pure point spectrum, which is accumulated to \( +\infty \). But every point of the spectrum of the operator with the fixed PM is the point of the spectrum of the energy operator of this system before fixation of PM. Consequently the operator \( H_{03}(Z_3) \) similar to the operator \( H_{03}(Z_3) \) has pure point spectrum. So the relation (1.17) holds and it means that for 3-particle systems of the type helium atom the conditions (1.18), (1.7) hold, i.e. Theorems 1.2, 1.4 can be applied to \( Z_1 = \{1, 2, 3\} \).

1.15 Unfortunately, even for 4-particle system the simple approach of §1.14 does not give the results. But when this paper was in preparing for print, S.Vugalter proved [12] using the other approach and Theorems of the type Theorems 1.5, 1.6, that the condition (1.17) holds for any neutral atom. So our Theorems 1.2-1.4 can be applied to hamiltonians of arbitrary atoms in magnetic fields and in particular we obtain for them the asymptotics of the discrete spectrum (Theorem 1.4).

\section{2. Proof of Theorem 1.2}

2.1 The basic idea of the proof consists in the construction of such subspaces of functions \( \mathcal{M}_i(\lambda) \ i = 1, 2 \), that for all \( \lambda < 0 \)

\[
(H_0^3 \psi, \psi) \leq (\mu^{(\alpha)} + \lambda) ||\psi||^2 \quad \text{if} \ \psi \in \mathcal{M}_1(\lambda),
\]

\[
(H_0^3 \psi, \psi) > (\mu^{(\alpha)} + \lambda) ||\psi||^2 \quad \text{if} \ \psi - \mathcal{M}_2(\lambda),
\]

and in the next estimate of the dimensions of these subspaces in the values \( N(\lambda; \hbar^{\frac{3}{2}} \mathbb{Z}_0) \). The construction of the space \( \mathcal{M}_1(\lambda) \), the estimate of its dimension and the proof of the inequality (2.1) are realized similar to [3]. That is why we present here only the definition \( \mathcal{M}_1(\lambda) \). The construction of the space \( \mathcal{M}_2(\lambda) \), the estimate of the value \( dim \mathcal{M}_2(\lambda) \) and the proof of the inequality (2.2) are realized with the simultaneous application of the approaches from [2] and from [3]. In this connection we have to remark that although the space \( \mathcal{M}_2(\lambda) \) is constructed similar to [3] on the base of linear span of the products of the eigenfunctions of one-dimensional effective operator (corresponding to its eigenvalues not exceeding \( \lambda \)) and of the eigenfunctions of the ground state of the system \( Z_2 \), determining the bottom of the continuous spectrum, but in the difference of [3] the supports of these functions are "cut of" by the cones of the relative motion of the decomposed system not in the direction of \( 3 \)-rd axes (as in [3]) but in all space and only after it we separate from these cones the regions, corresponding to the decompositions only in the direction of the third axes. Thus we do here double decomposition of the configuration space \( R_0 \). Let us note that the decomposition of the space \( R_0 \) only in the direction of \( z \)-axes can give the result only in the case \( Q[C_1]Q[C_2] > 0 \ [3] \), which is impossible for the neutral systems.

2.2 Let us begin the construction of the space \( \mathcal{M}_2(\lambda) \) with the needed properties. Since the operator \( H_0^{(\alpha)} \) depends only on the relative coordinates \( q \), then we may start similar to [4], where
magnetic field is absent. Namely by the arbitrary $\mathcal{E} > 0$ we construct such finite-dimensional space $\mathcal{M}_0^2$, that for $\psi(q) \in \mathcal{D}(H_0^{(\alpha)})$, $\psi - \mathcal{M}_0^2$ the following inequality holds

$$(H_0^{(\alpha)}\psi, \psi) \geq \sum_{Z_2 \in \mathcal{O}(\alpha)} \{(H_0^{(\alpha)}\psi_{Z_2}, \psi_{Z_2}) - \mathcal{E}\|\psi_{Z_2}\|_{2(\gamma + 2)/2}\|^2 - C(\mathcal{E})\|\psi_{Z_2}\chi(t_{Z_2})\|_{2(\gamma + 2)/2}\|^2 + \mu^{(\alpha)}(\|\psi\|^2 - \sum_{Z_2 \in \mathcal{O}(\alpha)} \|\psi_{Z_2}\|^2)\}$$

where

$$\psi_{Z_2} = \psi u_{Z_2} \omega_b, \quad u_{Z_2} = u(t_{Z_2}), \quad \omega_b = \omega_b(|\zeta_{Z_2}|),$$

$$(t_{Z_2}) = |P_0(Z_2)q|_1 \cdot |P_c(Z_2)q|_1^2,$$

$P_0(Z_2)$ and $P_c(Z_2)$ are the projectors in the sense of the inner product $(\cdot, \cdot)_1$, (see $1.3$) respectively on the subspaces

$$R_0(Z_2) = \{q^0(Z_2)\} = \{q \in R_0, \sum_{j \in C_1} m_j q_j = 0 \ t = 1, 2\},$$

$$R_c(Z_2) = \{q^c(Z_2)\} = \{q \in R_0, \ (q, q^0)_1 = 0 \ \text{for } \forall q^0 \in R_0(Z_2)\},$$

$$|P_0(Z_2)q|_1^2 = \frac{1}{2} \sum_{i=1}^2 \frac{1}{M[C_i]} \sum_{i, j \in C_i} m_i m_j |q_i - q_j|^2,$$

$$(P_c(Z_2)q)_1^2 = (q, q) - |P_0(Z_2)q|_1^2 = M(Z_2) \zeta_{Z_2}^2$$

(see $[10]$, p. 125),

$$q_i - q_j = q_i^0(Z_2) - q_j^0(Z_2), \quad q_j^0(Z_2) = q_j - \zeta[C_i] \ j \in C_i,$$

$$\zeta_{Z_2} = (\zeta_1, \zeta_2, \zeta_3) = (\zeta_1, \zeta_3) = \zeta[X_1] - \zeta[X_2],$$

$$\zeta[C_i] = M[C_i] \sum_{j \in C_i} m_j q_j \ t = 1, 2;$$

$u(s), \omega_b(s) \in C^1[0, +\infty), \ u(s) \equiv 1 \ w e n 0 \leq s \leq \delta, \ u(s) \equiv 0 \ w e n s > \delta, \omega_b(s) \equiv 0 \ w e n 0 \leq s \leq b, \ \omega_b(s) = 1 \ w e n s > b, \ \chi(q) \equiv 1 \ w e n q \in \Omega = \{q \in Z_2 \in [\delta, \beta], \ |\zeta_{Z_2}| \geq \beta\}, \chi(q) \equiv 0 \ w e n q \in \Omega.$

In $\S 2.3$–2.8 we estimate from below the right hand of the inequality (2.3). Using the obtained estimate in $\S 2.9$ we define the space $\mathcal{M}_2(\lambda)$ in this way, that the space $\mathcal{M}_2(\lambda) := \mathcal{M}_0^2 \oplus \mathcal{M}_2(\lambda)$ will have the needed properties.

