Some Questions of a General Spectral Theory of Banach Modules

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Supported by Federal Ministry of Science and Research, Austria
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Abstract

When solving the problem on multiplication of generalized functions physicists and mathematicians are considering a generalized function as an equivalence class of sequences of $C^\infty$-functions. The equivalence relation is defined by approximation. It follows that these classes form an algebra which is called the algebra of mnemofunctions. The general method of constructing such algebras was obtained by A. B. Antonevich and Ya. V. Radyno in [2]. It has been found that this method could be applied not only to generalized functions but also to objects of different character, for example, to unbounded operators. In this paper we are attempting to do so. We expect that the proposed ideas and approaches will permit to solve nonlinear problems and in particular, certain problems of Quantum field theory.

Key words and phrases. generalized functions (mnemofunctions), generalized operators, spectrum, resolvent set, non-Archimedean normed space.

AMS subject classification. 46F99

1 Generalized normed spaces and pseudo-norms

Let $X$ be a normed complex space. For a given $m \in \mathbb{Z}$ let us denote by $X^{[m]}$ the space of sequences $x = (x_k)$ (where $x_k \in X$) such that

$$\sup_{k \geq 1} \left( ||x_k|| k^{-m} \right) < +\infty,$$

or alternatively $X^{[m]} = l_\infty(X; k^{-m}).$

The following is an easy exercise:

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Proposition 1.1. The set $X^{[m]}$ (with coordinate-wise addition and scalar multiplication) is a normed space. By the norm $\|x\|_m$ of a element $X^{[m]}$ we mean the quantity

$$\|x\| = \sup_{k \geq 1} \left(\|x_k\|^{k^{-m}}\right).$$

Furthermore, if $X$ is a Banach space then $X^{[m]}$ a Banach space.

Let us introduce the following spaces:

$$X^{[\infty]} = \bigcup_{m \in \mathbb{Z}} X^{[m]}, \quad X^{[-\infty]} = \bigcap_{m \in \mathbb{Z}} X^{[m]}, \quad X_* = X^{[+\infty]} / X^{[-\infty]}.$$ 

**Proposition 1.2.** If $X$ is a normed algebra (in particular, $X$ is a Banach algebra) then $X^{[+\infty]}$ and $X^{[-\infty]}$ are algebras where multiplication is defined coordinate-wise, and $X^{[-\infty]}$ is an ideal in $X^{[+\infty]}$. Hence $X_*$ is an algebra. It should be observed that $X^{[m]}$ (except $X^{[0]}$) is not an algebra.

If $X = \mathbb{C}$ then we obtain the ring $\mathbb{C}_*$ which we shall call the ring of generalized numbers (or generalized numbers).

Let $x = (x_k)$ be in $X^{[+\infty]}$. The symbol $[x]$ will denote the class of equivalent sequences in $X_*$ which contain the element $x$.

The field $\mathbb{C}$ is embedded in $\mathbb{C}_*$ by the natural injection:

$$\mathbb{C} \ni \lambda \mapsto [(\lambda, \lambda, \ldots)] \in \mathbb{C}_*.$$

Let $[\lambda] = [(\lambda_k)]$ be in $\mathbb{C}_*$ and $[x] = [(x_k)] \in X_*$. We define a “generalized scalar multiplication” by the equality

$$[\lambda] \cdot [x] = [(\lambda_k x_k)].$$

Then $X_*$ will be called $\mathbb{C}_*$-module and $\mathbb{C}_*$-algebra, if $X$ is an algebra.

**Remark 1.1.** The element $\tilde{x} = (x_1, x_2, \ldots, x_{n_0}, 0, 0, \ldots)$ belongs to $X^{[-\infty]}$. Thus, for $\tilde{x}$ and $\tilde{y}$ in $X_*$, we have that $\tilde{x} = \tilde{y}$, if the inequality $\|x_n - y_n\| \leq c \cdot n^{-m}$ is true for every $n \geq n_0$, $n \in \mathbb{N}$.

**Proposition 1.3.** The ring $\mathbb{C}_*$ is not a field.

Since the ring of generalized complex numbers $\mathbb{C}_*$ is not a field we shall describe which elements from $\mathbb{C}_*$ are invertible and which elements are not. The following proposition explains this.

**Proposition 1.4.** Let $X$ be a Banach algebra. If an element $\tilde{x} = [(x_k)] \in X_*$ is invertible in $X_*$ then for any representative $(x_k)$ there exist an integer $n_0 > 0$ and two constants $c > 0$ and $m$ such that $x_k$ is invertible in $X$ for $k \geq n_0$ and

$$\|x_k^{-1}\| \leq c k^m, \quad k \geq n_0.$$

Conversely, if there is some representative of $\tilde{x}$ for which the above property holds, the element $\tilde{x} = [(x_k)] \in X_*$ is invertible in $X_*$. 


Corollary 1.1. If an element \( \tilde{\lambda} \in C_* \) is invertible in \( C_* \) then for any representative \((\lambda_k)\) there exist an integer \( n_0 \) and two constants \( c > 0 \) and \( m \) such that \( \lambda_k \) is invertible in \( C \) for all \( k \geq n_0 \) and
\[
|\lambda_k^{-1}| \leq c k^m, \text{ for all } k \geq n_0.
\]
Conversely, if there is some representative \((\lambda_k)\) for which the above property holds, the element \( \tilde{\lambda} \in C_* \) is invertible in \( C_* \).

