On Poincaré Transformations and the Modular Group of the Algebra Associated with a Wedge

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On Poincaré transformations and the modular group of the algebra associated with a wedge

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Abstract:

It will be shown that in a theory of local observables the modular group of the algebra of any wedge domain acts local if the theory is Poincaré covariant, fulfills wedge duality and moreover the observables fulfill some reality condition with respect to the representation of the Lorentz group.

If in addition to this representation of the Poincaré group the theory happens to be covariant with respect to a second representation of the Poincaré group then both representations differ only by a representation of the Lorentz group, which is a gauge transformation, i.e. it maps every local algebra onto itself and commutes with the above standard representation of the Poincaré group.

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I. Introduction:

In this note we deal with a theory of local observables fulfilling the conditions of isotony, locality translational covariance and the Reeh–Schlieder property. We also assume that we are dealing with a vacuum representation of this theory. In case where the local algebras are generated by a finite number of Poincaré covariant Wightman fields Bisognano and Wichmann [BW75,76] have shown that the modular group of (\mathcal{M}(\mathcal{W}), \Omega), the algebra of the wedge, coincides with the Lorentz boosts which map the wedge onto itself.

But not every quantum field theory is of this kind. There are examples of Oksak and Todorov [OT68] of infinite component fields which are still Poincaré covariant but where Josts TCP-theorem [Jo57] fails. By a result of Borchers [Bch92] one knows that the modular group of the wedge acts local in the characteristic two–plain of the wedge. However,

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this result does not say anything about the behaviour in the directions perpendicular to the characteristic two-plane. Indeed, examples of Yngvason [Yng94] show that the action in the perpendicular direction does not need to be local if the theory fails to be covariant under the Lorentz group.

If the modular group of every wedge acts local in all directions then it is possible to construct a representation of the whole Poincaré group and to show that one is dealing with a covariant theory in the vacuum sector. (For this it is necessary to assume that the algebras of the double cones coincide with the intersection of the algebras of all wedges containing this double cone. Since this does not change the algebras of the wedges this is only a technical device, and the results are independent of this procedure.) The first construction of the Poincaré group is due to Brunetti, Guido and Longo [BGL95] and Guido and Longo [GL95]. In the first paper they only constructed a representation of the covering group of the Poincaré group, in the second paper it was shown that one is dealing indeed with a representation of the Poincaré group. A direct construction of the Poincaré group was given by the author [Bch76]. By the Poincaré group we mean always the proper orthochronous part of the Poincaré group. If one assumes that the action of the modular groups of every wedge is local then the theory is not only Poincaré covariant but it fulfills also wedge duality. Having these results in mind it is natural to ask which theories fulfill the principle of local action for the algebras of the wedges. It is clear that one can show local action only if one compares the modular groups with given Poincaré transformations. Therefore we will deal only with Poincaré covariant local field theories. From examples we know that not every Poincaré covariant theory will have the property that the modular group corresponds to the associated group of Lorentz boosts. Therefore, it is not sufficient to assume wedge duality. It turns out that some reality condition between the local observables and the representation of the Lorentz group must be fulfilled. If the wedge duality and the reality condition are fulfilled then we call it the standard representation.

In section II we will recapitulate the results about the wedge duality and introduce the new reality condition. With these assumptions we will show that the modular groups of the wedges act local. In section III we investigate the case where the theory of local observables is covariant under two different representations of the Poincaré group. One of these representations is assumed to be the standard representation.

In Jost’s proof of the CPT-theorem one has to require that with every charged field also its charge-conjugated field is present. Our reality condition is of similar nature. Although we know from the results of section II and the investigation of Guido and Longo [GL95] that the wedge duality and reality condition imply the CPT-theorem I did not see any direct way to derive from these conditions the CPT-theorem. The reason for this failure seems to be that there is no replacement of the Bargmann-Hall-Wightman theorem [HW57].

II. Local action of the modular groups associated with wedges

In this section we first will fix the notations and then recapitulate the results from [Bch95]. Then we formulate the reality condition and prove the identity of the modular
A wedge is characterized in two ways.