### 2.3

Let us denote by $\varphi_i (q_{Z_2}) \ i = 1, 2, \ldots, i_0$ the orthonormal eigenfunctions, forming the canonical basis of the representation of the type $\lambda_0$ (see $\S 1.7$) in the eigenspace $W(\alpha; Z_2)$ of the operator $H_0(\alpha; Z_2)$, corresponding to the eigenvalue $\mu^0$; here $i_0 = \dim W(\alpha; Z_2)$, $q_{Z_2} = \{q_1(Z_2), \ldots, q_n(Z_2)\} \in R_0(Z_2), \ q_i(Z_2) = \{q_i, q_{i2}, q_{i3}(Z_2)\}$. In further estimates we shall use many times two following properties of the functions $\varphi_i (q_{Z_2})$:

$$\varphi(q_{Z_2})|q_{Z_2}|^k \in \mathcal{L}_2(R_0(Z_2)) \ k = 1, 2$$

and

$$(L \varphi_i, \varphi_j)\zeta_t(R_0(Z_2)) = 0 \ i \neq j$$

for arbitrary function $L = L(q_{Z_2}, \zeta_{Z_2})$, which is invariant with respect to the permutations of the group $\hat{S}(Z_2)$ and for which $L \varphi_j \in \mathcal{L}_2(R_0(Z_2))$ at the arbitrary fixed $\zeta_{Z_2}$. The relation (2.4)
is proved similar to Lemma 3.1 [3]. The equality (2.5) follows from the fact, that the functions \( L \varphi_i \) and \( \varphi_j \) belong to the different lines of the irreducible representation of the type \( \hat{\alpha}_0 \): the functions \( \varphi_i, \varphi_j \) have this property if \( i \neq j \), and it remains after multiplication \( \varphi_j \) by the function \( L(qZ_2, \zeta Z_2) \), which is invariant concerning of the permutations from \( S(Z_2) \).

Further we shall use the expansion of the function \( \psi Z_2 \) in the functions \( \varphi_i(qZ_2) \)

\[
\psi Z_2(q) = \sum_{i=1}^{i_0} \varphi_i(qZ_2) \Phi_i(\zeta) + g(q),
\]

where \( \zeta = \zeta(Z_2) \), \( \Phi_i(\zeta) = (\psi Z_2, \varphi_i) c_2(R_{oZ}(Z_2)) \) and evidently,

\[
(g(q), \varphi_i(qZ_2)) c_2(R_{oZ}(Z_2)) = 0 \quad \text{for } \forall \zeta.
\]

Let \( \chi_1(q), \chi_2(q) \) and \( \chi_0(q) \) be the characteristic functions of the regions

\[
\Omega_1 = \Omega_1(\beta) = \{ q \in R_0, \ t Z_2 \leq \beta \}, \ \Omega_2 = \Omega_2(b) = \{ q \in R_0, |\zeta Z_2| \geq b \},
\]

\[
\Omega_0 = \Omega_0(\beta, b) = \Omega_1 \cap \Omega_2.
\]

It is clear that \( \chi_0 = \chi_1 \chi_2 \). In virtue of (2.6)

\[
\psi Z_2(q) = \sum_{i=1}^{i_0} \varphi_i(qZ_2) \Phi_i(\zeta) \chi_j + g(q) \chi_j \quad j = 0, 2.
\]

Let us consecutively estimate the terms in the right hand of the equality (2.3). We fix \( Z_2 \) and everywhere write for brevity \( \zeta \) instead of \( \zeta Z_2 \). Evidently

\[
|\zeta|^{k_1} \chi_j(q) \leq B_0(\zeta) \quad j = 0, 2,
\]

where \( B_0(\zeta) = |\zeta|^{k_1} \) if \( |\zeta| \geq b \), \( B_0(\zeta) = b^{k_1} \) if \( |\zeta| \leq b \). Therefore and by (2.5), (2.8) for large \( b = b(\varepsilon) \)

\[
\| \psi Z_2(\zeta) \chi_j^{(1/2)/2} \| \leq \varepsilon \| g \|^2 + \sum_{i,j=1}^{i_0} (B_0^{k_1+2} \chi_j \Phi_i, \varphi_j \Phi_j) \leq
\]

\[
\leq \varepsilon \| g \|^2 + \sum_{i=1}^{i_0} \| B_0^{k_1+2} \chi_j \Phi_i(\zeta) \|^2.
\]

Since \( q_0(Z_2) = P_0(Z_2)q = \{ q_0^i(Z_2), \ldots, q_0^n(Z_2) \} \) (where \( q_0^i(Z_2) = q_i - \zeta[C_i] \), when \( i \in C_i \)), then

\[
|P_0(Z_2)q|_1 \leq |q Z_2|_1
\]

and if \( q \in supp Z_2 \)

\[
\frac{|q Z_2|}{\delta \cdot M(Z_2) \cdot |\zeta|} \geq \frac{|P_0(Z_2)q|_1}{\delta |q P_0(Z_2)q|_1} \geq 1.
\]

At \( b > b(a) \) by (2.8), (2.9), (2.11) and (2.4) (for \( k = 1 \))

\[
\| \psi Z_2(\zeta) \chi_j \|^{k_1+2} \| \leq 2 \| \sum_{i=1}^{i_0} \varphi_i \cdot \Phi_i \cdot B_0 \chi \chi \|^{2} + 2 \| g B_0 \|^{2} \leq
\]

\[
\leq e \sum_{i=1}^{i_0} \| \Phi_i(\zeta) B_0^{k_1+2} \chi_j \|^{2} + \varepsilon \| g \|^2.
\]
Here and everywhere we denote by $c$ any constants (may be depending on $\mathcal{E}$), the values of which do not play any role for the proof.

