Riemann Surfaces with Large First Eigenvalue

Robert Brooks
Eran Makover


Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
Riemann Surfaces with Large First Eigenvalue

Robert Brooks\(^1\)
Department of Mathematic
The Technion – Israel Institute of Technology
32000 Haifa, Israel
rbrooks@tx.technion.ac.il

Eran Makover\(^2\)^\(^3\)
Department of Mathematics
Hebrew University
Jerusalem, Israel
Eran.Makover@Dartmouth.edu

March, 1998
revised May, 2000

\(^1\)Partially supported by a grant from the Israel Science Foundation, the Fund for the Promotion of Research at the Technion, and the C. Wellner Fund

\(^2\)Partially supported by US-Israel Binational Science Foundation grant BSF 95-348 and the Edmund Landau Center for Research in Mathematical Analysis

\(^3\)Current address: Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755
In this paper, we consider the problem of constructing compact Riemann surfaces $S$ of arbitrary genus with the property that the first eigenvalue $\lambda_1(S)$ is large. Throughout this paper, by the phrase “Riemann surface” we understand an oriented surface with a complete metric of constant curvature $-1$.

To state our results precisely, let us set

$$\Lambda(g) = \text{the maximum value of } \lambda_1(S_g),$$

where $S_g$ ranges over compact Riemann surfaces of genus $g$.

The question of the behavior of $\Lambda(g)$ for large $g$ was raised by Buser [Bu1] in 1978, where he conjectured that $\lim_{g \to \infty} \Lambda(g) = 0$. He subsequently showed in [Bu2] that $\limsup_{g \to \infty} \Lambda(g) \geq 3/16$, using heavy machinery from number theory. Much of this heavy machinery was later removed in [BBD], but both [Bu2] and [BBD] construct surfaces of large $\lambda_1$ only in certain genera arising from number-theoretic considerations.

The main result of this paper, Theorem 1.4 below, gives a method for modifying a family of Riemann surfaces with large first eigenvalue to obtain a much larger family which retains the property that the first eigenvalue is large. In particular, the family constructed in this way may include surfaces of all genera, even though the first family contains surfaces of only very special genera.

In Theorem 1.2, we apply this construction to the noncompact modular surfaces $P(k) = \mathbb{H}^2 / \Gamma_k$, to be described below. The genus of $P(k)$ is approximately cubic in $k$, while the number of cusps is approximately quadratic in $k$; see Lemma 6.1 for details.

For

$$C = \inf_k \lambda_1(P(k)),$$

we have

**Theorem 1.2** $\liminf_{g \to \infty} \Lambda(g) \geq C$.

According to a famous theorem of Selberg [Sel], $C \geq 3/16$. This has been improved by Luo-Rudnick-Sarnak [LRS] to $C \geq 171/784$. Selberg conjectured in [Sel] that $C = 1/4$.

We remark that the upper bound

$$\limsup_{g \to \infty} \Lambda(g) \leq 1/4$$

1
was already observed in [Bu1].

In Section §1, we also give a version of this result for Riemann surfaces with cusps.

The results of this paper have been announced in [BM2].

A quite different construction of Riemann surfaces with large $\lambda_1$, which relies on no number theory whatsoever, but which produces a smaller and non-explicit bound for $\liminf_{g \to \infty} \Lambda(g)$, is given by the authors in [BM].

**Acknowledgements:** It is a pleasure to thank a number of colleagues for helpful remarks and suggestions, including Bruno Colbois, Yves Colin de Verdière, and Laurent Saloff-Coste. In particular, the suggestion to apply the techniques of [PS] to establish Theorem 1.2 comes from Colbois.

The first author would also like to thank the Erwin Schrödinger Institute of the University of Vienna for its hospitality while this paper was completed.

## 1 Some Background

The problem of constructing Riemann surfaces with large first eigenvalue has been considered by a number of authors. The best results in this direction are the Selberg 3/16 Theorem [Sel] and its strengthening by Luo-Rudnick-Sarnak [LRS]. To state this result, we denote by $\Gamma_k$ the $k$-th congruence subgroup of $PSL(2, \mathbb{Z})$, given by

$$\Gamma_k = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{Z}) : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{k} \right\}.$$

Let $P(k)$ denote the noncompact surface

$$P(k) = \mathbb{H}^2 / \Gamma_k,$$

and let $C$ denote the infimum

$$C = \inf_{k \in \mathbb{Z}_+} \lambda_1(P(k)).$$

It was shown by Selberg in [Sel] that

**Theorem 1.1 ([Sel])** $C \geq 3/16.$
He also conjectured there that $C = 1/4$, the largest possible value. Selberg’s result was improved by Luo-Rudnick-Sarnak ([LRS], see also [Sa]), who showed that $C \geq 171/784$.

The problem of constructing compact surfaces with large $\lambda_i$ was considered by Burger, Buser, and Dodziuk [BBD]. Their main technique, which is crucial for us, may be summarized in the following result.