1) Assume \( x_0, x_1 \) are two given vectors with \( x_0^2 = 1, x_1^2 = -1 \) and \( (x_0, x_1) = 0 \) then we define:
\[
W(x_0, x_1) = \{ y \in \mathbb{R}^d; |(x_0, y)| < (x_1, y) \}.
\]

2) For \( \ell_1, \ell_2 \in \partial V^+ \) both non-zero we define:
\[
W(\ell_1, \ell_2) = \{ \lambda \ell_1 + \mu \ell_2 + \ell^\perp; \lambda > 0, \mu < 0, (\ell^\perp, \ell_1) = 0, i = 1, 2 \}.
\]

These wedges coincide if the two-planes spanned by \( (x_0, x_1) \) and by \( (\ell_1, \ell_2) \) coincide and if the scalar product \( (\ell_1, x_1) \) is negative. Therefore we call this two-plane the characteristic two plane of the wedge.

Let us restrict to wedges which are characterized by the \((0,1)\)-plane. These wedges will be denoted by \( W_r \) and \( W_L \) respectively. For the Lorentz boosts mapping this wedge onto itself we get
\[
\Lambda(t) = \begin{pmatrix}
\cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\
-\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Let \( K_0 \) be a double cone in the characteristic two-plane of the wedge with center at the origin and \( K \) be the cylindrical set with the same cylindrical direction as that of \( W \) such that the intersection with the characteristic two-plane is \( K_0 \). Let \( A \in \mathcal{M}(K) \) then we denote by \( A(K, x) \) the translated operator \( T(x)AT(-x) \) where \( T(x) \) denotes the given representation of the translations. With this notation we introduce the following set:

Let \( \mathcal{A}_r \) be the set of operators \( A(K, 0) \) with the properties:

(i) the operator \( A(K, x) \) with \( K + x \subset W_r \) is such that \( U(\Lambda(t))A(K, x)\Omega \) has a bounded analytic continuation into the strip \( S(-\frac{1}{2}, 0) \) with continuous boundary-values and

(ii) \( A^*(K, x) \) with \( K + x \subset W_l \) is such that \( U(\Lambda(t))A^*(K, x)\Omega \) has a bounded analytic continuation into the strip \( S(0, \frac{1}{2}) \) with continuous boundary-values.

The set \( \mathcal{A}_r \) will be denoted by \( \mathcal{A}_l \). It has the corresponding property with respect to \( \mathcal{M}(W_l) \).

Recall the main result about wedge duality:

We consider a Lorentz covariant theory of local observables in the vacuum-sector. This theory fulfills wedge-duality exactly if
\[
\{ A(K, x); A(K, 0) \in \mathcal{A}_r, \ K + x \subset W_r \}
\]
\[
\{ A(K, x); A(K, 0) \in \mathcal{A}_l, \ K + x \subset W_l \}
\]
are \(*\)-strong dense in \( \mathcal{M}(W_r) \) and \( \mathcal{M}(W_l) \) respectively.

In the above result it has not been mentioned what one knows about the structure of the elements \( U(\Lambda(\frac{1}{2}))A(K, x)\Omega \) where \( K + x \) is in the right wedge. This question has been answered in Them. 3.6 of [Bch95].
(i) For every $A(K,0) \in \mathcal{A}_r$ and every $x$ with $K + x \subset W_r$ there exists an element $\hat{A}(K,0) \in \mathcal{A}_l$ such that the following relation holds
\[ U(\Lambda(-i/2))A(K,x)\Omega = \hat{A}(K,P_W x)\Omega, \]
with $P_W$ the reflection in the characteristic two–plane which does not change the perpendicular directions.

(ii) For every $y$ with $K + y \in W_l$ and $A(K,0) \in \mathcal{A}_l$ there exists an element $\tilde{A}(K,0) \in \mathcal{A}_r$ fulfilling the relation
\[ U(\Lambda(i/2))A(K,y)\Omega = \tilde{A}(K,P_W y)\Omega. \]

Having collected the notations and results from [Bch95] we can formulate the

**Reality condition:**

We say a Poincaré covariant theory of local observables in the vacuum sector with the property of wedge duality fulfills the reality condition if:

(i) Every $A(K,0) \in \mathcal{A}_r \cap \mathcal{A}_l$ and every $x$ such that $K + x \subset W_r$ fulfills the relation
\[ \widehat{A^*}(K,P_W x) = \{ \hat{A}(K,P_W x) \}^*. \]

(ii) $\Omega$ is cyclic for the set
\[ \{ A(K,x); A(K,0) \in \mathcal{A}_r \cap \mathcal{A}_l, \text{ and } K + x \subset W_r \}. \]

With this notation we obtain:

**II.1 Theorem:**

In a representation of a Poincaré covariant theory of local observables in the vacuum sector the modular group associated with the algebra of any wedge coincides with the corresponding Lorentz boosts iff the theory fulfills wedge duality and the above reality condition with respect to the Lorentz transformations.