2.5 Now we are beginning to estimate the principal term in the inequality (2.3), namely the quadratic form $(H_0(a)\psi, \psi)$. Obviously,

$$(H_0\psi, \psi) = \left( H_0(Z_2) + I_{Z_2} - \frac{1}{M(Z_2)} \frac{d^2}{d\zeta^2} \psi, \psi \right),$$

(2.13)

where $I_{Z_2} = V(q) - V_{Z_2}(q)$. In virtue of (2.6)

$$(H_0(Z_2), \psi, \psi) \geq \mu^{(a)} \| \psi \|^2 + \delta_0 \| g \|^2,$$

(2.14)

where $\delta_0 > 0$ is the distance from $\mu^{(a)}$ to nearest point of the spectrum of the operator $H_0(\alpha; Z_2)$.

Further it is clear that,

$$-\frac{1}{M(Z_2)} \left( \frac{d^2}{d\zeta^2} \psi, \psi \right) \geq -\frac{1}{M(Z_2)} \sum_{i=1}^{\infty} \left( \frac{d^2}{d\zeta^2} \psi_i, \psi_i \right).$$

(2.15)

The principal difficulties of the estimate of the value $(H_0(a)\psi, \psi)$ are connected with the estimate of the form $(I_{Z_2}\psi, \psi)$. Using (2.8) at $j = 0$ we have

$$(I_{Z_2}, \psi, \psi) = (\chi_0 I_{Z_2} g, g) + \left( \chi_0 I_{Z_2} \sum_{i=1}^{\infty} \varphi_i \psi_i, \sum_{i=1}^{\infty} \varphi_i \psi_i \right) + 2Re \left( \chi_0 I_{Z_2} \sum_{i=1}^{\infty} \varphi_i \psi_i, g \right).$$

(2.16)

2.6 Let $i \in C_1$, $j \in C_2$. Then

$$|q_i - q_j| = |q_i - \zeta |[C_1] + \zeta [C_2] - q_j + \zeta Z_2| = |q_j^0(Z_2) - q_j^0(Z_2) + \zeta|.$$ 

It follows from §2.2 that at $q \in \Omega_1(\beta)$

$$|q_j^0(Z_2)| \leq c|\beta| \zeta$$

(2.17)

and consequently for small $\beta$ and $\forall q \in \Omega_1(\beta)$

$$|q_i - q_j| \geq |\zeta|/2$$

(2.18)

For $q \in \Omega_0$ the inequality $|\zeta| \geq b$ holds. Hence and by (2.17) for $b \geq 2a$ (see §1.1)

$$\chi_0 V_{ij}(|q_i - q_j|) = c_1 c_2 |q_i - q_j|^{1+\gamma} \chi_0.$$  

(2.19)

Therefore at the large $b$ and at the small $\beta$

$$(\chi_0 I_{Z_2} g, g) \leq \mathcal{E} \| g \|^2.$$  

(2.20)

To estimate the other terms in (2.16) we expend the function $|q_i - q_j|^{1+\gamma} = q_j^0(Z_2) - q_j^0(Z_2) + \zeta^{1+\gamma}$

for $q \in \Omega_0(\beta, b)$, $\beta < < 1$ in the powers of the function $|\zeta|^{1+1}$, and after it we expend the principal term of the obtained expression in the powers of the function $|\zeta|^{1+1}$. We have :

$$\chi_0 |q_j^0(Z_2) - q_j^0(Z_2) + \zeta^{1+\gamma} = \chi_0 |\zeta|^{1+\gamma} + \gamma (\zeta, q_j^0 - q_j^0) |\zeta|^{1+1} \chi_0 + F_1,$$

(2.21)

where by (2.9), (2.11)

$$|F_1| \leq c_\chi \alpha (|q_j^0(Z_2)|^2 + |q_j^0(Z_2)|^2) |\zeta|^{1+1} \leq c|q(Z_2)|^2 B_0^{\gamma+2} (\zeta).$$
Further let $\Omega_3(\beta) = \{ q | q \in \Omega_3, |\zeta_1| \leq \beta_1 |\zeta_3| \}$, $\Omega_4 = \Omega_0 - \Omega_3$ and $\chi_j$ be the characteristic function of the region $\Omega_j(\beta)$, $j = 3, 4$. Then for any $p > 0$

$$\chi_0|\zeta|^{4p} = \chi_0 (\chi_3 + \chi_4)|\zeta|^{4p} = \chi_0 \chi_3 |\zeta_3|^{4p} + \chi_0 \chi_4 |\zeta|^{4p} + F_2.$$  

(2.22)

where in virtue of (2.9), (2.11)

$$\chi_0 \chi_4 |\zeta|^{4p} \leq \chi_4 B_0^p (\zeta_3) \leq B_0^{p+2} (c_3) b_3^2 |\zeta_1|^2 \beta_1^{2},$$

$$|F_2| \leq \frac{p}{2} \chi_0 \chi_3 |\zeta_1|^2 B_0^{p+2} (c_3).$$

Let us substitute in the right hand of the relation (2.21) the expression for $\chi_0 |\zeta|^{4p}$ at $p = \gamma$ from (2.22), multiply the obtained equality by $e_i e_j$ and sum it over all $i \in C_1, j \in C_2$. Then we will obtain

$$\chi_0 I_{z_2} = Q |\zeta_3|^{4\gamma} \chi_0 \chi_3 + |\zeta|^{4\gamma} \sum_{i \in C_1, j \in C_2} e_i e_j \zeta_3 (q_{i3}^0 (z_2) - q_{j3}^0 (z_2)) \chi_0 + F_3,$$

(2.23)

where $|F_3| \leq c B_0^{\gamma+2} (c_3) q (z_2)^2$.

