Inverse Limit of M–cocycles and Applications

Jan Kwiatkowski

Vienna, Preprint ESI 494 (1997)  
October 6, 1997

Supported by Federal Ministry of Science and Research, Austria  
Available via http://www.esi.ac.at
**Inverse limit of M-cocycles and applications.**

Jan Kwiatkowski

**Abstract**

For any $m$, $2 \leq m < \infty$ we construct an ergodic dynamical system having spectral multiplicity $m$ and infinite rank. Given $r > 1$, $0 < b < 1$ such that $rb > 1$ we construct a dynamical system $(X, B, \mu, T)$ with simple spectrum such that $r(T) = r$, $F^{*}(T) = b$, $C(T)/\text{wcl}\{T^n, n \in \mathbb{Z}\} = \infty$.

\hspace{1cm}

\textsuperscript{1}1991 Mathematics Subject Classification: Primary 28D05, 54H20. Key words and phrases: Multiplicity, rank, compact group extension, Morse cocycle.
1 Introduction

It was conjectured in [M1] that for any pair of integers or \( \infty (m, r) m < r \), there exists an ergodic dynamical system \( (X, \mu, T) \) whose rank \( r(T) = r \) and whose spectral multiplicity \( m(T) = m \). Partial solutions of this question were obtained by several authors: [Ch] (the pair \( (1,1) \)), [dJ] (1,2), [M1] (1,r), [Gole] (2,r), [R1,2] (r,r), [M2] (r,2r), [FeKw] (p,1,p), p-prime, and [Fe1] (1,\infty), [FeKwMa] (given \( m \), the set \( r \) such that \( m(T) = m \) and \( r(T) = r \) has density 1). The latest result of this series [KwLa1] says that for any pair \( (m, r) \), \( 2 \leq m \leq r < \infty \) there is an ergodic automorphism \( T \) with \( r(T) = r \) and \( m(T) = m \). Thus, together with [M1] every finite pair \( (m, r) \) \( m < r \), is obtainable.

The solution of the problem (multiplicity, rank) would be complete if for any finite \( m \geq 1 \) and \( r = \infty \) we can find an ergodic automorphism realizing \( (m, \infty) \). The pair \((1, \infty)\) is realized by Gaussian-Kronecker system [dLR]. In this note we construct an ergodic automorphism realizing the pairs \( (m, \infty) \) for every \( m \geq 2 \).

By \( C(T) \) we mean the set of all measure-preserving automorphisms of \( (X, \mathcal{B}, \mu) \) which commute with \( T \). We say that a sequence \( \{ S_n \} \subset C(T) \) tends weakly to \( S \in C(T) \) if for every \( A \in \mathcal{B} \)

\[
\mu(S_n A \triangle SA) \to 0.
\]

\( C(T) \) is a Polish group. By \( wcl \{ T^n, n \in \mathbb{Z} \} \) we mean the weak closure of the set \( \{ T^n, n \in \mathbb{Z} \} \). The weak closure theorem [Kin] says that \( C(T) = wcl \{ T^n, n \in \mathbb{Z} \} \) if \( r(T) = 1 \). If turns out that it is the only relations between rank and the cardinality of the quotient group \( C(T)/wcl \{ T^n, n \in \mathbb{Z} \} \) in the class of ergodic dynamical systems. In [KwLa2] examples of ergodic automorphisms \( T \) are constructed such that \( r(T) = r \geq 2, \# \overline{C(T)_{wcl \{ T^n, n \in \mathbb{Z} \}}} = m \geq 1 \), where \( r, m \)
are given. We construct an example of ergodic automorphisms $T$ such that $T$ has simple spectrum, $r(T) = r, F^\ast(T) = b, \# \frac{C(T)}{w_{c} \{ T^n, n \in \mathbb{Z} \}} = \infty$, where $r, b$ are given and $r \geq 2, 0 < b < 1, br > 1$.

In [KwLa1] we used Morse automorphisms over a finite abelian group. Now, we use the class of inverse limits of Morse automorphism over a compact metric abelian groups. Similarly as in [KwLa1] the automorphisms we construct here can be obtained within the class of the weakly mixing transformations.

\section{Preliminaries}\nonumber

Let $(X, B, \mu, T)$ be an ergodic dynamical system. We can look at the associated spectral operator $U_T : L^2_0(X, \mu) \longrightarrow L^2_0(X, \mu)$, $U_T f = f \circ T, \; f \in L^2_0(X, \mu)$, where $L^2_0(X, \mu)$ consists of those functions of $L^2(X, \mu)$ that $\int_X f d\mu = 0$. By the spectral multiplicity $m(T)$ of $T$ we mean the supremum of all essential spectral multiplicities of $T$ on $L^2_0(X, \mu)$. We refer the reader to [Fe2] for the definition of the rank $r(T)$ and the covering number $F^\ast(T)$ of $T$ and for more information on those notions.