Definition 1.1. We call an element \( \tilde{\lambda} = [(\lambda_k)] \in C_* \) non-negative and write \( \tilde{\lambda} \geq 0 \), if there is a representative \((\lambda_k')\) of \( \lambda \) such that \( \lambda_k' \geq 0 \) for all \( k \). We call an element \( \tilde{\lambda} \in C_* \) positive and write \( \tilde{\lambda} > 0 \), if \( \tilde{\lambda} \geq 0 \) and \( \tilde{\lambda} \) is invertible in \( C_* \).

If \( \tilde{\lambda} > 0 \), then for any of its representatives \((\lambda_k)\) there exists an integer \( n_0 > 0 \) and two constants \( c > 0 \) and \( m \) such that \( \lambda_k \) is invertible in \( C \) for \( k \geq n_0 \) and
\[
\lambda_k^{-1} > \frac{c}{k^m}, \text{ for any } k \geq n_0.
\]

It is possible that \( \tilde{\lambda} \geq 0 \) but \( \tilde{\lambda} \) not positive. For example take as a representative the sequence
\[
1, e^{-1}, 1, e^{-2}, 1, \ldots, e^{-k}, 1, \ldots
\]

The positive generalized numbers will play an important role in our calculations. Positive numbers in \( C_* \) play the same role as the positive real numbers of \( C \).

Since \( X_* \) is a \( C_* \)-module, the functions on \( X_* \) are considered \( C_* \)-valued.

We define a function by the formula
\[
\| \cdot \|_* : X_* \ni \tilde{x} = [(x_k)] \mapsto \|\tilde{x}\|_* = [(\|x_k\|)] \in C_*.
\]

We will use this function for defining a topology on \( X_* \). We, thus give some more details about \( \| \cdot \|_* \).

Proposition 1.5. The function \( \| \cdot \|_* \) has the following properties:
For any \( \tilde{x}, \tilde{y} \in X_* \), \( \tilde{\lambda} \in C_* \) we have
1) \( \|\tilde{x}\|_* \geq 0 \),
2) If \( \|\tilde{x}\|_* = 0 \), then \( \tilde{x} = 0 \),
3) \( \|\tilde{x} + \tilde{y}\|_* \leq \|\tilde{x}\|_* + \|\tilde{y}\|_* \),
4) \( \|\lambda \tilde{x}\|_* = |\lambda|_* \|\tilde{x}\|_* \),
5) \( \|\tilde{x}\|_* \leq \|\tilde{x}\| \|\tilde{y}\|_* \).

Since these properties are completely similar to the properties of a norm function, we call \( \| \cdot \|_* \) pseudo-norm.

Proposition 1.6. Let \( X \) be a normed algebra and \( \tilde{x} \in X_* \) an invertible element. Then \( \|\tilde{x}\|_* \) is a positive element of \( C_* \).
2 The Topology on $X_\ast$ Generated by the Pseudo-norm

We define a topology on $X_\ast$ which we believe is natural for our objects.

**Definition 2.1.** The open ball with center $\bar{x}_0 \in X_\ast$ and radius $0 < \bar{r}$ in $(X_\ast, \| \cdot \|_\ast)$ is the following set

$$B(\bar{x}_0, \bar{r}) = \{ \bar{x} \in X_\ast : \| \bar{x} - \bar{x}_0 \|_\ast < \bar{r}, \quad \bar{r} > 0 \}.$$ 

The set $U \subset X_\ast$ is open if for any $\bar{x}_0 \in U$ there exists $\bar{r} > 0$ such that $B(\bar{x}_0, \bar{r}) \subset U$.

**Proposition 2.1.** This family of open sets defines a Hausdorff topology on $X_\ast$.

In this topology $X_\ast$ is topological group with respect to addition. If $X$ is a normed algebra, then $X_\ast$ is a topological ring (for more details see [3], [4], [12], [8]). As soon as, $X_\ast$ becomes a topological group we can consider the concept of Cauchy sequence.

**Definition 2.2.** A sequence $(\bar{x}_n) \in X_\ast$ is called Cauchy sequence if, for each $\bar{\varepsilon} > 0$, there is an $N$ such that $\| \bar{x}_n - \bar{x}_m \|_\ast < \bar{\varepsilon}$ for all $n, m \geq N$.

**Proposition 2.2.** If $X$ is a Banach space, then $X_\ast$ is a complete group and the topological ring $C_\ast$ is complete as well.

**Definition 2.3.** Let $\bar{a} \in X_\ast$ and $\bar{\varepsilon}$ be the unit of $X_\ast$. The resolvent set of $\bar{a}$ is

$$\rho(\bar{a}) = \{ \bar{\lambda} \in C_\ast : \bar{\lambda} \bar{\varepsilon} - \bar{a} \text{ is invertible in } X_\ast \}.$$ 

The spectrum of $\bar{a}$ is the set $\sigma(\bar{a}) = C_\ast \setminus \rho(\bar{a})$.

In the next section we classify the points of a resolvent set. Now, we consider an analogue to one important theorem of Banach algebras [5], [11], [9].

**Theorem 2.1.** Let $X$ be a Banach algebra, $\bar{x} \in X_\ast$, and $\| \bar{x} \|_\ast < 1$. Then the element $\tilde{y} = \bar{\varepsilon} - \bar{x}$ is invertible in $X_\ast$. Furthermore, the set of invertible elements, $X_\ast^{-1}$, of the algebra $X_\ast$ is open in $X_\ast$ and the mapping $\bar{x} \mapsto \bar{x}^{-1}$ is continuous.