2.7 Now we are returning to the estimate of the terms in (2.16). By (2.5) with $L = \chi_0 I_{z_2}$ we have

$$\left( \chi_0 I_{z_2} \sum_{i=1}^{i_0} \varphi_i \Phi_i, \sum_{j=1}^{i_0} \varphi_j \Phi_j \right) = \sum_{j=1}^{i_0} (\chi_0 I_{z_2} \varphi_j \Phi_j, \varphi_j \Phi_j).$$

(2.24)

Further in virtue of the condition $A_2$ (§1.6) and since hamiltonian $H_{03}(z_2)$ is invariant with respect to the inversion $q_{i3}^0 \leftrightarrow -q_{i3}^0$ (but not to the inversion $q_{i3}^0 \leftrightarrow -q_{i3}^0$), the functions $\varphi_j$ have the definite parity and the functions $|\varphi_j|^2$ are even with respect to change of variables $q_{i3}^0 \rightarrow -q_{i3}^0$. Consequently

$$\int \chi_0 \zeta_3 |\zeta|^{4\gamma} (q_{i3}^0 (z_2) - q_{j3}^0 (z_2)) |\varphi_i q (z_2)|^2 dR_{03}(z_2) = 0.$$

Therefore, by (2.23)

$$(I_{z_2} \chi_0 \varphi_j \Phi_j, \varphi_j \Phi_j) = Q (z_2) \int |\zeta_3|^{4\gamma} \chi_0 \chi_3 |\varphi_j \Phi_j|^2 dR_{03}(z_2) +$$

$$+ (F_3 \varphi_j \Phi_j, \varphi_j \Phi_j).$$

Since the initial system was neutral, then the subsystems $C_1, C_2$ from $z_2$ are neutral ($Q (z_2) = 0$) or have the charges of different signs ($Q (z_2) < 0$). That is why and in virtue (2.4)

$$(I_{z_2} \chi_1 \varphi_j \Phi_j, \varphi_j \Phi_j) \geq Q (z_2) \int |\Phi_j (\zeta_3)|^2 |\zeta_3|^{4\gamma} d\zeta_3 -$$

$$- c \int |\Phi_j (\zeta_3)|^2 B_0^{\gamma+2} (c_3) d\zeta_3,$$

(2.25)

where $c$ is a constant, $b_0 = (1 + \beta_3^2)^{1/2} b_3$. Here and next we use the fact, that when $\chi_0 \chi_3 = 1$, then $b_2^2 \leq \zeta_3^2 + \zeta_4^2 \leq \zeta_3 (1 + \beta_4^2)$, that is $|\zeta_3| \geq b_0$.

2.8 At last let us estimate the cross terms in (2.15). Let $\chi_5 (\zeta_3) = 0$ at $|\zeta_3| < b_0$, $\chi_5 (\zeta_3) = 1$ at $|\zeta_3| \geq b_0$. Then $\chi_0 \chi_3 \chi_5 = \chi_0 \chi_3$ and by virtue of (2.23)

$$(\chi_0 I_{z_2} \varphi_j \Phi_j, g) = Q (|\zeta_3|^{4\gamma} \chi_5 (\zeta_3) \varphi_j \Phi_j, g) + (F_4 \varphi_j \Phi_j, g) + (F_5 \varphi_j \Phi_j, g),$$

(2.26)
where 
\[ F_4 = Q(x_1 x_3 - 1) x_5 \zeta_3^{1+y}, \quad |F_5| \leq c|q'(Z_2)|^2 B_0^{1+y}. \]

In consequence of (2.7)
\[ (|\zeta_3|^{1+y} \zeta_3 \Phi_j, q_j) = 0. \tag{2.27} \]

The second and the third addends in (2.26) are estimated by Buniakovsky inequality:
\[ |(F_i \varphi_j \Phi_j), g_i| \leq \mathcal{E}||g||^2 + c||F_i \varphi_j \Phi_j||^2, \quad t = 4, 5. \tag{2.28} \]

By (2.4)
\[ \|F_5 \varphi_j \Phi_j\|^2 \leq c\|B_0^{1+y} (\zeta_3) \Phi_j\|^2. \tag{2.29} \]

It is clear that \( F_4 \neq 0 \) only if \( \chi_0 = 0, \chi_5 = 1 \) or if \( \chi_3 = 0, \chi_5 = 1 \). If \( |\zeta_3| \geq b \) (and consequently \( |\zeta| \geq b \)), then the equalities \( \chi_0 = 0, \chi_5 = 1 \) mean that \( \chi_1 = 0 \) (see §2.3), that is
\[ |F_0(Z_2)q_1| \geq \beta|F_2(Z_2)q_1| = \beta M(Z_2)|\zeta_3|, \]
and the equalities \( \chi_3 = 0, \chi_5 = 1 \) are equivalent to the condition
\[ |\zeta| \geq \beta_1 |\zeta_3| \geq \beta_1 b; \]
if \( b_0 \leq |\zeta_3| < b \) (when \( |\zeta_3| < b_0 \) then \( F_4 \equiv 0 \)), at this time, obviously, \( b \cdot |\zeta_3|^{1+y} \geq \beta_1. \) According to above at \( q \in \text{supp} F_4 \)
\[ \tau := (|F_0(Z_2)q_1| + |\zeta_3| + b)|\zeta_3|^{1+y} \geq \beta_2 := \min \{\beta_1, 1, \beta M(Z_2)\} \]
and by (2.4)
\[ \|F_4 \varphi_j \Phi_j\|^2 \leq \|F_4 \tau \beta_2^{1+y} \varphi_j \Phi_j\|^2 \leq c\|x_5 |\zeta_3|^{1+y} \Phi_j\|^2. \tag{2.30} \]