Now let $T : (X, B, \mu) \longrightarrow (X, B, \mu)$ be the $(p_t)$-adic adding machine i.e. $p_t/n+1, \lambda_t+1 = p_t+1/p \geq 2$ for $t \geq 0, p_0 = \lambda_0 = 2$,

$$X = \{x = \sum_{t=1}^{\infty} q_t p_{t-1}, 0 \leq q_t \leq \lambda_t - 1, p_{-1} = 1\}$$

is the group of $(p_t)$-adic integers and $Tx = x + \hat{1}, \; \hat{1} = (1, 0, 0, \ldots)$. The space $X$ has a standard sequence $(\xi_t)$ of $T$-towers. Namely

$$\xi_t = (D^0_0, D^1_1, \ldots, D^t_{p_t-1}),$$

where $D^0_t = \{x \in X; q_0 = \cdots = q_t = 0\}, \; D^j_t = T^j(D^0_0), j = 0, \ldots, p_t - 1, X = \bigcup_{j=0}^{p_t-1} D^j_t$.

The tower $\xi_{t+1}$ refines $\xi_t$ and the sequence of partitions $(\xi_t)$ converges to the point partition. Let $G$ be an abelian compact metric group and let $m_G$ be normalized Haar measure of $G$. A cocycle is a measurable function $\varphi : X \longrightarrow G$, a cocycle $\varphi$ defines an automorphism $T_\varphi$ on $(X \times G, B \times m_G)$.

$$T_\varphi(x, y) = (Tx, g + \varphi(x)), x \in X, g \in G,$$

where $B$ is the product $\sigma$-algebra $B$ and the $\sigma$-algebra of the borelian subsets of $G$.

Then $T_\varphi(x, y) = (T^n x, g + \varphi^{(n)}(x)), n = 0, \pm 1, \ldots$, where

$$\varphi^{(n)}(x) = \begin{cases} 
\varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x), & n \geq 1 \\
0, & n = 0 \\
-\varphi(T^{-1}(x)) - \cdots - \varphi(T^n x), & n \leq -1 
\end{cases} \quad (1)$$
The dynamical system $(X \times G, \mathcal{B}, \mu \times m_G, T_\varphi)$ is called a group extension of $(X, B, \mu, T)$.

$T_\varphi$ is ergodic if for every non-trivial $\gamma \in \hat{G}$ ($\hat{G}$ is the dual group), there is no measurable solution $f : X \rightarrow S^1$ (the unit complex circle) to the functional equation

$$\gamma(\varphi(x)) = \frac{f(Tx)}{f(x)}, \quad x \in X \, [\mathbb{P}a].$$

(2)

We say that $\varphi : X \rightarrow G$ is a $M$-cocycle if for every $t \geq 1$, $\varphi$ is constant on each level $D^t_i$, $i = 0, \ldots, m - 2$ (except of the top $D^t_{p_t - 1}$). Such a cocycle is defined by a sequence a blocks $b^{(0)}, b^{(1)}, \ldots$ over $G$. By a block $B$ over $G$ we mean a finite sequence

$$B = B[0] \ldots B[k - 1],$$

where $k \geq 1$ and $B[i] \in G$, $i = 0, \ldots, k - 1$. The number $k$ is called the length of $B$ and it is denoted by $|B|$. If $C = C[0] \ldots C[m - 1]$ is another block then the concatenation of $B$ and $C$ is the block

$$BC = B[0] \ldots B[k - 1]C[0] \ldots C[m - 1].$$

We can concatenate more than two blocks in the obvious way. If $\tau : G \rightarrow G$ is a continuous group automorphism then we let $\tau(B)$ be the block

$$\tau(B) = \tau(B[0]) \ldots \tau(B[k - 1]).$$

By $B(g)$, $g \in G$ we will denote the block

$$B(g) = (B[0] + g) \ldots (B[k - 1] + g)$$

and by $\hat{B}$ the block $\hat{B} = (B[1] - B[0]) \ldots (B[k - 1] - B[k - 2])$, $k \geq 2$. Now, we can define the product $B \times C$ of $B$ and $C$ as follows

$$B \times C = B(C[0]) \ldots B(C[m - 1]).$$

Of course

$$|B \times C| = |B| \cdot |C| \quad \text{and} \quad \tau(B \times C) = \tau(B) \times \tau(C).$$