**Lemma 1.1 (Handle Lemma, [BBD])** Let $S$ be a Riemann surface with finite area, and let $\{C_i \}$ be a collection of an even number of cusps of $S$. Let $S_i$ be the surface formed from $S$ by first replacing the punctures corresponding to the cusps $C_i$ with geodesics of length $t$ and then gluing the geodesic corresponding to $C_{2i-1}$ with the geodesic corresponding to $C_{2i}$.

Then

$$\limsup_{i \to \infty} \lambda_i(S_i) \geq \lambda_i(S).$$

The emphasis of the present paper is on constructing surfaces of arbitrary genus with $\lambda_i$ large.

Our main result is

**Theorem 1.2** For each $\varepsilon > 0$, there exists $N$ such that, for every genus $g \geq N$, there exists a compact Riemann surface $S_g$ of genus $g$ with

$$\lambda_i(S_g) \geq C - \varepsilon.$$

Regarding surfaces with cusps, we prove

**Theorem 1.3** For each $\varepsilon > 0$ there exist constants $K = K(\varepsilon)$ and $N = N(\varepsilon)$ such that, for $g > N$ and all $k < Kg$, there exists a Riemann surface $S$ of genus $g$ with $k$ cusps satisfying

$$\lambda_i(S) \geq C - \varepsilon.$$
Lemma 1.2 (Cusp-Closing Lemma) For all $l$ sufficiently large and for all $\varepsilon > 0$, there exists $C(l, \varepsilon) > 0$ such that

$$\lambda_1(S^C) \geq \min \left[(1/4 - \varepsilon), C(l, \varepsilon) \lambda_1(S^O)\right],$$

whenever the collection of cusps has length $\geq l$.

Furthermore, $C(l, \varepsilon) \to 1$ as $l \to \infty$.

Lemma 1.3 (Cusp-Opening Lemma) For all $r$ sufficiently large and for all $\varepsilon > 0$, there exists $C(r, \varepsilon) > 0$ with the following properties. Let $S^C$ be a hyperbolic Riemann surface and $\{x_i\}$ a collection of points of $S^C$ such that the injectivity radii at the points $x_i$ are all $\geq r$ and the balls $B(x_i, r)$ of radius $r$ about the $x_i$’s are all disjoint. Let $S^O$ be the surface obtained from $S^C$ by replacing the points $x_i$ by punctures.

Then

$$\lambda_1(S^O) \geq C(r, \varepsilon) \min \left[(1/4 - \varepsilon), \lambda_1(S^C)\right].$$

Furthermore, $C(r, \varepsilon) \to 1$ as $r \to \infty$.

With the first eigenvalue $\lambda_1$ replaced by the Cheeger constant, Lemmas 1.2 and 1.3 were proved in [PS]. The essential point here is the systematic use of the Ahlfors-Schwarz Lemma [A] in a manner suitable for spectral theory; see [GA] for details.

One could imagine attempting to prove Theorem 1.2 in the following way. Consider the surfaces $P(k) = \mathbb{H}^2/\Gamma_k$, and then pick some even number of cusps to which we apply the Handle Lemma 1.1. To the remaining cusps we apply Lemma 1.2. In this way, we may construct large numbers of compact surfaces with $\lambda_1 \geq C - \varepsilon$, of varying genus.

Unfortunately, one cannot cover all genera in this manner. Among $g$ less than $1,000$, this method covers all $g$ except those in the interval $[731, 805]$. The method continues to miss intervals of genera going out to infinity. However, one does in this way construct compact surfaces with large $\lambda_1$ for most genera, without using Lemma 1.3.

To capture the remaining genera, we use Lemma 1.3 to introduce new cusps into the surfaces $P(k)$, without disturbing $\lambda_1$ greatly. The number of new cusps we can introduce this way is of the order $(\text{const}(r, \varepsilon)) \text{vol}(P(k))$. We then have more cusps with which to form handles or close off, and we shall see in §5 below that this allows us to cover the remaining genera, establishing Theorem 1.2.
Theorem 1.3 is proved in exactly the same way, except that we introduce more new cusps, so that some of them can remain after we close the others.

More generally, we prove

**Theorem 1.4** Let $R_j$ be a family of compact Riemann surfaces with the following properties.

(i) The injectivity radius of $R_j \to \infty$ as $j \to \infty$.

(ii) For all $\delta > 0$, there exists $N$ such that the union of intervals

$$\cup_j \left[ \text{vol}(R_j), (1 + \delta)(\text{vol}(R_j)) \right]$$

covers $[N, \infty)$.

(iii) $\inf_j (\lambda_1(R_j)) \geq C$.

Then for each $\varepsilon > 0$, there exists $N$ such that if $g \geq N$, there is a Riemann surface $S_g$ of genus $g$ with

$$\lambda_1(S_g) \geq C - \varepsilon.$$

## 2 The Ahlfors-Schwarz Lemma

In this section, we summarize and extend some results needed from [PS]; see also [GA].