It follows from the relations (2.26)-(2.30), that
\[ |(\chi_0 I_{Z_2} \varphi_j, g)| \leq \mathcal{E}||g||^2 + c\|B_0^{1+y} (\zeta_3) \Phi_j\|^2. \tag{2.31} \]

\[ \textbf{2.9} \] Using (2.3),(2.10),(2.12)–(2.14),(2.16),(2.20) and (2.31), we obtain that for \( \psi = M_0^0(\lambda) \) (see §2.2)
\[ (H_0^0 \psi, \psi) \geq \mu^0 \|\psi\|^2 + \sum_{Z_5 \in O(z_1)} \sum_{j=1}^{i_0} (h_{Z_2}^j \Phi_j, \Phi_j), \tag{2.32} \]
where the arguments \( \zeta_3 \) of the functions \( \Phi_j(\zeta_3) \) depend on \( Z_2 \), but we do not indicate this dependence in order not to complicate the notations.\(^6\) The inequality (2.32) suggests, how to define such space \( \mathcal{M}_2(\lambda) \), that for \( \psi = (\mathcal{M}_2(\lambda) \oplus \mathcal{M}_2^0) \) the inequality (2.2) holds. Namely we put
\[ \mathcal{M}_2(\lambda) = \left\{ \tilde{\psi} | \tilde{\psi} = P(a) \sum_{g \in S} \sum_{i,j} c_{ij}(g) T_{g} \varphi_i(q_{Z_2}) f_j(\zeta_3) u_{Z_2} \nu_k \right\}, \]
where \( f_j(\zeta_3) \) are the eigenfunctions of the operator \( h_{Z_2}^j \), corresponding to its eigenvalues, which are not greater than \( \lambda \); the functions \( u_{Z_2} = u(t_{Z_2}) \), \( \omega_0 = \omega_b(\zeta_3), \varphi_i(q_{Z_2}) \) are the same as in §2.2; \( c_{ij}(g) \) are arbitrary numbers. Let
\[ \mathcal{M}_2(\lambda) = \mathcal{M}_2^0 + \mathcal{M}_2(\lambda). \]

\(^6\)if the condition \( A_1 \) does not hold, then the form of the function \( \Phi_j \) depends on \( Z_2 \) also.
Then for $\psi - M_2(\lambda)$ the expansions (2.6) of the functions $\psi Z_2 \equiv \psi u Z_2 \omega_1$ at any $Z_2 \in O(\alpha)$ can not contain the terms $\varphi_j(qZ_2)\Phi_j(\zeta_3)$ with such functions $\Phi_j(\zeta_3)$, for which
$$(h Z_2^+ \Phi_j, \Phi_j) < \lambda \|\Phi_j\|^2.$$ That is why and since
$$\|\psi Z_2\|^2 \geq \sum_{j=1}^{i_0} \|\Phi_j\|^2, \quad \|\psi\|^2 \geq \sum_{Z_2 \in O(\alpha)} \|\psi Z_2\|^2$$
we obtain, that
$$\sum_{Z_2 \in O(\alpha)} \sum_{j=1}^{i_0} (h Z_2^+ \Phi_j, \Phi_j) \geq \lambda \|\psi\|^2.$$ Hence and from (2.32) the validity of the inequality (2.2) follows.

The estimate
$$\dim \mathcal{M}_2(\lambda) \leq d Z_2(\alpha; \tilde{a}_0) N(\lambda; h Z_2^+) + \text{const}$$
follows from Lemma 4.2 [3].  

2.10 To end the proof of Theorem 1.2 we construct such space $\mathcal{M}_1(\lambda)$, that for $\psi \in \mathcal{M}_1(\lambda)$ the inequality (2.1) holds and
$$\dim \mathcal{M}_1(\lambda) \geq d Z_2(\alpha; \tilde{a}_0) N(\lambda; h Z_2^+) - c.$$ (2.33)
Let
$$\mathcal{M}_1(\lambda) = \{\psi | \psi \in H^{(a)} \sum_{i,j} \sum_{g \in S} c_{ij}(g) T_3(\varphi_i(qZ_2) f_j(k_0, \zeta_3(Z_2))) u(\tau_3(Z_2))\},$$ (2.34)
where $f_j(k_0, \zeta_3(Z_2))$ are the eigenfunctions of the effective operator $h Z_2^+$ in the region $|\zeta_3(Z_2)| \geq b_0$ with zero boundary condition at $|\zeta_3(Z_2)| = b_0$, corresponding to all its eigenvalues, which are less than $\lambda$, $\varphi_i(qZ_2)$ $i = 1, 2, \ldots, i_0$ are the same as in §2.3, $a(s)$ is the same as in §2.2, $\tau_3(Z_2) = |P_{03}(Z_2) g^{(3)}|$, $|\zeta_3(Z_2)|^{1}$, $q^{(3)} = (q^{(3)}_1, \ldots, q^{(3)}_n) \in R_0$, $q^{(3)}_j = (0, 0, q^{(3)}_j)$ $j = 1, \ldots, n$, $c_{ij}(g)$ are arbitrary numbers. The space (2.34) coincides (up to the notations) with the space $\mathcal{M}_1(\lambda)$ from [3] (see formula (2.5)) and the verification of the properties (2.33) and (2.1) can be realized exactly similar to [3]. Theorem 1.2 is proved.

§3. Proofs of Theorems 1.5, 1.6.

3.1 The external scheme of the proof of Theorem 1.5 (§3.2-3.8) is the same as the scheme of the proof of Theorem 2.1 [11]. As in [11] to estimate the quadratic form of the considered operator over the sequences $g_k$, describing the leaving of the system from any compact region of the configuration space, at first we make the decomposition of this space in the direction of the axes $z$, and after it we decompose the space in the plane $(x, y)$. But here the filling of this scheme is the other one. In the direction of $z$-axes we do at first the decomposition in the configuration space of the cluster $C_1$ (independently on the state of the cluster $C_2$), and after it we decompose the configuration space of the cluster $C_2$, when the cluster $C_1$ is in a compact region. In the plane $(x, y)$ we make the standard decomposition, but here we prove at the first time, that in the region, corresponding to the unlimited motion of the clusters $C_1, C_2$ one from other in the plane $(x, y)$ without their decomposition, the norms of the wave functions $g_k$ tend to zero, that is such motion is impossible.
Theorem 1.6 is proved by the standard method, but with the decisive application of Theorem 1.5 from §1.