This multiplication operation "$\times$" is associative so it can be extended to more than two blocks. If $|B| = |C| = k$ then we define

$$\tilde{d}(B, C) = k^{-1} \# \{0 \leq i \leq k - 1; B[i] \neq C[i]\}.$$

Now we describe Morse sequences ($M$-sequences). Let $b^{(0)}, b^{(1)}, \ldots$ be finite blocks over $G$ with $|b^{(t)}| = \lambda_t$, $b^{(t)}[0] = 0$, $t \geq 0$. Then we may define a one-sided sequence over $G$ by

$$\omega = b^{(0)} \times b^{(1)} \times \ldots$$
Such a sequence $\omega$ allows one to define a $M$-cocycle $\phi = \varphi_\omega$ on $X$ as follows: let

$$B_t = b^{(1)} \times \cdots \times b^{(t)}, t \geq 0.$$ 

Then $|B_t| = p^t, |\bar{B}_t| = p^t - 1.$

Let

$$\phi(x) = \bar{B}_t[j] \text{ if } x \in D_j^t, j = 0, \ldots, p^t - 2.$$ 

Of course, $\phi$ is a $M$-cocycle. It is easy to remark that each $M$-cocycle can be obtained as described above. As a consequence of the definition of $\phi$ and (1) we get

$$\phi^{(n)}(x) = B_t[j + n] - B_t[j],$$

if $x \in D_j^t$ and $j = 0, \ldots, p^t - n - 1.$ If we examine $\phi^{(p_1)}(x), 1 \leq k \leq \lambda_{t+1} - 1$ on the tower $\varepsilon_{t+1}$ then (3) implies

$$\phi^{(p_1)}(x) = b^{(t+1)}[q + k] - b^{(t+1)}[q]$$

if $x \in D_{q+1}^{(t+1)}$, $0 \leq q \leq \lambda_{t+1} - k - 1$, $j = 0, \ldots, p^t - 1$.

3 Spectral analysis of $M$-cocycles and their inverse limit.

3.1 Spectral calculations.

It is known that

$$L^2(X \times G, \mu \times m_G) = \bigoplus_{\gamma \in \hat{G}} L_\gamma,$$

where

$$L_\gamma = \{ f \in L^2(X \times G, \mu \times m_G), f \in L^2(X, \mu) \}.$$ 

Moreover, the subspaces $L_\gamma$ are $U_{T_\gamma}$-invariant and using the same arguments as in [KwSi] we get that $U_{T_\gamma}$ on $L_\gamma$ has simple spectrum.

Let $\mu_\gamma$ be the spectral measure of $U_{T_\gamma}$ on $L_\gamma$. The subspace $L_\epsilon$ ($\epsilon$ is the trivial character) is generated by the eigenfunctions of $T_\overline{\epsilon}$ (in fact of $T$) corresponding to all $p^t$-roots of unity. A $M$-cocycle $\phi = \varphi_\omega$ is called continuous if $L_\epsilon$ contains all eigenfunctions of $T_\overline{\epsilon}$, or equivalently if each measure $\mu_\epsilon, \gamma \neq \epsilon$, is continuous.

We shall use the following criteria to know whether two measures $\mu_\gamma, \mu_{\gamma'}, \gamma, \gamma' \in \hat{G}$, $\gamma \neq \gamma'$, are orthogonal or equivalent.

**Proposition 1** [KwRo], [FeKw], [GoKuLeLi]

If $\upsilon : G \rightarrow G$ is a group automorphism and blocks $b^{(0)}, b^{(1)}, \ldots$ satisfy

(a) $$\sum_{t=0}^{\infty} d(b^{(t)})[k_t \lambda_t - 1, r(b^{(t)})[0, \lambda_t - k_t - 1]] < \infty$$

for a sequence $(k_t)_t^{\infty}$, $0 \leq k_t < \lambda_t$ for which
(b) \( \sum_{i=0}^{\infty} \frac{b_i}{i!} < \infty. \)

then \( \mu_\gamma \simeq \mu_\hat{\gamma} \) for all \( \gamma \) in \( \hat{G} \), where \( \hat{\gamma} \) is the dual automorphism.

\[ \text{Proposition 2 [GoKwLeLi]} \]

If for given \( \gamma, \gamma' \in \hat{G} \)

\[
\begin{align*}
\lim_{t \to \infty} \int_X \gamma(\varphi(\gamma, t)x) \mu(dx) & \text{ and } \lim_{t \to \infty} \int_X \gamma'(\varphi(\gamma, t)x) \mu(dx) \\
& \text{ exists along a subsequence } \{N_k\} \text{ and are different} \\
& \text{ then } \mu_\gamma - \mu_\gamma', \text{ where } \sum_{i=1}^{\infty} \frac{\delta_{i|\gamma}}{i!} < \infty \end{align*}
\]

(note that \( T^{n_{\gamma, t}} \to \text{Id} \) in the weak topology).

