On Joint Recurrence

Klaus Schmidt


Supported by Federal Ministry of Science and Research, Austria
Available via http://www esi.ac.at
ON JOINT RECURRENCE

KLAUS SCHMIDT

Abstract. Let $T$ be a measure-preserving and ergodic automorphism of a probability space $(X, \mathcal{S}, \mu)$. By modifying an argument in [3] we obtain a sufficient condition for recurrence of the $d$-dimensional stationary random walk defined by a Borel map $f : X \rightarrow \mathbb{R}^d$, $d \geq 1$, in terms of the asymptotic distributions of the maps $(f + fT + \cdots + fT^{n-1})/n^{1/d}$, $n \geq 1$. If $d = 2$, and if $f : X \rightarrow \mathbb{R}^2$ satisfies the central limit theorem with respect to $T$ (i.e., if the sequence $(f + fT + \cdots + fT^{n-1})/\sqrt{n}$ converges in distribution to a Gaussian law on $\mathbb{R}^2$), then our condition implies that the two-dimensional random walk defined by $f$ is recurrent.

1. Recurrence of $d$-dimensional stationary random walks

Let $T$ be a measure preserving and ergodic automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, $d \geq 1$, and let $f = (f_1, \ldots, f_d) : X \rightarrow \mathbb{R}^d$ be a Borel map. For every $n \in \mathbb{Z}$ and $x \in X$ we set

$$f(n, x) = \begin{cases} \sum_{k=0}^{n-1} f(T^kx) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -f(-n, T^nx) & \text{if } n < 0. \end{cases} \quad (1.1)$$

The resulting map $f : \mathbb{Z} \times X \rightarrow \mathbb{Z}^d$ satisfies that

$$f(m, T^nx) + f(n, x) = f(m + n, x) \quad (1.2)$$

for every $m, n \in \mathbb{Z}$ and $\mu$-a.e. $x \in X$. If $\| \cdot \|$ denotes the maximum norm on $\mathbb{R}^d$ then the map $f : X \rightarrow \mathbb{R}^d$ is recurrent (or the individual components $f_1, \ldots, f_d$ of $f$ are jointly recurrent) if

$$\liminf_{n \rightarrow \infty} \| f(n, x) \| = 0 \quad (1.3)$$

for $\mu$-a.e. $x \in X$. If $f$ is not recurrent it is called transient (for terminology and background we refer to [3]).
Proposition 1.1 ([3]). Let $f: X \mapsto \mathbb{R}^d$ be a Borel map. The following conditions are equivalent.

1. $f$ is recurrent;
2. $\mu(\{x \in X : \lim \inf_{|n| \to \infty} \|f(n, x)\| < \infty\}) > 0$;
3. For every $B \in \mathcal{S}$ with $\mu(B) > 0$ and every $\varepsilon > 0$,
   \[ \mu(B \cap T^{-m}B \cap \{x \in X : \|f(m, x)\| < \varepsilon\}) > 0 \]
   for some nonzero $m \in \mathbb{Z}$.

For every $k \geq 1$ we define probability measures $\sigma_k^{(d)}$ and $\tau_k^{(d)}$ on $\mathbb{R}^d$ by setting

\[
\sigma_k^{(d)}(A) = \mu(\{x \in X : f(k, x)/k^{1/d} \in A\}),
\]

\[
\tau_k^{(d)}(A) = \frac{1}{k} \sum_{i=1}^{k} \sigma_i^{(d)}(A) \tag{1.4}
\]

for every Borel set $A \subset \mathbb{R}^d$, where $1_A$ is the indicator function of $A$. In [2] and [3] it was shown that the recurrence of $f$ can be deduced from certain properties of these probability measures. For example, if $d = 1$ and

\[
\lim_{k \to \infty} \sigma_k^{(1)} = \delta_0
\]

in the vague topology, where

\[
\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{otherwise}, \end{cases}
\]

then $f$ is recurrent by [2] or [3]. In [3] it was also shown that a map $f: X \mapsto \mathbb{R}$ is recurrent whenever

\[
\liminf_{\eta \to 0} \liminf_{k \to \infty} \sigma_k^{(1)}([-\eta, \eta])/2\eta > 0.
\]

The purpose of this paper is to prove the following extension of this result to higher dimensions.