**Corollary 2.1.** Let $\bar{a} \in X_\ast$. The resolvent set $\rho(\bar{a})$, is open in $C_\ast$.

3 Non-Archimedean topology on $X_\ast$

In this section we re-introduce the topology on the space $X_\ast$ in a different way [5], [10]. Let us denote

$$X_m = X^{[m]} / X^{-\infty}.$$
We have
\[ 0 = X_{-\infty} \subset \ldots \subset X_m \subset X_{m+1} \subset \ldots \subset X_* . \]

**Definition 3.1.** The order of an element \( x \in X_* \) is
\[ \nu(x) = \min \{ m \in \mathbb{Z} : x \in X_m \} . \]

The order \( \nu(x) \) has the following properties:
1) \( \nu(x) = \nu(-x) \)
2) \( \nu(x + y) \leq \max \{ \nu(x), \nu(y) \} \)
3) \( \nu(xy) \leq \nu(x) + \nu(y) \)
4) \( \nu(x) = -\infty \) is equivalent to \( x = 0 \)
5) \( \nu(\|x\|_*) = \nu(x) \)
6) \( \nu(\lambda x) \leq \nu(\lambda) + \nu(x) \)

where \( x, y \in X_* \) and \( \lambda \in C_* \).

**Remark 3.1.** There are \( x, y \) such that \( \nu(xy) \neq \nu(x) + \nu(y) \).

For example, take \( x = (1, 1/2, 3, 1/4, 5, \ldots) \) and \( y = x^{-1} = (1, 2, 1/3, 4, 1/5, \ldots) \). It is clear that \( \nu(x) = \nu(x^{-1}) = 1 \), but \( xx^{-1} = 1 \) and
\[ \nu(xx^{-1}) = 0 < 2 = \nu(x) + \nu(x^{-1}) . \]

The following definition classifies the points of the resolvent set. An example is given in the next section.

**Definition 3.2.** Let \( \tilde{x} \in X_* \) be invertible and \( \tilde{y} = [\|\tilde{y}\|] \) be its inverse. The order \( \nu(\tilde{y}) \) of the element \( \tilde{y} \) is called the order of reversibility of \( \tilde{x} \).

We denote by \( \rho^m(\tilde{a}) \) the set of all \( \tilde{\lambda} \in C_* \) such that the element \( \tilde{\lambda} \tilde{e} - \tilde{a} \) has order of reversibility \( m \). If \( \tilde{\lambda} \in \rho^m(\tilde{a}) \), we say that \( \tilde{\lambda} \) has order of regularity \( m \). It is easy to see that
\[ \rho(\tilde{a}) = \bigcup_{m \in \mathbb{Z}} \rho^m(\tilde{a}) . \]

**Definition 3.3.** Let \( p > 1 \) and \( x \in X_* \). The function \( \| \cdot \|_p : X_* \to R_+ \) is defined by
\[ \|x\|_p = p^{\nu(x)} . \]

**Proposition 3.1.** The function \( \| \cdot \|_p \) is a norm on \( X_* \) which defines a metric on \( X_* \) so that \( X_* \) is a non-Archimedean normed group (NN-group) (or NN-ring, if \( X \) is a normed algebra). Norms corresponding to different \( p \)'s are equivalent.

**Proposition 3.2.** The topology on \( X_* \) determined by \( \| \cdot \|_p \) is the same as the one determined by the pseudo-norm \( \| \cdot \|_* \). If \( X \) is a Banach space then \( X_* \) is a non-Archimedean Banach \( C_* \)-module.
Theorem 3.1. Let \( X \) be a Banach algebra. Then \( X_* \) is a non-Archimedean Banach \( C_* \)-algebra and the spectrum of any element \( a \in X_* \) is nonempty and closed in \( C_* \).

By \( C_* \)-algebra we mean a \( C_* \)-module with an operation of multiplication.

4 Examples of Generalized Operators

In this section we give some examples that illustrate some aspects of the previous theory. First we present an algebra \( X_* \) which, in some sense, contains unbounded operators.

Example 4.1. Consider the sequence \( \{A_n\} \) of bounded operators on the space of continuous functions \( C[0,1] \) defined by

\[
A_n x(t) = nx(t) - n^2 \int_0^t x(s)e^{n(t-s)} ds.
\]

The norms of such operators \( \|A_n\| \) grow not faster than some power of \( n \). Indeed,

\[
A_n = nI - nT_n, \quad \text{where} \quad T_n x(t) = n \int_0^t x(s)e^{n(t-s)} ds.
\]

Hence

\[
\|A_n\| = \|nI - nT_n\| \leq n + n \|T_n\|
\]

and

\[
\|T_n\| = \max_{t \in [0,1]} \left| n \int_0^t e^{n(t-s)} ds \right| = \max_{t \in [0,1]} (1 - e^{-nt}) = 1.
\]

Therefore

\[
\|A_n\| \leq 2n.
\]

Thus, the sequence of the operators \( A_n \) defines some element \( \tilde{A} \) of \( X_* \), if \( X \) is the algebra of the bounded operators on \( C[0,1] \). Let us take the operator \( A = d/dt \) with domain

\[
D(A) = \{ x(t) \in C[0,1] : x'(t) \in C[0,1], \quad x(0) = 0 \}.
\]

We shall show that a sequence \( \{A_n\} \) strongly converges to the operator \( A \).

Let us consider a function \( x \in C[0,1] \) such that \( A_n x \) converges uniformly to some function \( y \) in \( C[0,1] \). We claim that \( x(0) = 0 \). Indeed, if \( x(0) \neq 0 \) then

\[
A_n x(0) = nx(0) \to \pm \infty.
\]

Furthermore

\[
A_n x(0) = y(0) = 0.
\]

Next consider the sequence of functions

\[
w_n(t) = \int_0^t A_n x(s) ds.
\]

Then

\[
w_n(t) = \int_0^t n x(s) ds - n^2 \int_0^t \left[ \int_0^s x(z)e^{n(z-s)} dz \right] ds =
\]
The first term converges to zero uniformly. Indeed, $e^{-nt}$ converges to 0 as $t \to \infty$.