Let \( H_0 \) be a subgroup of \( G \) and \( H = \frac{G}{H_0} \) be the quotient group. Let \( \pi : G \to H \) be the quotient map and let \( m_H \) be Haar measure on \( H \). We can define a map \( P = 1dx \times \pi \) of the dynamical system \((X \times G, T_\varphi, \mu \times m_G)\) onto \((X \times H, T_\varphi, H, \mu \times m_H)\), where \( \varphi_H(x) = \pi(\varphi(x)) \). The systems \((X \times H, T_\varphi, H, \mu \times m_H)\) are called the natural factors of \((X \times G, T_\varphi, \mu \times m_G)\). If \( B \) is a block over \( G \) then by \( \pi(B) \) denote a block over \( H \) defined by

\[
\pi(B) = \pi(B[0]) \ldots \pi(B[k-1]), k = |B|.
\]

Using the obvious equality \( \pi(B \times C) = \pi(B) \times \pi(C) \), it is not hard to see that if \( \varphi \) is a \( M \)-cocycle defined by the sequence of blocks \( b^{(0)}, b^{(1)} \ldots \) over \( G \) then \( \varphi_H \) is a \( M \)-cocycle determined by the blocks \( \pi(b^{(0)}), \pi(b^{(1)}), \ldots \).

It is known that \( \hat{H} \) can be identified with a subgroup of \( \hat{G} \), namely with the subgroup of those \( \gamma \in \hat{G} \) that \( \gamma[H_0] = 1 \). Let

\[
L_{\gamma, H} = \{ f \circ \gamma \in L^2(X \times H, \mu \times m_H), f \in L^2(X, \mu) \}
\]

\( \gamma \in \hat{H} \). Then

\[
L^2(X \times H, \mu \times m_H) = \bigoplus_{\gamma \in \hat{H}} L_{\gamma, H}
\]

and the unitary operator \( U_{T_\varphi, H} \) on \( L_{\gamma, H} \) is spectrally isomorphic to the unitary operator \( U_{T_\varphi} \) on \( L_{\gamma} \). Thus \( U_{T_\varphi, H} \) has simple spectrum on \( L_{\gamma, H} \) and its spectral measure is \( \mu_\gamma \).

### 3.2 Inverse limit of \( M \)-cocycles.

Let \((X, \mathcal{B}, \mu, T)\) and \((X_{\gamma}, \mathcal{B}_{\gamma}, \mu_{\gamma}, T_{\gamma})\), \( \gamma = 1, 2, \ldots \), be dynamical systems. We say that \((X, \mathcal{B}, \mu, T)\) is an inverse limit of \((X_{\gamma}, \mathcal{B}_{\gamma}, \mu_{\gamma}, T_{\gamma})\) if there exist homomorphisms \( V_\gamma : (X, \mathcal{B}, \mu, T) \to (X_{\gamma}, \mathcal{B}_{\gamma}, \mu_{\gamma}, T_{\gamma}) \) such that \( V_{\gamma, \gamma'}^{-1}(\mathcal{B}_{\gamma'}) \subset V_{\gamma, \gamma'}^{-1}(\mathcal{B}_{\gamma'}) \subset V_{\gamma, \gamma'}^{-1}(\mathcal{B}_{\gamma'}) \).
and the \( \sigma \)-algebras \( V_{s-1}(B_j) \) generate \( B \). For each \( s \geq 0 \) we have a homomorphism \( W_s : (X_{s+1}, B_{j+1}, \mu_{j+1}, T_j) \to (X_j, B_j, \mu_j, T_j) \) and \( W_s \circ V_{s+1} = V_s \).

We will write \( T = \lim_{s} T_j \). It follows from the definitions of the spectral multiplicity, the rank and the covering number that \( m(T_s) = \lim m(T_j), \ r(T_j) = \lim r(T_j), \ F^*(T) = \lim F^*(T_j) \) and moreover \( M(T_s) \leq m(T_{s+1}), \ r(T_s) \leq r(T_{s+1}), \ F^*(T_s) \geq F^*(T_{s+1}) \).