Theorem 1.2. Let $T$ be a measure preserving and ergodic automorphism of a probability space $(X, \mathcal{S}, \mu)$, $d \geq 1$, $f: X \mapsto \mathbb{R}^d$ a Borel map, and define the probability measures $\tau_k^{(d)}$, $k \geq 1$, on $\mathbb{R}^d$ by (1.4). We denote by $\lambda$ the Lebesgue measure on $\mathbb{R}^d$ and set, for every $\eta > 0$, $B(\eta) = \{v \in \mathbb{R}^d : \|v\| < \eta\}$. If $f$ is transient then

\[
\sup_{\eta > 0} \limsup_{k \to \infty} \tau_k^{(d)}(B(\eta))/\lambda(B(\eta)) < \infty \tag{1.5}
\]

and

\[
\lim_{\eta \to 0} \liminf_{k \to \infty} \tau_k^{(d)}(B(\eta))/\lambda(B(\eta)) = 0. \tag{1.6}
\]

The interesting cases are, of course, $d = 1$ and $d = 2$. The case $d = 1$ was discussed in [3]; in order to explain the significance of Theorem 1.2 for $d = 2$ we say that a Borel map $\phi: X \mapsto \mathbb{R}^d$ satisfies the central limit theorem with respect to $T$ if the distributions of the functions $f(n, \cdot)/\sqrt{n}$, $n \geq 1$, converge to a (possibly degenerate) Gaussian probability measure on $\mathbb{R}^d$ as $n \to \infty$ (for the existence of such functions see [1]).
Corollary 1.3. Let $T$ be a measure preserving and ergodic automorphism of a probability space $(X, \mathcal{S}, \mu)$, and let $f: X \rightarrow \mathbb{R}^2$ be a Borel map satisfying the central limit theorem with respect to $T$. Then $f$ is recurrent.

Proof of Corollary 1.3. If $\lim_{n \rightarrow \infty} f(n, x)/\sqrt{n}$ is $\mu$-a.e. to a constant, then this constant has to be zero by (1.2). This shows that, if $f$ satisfies the central limit theorem with respect to $T$, then there exists a positive constant $c$ such that $\liminf_{k \rightarrow \infty} \tau_k^{(d)}(B(\eta)) > c\eta^2$ for all sufficiently large $k$ and all sufficiently small $\eta > 0$. According to (1.6) this means that $f$ is recurrent. $\square$

Note that Theorem 1.2 and Corollary 1.3 make no assumptions concerning the integrability of $f$.

2. The proof of Theorem 1.2

The proof of Theorem 1.2 differs from that of Theorem 3.6 in [3] only by avoiding the use of the total order of $\mathbb{R}$ (which is, of course, not available if $d > 1$).

Let $T$ be a measure preserving and ergodic automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, $d \geq 1$, and let $f: X \rightarrow \mathbb{R}^d$ be a transient Borel map. For the definition of the probability measures $\sigma^{(d)}_k$, $\tau^{(d)}_k$ on $\mathbb{R}^d$ we refer to (1.4).

Proposition 1.1 implies that there exist a Borel set $C \subset X$ with $\mu(C) > 0$ and an $\varepsilon > 0$ with

$$\mu(C \cap T^{-k}C \cap \{x \in X : \|f(k, x)\| < \varepsilon\}) = 0$$

whenever $0 \neq k \in \mathbb{Z}$. By decreasing $C$, if necessary, we may assume that $\mu(C) = 1/L$ for some $L \geq 1$.