3.2 Let the decomposition $Z_2 = \{C_1, C_2\}$ be fixed and $g_k(q)$ be such sequence from $P^{(a)}C_0^2(R_{03}(Z_2))$, that

$$\|g_k\| = 1,$$ $\sup(H_{03}Z_2)g_k, g_k) < +\infty$ and $g_k(q) \to 0$ $\mathcal{L}_2(R_{03}(Z_2)).$

We shall prove, that

$$\lim(H_{03}(a; Z_2)g_k, g_k) \geq \mu(a; Z_2), \quad (3.1)$$

where

$$\mu(a; Z_2) = \min_{Z_2, Z_2 < Z_2, s \geq 3} \inf \mathcal{H}_{03}(a, Z_s).$$

It will follow from this relation, that

$$\sigma_{ess}(H_{03}(a; Z_2)) \leq [\mu(a, Z_2), +\infty). \quad (3.2)$$

The proof of the contrary may be realized by standard method similar to [13] (§§2.7,2.8) and we miss it here, all the more for the application it is sufficient to know only (3.2)

3.3 Let us introduce the needed notations. Let $Y_{s_j} = \{C_{1j}, \ldots, C_{sj}\}$ be the arbitrary decomposition of the cluster $C_j$ into $s$ clusters $C_{pj}$, non interacting one with other, $1 \leq p \leq s \leq n_j$, $\bigcup_{p} C_{pj} = C_j$, $C_{p_j} \cap C_{l_j} = \emptyset$ $p \neq l$. Let us introduce the spaces

$$R_{03}[Y_{s_j}] = \{q|q = (q_1 \ldots q_n), q_i = (0,0,0) \in C_{j}, q_i = (0,0,q_3) i \in C_{j},$$

$$\sum_{i \in C_{pj}} m_i q_3 = 0 \quad p = 1,2, \ldots, s\}$$

$$R_{03}[Y_{s_j}] = \{q|q = (q_1 \ldots q_n), q_i = (0,0,0) \in C_{j}, q_i = (0,0,q_3) i \in C_{j},$$

$$(q, q')_1 = 0 \quad \text{for} \quad \forall q' \in R_{03}[Y_{s_j}]$$

and the projectors (in the sense $(\cdot, \cdot)_1$) $P_{03}[Y_{s_j}]$ in the space $R_{03}(Z_2)$ onto $R_{03}[Y_{s_j}]$, $\kappa = 0, c$. Let

$$\tau(Y_{s_j}) = \sum_{p \in C_{j}} m_p q_{3p}^2, \quad \tau(Y_{s_j}) = |P_{03}(Y_{s_j})q|_1 \cdot |P_{c3}(Y_{s_j})q|_1 s \geq 2.$$}

Further we choose the numbers $b_1 > a_1 \gg 1$ and $a_j < b_j \ll 1$, $2 \leq j \leq n_i = |C_i|$, as in §2.4 [13], yet not for all system, but for its subsystem $C_i$, and determine such real twice continuously differentiable functions $u_{s_j}(\tau)$, $v_{s_j}(\tau)$, that $0 \leq u_{s_j}(\tau)$, $v_{s_j}(\tau) \leq 1$, $u_{s_j}^2(\tau) + v_{s_j}^2(\tau) = 1$, $u_{s_j}(\tau) = 1$ at $0 \leq \tau \leq a_j$, $u_{s_j}(\tau) = 0$ at $\tau \geq b_j$. Let

$$u_{Y_{s_j}} = u_{s_j}(\tau(Y_{s_j})), \quad v_{Y_{s_j}} = v_{s_j}(Y_{s_j}).$$

3.4 Further we estimate the quadratic form $(H_{03}(a; Z_2)g_k, g_k)$ (see §3.2), using the decomposition of the space of the relative motion of the cluster $C_1$ in the direction of 3-rd axes with the help of the introduced functions $u_{Y_{1j}}, v_{Y_{1j}}$. Let

$$\psi(q) = g_k(q), \quad \hat{\psi}_0 = \psi, \quad \hat{\psi}_{s_j} = \hat{\psi}_{s_j, 1} \cdot (1 - \sum_{Y_{1j}} u_{Y_{1j}}^2)^{1/2}, \quad \hat{\psi}_{s_j, 1, Y_{1j}} = \hat{\psi}_{s_j, 1, Y_{1j}}, \quad j = 1,2, \ldots, n_1.$$

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$\psi_j = \sum_{i_{j1}} \psi_{j \pm 1, i_{j1}}$, $j = 1, 2, \ldots, n - 1$; $Z_{s1} = \{Y_{s1}, C_2\}$, $Z_{s2} = \{C_1, Y_{s2}\}$.

By the analogy with the relations (2.5), (2.6) from [13] we obtain, that for arbitrary $E > 0$ and $a_1 = a_1(E)$, $b_1 = b_1(E)$

$$\sum_{j=0}^{n_{1-1}} \sum_{Z_{j+1,1}} (H_{0\delta}(Z_{j+1,1}) \psi_j, Y_{j+1,1} \psi_j, Y_{j+1,1}) - 2E (3.3)$$

and

$$\sum_{j=0}^{n_{1-1}} \sum_{Z_{j+1,1}} (H_{0\delta}(Z_{j+1,1}) \psi_j, Y_{j+1,1} \psi_j, Y_{j+1,1}) \geq \mu(\alpha; Z_{j+1,1}) \psi_j, Y_{j+1,1} \psi_j, Y_{j+1,1} \geq 1, \quad j \geq 1, \quad (3.4)$$

since at $j \geq 1$ the decomposition $Z_{j+1,1}$ is the decomposition of two-clusters system $Z_2 = (C_1, C_2)$ at least into 3 clusters in consequence of the partition of the cluster $C_1$.

3.5 To estimate in (3.3) the term with $j = 0$ next we make the decomposition of the space of the relative motion of the cluster $C_1$ in the direction of $z$-axes (under condition that for the particles of non decomposed cluster $C_1$ the inequality $\sum_i \alpha_i \leq b_i$ holds). To do it we use the functions $w_{y_{j,2}}$. Let

$$\omega_0 = \psi_0, \quad \omega_j = \psi_{j-1} \psi_{j+1}, \quad \omega_j = \sum_{Y_{j2}} w_{y_{j,2}} \psi_{y_{j,2}} \quad j = 1, 2, \ldots, n_2 - 1$$

and similar to (3.3), (3.4) we obtain

$$\sum_{i=0}^{n_{2-1}} \sum_{Z_{i+1,2}} (H_{0\delta}(Z_{i+1,2}) \omega_i, Y_{i+1,2} \omega_i, Y_{i+1,2}) \geq \mu(\alpha; Z_{i+1,2}) \omega_i, Y_{i+1,2} \omega_i, Y_{i+1,2} \geq 1 \quad (3.5)$$

$$\sum_{i=0}^{n_{2-1}} \sum_{Z_{i+1,2}} (H_{0\delta}(Z_{i+1,2}) \omega_i, Y_{i+1,2} \omega_i, Y_{i+1,2}) \geq \mu(\alpha; Z_{i+1,2}) \omega_i, Y_{i+1,2} \omega_i, Y_{i+1,2} \geq 1 \quad (3.6)$$