Hence
\[
\int_0^t x(z)e^{-nt}dz = T_n x(t).
\]

**Lemma 4.1.** If $x(t) \in C[0,1]$ and $x(0) = 0$ then the sequence
\[
T_n x(t) = n \int_0^t x(s)e^{-n(t-s)}ds
\]
converges uniformly to $x(t)$.

**Proof.**
\[
T_n x(t) = \int_0^{t-1/\sqrt{n}} ne^{-n(t-s)}x(s)ds + \int_{t-1/\sqrt{n}}^t ne^{-n(t-s)}x(s)ds.
\]
The first term converges to zero uniformly. Indeed, $t - s \geq 1/\sqrt{n}$ when $s \in [0,t-1/\sqrt{n}]$.

Hence $ne^{-n(t-s)} \leq ne^{-\sqrt{n}}$.

Next we show that the sequence of functions
\[
\int_{t-1/\sqrt{n}}^t ne^{-n(t-s)}x(s)ds
\]
converges to $x(t)$ in $C[0,1]$. We apply the mean value theorem, namely
\[
\int_{t-1/\sqrt{n}}^t ne^{-n(t-s)}x(s)ds = x(\tau)e^{-nt}\int_{t-1/\sqrt{n}}^t ne^{-ns}ds = x(\tau)(1 - e^{-\sqrt{n}}),
\]
where the point $\tau$ belongs to $[t-1/\sqrt{n}, t]$. Thus, we can write
\[
\left| \int_{t-1/\sqrt{n}}^t ne^{-n(t-s)}x(s)ds - x(t) \right| = \left| x(\tau)(1 - e^{-\sqrt{n}}) - x(t) \right| \leq
\]
\[
\leq \max_{\tau \in [t-1/\sqrt{n}, t]} \left| x(\tau) - x(t) \right| e^{-\sqrt{n}} \leq
\]
\[
\leq \max_{\tau \in [t-1/\sqrt{n}, t]} \left| x(\tau) - x(t) \right| + \max_{\tau \in [t-1/\sqrt{n}, t]} \left| x(\tau) \right| e^{-\sqrt{n}}.
\]
The function $x(t)$ is continuous on $[0,1]$. Hence it is uniformly continuous on $[0,1]$. Thus, from the last estimations follows that
\[
\max_{t \in [0,1]} \left| \int_{t-1/\sqrt{n}}^t ne^{-n(t-s)}x(s)ds - x(t) \right| \leq
\]
\[
\leq \max_{t \in [0,1]} \max_{\tau \in [t-1/\sqrt{n}, t]} \left| x(\tau) - x(t) \right| + \max_{t \in [0,1]} \left| x(t) \right| e^{-\sqrt{n}}.
\]
From this estimate the lemma follows easily.

According to Lemma 4.1 the sequence of functions
\[
w_n(t) = T_n x(t)
\]
converges uniformly to $x$. Since the sequence $w_n(t) = A_n x(t)$ converges uniformly to $y$, $x(t)$ is continuously differentiable function and $x'(t) = y(t)$. Thus, for the convergence of the sequence $A_n x$ in $C[0,1]$ it is necessary that $x$ is continuously differentiable and $x(0) = 0$.

Finally, we show that these two conditions are sufficient for the convergence of $A_n x$. Indeed, for such an $x$ it is possible to write $A_n$ as

$$A_n x(t) = n \int_0^t x'(s)e^{n(s-t)}ds = T_n x'(t).$$

According to Lemma 4.1, the sequence $T_n x'(t)$ converges to $x'(t)$.

So, it is possible to consider the element $\tilde{A} \in X*$ (we call such elements generalized operators) which is represented by the sequence $A_n$. We can assume that $\tilde{A}$ corresponds to the unbounded operator $A$.

The spectrum of the operator $A$ is empty in $C$. The existence of the inverse operator $(A - \lambda I)^{-1}$ is equivalent to the solvability of the initial value problem

$$\begin{cases} u'(t) - \lambda u(t) = f(t); \\ u(0) = 0 \end{cases}$$

and this problem is solvable for all $\lambda \in C$.

However, according to Theorem 3.1 the spectrum of the generalized operator $\tilde{A}$ is not empty in $C_*$. For example, the generalized complex number generated by the sequence $\lambda_n = n$ belongs to it, since the spectrum of the operator $A_n$ is $\{n\}$.

Let us now investigate the spectrum of $\tilde{A}$. We need to estimate the norms of the operator

$$(A_n - \mu_n I)^{-1} = \frac{1}{\mu_n} \left(G + \frac{1}{n} I\right) \left(\frac{1}{\mu_n} - \frac{1}{n} - G\right)^{-1}$$

where

$$Gx(t) = \int_0^t x(s)ds.$$ 

We now compute the inverse operator to $(\lambda_n I - G)$, where

$$\lambda_n = \frac{1}{\mu_n} - \frac{1}{n}.$$ 

First we need to solve the equation

$$\lambda_n x(t) - \int_0^t x(s)ds = y(t).$$

Formal differentiation gives

$$\begin{cases} x'(t) - \lambda_n^{-1} x(t) = \lambda_n^{-1} y'(t); \\ x(0) = \lambda_n^{-1} y(0). \end{cases}$$