It is clear that \( T \) is ergodic (weakly mixing, mixing) iff \( T_j \) fulfills the same conditions for every \( s \geq 0 \). Given an ergodic dynamical system \( (X, B, \mu, T) \) and sequences of metric compact abelian groups \( (G_j)_{j=0}^{\infty} \) and group homomorphisms \( \pi_j : G_{j+1} \to G_j \) with \( \pi(G_{j+1}) = G_j \). The sequence \( (G_j, \pi_j), s \geq 0 \) defines the inverse limit \( G = \lim \pi_j(G_j) \) and the homomorphisms \( \psi : G \to G_j \) such that \( \pi_j \circ \psi = \psi_j \). \( G \) is a metric compact abelian group. Assume that \( \varphi_j : X \to G_j \) are cocycles such that \( \pi_j \circ \varphi_j = \varphi_j \). The cocycles \( \varphi_j \) define a unique cocycle \( \varphi : X \to G \) satisfying \( \psi \circ \varphi = \varphi \). Then \( T_{\varphi} = \lim_{s} T_{\varphi_j} \).

Now, let \( (X, B, \mu, T) \) be a \((p_t)\)-adic adding machine, \( p_t = \lambda_0 \cdots \lambda_t, t \geq 0 \). We describe special inverse limits of group extensions \( T_{\varphi_j} \) determined by \( M \)-cocycles. To do this assume additionally that we have one-to-one mappings \( \tau_j : G_j \to G_{j+1} \) such that \( \tau_j \circ \tau_{j-1} = \text{id}, s \geq 0 \). Let us denote \( H_j = \tau_j(G_j) \). There exist unique subsets \( H_j \subset G \) such that

\[
(\tau_{j-1} \circ \cdots \circ \tau_j \circ \psi_j)(g) = \psi_t(g)
\]

whenever \( g \in H_j \) and \( t > s \). Given blocks \( b(t), t \geq 0 \) over \( G_t \). Each block \( b(t) \) can be treated as a block over \( G_t \) if we identify the members of \( b(t) \) with the corresponding elements of \( H_t \). The sequence \( (b(t))_{t=0}^{\infty} \) defines a cocycle \( \varphi : X \to G \). Let \( m \) and \( m_j \) be normalized Haar measures of \( G \) and \( G_j \). The dynamical system \( (X \times G, B, T_{\varphi_j}, \mu \times \mu_j) \) has natural factors

\[
(X \times G_j, B_j, T_{\varphi_j}, \mu \times \mu_j), \ j \geq t
\]

where \( \varphi_s = \psi_s \circ \varphi \) and the mappings

\[
W_s = 1d_X \times \psi_s : X \times G \to X \times G_j
\]

are homomorphisms of those systems. Each cocycle \( \varphi_s \) is a \( M \)-cocycle determined by the blocks \( (b(t))_{t=0}^{\infty} \), where \( b(t) = \psi (b(t)) \), \( t \geq s \) and \( b(t) = \tau \circ \cdots \circ \tau_{s-1} b(t) \) if \( t < s \).

4 Example 1.

In this section we describe an example of a \( M \)-cocycle \( \varphi \) such that \( T_{\varphi} \) has infinite rank and the spectral multiplicity equal to \( r \geq 1 \).
4.1 Definition of the cocycle.

Let \( r_t = r^{2^t}, t \geq 0 \) and \( n \geq 2 \). Select a sequence \( (b_i)_0^\infty \) of positive integers such that \( n|b_i, b_i \not\rightarrow \infty \) and

\[
(1 - \frac{n}{b_i})^{b_i} \rightarrow 1.
\]  

(8)

(8)

Let \( Z_n = \{0, 1, \ldots, n - 1\} \simeq \mathbb{Z}/n\mathbb{Z} \),

\[
G_t = \overbrace{Z_n \oplus \cdots \oplus Z_n}^{r_t}
\]

be the direct products of \( r_t \) copies of \( Z_n \)'s, \( t = 0, 1, \ldots \). For \( g \in G_t \) we write \( g = (g_0, g_1, \ldots, g_{r_t - 1}) \), \( g_i \in \mathbb{Z}/n\mathbb{Z} \).

We let

\[
e_i^{(t)} = e_i = (0, \ldots, 0, 1, 0, \ldots, 0), i = 1, \ldots, r_t - 1.
\]

Define homomorphisms \( \pi_t : G_{t+1} \rightarrow G_t \) by

\[
\pi_t(e_j^{(t+1)}) = e_i^{(t)}, \quad \text{where } j = 0, 1, \ldots, r_{t+1} - 1; i = 0, 1, \ldots, r_t - 1 \text{ and } i \equiv j (\text{mod } r_t).
\]

We have the natural mappings \( \tau_t : G_t \rightarrow G_{t+1} \) defined by

\[
\tau_t = \left( \sum_{i=0}^{r_t-1} g_i e_i^{(t)} \right) = \sum_{i=0}^{r_t-1} g_i e_i^{(t+1)},
\]

\[
g_0, \ldots, g_{r_t-1} = 0, 1, \ldots, n - 1.
\]