Let $S^O$ be a Riemann surface of finite area. Then, from the point of view of conformal structure, we may consider $S^O$ as arising from a compact surface $S^C$ by replacing finitely many points $\{p_k\}$ by punctures. $S^C$ may be recovered from $S^O$ by noting that each cusp $C_i$ of $S^O$ has a neighborhood conformally equivalent to a punctured disk. We may obtain $S^C$ from $S^O$ by replacing these punctured disks by non-punctured disks.

A natural question is to relate the *hyperbolic geometry* of $S^O$ to that of $S^C$. This would at first glance appear to be quite difficult, since $S^C$ may not even carry a metric of constant negative curvature.

However, the main result of [PS] asserts that if the cusps $C_i$ are all large, in a sense to be defined below, then there are neighborhoods $B_i$ of the cusps $C_i$ and corresponding neighborhoods $B(p_k, r)$ of the points $p_k$, such
that outside these neighborhoods, the two metrics are close to each other. Furthermore, the size of these neighborhoods depends only on the size of the cusps.

To be more precise, we may let $\mathcal{C}$ denote the quotient of the upper-half plane $\mathbb{H}^2$ in its natural hyperbolic metric by the isometries $z \to z + 1$. We then have the natural holomorphic map $\mathcal{C} \to D = \{z : |z| < 1\}$ given by $z \to e^{2\pi iz}$.

If we denote by $B_y$ the subset of $\mathcal{C}$ given by $\{z : \Im(z) \geq y\}$, we may characterize $B_y$ intrinsically as the set of points which are contained in a closed horocycle of length $\leq l = 1/y$. It is clear that the image of $B_y$ in $D$ is the punctured disk about 0 of radius $e^{-2\pi y}$. If we denote the radius of this disk in the hyperbolic metric on $D$ by $r$, then $r$ and $l$ are related by the formula

$$l = \frac{1}{y} = \frac{2\pi}{\log(\frac{e^{2\pi y}}{r-1})}.$$  

In particular, $r$ and $l$ are functions of each other, and $r \to 0$ (resp. $\infty$) as $l \to 0$ (resp. $\infty$).

The result of [PS] tells us that, if $l$ is sufficiently large, we may modify the metric in each cusp conformally, by a function of $r$ which is close to 1, to obtain a new metric on $S^O$ which continues across the cusps to give a metric of curvature close to $-1$ on $S^C$. The Ahlfors-Schwarz Lemma [A] as given in [PS] then shows that this metric must be close to the hyperbolic metric on $S^C$.

It follows that the image of $B_y(C_i)$ is contained in the ball $B(r',p_i)$, for some $r'$ close to the value of $r$ given in the equation above, and contains a ball $B(r'',p_i)$ of radius $r''$ close to $r$. In particular, the injectivity radius about each point $p_i$ in $S^C$ must be large, depending on $l$.

In §3 below, we will need to reverse this procedure. Namely, suppose that we are given finitely many points $p_1, \ldots, p_k$ in a compact hyperbolic surface $S^C$ and an $r$ such that the injectivity radius at each $p_i$ is at least $r$ and such that the balls $B(r, p_i)$ are disjoint. Let $S^O$ denote the surface

$$S^O = S^C - \{p_1, \ldots, p_k\},$$

with its hyperbolic metric. We wish to show that the cusps $C_1, \ldots, C_k$ all have length $\geq l$, for a value of $l$ close to $\frac{2\pi}{\log(\frac{e^{2\pi y}}{r-1})}$. Set $y = \frac{1}{7}$. 

6
The proof of [PS] carries over without change, reversing the roles of $S^O$ and $S^C$ to show this, except for one point. We construct a metric $ds_{2_i}^2$ on $S^O$ whose curvatures all lie between $-(\frac{1}{1+\varepsilon})$ and $-(1+\varepsilon)$ and which agrees with $ds_{\mathcal{S}^C}$ outside $\bigcup_i B_{\eta}(C_i)$. We wish to compare this with the hyperbolic metric $ds_{\mathcal{S}^O}$ on $S^O$.

Unfortunately, the version of the Ahlfors-Schwarz Lemma [A] given in [W] does not apply, because $S^O$ is non-compact. However, the argument in [GA] shows how to carry out this comparison when the compactness condition of [W] is replaced by a completeness condition and the pointwise curvature bounds are replaced by global $L^\infty$ curvature bounds.

The rest of the argument in [PS] then proceeds without change to show that the cusps are large, as desired.

3 The Cusp-Closing Lemma

Let $S^O$ be a finite-area Riemann surface, and $\{C_i\}$ a collection of cusps of $S^O$. The collection $\{C_i\}$ has length $\geq l$ if for each $C_i$ there is a closed horocycle of length $l$ about $C_i$ such that the $\gamma_i$'s are mutually disjoint. Let $S^C$ denote the Riemann surface obtained from $S^O$ by conformally filling in the cusps of $\{C_i\}$.