Thus

$$x(t) = \frac{1}{\lambda_n} y(0)e^{\frac{1}{\lambda_n} t} + \frac{1}{\lambda_n^2} \int_0^t e^{-\frac{1}{\lambda_n}(z-t)} y(z)dz =$$

$$= \frac{1}{\lambda_n} y(t) + \frac{1}{\lambda_n^2} \int_0^t e^{-\frac{1}{\lambda_n}(z-t)} y(z)dz = (\lambda_n I - G)^{-1} y(t).$$
Then
\[
\begin{align*}
(A_n - \mu_n I)^{-1} y(t) &= \\
&= \frac{1}{\lambda_n \mu_n} \left[ \int_0^t y(s) ds + \frac{1}{n} \int_0^t \left( \int_0^s e^{-(z-t)/\lambda_n} y(z) dz \right) ds + \frac{1}{n} y(t) + \frac{1}{n \lambda_n} \int_0^t e^{-(z-t)/\lambda_n} y(z) dz \right] = \\
&= \frac{1}{\lambda_n \mu_n} \left[ \int_0^t y(s) ds + \frac{1}{n} \int_0^t \int_s^t e^{z/\lambda_n} e^{-z/\lambda_n} ds dz + \frac{1}{n} y(t) + \frac{1}{n \lambda_n} \int_0^t e^{-(z-t)/\lambda_n} y(z) dz \right] = \\
&= \frac{1}{\lambda_n \mu_n} \left[ \int_0^t e^{-(z-t)/\lambda_n} y(z) dz + \frac{1}{n} y(t) + \frac{1}{n \lambda_n} \int_0^t e^{-(z-t)/\lambda_n} y(z) dz \right] = \\
&= \frac{1}{\lambda_n \mu_n} \left[ (1 + \frac{1}{n \lambda_n}) \int_0^t e^{-(z-t)/\lambda_n} y(z) dz + \frac{1}{n} y(t) \right].
\end{align*}
\]

Thus,
\[
(A_n - \mu_n I)^{-1} y(t) = \frac{1}{\lambda_n \mu_n} \left( 1 + \frac{1}{n \lambda_n} \right) \int_0^t e^{-(z-t)/\lambda_n} y(z) dz + \frac{1}{n \mu_n \lambda_n} y(t),
\]

where
\[
\lambda_n = \frac{1}{\mu_n} - \frac{1}{n}.
\]

Finally,
\[
(A_n - \mu_n I)^{-1} y(t) = \frac{n^2}{(n - \mu_n)^2} \int_0^t e^{-\frac{n \mu_n}{n - \mu_n} (z-t)} y(z) dz + \frac{1}{n - \mu_n} y(t).
\]

Now we are ready to estimate the norm of the operator \((A_n - \mu_n I)^{-1}\)
\[
\left| \frac{1}{n - \mu_n} - \frac{n^2}{n - \mu_n^2} \right| \leq \left\| (A_n - \mu_n I)^{-1} \right\| \leq \frac{1}{n - \mu_n} + \frac{n^2}{(n - \mu_n)^2} \left\| D_{\mu_n} \right\|
\]

where
\[
D_{\mu_n} y(t) = \int_0^t e^{-\frac{n \mu_n}{n - \mu_n} (z-t)} y(z) dz,
\]
\[
\left\| D_{\mu_n} \right\| = \max_{t \in [0,1]} \left| e^{-\frac{n \mu_n}{n - \mu_n} (z-t)} \right| dz = \frac{e^{Re \left( \frac{n \mu_n}{n - \mu_n} \right)} - 1}{Re \left( \frac{n \mu_n}{n - \mu_n} \right)}.
\]

Hence
\[
\left\| (A_n - \mu_n I)^{-1} \right\| \leq \left| \frac{1}{n - \mu_n} \right| + \frac{n^2}{(n - \mu_n)^2} \frac{e^{Re \left( \frac{n \mu_n}{n - \mu_n} \right)} - 1}{Re \left( \frac{n \mu_n}{n - \mu_n} \right)}.
\]

From the last estimation for the norm of the operators \((A_n - \mu_n I)^{-1}\) it follows that the points in the resolvent set \(\rho \left( \tilde{A} \right)\) of the generalized operator \(\tilde{A}\) will be those generalized complex numbers \(\mu \in \mathbb{C}_*\) for which there exists a representative \((\mu_n)\) and some constants \(c_1\) and \(c_2\) such that
\[
Re \left( \frac{n \mu_n}{n - \mu_n} \right) \leq c_1 + c_2 \ln n. \tag{4.1}
\]

For example, it is not hard to see that the generalized complex number \(\mu\) having the representative \(\mu_n = \ln n\) will be in \(\rho \left( \tilde{A} \right)\). Generalized complex numbers which have real part that behaves like \(n^\varepsilon, 0 < \varepsilon \leq 1\), will belong to the spectrum of the operator \(\tilde{A}\). The sequence of numbers \(n - 1/n\) will represent a generalized number in the spectrum of \(\tilde{A}\), and the sequence \(n + 1/n\) will represent a point in \(\rho \left( \tilde{A} \right)\).
We now exhibit a set of points \( \mu_n \) in \( \mathbb{C} \), that satisfy the condition (4.1). Let

\[
a_n = c_1 + c_2 \ln n
\]

\[
z = u + iv = \frac{wn}{n + w}.
\]

(4.2)

It is clear that the straight line \( w = a_n + iy \) transforms to a circle by the map \( z(w) \). After a standard calculation we find that the equation of this circle is

\[
\left( u - \left( n - \frac{n^2}{2(n + a)} \right) \right)^2 + v^2 = \left( \frac{n^2}{2(n + a)} \right)^2
\]

and \( \mu_n \) must lie inside this circle (including the boundary).