Lemma 2.1. For every $\eta > 0$ and $N \geq 1$,

$$\limsup_{k \rightarrow \infty} \tau_k^{(d)}(B(\eta)) \leq 2^d L \varepsilon^{-d} \lambda(B(\eta)), \quad \limsup_{k \rightarrow \infty} \sum_{n=0}^N 2^n \tau_n^{(d)}(B(2^{-n/d}\eta)) \leq 2^{d+1} d L \varepsilon^{-d} \lambda(B(\eta)).$$

Proof. We modify $T$ on a null-set, if necessary, and assume without loss in generality that $T^n x \neq x$ for every $x \in X$ and $0 \neq n \in \mathbb{Z}$ and hence that (1.2) holds for every $m, n \in \mathbb{Z}$ and $x \in X$. Denote by

$$R_T = \{(T^n x, x) : x \in X, n \in \mathbb{Z}\} \subset X \times X$$

the orbit equivalence relation of $T$ and define a Borel map $f: R_T \rightarrow \mathbb{R}^d$ by setting

$$f(T^n x, x) = f(n, x)$$

for every $(T^n x, x) \in R_T$. Then (1.2) implies that

$$f(x, x') + f(x', x'') = f(x, x'')$$

whenever $(x, x'), (x, x'') \in R_T$.

We denote by $[T]$ the full group of $T$, i.e. the group of all measure preserving automorphisms $V$ of $(X, \mathcal{S}, \mu)$ with $V x \in \{T^n x : n \in \mathbb{Z}\}$ for every $x \in X$. Since $T$ is ergodic we can find an element $S \in [T]$ and a $T$-invariant $\mu$-null set $N \subset \mathcal{S}$ with the following properties:
(i) if \( C' = C \setminus N \) then the sets \( S^k C' \) are disjoint for \( k = 0, \ldots, L - 1 \) and 
and \( S^L C' = C' \).

(b) \( N = X \setminus \bigcup_{k=0}^{L-1} C'_k \),

(c) for every \( x \in C' \) the sets \( \{ j \geq 1 : S^j x \in C' \} \) and \( \{ j \geq 1 : S^{-j} x \in C' \} \)
are infinite and \( SL_x = T^{m_{C'}(x)} \) with 
\[ m_{C'}(x) = \min \{ j \geq 1 : T^j x \in C' \}. \]

Note that the restriction of \( S^L \) to \( C' \) is the automorphism of \( C' \) induced by \( T \), and that \( \{ S^n x : n \in \mathbb{Z} \} = \{ T^n x : n \in \mathbb{Z} \} \) for every \( x \in X \setminus N \).

Define a Borel map \( b: X \to \mathbb{R}^d \) by setting, for every \( x \in C' \), \( b(S^k x) = \mathbf{f}(S^k x, S^k x) \) for \( k = 1, \ldots, L \), and by putting \( b(x) = 0 \) for \( x \in N \). Then the map \( g(x) = \mathbf{f}(S x, x) - b(S x) + b(x) \) satisfies that
\[ g(x) = \begin{cases} 
\mathbf{f}(S^L x, x) = f(m_{C'}(x), x) & \text{if } x \in C', \\
0 & \text{otherwise.}
\end{cases} \]

Furthermore, if \( f'(x) = f(x) - b(T x) + b(x) \), and if \( f'(n, \cdot): X \to \mathbb{R}^d \) and \( \mathbf{f}': R_T \to \mathbb{R}^d \) are defined by (1.1) and (2.3) with \( f' \) replacing \( f \), then
\[ f'(n, x) = f(n, x) - b(T^n x) + b(x), \]
\[ \mathbf{f}'(x, x') = \mathbf{f}(x, x') - b(x) + b(x') \tag{2.5} \]
for every \( x \in X \setminus N, n \in \mathbb{Z} \) and \( x' \in \{ T^k x : k \in \mathbb{Z} \} = \{ S^k x : k \in \mathbb{Z} \} \).