3.6 Now we have to estimate only the value $(H_{0\delta}(Z_2) \omega_0, Y_{j2}, \omega_0, Y_{j2})$. To do it we decompose the support of the function $\omega_0, Y_{j2}$ into the regions corresponding to all possible decompositions $Z_{s} = \{C_1', \ldots, C_s'\}$ of the system $Z_1$ in the plane $(x, y)$. To do this we introduce from the beginning the spaces

$$R_{0\perp}(Z_{s}) = \{q| q = (q_1, \ldots, q_n), \quad \eta_{i\perp} = (q_{i1}, q_{i2}), \quad \sum_i m_i q_{i\perp} = 0 \quad p = 1, 2, \ldots, s\},$$

$$R_{\perp}(Z_{s}) = \{q| q = (q_1, \ldots, q_n), \quad \eta_{i\perp} = (q_{i1}, q_{i2}), \quad (q, q')_1 = 0$$

for $\forall q' \in R_{0\perp}(Z_{s})$}

and denote by $P_{0\perp}(Z_{s})$ the projector in the sense of the inner product $(\cdot, \cdot)_1$ onto the subspace $R_{0\perp}(Z_{s}) = 0, \perp, \quad s \geq 2$. Further let $q_{i\perp} = (q_{i1}, \ldots, q_{i\perp})$,

$$\beta_1 = |q_{\perp}|, \quad \beta(Z_{s}) = |P_{0\perp}(Z_{s}) q_{\perp}| \cdot |P_{0\perp}(Z_{s}) q_{\perp}|_{1\perp} \quad s \geq 2.$$
We determine the numbers $\tilde{a}_i, \tilde{b}_s$ and the functions $\tilde{u}_s(\beta), \tilde{v}_s(\beta)$ similar to §2.4 [13] and put

\[
\tilde{a}_s Z_i = \tilde{u}_s(\beta(Z_i')), \quad \tilde{v}_s(\beta(Z_i')) = s=1,2,\ldots , \quad \varphi_s = \omega_{0,Y_1}, \quad \hat{f}_i = f,
\]

\[
\hat{f}_i = f_{i+1}(1 - \sum_{Z_i'} \tilde{a}_s Z_i')^{1/2}, \quad f_{i+1,Z_i'} = \hat{f}_{i+1,\tilde{a}_s Z_i'}, \quad i=1, \ldots ,\quad n,
\]

\[
f_i = \sum_{Z_i'} f_{i+1,Z_i'}.
\]

By the analogy with above we obtain:

\[
(H_{03}(Z_2)f, f) \geq \sum_{j=0}^{\infty} \left( H_{03}(Z_2 \cap Z_{j+1}') f_{j,Z_i'}, f_{j,Z_i'} \right) - 2\mathcal{E} \| f \|^2, \quad (3.7)
\]

where we denote by $Z_2 \cap Z_{j+1}'$ the decomposition $Z_j = \{C_1, \ldots , C_p\}$, consisting of the such clusters $C_i \cap C_j$, $C_i \in Z_2$, $C_j \in Z'$, which are not empty. Evidently when $Z_i' \neq Z_1, Z_2$, we obtain $Z_p < Z_2$, $p \geq 3$ and it means that

\[
(H_{03}(Z_2 \cap Z_{j+1}') \varphi_{j,Z_i'}, \varphi_{j,Z_i'}) \geq \mu(\alpha, Z_2) \| \varphi_{j,Z_i'} \|^2
\]

at $Z_{j+1}' \neq Z_1, Z_2$. \quad (3.8)

If $Z_{j+1}' = Z_1, Z_2$, then evidently $Z_2 \cap Z_{j+1}' = Z_2$. Consequently in the relation (3.7) we have to estimate only the addends

\[
(H_{03}(Z_2) f_0, Z_i, f_0, Z_i) \quad \text{and} \quad (H_{03}(Z_2) f_1, Z_i, f_1, Z_i).
\]

3.7 By the construction

\[
f_0, Z_i(q) = f_0, Z_i(k, q) = g_k(q) u_{11}(\tau(Y_{11})) u_{12}(\tau(Y_{12})) \tilde{a}_1(\beta_1)
\]

and hence at $q \in supp f_0, Z_i(k, q)$ for all $k$

\[
|q|^2 \leq 2\tilde{a}_1^2(1) + \tilde{b}_1^2(1).
\]

The spectrum of the operator $H_{03}(\alpha, Z_2)$ in a compact region of the space $R_{03}(Z_2)$ with the zero boundary condition is pure discrete with the single limit point $+\infty$. Hence and from week convergence to zero of the sequence $g_k(q)$ in $L_2(R_{03}(Z_2))$

\[
\lim_{k \to \infty} (H_{03}(Z_2) f_0, Z_i(k, q), f_0, Z_i(k, q)) \geq \mu(\alpha, Z_2) \lim \| f_0, Z_i(k, q) \|.
\]

3.8 At last let us estimate the value $(H_{03}(Z_2) f_1, Z_i, f_1, Z_i)$. As a preliminary we remark that at $q \in supp f_1, Z_i$ by the construction

\[
|P_{0,1}(Z_2) q_{\perp}|, \quad |P_{0,1}(Z_2) q_{\perp}| \geq \tilde{a}_1,
\]

and since

\[
|q|^2 = |P_{0,1}(Z_2) q_{\perp}|^2 + |P_{1,1}(Z_2) q_{\perp}|^2,
\]

then for $q \in supp f_1, Z_i$

\[
|P_{0,1}(Z_2) q_{\perp}| \geq \tilde{a}_1(1 + \tilde{b}_2^2)^{1/2}. \quad (3.10)
\]

Furthermore by the lemma 2.3 ([10], see p.125)

\[
|P_{1,1}(Z_2) q_{\perp}| = \left[ M[C_1] \cdot M[C_2] \right] |\zeta_{\perp}|^2.
\]

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\[ |P_{0\perp}(Z_2)q_\perp|^2 \geq |q_i - \zeta_\perp[C_j]|^2 m_i \quad \text{for } i \in C_j, \]

where \( \zeta_\perp = \zeta_\perp[C_1] - \zeta_\perp[C_2], \quad \zeta_\perp[C_j] = \sum_{i \in C_j} m_i q_i \cdot M[C_j] \). 