Thus, we have shown that the generalized operator \( \tilde{A} \) generated by a sequence of bounded operators

\[
A_n x(t) = nx(t) - n^2 \int_0^t x(s)e^{n(t-s)} ds
\]

associates with the unbounded operator \( A \) with empty spectrum. We summarize our findings in the following theorem:

**Theorem 4.1.** The spectrum \( \tilde{A} \) is contained in set of the generalized complex numbers \( \mu \) having the representatives \( \mu_n \) which have one of the following properties:

(i) The sequence \( n - \mu_n \) represents a generalized complex number which is irreversible in \( \mathbb{C}_* \);

(ii) The sequence \( \mu_n \) satisfies the inequality

\[
\left| \mu_n - \left( n - \frac{n^2}{2(n + c_1 + c_2 \ln n)} \right) \right| \leq \frac{n^2}{2(n + c_1 + c_2 \ln n)}
\]

where \( c_1 \) and \( c_2 \) are constants.

**Example 4.2.** Here we will consider an operator \( A \) on space of two-sided sequences \( l_2(\mathbb{Z}) \). This operator assigns to a sequence \( x(k) \) a sequence with terms \( a(k)x(k + 1) \), where \( a(k) = 1 \) for \( k \neq 0 \) and \( a(0) = 0 \). The operator \( A \) is bounded in \( l_2(\mathbb{Z}) \) and its spectrum is \( \sigma(A) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) [6].

Let us consider the sequence of operators \( B_n \) defined by

\[
B_n : x(k) \mapsto b_n(k)x(k + 1),
\]

where

\[
b_n(k) = \begin{cases} 1, & \text{if } k \neq 0; \\ 1/n, & \text{if } k = 0. \end{cases}
\]

Let \( X \) be the algebra of bounded operators in \( l_2(\mathbb{Z}) \). The sequence \( (B_n) \) and the sequence \( (A, A, \ldots) \) represent some generalized operators in \( X_* \). Notice that \( \|A - B_n\| = 1/n \) and \( \sigma(B_n) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) for any \( n \). In other words, the operators \( A \) and \( B_n \) are “close”, but their spectra “differ” from each other.

Now, if \( |\lambda| > 1 \) then

\[
(\lambda I - B_n)^{-1} x(k) = \frac{1}{\lambda} \left[ x(k) + \frac{b_n(k)}{\lambda} x(k + 1) + \frac{b_n(k)b_n(k + 1)}{\lambda^2} x(k + 2) + \ldots \right]
\]
This series is absolutely convergent and furthermore the following estimate holds

\[ \| (\lambda I - B_n)^{-1} \| \leq \frac{1}{|\lambda| - 1}. \]

If \(|\lambda| = 1\) then \((\lambda I - B_n)\) is not invertible. Finally, if \(|\lambda| < 1\) then

\[ \| (\lambda I - B_n)^{-1} \| = \| (B_n (\lambda B_n^{-1} - I))^{-1} \| = \| (I + \lambda B_n^{-1} + \lambda^2 B_n^{-2} + \cdots) B_n^{-1} \| \leq (1 + |\lambda| n + |\lambda|^2 n + \cdots) n. \]

Thus

\[ \| (\lambda I - B_n)^{-1} \| \leq \left( \frac{|\lambda|}{1 - |\lambda|} + \frac{1}{n} \right) n^2. \]

Let \(\bar{B}\) be the generalized operator represented by \((B_n)_{n=1}^{\infty}\). The points of the resolvent set of \(\bar{B}\) inside and outside of the unit circle have different orders of regularity. Inside of the unit circle the order of regularity is greater than zero. For the generalized number \(\lambda\) with representative \((0)_{n=1}^{\infty}\) the order is \(m = 1\); for \(\lambda = [(1/2)_{n=1}^{\infty}]\), we have \(m = 2\); for \(\lambda = [(3)_{n=1}^{\infty}]\), \(m = 0\); for \(\lambda = [(1 + 1/n^3)_{n=1}^{\infty}]\), \(m = 3\); for \(\lambda = [(1 - 1/n^3)_{n=1}^{\infty}]\), \(m = 5\).

**Remark 4.1.** We see that the points of \(\rho(\bar{B})\) have various orders of regularity and characterize this generalized operator more accurately than just the sequence of spectra of the operators \(B_n\).

**Example 4.3.** Let us consider an example which demonstrates a discontinuity property of the spectral radius \([6],[1]\).

Consider the following sequences of functions in \(C(S^1)\) (namely continuous, 1-periodic functions):

\[
a_n(x) = \begin{cases} 
nx, & x \in [0, 1/n]; \\
1, & x \in [1/n, 1 - 1/n]; \\
n(1 - x), & x \in [1 - 1/n, 1]; 
\end{cases}
\]

\[
b_n(x) = \begin{cases} 
0, & x \in [0, 1/n^2]; \\
(n^2 x - 1)/(n - 1), & x \in [1/n^2, 1/n]; \\
1, & x \in [1/n, 1 - 1/n]; \\
n(1 - x), & x \in [1 - 1/n, 1]. 
\end{cases}
\]

Next we consider two sequences of bounded operators defined on \(C(S^1)\), depending on a parameter \(h\).

\[
A_{h,n} u(x) = a_n(x) u(x + h),
\]

\[
B_{h,n} u(x) = b_n(x) u(x + h).
\]

For \(n \in \mathbb{N}\) and \(h \in \mathbb{R} \setminus \mathbb{Q}\) the spectra of these operators are

\[
\sigma(A_{h,n}) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \exp \left[ \int_0^1 \ln a_n(x) \, dx \right] = e^{-2/n} \right\}
\]

and

\[
\sigma(B_{h,n}) = \{0\}.
\]
\[ \| A_{h,n} - B_{h,n} \| = 1/n. \]

The operators \( A_{h,n} \) and \( B_{h,n} \) are “close”, however their spectra “strongly” differ from each other.