Then \( \tau_t \circ \tau_t = \text{id} \). Set

\[
G = \lim_{\leftarrow \text{ in } G_t, \tau_t}.
\]

As preceding let \( \psi_t : G \rightarrow G_t \) be continuous homomorphisms such that

\[
\pi_t \circ \psi_t = \psi_t.
\]

Now, we are in a position to describe \( M \)-cocycles \( \varphi_t \) as in the part 3.2. To do this we define a sequence of blocks \( \{b^{(t)}\}_0^\infty \), each block \( b^{(t)} \) over \( G_t \). Put

\[
F_i = F_i^{(t)} = 0(e_i)(2e_i)\cdots(l-1(e_i)), \quad i = 0, 1, \ldots, r_t - 1; l = l_t, e_i = e_i^{(t)}.
\]

Then define block \( b_u^{(t)} = \beta_{u,k} = \beta_{u,k}, u = 0, 1, \ldots, 2^t - 1, k = 0, \ldots, r - 1 \) as follows

\[
\begin{cases}
\delta_{u,k} = F_{ur + k} \times F_{ur + (k \oplus 1)} \times \cdots \times F_{ur + (k \oplus r - 1)} \\
\text{where } a \oplus b \text{ is taken mod } r, a, b = 0, 1, \ldots, r - 1 \\
\text{and}
\delta_{u,k} = \beta_{u,k} \times \beta_{u \oplus 1,k} \times \cdots \times \beta_{u \oplus 2^t - 1,k},
\end{cases}
\]

(10)
Finally define
\[
\beta_{u}^{(t)} = \beta_{u} = \beta_{u,0} \beta_{u,1} \cdots \beta_{u,r-1}, \quad u = 0, 1, \ldots, 2^l - 1
\]  
(11)
and
\[
b^{(t)} = \beta_{0} \cdots \beta_{t} \beta_{t+1} \cdots \beta_{t_r} = \beta_{t(2^{r-1})} \cdots \beta_{t_r-1}
\]  
(12)
where \(q_{t,u}\) are positive integers such that
\[
\sum_{t=0}^{\infty} \frac{1}{q_{t,u}} < \infty, \quad q_{t} = \min(q_{0,0}, q_{1,1}, \ldots, q_{t,u}).
\]  
(13)
Some additionally conditions on \(q_{t,u}\)'s shall be specified later.

Of course, \(F^{(t)}_{x}, \beta^{(t)}_{x}, b^{(t)}\) are blocks over \(G_{t}\) and we have
\[
|F_{t}| = l_{t}, \quad |\beta_{t}^{(t)}| = t^{l_{t}}, \quad |\beta_{t}| = r l^{l_{t}}, \quad |b^{(t)}| = r l^{l_{t}} Q_{t}
\]
where
\[
Q_{t} = \sum_{u=0}^{t-1} q_{t,u}.
\]
Let \(v = v_{t} : G_{t} \longrightarrow G_{t}\) be a group automorphisms defined by
\[
v(\epsilon_{u r + k}) = \epsilon_{u r + (k \oplus 1)},
\]
u = 0, 1, \ldots, 2^l - 1, k = 0, 1, \ldots, r - 1, \epsilon_{u r + k} = \epsilon_{u r + k}^{(t)}.
Then we have
\[
v(F_{u r + k}) = F_{u r + (k \oplus 1)}, \quad v(\beta_{u,k}) = \beta_{u,k \oplus 1}.
\]  
(14)
Now, let \((X, B, \mu, T)\) be the \((p_{t})\)-adic adding machine, where \(p_{t} = \lambda_{0} \cdots \lambda_{t}, \lambda_{t} = |b^{(t)}| = r l^{l_{t}} Q_{t}, t \geq 0\). The sequence \(\{b^{(t)}\}_{t=0}^{\infty}\) determines the sequences of blocks \(\{b_{s}^{(t)}\}_{s=0}^{\infty}, s \geq 0\) and in consequence \(M\)-cocycles \(\varphi, \varphi_{s}, \varphi : X \longrightarrow G_{t}\).
\(\varphi_{s} : X \longrightarrow G_{s}\) described in the part 3.2.

We have a sequence of dynamical systems
\[
(X \times G_{0}, T_{\varphi_{0}}) \xleftarrow{W_{2}} (X \times G_{1}, T_{\varphi_{1}}) \xleftarrow{W_{2}} (X \times G_{2}, T_{\varphi_{2}}) \xleftarrow{W_{2}} \cdots
\]  
(15)
determined by the homomorphisms \(\pi_{t}\), the mappings \(\tau_{t}\) (in this case \(\tau_{t}\) are homomorphisms) and by the blocks (12).
4.2 Additional conditions.