We show

**Lemma 1.2 (Cusp-Closing Lemma)** For each $\varepsilon > 0$ and $l > 0$, there exists a constant $C(l,\varepsilon)$ such that, if $\{C_i\}$ has length $\geq l$, then

$$\lambda_1(S^C) \geq \max [1/4 - \varepsilon, C(l,\varepsilon)\lambda_1(S^O)].$$

Furthermore, $C(l,\varepsilon) \to 1$ as $l \to \infty$.

**Proof:** We begin with the following well-known characterization of the first eigenvalue.

$$\lambda_1(M) = \inf \int_M \frac{\|\text{grad}(f)\|^2}{(\int_M f^2 \, d\text{vol}) - \frac{1}{\text{vol}(M)}(\int_M f \, d\text{vol})^2} \, d\text{vol}.$$ 

Now let $g: S^O \to S^C$ denote the canonical conformal map. We may write

$$g_*(d\text{vol}_{S^O}) = h \cdot d\text{vol}_{S^C},$$

7
where $h$ is defined away from the images $p_i$ of the cusps $C_i$.

We then have the following theorem from [PS].

**Theorem 3.1 ([PS])**  For each $\rho > 0$, there exists a number $r_1 = r_1(\rho)$ such that, if $l$ is sufficiently large, then for all $x$ such that the distance from $x$ to the points $p_i$ is $\geq r_1$, then

$$(1 - \rho) \leq h(x) \leq (1 + \rho).$$

Now let $f_C$ be an eigenfunction of eigenvalue $\lambda = \lambda_1(S^C)$, such that $\|f_C\|_2 = 1$, and assume that $\lambda \leq 1/4 - \varepsilon$. Setting $f_0 = g^*(f_\varepsilon)$, we will show that if $l$ is sufficiently large, then

$$\frac{\int_{S^C} \|\operatorname{grad} f_0\|^2 \, d\text{vol}}{\int_{S^C} f_0^2 \, d\text{vol}} - \frac{1}{\operatorname{vol}(S^C)} (\int_{S^C} f_0 \, d\text{vol})^2 \leq \frac{1}{C(l)} \frac{\int_{S^C} \|\operatorname{grad} f_C\|^2 \, d\text{vol}}{\operatorname{vol}(S^C)} (\int_{S^C} f_C \, d\text{vol})^2.$$

It will then follow that $\lambda_1(S^C) \geq C(l)\lambda_1(S^0)$.

We first observe from the conformal invariance of the integrand of

$$\int \|\operatorname{grad} f\|^2 \, d\text{vol}$$

in dimension 2 that

$$\int_{S^0} \|\operatorname{grad} f_0\|^2 \, d\text{vol} = \int_{S^C} \|\operatorname{grad} f_C\|^2 \, d\text{vol}.$$

Since the denominator on the right-hand side is 1 by assumption, it will suffice to show that

$$\int_{S^0} f_0^2 \, d\text{vol} - \frac{1}{\operatorname{vol}(S^0)} (\int_{S^0} f_0 \, d\text{vol})^2 \geq C(l).$$

We rewrite this as an integral on $S^C$ as

$$\int_{S^C} f_C^2 h \, d\text{vol} - \frac{1}{\operatorname{vol}(S^0)} (\int_{S^C} f_C h \, d\text{vol})^2 \geq C(l).$$
Let $B$ denote the collection of points $x$ such that $d(x, p_i) < r_1$ for some $i$. Then
\[
\int_{S^c} f_C^2 \cdot h \, d\text{vol} = \int_B f_C^2 \, h \, d\text{vol} + \int_{S^c-B} f_C^2 \, h \, d\text{vol}.
\]

But on $S^c - B$, $|h - 1| \leq \rho$, so that
\[
|\int_{S^c-B} f_C^2 h \, d\text{vol} - \int_{S^c-B} f_C^2 \, d\text{vol}| \leq \rho \int_{S^c-B} f^2 \, d\text{vol} \leq \rho.
\]

It remains to bound the other terms that appear. We use the following standard technique (see e.g. [SGNT]).

Let $S_\lambda(r)$ denote the $\lambda$-th spherical function, that is
(i) $S_\lambda(0) = 1$.
(ii) $\Delta S_\lambda = \lambda \cdot S_\lambda$, where we regard $S_\lambda$ as a function on $\mathbb{H}^2$ via the formula
\[
S_\lambda(x) = S_\lambda(\text{dist}(x, x_0))
\]
for some fixed point $x_0$.

For $\lambda < 1/4$, $S_\lambda$ is a positive function, which is decreasing both in $\lambda$ and $r$.

We need the following lemma.