If we take \( X \) to be the algebra of bounded operators on \( C(S^1) \), then the sequences \( (A_{h,n})_{n=1}^\infty \) and \( (B_{h,n})_{n=1}^\infty \) represent some generalized operators \( \tilde{A}_h \) and \( \tilde{B}_h \) in \( X_* \). Let us calculate \( \rho(\tilde{B}_h) \).

The generalized operator \( \tilde{B}_h \) is the most interesting. It is possible to write down \( (\lambda I - \tilde{B}_{h,n})^{-1} \) explicitly. For \( \lambda \neq 0 \)

\[
(\lambda I - \tilde{B}_{h,n})^{-1} u(x) = \frac{1}{\lambda} \left( u(x) + \frac{b_n(x)}{\lambda} u(x + h) + \frac{b_n(x) b_n(x + h)}{\lambda^2} u(x + 2h) + \cdots \right. \\
\left. \cdots + \frac{b_n(x) b_n(x + h) \cdots b_n(x + (N - 1)h)}{\lambda^N} u(x + Nh) \right),
\]

where \( N \) is the smallest integer for which

\[ b_n(x) b_n(x + h) \cdots b_n(x + Nh) \equiv 0. \]

The number \( N \) depends on the irrational number \( h \). More precisely, \( N \) will depend on how well \( h \) approximated by rational numbers \([7]\).

If \( \lambda \neq 0 \), the norm of the operator \( (\lambda I - \tilde{B}_{h,n})^{-1} \) is estimated as follows:

\[
\| (\lambda I - \tilde{B}_{h,n})^{-1} u(x) \| = \\
= \max_{x \in [0,1]} \frac{1}{\lambda} \left| u(x) + \frac{b_n(x)}{\lambda} u(x + h) + \cdots + \frac{b_n(x) b_n(x + h) \cdots b_n(x + (N - 1)h)}{\lambda^N} u(x + Nh) \right| \\
\leq \| u \| \max_{x \in [0,1]} \frac{1}{|\lambda|} \left( 1 + \frac{b_n(x)}{|\lambda|} + \cdots + \frac{b_n(x) b_n(x + h) \cdots b_n(x + (N - 1)h)}{|\lambda|^N} \right).
\]

Using suitable functions \( u(x) \), we can conclude

\[
\| (\lambda I - \tilde{B}_{h,n})^{-1} \| = |\lambda|^{-1} \max_{x \in [0,1]} \left( 1 + \frac{b_n(x)}{|\lambda|} + \cdots + \frac{b_n(x) b_n(x + h) \cdots b_n(x + (N - 1)h)}{|\lambda|^N} \right).
\]

Next, we compute estimate for a given sequence of norms. Denote

\[ I_0 = \{ x \in \mathbb{R} : b_n(x) = 0 \}. \]

The function

\[
M_\lambda(x) = \frac{1}{|\lambda|} \left( 1 + \frac{b_n(x)}{|\lambda|} + \frac{b_n(x) b_n(x + 2h)}{|\lambda|^2} + \cdots + \frac{b_n(x) b_n(x + h) \cdots b_n(x + (N - 1)h)}{|\lambda|^N} \right)
\]

will reach its maximum value at some point of the set

\[ I_0 + Nh, \]

since, in this set there exist a point \( x_0 \) such that

\[ 0 < b_n(x_0) b_n(x_0 + h) \cdots b_n(x_0 + kh) \leq 1, \quad k = 0, 1, 2 \ldots N - 1. \]
For an upper estimate of the norm

\[ \| (\lambda I - B_{h,n})^{-1} \| = \max_{x \in [0,1]} M_\lambda(x) \]

we replace all products \( b_n(x)b_n(x+h) \cdots b_n(x+kh) \) in \( M_\lambda(x) \) by 1 and get

\[ \max_{x \in [0,1]} M_\lambda(x) \leq \frac{1/|\lambda|^{N+1} - 1}{1 - |\lambda|}, \quad \lambda \neq 0, \ |\lambda| \neq 1 \]

(if \( |\lambda| = 1 \), then \( \max_{x \in [0,1]} M_\lambda(x) \leq N + 1 \)).

Further, we denote

\[ I_1 = \{ x \in \mathbb{R} : 0 \leq b_n(x) < 1 \}. \]

We have \( I_0 \subset I_1 \).

Let \( m \) be the smallest integer for which

\[ \mathbb{R} = \bigcup_{k=0,1,\ldots,m} I_1 + kh, \quad m \leq N. \]

Since the function \( b_n(x) \) is equal to 1 outside \( I_1 \), there is a point \( x_1 \) in \( I_1 + mh \) such that

\[ b_n(x_1)b_n(x_1+h) \cdots b_n(x_1+kh) = 1, \quad \text{for all } k = 0,1,2 \ldots m - 1. \]

To get a lower estimate of the norm

\[ \| (\lambda I - B_{h,n})^{-1} \| = \max_{x \in [0,1]} M_\lambda(x) \]

we replace \( b_n(x)b_n(x+h) \cdots b_n(x+kh) \) in \( M_\lambda(x) \) by

\[ \begin{cases} 1, & \text{if } k = 0,1,\ldots,m - 1; \\
0, & \text{if } k = m,\ldots,N - 1. \end{cases} \]

From this we get the following inequalities

\[ \max_{x \in [0,1]} M_\lambda(x) \geq \frac{1/|\lambda|^{m+1} - 1}{1 - |\lambda|}, \quad \lambda \neq 0, \ |\lambda| \neq 1. \]

\[ \max_{x \in [0,1]} M_\lambda(x) \geq m + 1, \quad |\lambda| = 1. \]

Therefore, we have for all complex numbers \( \lambda \neq 0, \ |\lambda| \neq 1 \), the following double inequality

\[ \frac{1/|\lambda|^{m+1} - 1}{1 - |\lambda|} \leq \| (\lambda I - B_{h,n})^{-1} \| \leq \frac{1/|\lambda|^{N+1} - 1}{1 - |\lambda|}, \quad (4.3) \]

whereas, if \( |\lambda| = 1 \) we have

\[ m + 1 \leq \| (\lambda I - B_{h,n})^{-1} \| \leq N + 1, \quad (4.4) \]

To estimate numbers \( N \) and \( m \) we should reformulate the problem. We take the circle \( S \) with circumference 1 and an interval \( I_0 \) with length \( r_0 \) such that the function \( b_n(x) \) vanishes in \( I_0 \).