The blocks $b_t^{(i)}$, $t, s \geq 0$, can be obtained by a similar procedure as $b_t$'s. If $t \leq s$ then $b_t^{(s)} = b_t^{(t)}$ (with $e_t^{(s)}$ instead of $e_t^{(t)}$, $i = 0, \ldots, r_t - 1$). If $t > s$ we define the blocks $F_{t,s}^{(s)}$ by (9) for $i = 0, 1, \ldots, r_s - 1$ and $l = l_t$. We have

$$
\pi_i \circ \cdots \circ \pi_{i-1}(F_{t,s}^{(s)}) = F_{t,s}^{(t)}, \quad |F_{t,s}^{(t)}| = b_t
$$

for $j = 0, 1, \ldots, r_t - 1, i = 0, 1, \ldots, r_s - 1$ and $j \equiv i \pmod{r_s}$.

Then we define $\gamma_{u,k}^{(t,s)}, \gamma_{u}^{(t,s)}, u = 0, 1, \ldots, 2^t - 1, k = 0, 1, \ldots, s - 1$ by (10) and (11) using the blocks $F_{u,r+k,s}^{(s)}$.

Let

$$
\delta_0 = \beta_0 \cdots \beta_{a-1} \beta_k \cdots \beta_0 \beta_1 \cdots \beta_{2^s-1} \cdots \beta_{2^t-1}
$$

for $a = 0, 1, \ldots, 2^s - 1, \beta_u = \beta_{u,r+k}, u = 0, 1, \ldots, 2^t - 1$.

Then (16) implies $\gamma_{u,k}^{(t,s)} = \psi_s(\gamma_{0^t}^{(t)+u})$ for $u = 0, 1, \ldots, 2^t - 1$ and $a = 0, 1, \ldots, 2^s - 1$.

Now, comparing the blocks (12) and (17) we get

$$
b_{2^t}^{(t)} = \overline{d_0 \cdots d_1 \cdots d_{2^s-1}}.
$$

To finish the definition of $\varphi$ we must give conditions for the numbers $q,u,u = 0,1,\ldots,2^t-1$, $t \geq 0$. To do this let us consider the dual group $\hat{G}$. We have $\hat{G} = \bigcup_{s=0}^{r-1} \hat{G}_s$. The group automorphisms $\tau_s : G_s \longrightarrow G_s$ satisfy the conditions $\tau_s \circ \pi_s = \pi_s \circ \tau_{s+1}$ and they determine a continuous group automorphism $\tau : G \longrightarrow G$ such that $\tau \circ \pi_s = \pi_s \circ \tau$. The dual group automorphism $\hat{\tau} : \hat{G} \longrightarrow \hat{G}$ satisfies $\hat{\tau}(\hat{G}_s) = \hat{G}_s$. It is not hard to see that every $\hat{\tau}$-trajectory of $\hat{G}$ has the length $\leq r$ and there are $\hat{\tau}$-trajectories having the lengths $r$. Consider all possible pairs $(\gamma, \gamma')$, $\gamma, \gamma' \in \hat{G}$ such that $\gamma, \gamma'$ are from different $\hat{\tau}$-trajectories.

Let us divide the set $\mathcal{N} = \{0,1,\ldots\}$ on disjoint infinite subset $N(\gamma, \gamma')$. For every such a pair $(\gamma, \gamma')$ we choose $s \geq 0$, $s = s(\gamma, \gamma')$ such that $\gamma, \gamma' \in \hat{G}_s$. The functions

$$
A_\gamma = \frac{1}{r} \sum_{p=0}^{r-1} \hat{\varphi}^{(p)}(\gamma), \quad A_{\gamma'} = \frac{1}{r} \sum_{p=0}^{r-1} \hat{\varphi}^{(p)}(\gamma')
$$

are orthogonal in $L^2(G_s, m_s)$ so we can find $g \in G_s$, $g = g(\gamma, \gamma')$ such that

$$
A_\gamma(g) \neq A_{\gamma'}(g)
$$

Choose $c = c(\gamma, \gamma')$ in such a way that

$$
\frac{1}{2} < c < 1 \quad \text{and} \quad 2(1-c) < \frac{1}{2}|A_\gamma(g) - A_{\gamma'}(g)|.
$$
To find numbers $q_{t,u}$ we need probability vectors $\psi_{(t,s)} = \psi = \omega_{s}^{(0,s)}$ where $s < t$ and $z = 0, 1, \ldots, 2^s - 1$ defined as follows

$$
\omega_{z} = \sum_{a=0}^{2^{t-s}-1} \frac{q_{t,z+2^st}}{Q_{t}}, \quad Q_{t} = \sum_{u=0}^{2^{t}-1} q_{t,u}.
$$

(20)

Take $t \in N(\gamma, \gamma')$ and $t > s = s(\gamma, \gamma')$.