**Lemma 3.1** Let $x_0$ denote a point in $\mathbb{H}^2$, and $f$ a function defined on $B(x_0, r_1 + r_2)$ with $\Delta(f) = \lambda f$. Then
\[
\int_{B(x_0, r_1)} f^2 \, d\text{vol} \leq \left[ \frac{\text{vol}(B(x_0, r_1))}{\int_{B(x_0, r_2)} S_\lambda^2 \, d\text{vol}} \right] \int_{B(x_0, r_1 + r_2)} f^2 \, d\text{vol}.
\]

**Proof:** Let $f^{\text{av}}$ denote the average of $f$ under the group of rotations about $x_0$. Then
\[
f^{\text{av}} = f(x_0)S_\lambda,
\]
so that
\[
\int_{B(x_0, r_2)} f^2 \, d\text{vol} \geq \int_{B(x_0, r_2)} (f^{\text{av}})^2 \, d\text{vol} = f^2(x_0) \int_{B(x_0, r_2)} S_\lambda^2 \, d\text{vol}.
\]
Hence
\[
f^2(x_0) \leq \frac{\int_{B(r_0,r_2)} f^2 \, d\text{vol}}{\int_{B(r_0,r_2)} S^2_\lambda \, d\text{vol}} \leq \frac{\int_{B(r_0,r_1+r_2)} f^2 \, d\text{vol}}{\int_{B(r_0,r_2)} S^2_\lambda \, d\text{vol}}.
\]
In particular, it follows that
\[
\int_{B(r_0,r_1)} f^2 \, d\text{vol} \leq \frac{\text{vol}(B(r_1))}{\int_{B(r_2)} S^2_\lambda \, d\text{vol}} \int_{B(r_0,r_1+r_2)} f^2 \, d\text{vol}.
\]
We now apply this lemma with \( f = f_C \) and with \( x_0 = p_i \) to obtain
\[
\int_{B(p_i,r_1)} f^2_C \, d\text{vol} \leq \frac{\text{vol}(B(r_1))}{\int_{B(r_2)} S^2_\lambda \, d\text{vol}} \int_{B(p_i,r_1+r_2)} f^2_C \, d\text{vol}.
\]
Summing over the cusps and using that \( \|f_C\|_2 = 1 \) gives
\[
\int_B f^2_C \, d\text{vol} \leq \frac{\text{vol}(B(r_1))}{\int_{B(r_2)} S^2_\lambda \, d\text{vol}}.
\]
Noting that \( \int_{B(r_2)} S^2_\lambda \, d\text{vol} \to \infty \) as \( r \to \infty \), and that \( S_\lambda \) is decreasing in \( \lambda \), we may choose \( r_2 \) sufficiently large so that this last is less than, say, \( \rho \), provided that \( l \) is sufficiently large so that the injectivity radius about \( p_i \) is at least \( r_1 + r_2 \).

We similarly have that
\[
\int_{g^*(B)} f^2_\Omega \, d\text{vol} \leq \frac{\text{vol}(g^*(B(r_1)))}{\int_{B(r_2)} S^2_\lambda \, d\text{vol}},
\]
and we may similarly choose \( r_2 \) large enough so that this is \( \leq \rho \).

It follows that
\[
\int_{S_\Omega} f^2_\Omega \, d\text{vol} \geq 1 - 2\rho.
\]
It remains to estimate the term
\[
\frac{1}{\text{vol}(S_\Omega)} (\int_{S_\Omega} f_\Omega \, d\text{vol})^2.
\]
But
\[ |\int_B f_C \, d\text{vol} \leq (\int_B f_C^2 \, d\text{vol})^{1/2} (\text{vol}(B))^{1/2} \]
by Cauchy-Schwarz, and similarly for \(|\int_B f_O \, d\text{vol}|\), so both these terms are bounded by \(\sqrt{\rho} \sqrt{\text{vol}(g^*(B))}\). From
\[ \int_{S^C} f_C \, d\text{vol} = 0 \]
it follows that
\[ |\int_{S^C - B} f_C \, d\text{vol}| \leq \sqrt{\rho} \sqrt{\text{vol}(g^*(B))}. \]

But also
\[ |\int_{S^C - B} (h - 1) f_C \, d\text{vol}| \leq \rho (\int_{S^C} f_C^2)^{1/2} (\text{vol}(S^C - B))^{1/2} \leq \rho (\text{vol}(S^C))^{1/2}. \]
Using that, for \(\rho\) small, \(\rho < \sqrt{\rho}\), and that \(\text{vol}(S^C) \leq (\text{const}(l)) \text{vol}(S^O)\), so that
\[ \frac{1}{\text{vol}(S^O)} (\int_{S^O} f_O \, d\text{vol})^2 \leq (\text{const}(l)) \rho. \]
This completes the proof of the lemma.

4 The Cusp-Opening Lemma

In this section, we will show

**Lemma 1.3 (Cusp-Opening Lemma)** For each \(r\) and \(\varepsilon > 0\), there is a constant \(C(\varepsilon, r)\) with the following property:

Let \(S^C\) be a compact Riemann surface and \(\{p_i\}\) a collection of points in \(S^C\) such that the injectivity radius about each \(p_i\) is at least \(r\) and such that the balls \(B(r, p_i)\) are mutually disjoint.

Let \(S^O\) denote the Riemann surface obtained from \(S^C\) by introducing punctures at the points \(p_i\).