It is not hard to see that

\[ S = \bigcup_{k=0}^{N} \{ I_0 + kh \}. \]
All these reasonings also apply to the number \( m \). In this case we should consider an interval \( I_1 \) of \( S \) which has a length \( r_1 \) and \( b_n(x) = 1 \) outside of \( I_1 \).

Next we need to invoke some results from Number Theory. Let \( x_0, x_1, \ldots, x_N \) be a sequence of real numbers. We denote by \( A([\alpha, \beta]; N+1) \) the number of elements in the set \( \{x_0, x_1, \ldots, x_N\} \cap [\alpha, \beta] \).

The number

\[
D_{N+1}(x_0, \ldots, x_N) = \sup_{0 \leq \alpha < \beta < 1} \left| \frac{A([\alpha, \beta]; N+1)}{N+1} - (\beta - \alpha) \right|
\]

is the deviation of the given sequence. We have

\[
|A([\alpha, \beta]; N+1) - (N+1)(\beta - \alpha)| \leq (N+1)D_{N+1}
\]

\[
A([\alpha, \beta]; N+1) \leq (N+1)(\beta - \alpha) + (N+1)D_{N+1}
\]

In our particular case

\[
x_k = hk, \; k = 0, 1, \ldots, N.
\]

To cover the whole circle \( S \) by using \( N \) steps and the interval \( I_0 \) for \( N \), it is necessary that

\[
1 \leq A([\alpha, \beta]; N+1), \; \text{for } \beta - \alpha = r_0/2.
\]

To estimate \( A([\alpha, \beta]; N+1) \) we have to apply the theorem which will be formulated below. First we give a definition.

**Definition 4.1.** Let \( \nu \) be a positive real number or infinity. An irrational number \( h \) is of type \( \nu \), if \( \nu \) is a supremum of those \( \gamma \) for which

\[
\lim_{q \to \infty} q\gamma \langle qh \rangle = 0, \quad q \in \mathbb{N},
\]

where

\[
\langle t \rangle = \min_{n \in \mathbb{Z}} |t - n|.
\]

**Theorem 4.2.** [7] If a number \( h \) is of type \( \nu \), \( \nu < +\infty \), then the deviation \( D_{N+1} \) of the sequence \( (hk)_{k=0}^{N} \) has the following property

\[
D_{N+1}(x_0, \ldots, x_N) = O\left(N^{-1/\nu} + \epsilon\right), \quad \text{for any } \epsilon > 0.
\]

In the special case of this theorem where \( h \) is of type \( \nu = 1 \), we have \((N+1)D_{N+1} = O(N^\epsilon)\), for any \( \epsilon > 0 \). An important class of these numbers is the algebraic irrationals. It is known [7] that for algebraic irrationals we can use the following estimate

\[
(N+1)D_{N+1} = O\left(\ln^2 N\right).
\]

Thus, if \( h \) is an algebraic irrational, the number \( N+1 \) is estimated from a condition of the type

\[
1 \leq (N+1) \cdot \frac{r_0}{2} - C \ln^2(N+1),
\]

where \( C \) is some constant. Note that this yields a quite precise estimate of \( N+1 \). However, here we are mainly interesting in the dependence of \( N+1 \) on \( r_0 \).
From the above estimates it follows that there are constants \( K_0 \) and \( K_1 \) such that 
\[ m + 1 \geq K_0 n \quad \text{and} \quad N + 1 \leq K_1 n \]
Using these bounds and inequality (4.3), we can get

\[
\frac{1/|\lambda|^{K_0 n} - 1}{1 - |\lambda|} \leq \| (\lambda I - B_{h,n})^{-1} \| \leq \frac{1/|\lambda|^{K_1 n} - 1}{1 - |\lambda|},
\]

for any \( \lambda \in \mathbb{C}, \ 0 < |\lambda| < 1 \).

Thus, according to the Proposition 2.1 the spectrum of the generalized operator \( \tilde{B}_h = (B_{h,n})_{n=1}^{\infty} \) contains, at least, the generalized complex numbers which have representatives \((\lambda_n)_{n=1}^{\infty}\) such that \( |\lambda_n| = \mu \), where \( \mu \) is real number and \( 0 < \mu < 1 \). We, therefore have the following theorem:

**Theorem 4.3.** Let \( h \) be an algebraic irrational and \( \tilde{B}_h \) be the generalized operator represented by \( B_{h,n} \). All generalized complex numbers \( \lambda \in \mathbb{C}_* \) which have representatives \( \lambda_n \) such that, for any fixed \( \varepsilon > 0 \),

\[
|\lambda_n| \leq 1 - \varepsilon, \quad \text{for all } n,
\]

are in the generalized spectrum of \( \tilde{B}_h \).

**Acknowledgment.** The author wants to thank the Erwin Schrödinger International Institute for its support and Professor Vassilis Papanicolaou (Wichita State University, USA) for his help with translating this paper to English.

**References**