After these simple remarks let us estimate from below the term
\[ F = (\nu_1 + \mathcal{E}_1)^2 + (\nu_2 - \mathcal{E}_1)^2 \]

in the expression of the operator \( H_{03}(Z_2) \); here \( \mathcal{E}_j = 2B \sum_{i=1}^n q_i \epsilon_i \).

Since \( Q = \sum_{j=1}^n \epsilon_j = 0 \), then
\[ \mathcal{E}_j = 2B \sum_{s \in C_1} (q_{sj} - \zeta_\perp[C_1]) \epsilon_s + 2B \sum_{s \in C_2} (q_{sj} - \zeta_\perp[C_2]) \epsilon_s + 
+ 2B \sum_{s \in C_2} (\zeta_\perp[C_2] - \zeta_\perp[C_1]) \epsilon_s = G_j(q) - 2BQ[C_2] \zeta_j(Z_2), \]

where by the relations (3.9)-(3.12)
\[ |G_j(q)| \leq \tilde{b}_2 c |\zeta_\perp(Z_2)| \]

and \( c \) does not depend on \( \tilde{b}_2 \) and \( q \).

By virtue of (3.13)
\[ F = \nu_1^2 + \nu_2^2 + 2\nu_1 (G_2(q) - 2BQ[C_2] \zeta_2(Z_2)) - 2\nu_2 (G_1(q) + 
+ 2BQ[C_1] \zeta_1(Z_2) + G_2(q) - 4BG_2(q)Q[C_2] \zeta_2(Z_2) + 
+ 4BG_1(q)Q[C_1] \zeta_1(Z_2) + 4B^2 (Q[C_2]^2 \zeta_2^2(Z) + Q[C_1]^2 \zeta_1^2(Z)). \]

Since the numbers \( \tilde{b}_2 \) and \( \tilde{a}_1 \) can be taken from the beginning respectively as sufficiently small and sufficiently large, then by (3.14),(3.15)
\[ F \geq 4B^2 Q[C_1]^2 |\zeta_\perp|^2 - \tilde{b}_2 c |\zeta_\perp|^2 - c |\zeta_\perp| \geq d_0 |\zeta_\perp|^2 \geq d_1 \tilde{a}_1^2, \]

where the constants \( d_0, d_1 \) do not depend on \( \tilde{a}_1 \). That is why for sufficiently large \( \tilde{a}_1 \)
\[ (H_{03}(Z_2) f_1, z_2, f_1, z_2) \geq \mu(\alpha, Z_2) \| f_1, z_2 \|^2. \]

Combining together the estimates (3.3)-(3.8),(3.17) we obtain that
\[ \lim (H_{03}(Z_2) g_k, g_k) \geq \mu(\alpha, Z_2) - \mathcal{E}. \]

The assertion of Theorem 1.5 follows from this relation.

**Remark.** It follows from the estimate (3.16) that in the region \( \Omega_1 = \text{supp} \ f_1, z_2 \), corresponding to the decomposition of the initial system into nondecomposed clusters \( C_1, C_2 \),
\[ \lim \| g_k \|_{\Omega_1} = O(\tilde{a}_1^{-1}), \]

where \( |\tilde{a}_1| \leq \text{const} |\zeta_\perp[C_2] - \zeta_\perp[C_1]| \) and \( \tilde{a}_1 \) can be chosen arbitrarily large. Thus we proved (at the same time) that Weyl sequence \( q_3(q) \) for the operator \( H_{03}(\alpha; Z_2) \) can not correspond to the decomposition \( Z_2 \) into nondecomposed clusters \( C_1, C_2 \).

**3.9 The proof of Theorem 1.6.** Let (see §1.4)
\[ \hat{q}_{33} = q_{33}(Z_2(2)) = q_{33} - \sum_{j=1, j \neq 2}^n m_j q_j \cdot (M - m_2)^{-1} f(q_{33}), \]

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be the smooth normed function, which is equal to zero for \( |\hat{q}_{333}| \leq 1 \), \( \hat{q} = q(Z_3(2, 3)) \) be arbitrary vector from \( R_{33}(Z_3(2, 3)) \), \( \varphi(\hat{q}) \) be the normed eigenfunction of the operator \( H_{33}(a; Z_3(2, 3)) \), corresponding to its eigenvalue \( \mu(a; Z_3(2, 3)) \) and having the permutational symmetry of some type \( a'' = a''(Z_2(2, 3)) \), \( a'' < a', a' = a'(Z_2(2)) \) be such type of the permutational symmetry of the system \( Z_2(2) \), that \( a'' < a' < a \). Let \( f_k = f(\hat{q}_{333})k^{1/2}, \psi_k = \varphi f_k, \psi_k^{(a')} = P^{(a')}(Z_2(2))\psi_k \).

Substituting the function \( \psi_k^{(a')} \) in the quadratic form of the operator \( H_{33}(a; Z_3(2, 3)) \) and using the known expression of the operator \( P^{(a')} \), we obtain after changing of variables \( \tau = k\hat{q}_{333} \) and after the complicate calculations (which are similar to the calculations from [14] (§7)), that

\[
(H_{33}(a, Z_2(2))\psi_k^{(a')}, \psi_k^{(a')}) = (H_{33}(a, Z_2(2))\psi_k, P^{(a')}\psi_k) = \\
= \mu(a; Z_2(2))\|\psi_k^{(a')}\|^2 + k(p(k) + O(k)),
\]

where at \( k \to 0 \) \( \lim p(k) < 0 \), \( \lim\|\psi_k^{(a')}\|^2 > 0 \). That is why for small \( k \)

\[
in f(H_{33}(a; Z_2(2))\psi_k^{(a')}, \psi_k^{(a')}) < \mu(a; Z_2(2))\|\psi_k^{(a')}\|^2,
\]

that is

\[
\mu^{(a)} := inf H_{33}(a; Z_2(2)) \in \sigma_d(H_{33}(a; Z_2(2)),
\]

and all is proved


12. S.A. Vugalter, private communication.