Then
\[ \lambda_1(S^O) \geq \min(1/4 - \varepsilon, C(\varepsilon, r) \lambda_1(S^C)). \]

Furthermore, \(C(\varepsilon, r) \to 1\) as \(r \to \infty\).
The idea of the proof is similar to the idea of the proof of Lemma 1.2. Namely, we will let \( f_\partial \) be an eigenfunction of the Laplacian with eigenvalue \( \lambda \), and let \( g : S^O \to S^C \) be the natural map. We will show that if \( \lambda < 1/4 - \varepsilon \), then \( g_\star(f_\partial) = f_C \) is a test function on \( S^C \) showing that \( \lambda_1(S^C) \leq \frac{1}{C(\varepsilon, r)} \lambda_1(S^O) \).

Let \( \mathcal{C} \) denote the Riemann surface obtained by dividing the hyperbolic plane \( \mathbb{H}^2 \) by the parabolic transformation \( z \to z + 1 \), and for all \( y \), let \( B_y \) denote the set of \( z \in \mathcal{C} \) such that there is a closed horocycle containing \( z \) of length at most \( \frac{1}{y} \). We may think of \( B_y \) as

\[
B_y = \{ z \in \mathcal{C} : \Im(z) \geq y \}/(z \sim z + 1).
\]

We show below that Lemma 1.3 will follow from the argument of Lemma 1.2 once we can show the following

**Lemma 4.1** Let \( f \) be an \( L^2 \) eigenfunction on \( B_{y_0} \), with eigenvalue \( \lambda = 1/4 - s^2 \).

Then, for \( y_1 > y_0 \), we have

\[
\int_{B_{y_1}} f^2 \, dvol \leq \left( \frac{y_0}{y_1} \right)^{2s} \int_{B_{y_0}} f^2 \, dvol.
\]

To prove Lemma 1.3 from Lemma 4.1, we need to show that if \( ||f_\partial||_2 = 1 \), then

\[
\int_{S^C} f_C^2 \, dvol - \frac{1}{\text{vol}(S^C)} (\int_{S^C} f \, dvol)^2 \geq C(\varepsilon, r).
\]

We proceed to decompose the left-hand side exactly as in the proof of Lemma 1.2, reversing the roles of \( O \) and \( C \), and using the sets \( B_{y_0} \) and \( B_{y_1} \) in place of \( B(p_i, r_i + r_2) \) and \( B(p_i, r_1) \) respectively. We then use Lemma 4.1 in place of Lemma 3.1, and will also need the estimate

\[
\int_{g_\star(B_\delta)} f^2 \, dvol \leq \int_{B_\delta} f^2_\partial \, dvol.
\]

This follows immediately from the usual Schwarz Lemma that the map \( g \) is volume-decreasing.

The remainder of the proof is then identical with the proof of Lemma 1.4.

We now proceed with the proof of Lemma 4.1.

12
Let \( f \) be an eigenfunction of the Laplacian with eigenvalue \( \lambda = 1/4 - s^2 \), on \( \{ \Re(z) \geq y_0 \}/(z \sim z + 1) \). We may expand \( f \) into a Fourier series in \( x \) as
\[
f = a_0(y) + \sum_{n \neq 0} a_n(y) \cos(2\pi nx) + \sum_{n \neq 0} b_n(y) \sin(2\pi nx).
\]

Then, noting that
\[
\Delta(f) = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),
\]
the \( a_n \)'s satisfy the differential equation
\[
a''_n = (4\pi n^2 - \frac{\lambda}{y^2})a_n.
\]

From the orthogonality of \( \sin \) and \( \cos \), we have that
\[
\int_{B_\infty} f^2 \, d\text{vol} = \sum_n \int_{y_0}^\infty a^2_n(y)y^{-2} \, dy + \sum_{n \neq 0} \int_{y_0}^\infty b^2_n y^{-2} \, dy,
\]
and similarly for \( \int_B y f^2 \, d\text{vol} \).

We prove Lemma 4.1 by comparing \( \int_{y=y_0}^\infty a_n y^{-2} \, dy \) with \( \int_{y=y_1}^\infty a_n y^{-2} \, dy \), and similarly for the \( b_n \)'s.

We first observe that the equation
\[
a''_0(y) = -\frac{\lambda}{y^2}a_0
\]
has as solutions
\[
a_0 = c_1y^{1/2-s} + c_2y^{1/2+s}.
\]
In order for \( f \) to be \( L^2 \), \( c_2 \) must vanish.

We now investigate the solutions to the equations
\[
a''_n(y) = 4\pi n^2 - \frac{\lambda}{y^2}
\]
for \( n \neq 0 \).

Using that \( \frac{\lambda}{y^2} \to 0 \) as \( y \to \infty \), and using standard comparison arguments, one sees that the general solution is
\[
a_n(y) = c_1 F_n(y) + c_2 G_n(y),
\]
where \( F_n \) grows like \((\text{const})e^{-2\pi ny}\) and \( G_n \) grows like \((\text{const})e^{2\pi ny}\). In order for \( f \) to be \( L^2 \), \( c_2 \) must vanish. The identical argument also gives that \( b_n(y) = cF_n(y) \), with the same function \( F_n \).