**Proof:** If we know that $U(\Lambda(t))$ and $\Delta_W^t$ coincide then one has wedge duality ([Bch96] Prop. 3.2). Moreover the reality condition is fulfilled because for every $A(K,x) \in \mathcal{M}(W_r)$ one has
\[ \widehat{A^*}(K,x)\Omega = U(\Lambda(-i/2))A^*\Omega = \Delta_W^{1/2}A^*\Omega = JA^*\Omega, \]
and
\[ \hat{A}(K,x)\Omega = JA^*J\Omega = \{ JA^* \}^*\Omega. \]

Hence the reality condition is fulfilled.
Next assume wedge duality and the reality condition. Let \( A(K,0) \in \mathcal{A}_r \cap \mathcal{A}_l \) and \( A(K,x) \in \mathcal{M}(W_r) \). Take an element \( B \in \mathcal{M}(W_l) \) and look at the matrix elements

\[
F^+(s,t) = (\Omega, B \Delta_r^{is} U(\Lambda(t))A(K,x)\Omega) \\
F^\perp(s,t) = (\Omega, A(K,x)U(\Lambda(-t))\Delta_r^{\perp is} B\Omega)
\]

Bringing \( B \) and \( \Delta_r^{is} \) to the left hand side we see that \( F^+(s,t) \) can be analytically continued into the tube domain \( (s,t) \in S(0, \frac{i}{2}) \times S(-\frac{1}{2},0) \). Correspondingly \( F^\perp(s,t) \) has an analytic continuation into the domain \( (s,t) \in S(-\frac{1}{2},0) \times S(0, \frac{1}{2}) \). Next we look at the coincidence domains. Since \( A(K,x) \in \mathcal{M}(W_r) \) and \( B \in \mathcal{M}(W_l) \) we have by wedge duality \( F^+(s,t) = F^\perp(s,t) \) for all \( (s,t) \in \mathbb{R}^2 \). Next we look at \( F^+(s + \frac{i}{2}, t - \frac{i}{2}) \). The modular theory yields \( \Omega B \Delta_r^{is} \Delta_r^{\perp is} = \Omega J_r B^* J_r \Delta_r^{is} \). By the above result about the analytic continuation of \( U(\Lambda(t))A(K,x)\Omega \) we know that there exists an element \( \hat{A}(K, P_W x) \in \mathcal{M}(W_l) \) with \( U(\Lambda(\frac{1}{2}))A(K,x)\Omega = \hat{A}(K, P_W x)\Omega \). Hence we find:

\[
F^+(s + \frac{i}{2}, t - \frac{i}{2}) = (\Omega, J_r B^* J_r U(\Lambda(t))\hat{A}(K, P_W x)\Omega).
\]

Next we want to compute \( F^\perp(s - \frac{i}{2}, t + \frac{i}{2}) \). We start with

\[
F^\perp(s,t) = (U(\Lambda(t))A^* (K,x)\Omega, \Delta_r^{\perp is} B\Omega).
\]

From this we obtain:

\[
F^\perp(s - \frac{i}{2}, t + \frac{i}{2}) = (U(\Lambda(t - \frac{i}{2}))A^* (K,x)\Omega, \Delta_r^{\perp is} B\Omega) \\
= (U(\Lambda(t))\hat{A}^*(K, P_W x)\Omega, \Delta_r^{\perp is} J_r B^* J_r \Omega).
\]

Because of the reality condition we find:

\[
= (\Omega, \hat{A}(K, P_W x)U(\Lambda(-t))\Delta_r^{\perp is} J_r B^* J_r \Omega).
\]