We denote by \( \sigma_k', \tau_k' \) the probability measures defined by (1.4) with \( f' \) replacing \( f \) and obtain as in Lemma 3.4 in [3] that
\[ \lim_{|k| \to \infty} \inf \left( \sigma_k'(B(\eta + \eta')) - \sigma_k'(B(\eta)) \right) \geq 0, \]
\[ \lim_{|k| \to \infty} \inf \left( \tau_k'(B(\eta + \eta')) - \sigma_k'(B(\eta)) \right) \geq 0 \tag{2.6} \]
for all \( \eta, \eta' > 0 \). In particular, the inequalities (2.2) will be satisfied if
\[ \limsup_{k \to \infty} \tau_k'(B(\eta)) \leq L 2^d \eta^d \varepsilon^{-d}, \]
\[ \limsup_{k \to \infty} \sum_{n=0}^{N} 2^n \tau_{2^n k}(B(2^{-n/d} \eta)) \leq d L^d 2^{d+1} \eta^d \varepsilon^{-d} \tag{2.7} \]
for every \( \eta > 0 \) and \( N \geq 1 \).

The equations (2.1) and (2.5) yield that
\[ C' \cap T^{-k} C' \cap \{ x \in X : \| f'(k, x) \| < \varepsilon \} \]
\[ = C' \cap V^{-1} C' \cap \{ x \in X : V x \neq x \} \text{ and } \| \mathbf{f}'(V x, x) \| < \varepsilon \} = \emptyset \]
whenever \( k \neq 0 \) and \( V \in [T] \). We set \( Y = X \times \mathbb{R}^d, \nu = \mu \times \lambda \), denote by \( S: Y \to Y \) the skew product transformation
\[ S(x, t) = (S x, t + \mathbf{f}'(S x, x)) = (S x, t + g(x)), \]
and obtain that the set
\[ D = C' \times B(\varepsilon/2) \]
is wandering under \( S \), i.e. that \( S^n D \cap D = \emptyset \) whenever \( 0 \neq m \in \mathbb{Z} \). For every \( x \in X \setminus N \) we denote by \( V_x \subset \mathbb{R}^d \) the discrete set
\[ \{ f'(k, x) : k \in \mathbb{Z} \} = \{ \mathbf{f}'(S^k x, x) : k \in \mathbb{Z} \} \]
and observe that
\[ |\{k \in \mathbb{Z} : f'(k, x) = v\}| = |\{k \in \mathbb{Z} : f'(S^k x, x) = v\}| = L \]
for every \( v \in V_x \) and \( x \in X \setminus N \), and that
\[ \|v - v'\| \geq \varepsilon \]
whenever \( v, v' \in V_x \) and \( v \neq v' \). Hence
\[ |\{0 < l \leq k : 0 < \|f'(l, x)\| \leq l^{1/d}/\eta\}| \]
\[ \leq \{|0 < l \leq k : 0 < \|f'(l, x)\| \leq k^{1/d}/\eta\}| \]
\[ < (k^{1/d} + \varepsilon/\eta)^d \nu(X \times B(\eta))/\nu(D) \]
\[ = (k^{1/d} + \varepsilon/\eta)^d L 2^d \eta^d \varepsilon^{-d}, \]

since \( S^{j+1} D \subset X \times B(k^{1/d} \eta + \varepsilon) \) whenever \( x \in S \mathcal{J} C^d \) and \( \|f'(l, x)\| \leq k^{1/d} \eta \) for some \( j \in \{0, \ldots, L - 1\} \) and \( l \in \{1, \ldots, k\} \), and since the sets \( S^m D, m \in \mathbb{Z} \), are all disjoint. By integrating we obtain that
\[ \tau'_k(B(\eta)) = \frac{1}{k} \sum_{l=1}^{k} \tau'_l(B(\eta)) = \frac{1}{k} \sum_{l=1}^{k} \mu(\{x \in X : \|f'(l, x)\| \leq l^{1/d}/\eta\}) \]
\[ \leq \frac{L}{k} + \frac{1}{k} \int \{|0 < l \leq k : 0 < \|f'(l, x)\| \leq l^{1/d}/\eta\}| d\mu(x) \]
\[ < \frac{L}{k} + \frac{(k^{1/d} + \varepsilon/\eta)^d}{k} \cdot L 2^d \eta^d \varepsilon^{-d}, \]
and by letting \( k \to \infty \) we have proved the first inequality in (2.7).