We may normalize the functions \( F_n \) in some way, for instance by setting \( F_n(y_0) = 1 \). We remark that \( F_n(y) \) is positive for all \( y \), and that \( F_0(y) = y^{1/2} \).

The lemma then clearly follows from the following inequality:

\[
\frac{\int_{y=y_1}^{\infty} F_n^2 y^{-2} \, dy}{\int_{y=y_0}^{\infty} F_n^2 y^{-2} \, dy} \leq \frac{\int_{y=y_1}^{\infty} F_0^2 y^{-2} \, dy}{\int_{y=y_0}^{\infty} F_0^2 y^{-2} \, dy} = \left( \frac{y_0}{y_1} \right)^{2s}.
\]

To establish the inequality, we rewrite the above as

\[
\frac{\int_{y=y_1}^{\infty} F_n^2 y^{-2} \, dy}{\int_{y=y_0}^{\infty} F_n^2 y^{-2} \, dy} \leq \frac{\int_{y=y_0}^{\infty} F_0^2 y^{-2} \, dy}{\int_{y=y_0}^{\infty} F_0^2 y^{-2} \, dy}.
\]

This in turn follows once we establish that \( \frac{F_n}{F_0} \) is a decreasing function of \( y \). But this follows from the following standard comparison argument. Taking derivatives, it suffices to show that \( \frac{F_n'F_0 - F_nF_0'}{F_0^2} \) is negative. Since \( F_n \) decays like \( e^{-2\pi ny} \) for \( y \) large, the numerator is negative for some large values of \( y \). Taking the derivative of the numerator gives

\[
F_n''F_0 - F_n F_0'' = F_n F_0(4\pi n^2),
\]

which is positive. Therefore, \( F_n'F_0 - F_nF_0' \) is negative for all values, and the inequality is established.

This completes the proof of Lemma 1.3.

## 5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4.

Let \( \{R_j\} \) be a family of compact Riemann surfaces, with injectivity radius \( r_j \) tending to infinity as \( j \to \infty \).

For fixed \( r \), choose \( j \) sufficiently large so that \( r_j \geq r \), and denote by \( N_j(r) \) the maximum number of disjoint balls of radius \( r \) which may be placed in \( R_j \). The following elementary estimate is well-known.
Lemma 5.1 \( \frac{\text{vol}(R_j)}{\text{vol}(B(2r))} \leq N_j(r) \leq \frac{\text{vol}(R_j)}{\text{vol}(B(r))} \).

**Proof:** The fact that the balls of radius \( r \) are disjoint gives the right-hand inequality, while the maximality condition insures that the balls of radius \( 2r \) cover \( R_j \), giving the left-hand inequality.

Given \( \varepsilon \), we may choose \( r \) sufficiently large so that
\[
\min(1/4 - \varepsilon/2, C(\varepsilon/2, r) \lambda_1(R_j)) \geq \lambda_1(R_j) - \varepsilon/2.
\]

Lemma 1.3 now tells us that for any subset of the set of centers of balls in this maximal set, we may introduce punctures at these points to obtain finite-area surfaces with \( \lambda_1 \geq \min(1/4 - \varepsilon/2, \lambda_1(R_j) - \varepsilon) \). By choosing the cardinality of this subset, we can obtain a surface whose genus has any value between \( g_j = \) the genus of \( R_j \) and \( g_j + \frac{N_j(r)}{2} \). Choosing \( \delta = \frac{1}{2\text{vol}(B(r))} \) in condition (ii) of Theorem 1.4 allows us to conclude that the set of genera of surfaces constructed in this way covers all genera larger than some number \( N \).

This concludes the proof of Theorem 1.4.

6 The Platonic Surfaces

Let \( P(k) = \mathbb{H}^2 / \Gamma_k \), and let \( \overline{P}(k) \) denote the compact Riemann surface obtained from \( P(k) \) by applying Lemma 1.2 to all the cusps. In [PS], these surfaces \( \overline{P}(k) \) are called the Platonic surfaces.

We show here that the surfaces \( \overline{P}(k) \) satisfy the assumptions of Theorem 1.4. It then follows that Theorem 1.2 may be established from surfaces derived from \( \overline{P}(k) \) by opening cusps (Lemma 1.3) and closing handles (Lemma 1.1).

We begin by remarking that the lengths of all the cusps of \( P(k) \) are at least \( k \), since the cusps contain \( k \) copies of fundamental domains for \( \text{PSL}(2, \mathbb{Z}) \), each of which has width 1. It then follows from Lemma 1.2 that, given \( \varepsilon \), for all \( k \) sufficiently large we have \( \lambda_1(\overline{P}(k)) \geq C - \varepsilon \), where \( C = \inf_k \lambda_1(P(k)) \).

It further follows from Theorem 3.1 of [PS] that the injectivity radius of \( \overline{P}(k) \) tends to \( \infty \) as \( k \to \infty \). In effect, Theorem 3.1 of [PS] allows one to
deduce this from a calculation of lengths of geodesics on $P(k)$, which can be carried out by simple matrix calculations.