By the wedge duality we obtain \( J_r B^* J_r \in \mathcal{M}(W_r) \). Since \( \hat{A}(K, P_W x) \) belongs to \( \mathcal{M}(W_l) \) we obtain

\[
F^+(s + \frac{i}{2}, t - \frac{i}{2}) = F^\perp(s - \frac{i}{2}, t + \frac{i}{2}).
\]

By both coincidences and the edge of the wedge theorem we obtain a bounded periodic function \( F(s,t) = F(s-i,t+i) \) since bounded entire functions are constant we find

\[
F(s,-s) = \text{const} = F(0,0), \\
(\Omega, B \Delta^{is} U(\Lambda(-s))A(K,x)\Omega) = (\Omega, BA(K,x)\Omega).
\]

Since \( \mathcal{M}(W_l)\Omega \) and \( \{A(K,x)\Omega\} \) are dense in \( \mathcal{H} \), where \( A(K,0) \) fulfils the reality condition, we obtain \( \Delta^{is} U(\Lambda(-s)) = 1 \).
III. The case of two different representations of the Poincaré group

Assume we are dealing with a Wightman field theory containing an infinite number of copies of the same field, then one can change the standard representation of the Poincaré group by adding a representation of the Lorentz group which acts in the index space. From this example we see that in a Poincaré covariant theory of local observables the representation of this group must not be unique. In this section we want to investigate the case where the theory is covariant under two different representations of the Poincaré group. We will assume that one of the representations is the standard representation described in the last section. This implies that we assume wedge duality also in this section. We show:

III.1 Theorem:
Assume we are dealing with a local quantum field theory in the vacuum sector, which is covariant under two different vacuum representations of the Poincaré group. Let \( U_0(\Lambda, a) \) be the standard representation and \( U_1(\Lambda, a) \) the second representation. Then exists a local gauge transformation of the Lorentz group \( G(\Lambda) \) with

\[
U_1(\Lambda, a) = U_0(\Lambda, a)G(\Lambda).
\]

Moreover \( G(\Lambda) \) commutes with \( U_0(\Lambda', a) \) for all \( a, \Lambda, \Lambda' \). In addition \( G(\Lambda) \) is a gauge transformation, i.e. it maps every local algebra onto itself.

Proof: We define:

\[
G(\Lambda, a) = U_1(\Lambda, a)U_0^{-1}(\Lambda, a).
\]

Since \( U_0(\Lambda, a) \) and \( U_1(\Lambda, a) \) map the algebra of the domain \( O \) onto the algebra of the domain \( O_\Lambda + a \) we obtain

\[
G(\Lambda, a)\mathcal{M}(O)G^{-1}(\Lambda, a) = \mathcal{M}(O).
\]

This is true also for every wedge \( W + b \). Hence we get

\[
[G(\Lambda, a), \Delta^W_{W+b}] = 0
\]

for every \((\Lambda, a)\) in the Poincaré group and every translated wedge. Since the modular groups of all wedges generate the representation \( U_0(\Lambda, a) \) we obtain:

\[
[G(\Lambda, a), U_0(\Lambda', a')] = 0.
\]

Since \( U_0(\Lambda, a) \) and \( U_1(\Lambda, a) \) are group representations we obtain by the commutativity that also \( G(\Lambda, a) \) is a representation of the Poincaré group. It remains to show that \( G(\Lambda, a) \) does not depend on the translations \( a \).

Let \( W(\ell, \ell') \) be a wedge then both \( U_0(1, \lambda \ell) \) and \( U_1(1, \lambda \ell) \) fulfil the condition of half-sided translations. Hence by the theorem of the author [Bch92] we obtain

\[
\text{Ad} \Delta^W_{W} U_i(1, \lambda \ell) = U_i(1, e^{i2\pi i} \lambda \ell), \quad i = 0, 1.
\]
This implies
\[ \text{Ad} \Delta_W^i G(1, \lambda \ell) = G(1, e^{i2\pi \ell} \lambda \ell). \]
On the other hand \( \Delta_W^i \) commutes with \( G(1, \lambda \ell) \) because \( G \) defines an automorphism of \( \mathcal{M}(W) \). This is only possible if \( G(\Lambda, a) \) does not depend on \( a \). □

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References