Similarly one sees that
\[ \sum_{n \geq 0} |\{0 < l \leq 2^n k : 0 < \|f'(l, x)\| \leq l^{1/d} 2^{-n/d}/\eta\}| \]
\[ = L \cdot \sum_{0 \neq v \in V_x} |\{n \geq 0 : v = f'(l, x) \text{ for some } l \}
\text{ with } 0 < l \leq 2^n k \leq k \eta^d/\|v\|^d\} \]
\[ \leq L \cdot \sum_{0 \neq v \in V_x} (|\{n \geq 0 : 1 \leq 2^n k \leq k \eta^d/\|v\|^d\}| + 1) \]
\[ \leq L \cdot \sum_{j \geq 1} \sum_{v \in V_x \cap (B((j+1)\eta) \setminus B(j \eta))} (|\{n \geq 0 : 1 \leq 2^n k \leq k \eta^d/j^d \varepsilon^d\}| + 1) \]
\[ \leq L^d 2^{d-1} d \cdot \sum_{j=1}^{k^{1/d} \eta/j \varepsilon} j^{d-1} \left( \frac{\log \eta^d/j^d \varepsilon^d}{\log 2} + 1 \right) \]
\[ < L^d 2^{d-1} d \cdot 4k \eta^d \varepsilon^d. \]

Hence
\[ \sum_{n=0}^{N} 2^n \tau'_{2^n k}(B(2^{-n/d} \eta)) = \sum_{n=0}^{N} \left( \frac{L}{k} + 2^n \tau'_{2^n k}(B(2^{-n/d} \eta) \setminus \{0\}) \right) \]
\[ \leq \frac{(N+1)L}{k} + L^d d 2^{d+1} \eta^d \varepsilon^d, \]
and by letting \( k \to \infty \) we obtain the second inequality in (2.7). Since (2.7) is equivalent to (2.2) we have proved the lemma. \( \square \)
Proof of Theorem 1.2. Suppose that $f: X \rightarrow \mathbb{R}^d$ is transient. Lemma 2.1 yields a constant $c > 0$ such that

$$\limsup_{k \to \infty} \sum_{n=0}^{N} 2^n \tau^{(d)}_{2^{2n}k}(B(2^{-n/d} \eta)) \leq c \lambda(B(\eta))$$

for every $\eta > 0$ and $N \geq 1$. It follows that there exists, for every $\eta > 0$ and $N \geq 1$, an integer $n \in \{0, \ldots, N\}$ with

$$\liminf_{k \to \infty} \tau^{(d)}_{2^{2n}k}(B(2^{-n/d} \eta)) \leq \frac{c}{N+1} \cdot \lambda(B(2^{-n/d} \eta)).$$

We conclude that

$$\liminf_{\eta \to 0} \liminf_{k \to \infty} \tau^{(d)}_{2^{2n}k}(B(2^{-n/d} \eta)) \leq \frac{c}{N+1} \cdot \lambda(B(2^{-n/d} \eta))$$

for some $n \in \{0, \ldots, N\}$, and hence that

$$\liminf_{\eta \to 0} \liminf_{k \to \infty} \tau^{(d)}_{k}(B(\eta)) \leq \frac{c}{N+1} \cdot \lambda(B(\eta)).$$

As $N \geq 1$ was arbitrary this proves (1.6). The inequality (1.5) is an immediate consequence of the first inequality in (2.2). \qed

References


Mathematics Institute, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria,
and
Erwin Schrödinger Institute for Mathematical Physics, Boltzmannsgasse 9, A-1090 Vienna, Austria
E-mail address: klaus.schmidt@univie.ac.at