It remains to show that the surfaces $\overline{P}(k)$ satisfy the property that, for any $\delta$, there is an $N$ such that the union of the intervals $[\text{vol}(\overline{P}(k)), (1 + \delta)\text{vol}(\overline{P}(k))]$ cover $[N, \infty)$.

We begin with the following standard result.

**Lemma 6.1** The genus $g_k$ of the surface $\overline{P}(k)$ is given by

$$g_k = \frac{k^2(k - 6)}{24} \Pi_{p\mid k} \frac{p^2 - 1}{p^2}.$$  

**Proof:** Let $N_k$ denote the number of cusps of $P(k)$. For $k > 2$, $N_k$ is given by the formula

$$N_k = \frac{k^2}{2} \Pi_{p\mid k} \frac{p^2 - 1}{p^2},$$

as can be seen from the fact that $N_k$ is half the number of vectors $(a, b)$ in $\mathbb{Z}/k \times \mathbb{Z}/k$ such that $a$ and $b$ are relatively prime in $\mathbb{Z}/k$.

Then $\overline{P}(k)$ has a triangulation with one vertex for each cusp of $P(k)$, and with $k$ edges meeting at each point. It follows that the number of edges in this triangulation is $\frac{N_k}{2}$, and the number of triangles is $\frac{N_k}{3}$. The lemma follows immediately.

Let us denote by $\mathcal{P}$ the collection of prime numbers, and for given $k$, let $\mathcal{P}(k)$ denote the primes dividing $k$. We now observe that the product $\Pi_{p\in\mathcal{P}} \left(\frac{p^2 - 1}{p^2}\right)$ satisfies

$$\Pi_{p\in\mathcal{P}} \left(\frac{p^2 - 1}{p^2}\right) = \frac{6}{\pi^2},$$

and thus, in particular, is finite.

It follows that, for every $\eta > 0$, there is a finite collection of primes $\mathcal{C}(\eta)$, such that

$$\Pi_{p\in\mathcal{C}(\eta)} \left(\frac{p^2 - 1}{p^2}\right) \geq \frac{1}{1 + \eta}.$$  

We now claim:

**Lemma 6.2** For all $\eta > 0$ and for all $k, l$ such that the set $[\mathcal{P}(l) - \mathcal{P}(k)] \cup [\mathcal{P}(k) - \mathcal{P}(l)]$ is disjoint from $\mathcal{C}(\eta)$, we have

$$\left(\frac{1}{1 + \eta}\right) \frac{k^2(k - 6)}{l^2(l - 6)} \leq \frac{\text{vol}(\overline{P}(k))}{\text{vol}(\overline{P}(l))} \leq (1 + \eta) \frac{k^2(k - 6)}{l^2(l - 6)}.$$
Proof: Indeed,

$$\frac{\text{vol}(\mathcal{P}(k))}{\text{vol}(\mathcal{P}(l))} = \frac{k^2(k-6)}{l^2(l-6)} \prod_{p \in \mathcal{P}(k), p \notin \mathcal{P}(l)} \left(\frac{p^2-1}{p^2}\right).$$

By assumption, the numerator lies between $\frac{1}{1+\eta}$ and 1, and the same for the denominator. The lemma follows.

Given $\delta > 0$, we may write $1 + \delta$ as a product

$$1 + \delta = (1 + \eta)(1 + \mu)$$

for some $\eta$ and $\mu$, $\eta, \mu > 0$.

Now let $n_j = a + b \cdot j$ be an arithmetic progression such that the symmetric difference of the primes dividing any two terms of the sequence do not meet $\mathcal{C}(\eta)$. We can choose such a sequence by choosing $a$ arbitrarily and choosing $b$ to be the product of the primes contained in $\mathcal{C}(\eta)$.

We show that for $N$ sufficiently large, the union of the intervals

$$[\text{vol}(\mathcal{P}(n_j)), (1 + \delta)(\text{vol}(\mathcal{P}(n_j)))]$$

covers $[N, \infty)$. It suffices to show that

$$(1 + \delta)(\text{vol}(\mathcal{P}(n_j))) > \text{vol}(\mathcal{P}(n_{j+1})).$$

By the lemma, this will hold provided

$$\frac{(n_{j+1}^2)(n_{j+1} - 6)}{(n_j^2)(n_j - 6)} < (1 + \mu),$$

or

$$(n_{j+1})^2(n_{j+1} - 6) < (1 + \mu)(n_j)^2(n_j - 6).$$

We now substitute our expressions for $n_j$ and $n_{j+1}$ into both sides, and subtract from both sides the terms $(a + b \cdot j)^2(a + b \cdot j)$. What remains on the left is a quadratic polynomial in $j$, and on the right a cubic polynomial in $j$ with positive leading coefficient $ab^3$. It then follows that we have the desired inequality for $j$ sufficiently large.

This concludes the proof of the lemma.
REFERENCES