Choose numbers $q_{t,u}, u = 0, 1, \ldots, 2^t - 1$ such a way that

$$
\omega_{u}^{(t,s)} \geq c(\gamma, \gamma')
$$

(21)

$$
\lim_{t \to \infty} \omega_{u}^{(t,s)} = c(\gamma, \gamma')
$$

(22)

$$
\omega_{z}^{(t,s)} = \omega_{z}^{(0,s)} \text{ for } z, z' = 1, \ldots, 2^t - 1
$$

(23)

If $t \in N(\gamma, \gamma')$ and $t \leq s(\gamma, \gamma')$ then we pick $q_{t,u}$ satisfying (23) for every $z, z' = 0, 1, \ldots, 2^t - 1$.

4.3 Propositions.

In the sequel let $T_{\varphi}$ be a group extension of $T$ defined by the cocycle $\varphi$ described in 4.1 and 4.2.

**Proposition 3** $T_{\varphi}$ is ergodic and $\varphi$ is continuous.

**Proof.** Take $\gamma \in \widehat{G}$, and assume that

$$
\frac{f(Tx)}{f(x)} = \gamma(\varphi_{s}(x))
$$

for a.e. $x \in X$, where $f : X \to S^1$ is a measurable function (see (2)). Using the same arguments as in [FeKwMa] we get

$$
\gamma(\varphi_{s}^{(p_i)}(x)) \xrightarrow{t} 1
$$

(24)

in measure. The definition of $b_{s}^{(t)}, (4), (19), (21), (22)$ and (23) imply that $\varphi_{s}^{(p_i)}(x)$ is equal to $e_{s}^{(t)}, \ldots, e_{r_{s}-1}$ on set $E_{t} \subset X, \mu(E_{t}) \to 1$.

Moreover, if

$$
E_{t,i} = \{ x \in E_{t}, \varphi^{(p_i)}(x) = e_{i}^{(s)} \}, \quad i = 0, 1, \ldots, r_{s} - 1
$$

then

$$
\mu(E_{t,i}) \geq \frac{1}{2} c(\gamma, \gamma')
$$
if \( t \in N(\gamma, \gamma') \) and \( \gamma' \) comes from a different \( \hat{t} \)-trajectory than \( \gamma \). It is obvious that the last inequality and (24) imply \( \gamma = 1 \). Thus \( T_\varphi \) is ergodic and then \( T_\varphi \) is ergodic because of \( T_\varphi = \lim T_\varphi \).

To show the continuity of \( \varphi \) we must prove that the only eigenvalues of \( T_\varphi \) are \( p_0 \)-roots of unity. Let \( F(x, g) \) be an eigenfunction with an eigenvalue \( \lambda \). We have

\[
F(x, g) = \sum_{\gamma \in \hat{G}} f_\gamma(x) \gamma(g),
\]

\( f_\gamma \in L^2(X, \mu) \). Using again the same arguments as in [FeKwMa] we get

\[
\gamma(\varphi_s^{(p_1)}(x))\lambda^{-p_1} \rightarrow 1 \text{ in measure}
\]

for every \( \gamma \in \hat{G} \) such that \( f_\gamma \neq 0 \) in \( L^2(X, \mu) \). Then \( \gamma \in \hat{G} \), for some \( s \geq 0 \) so (25) can be rewritten as

\[
\gamma(\varphi_s^{(p_1)}(x))\lambda^{-p_1} \rightarrow 1.
\]

Taking again \( \gamma' \) as before and \( t \rightarrow \infty, t \in N(\gamma, \gamma') \) we get that \( \gamma(\epsilon_t^{(p)}) \) is constant for \( i = 0, 1, \ldots, r - 1 \). Thus \( \gamma = 1 \). This means that \( F(x, y) = f_0(x) \) and \( \lambda \) is eigenvalue of \( T \) i.e. \( \lambda \) is a \( p_0 \)-roots of unity. We have proved the continuity of \( \gamma \). ■

**Proposition 4**

\( m(T_\varphi) = r \)

**Proof.** Let \( \mu_\gamma \) be the spectral measure defined in the part 3.1 \( \gamma \in \hat{G} \). We will show that

\[
\mu_\gamma \simeq \mu_{\gamma \rightarrow}(\gamma)
\]

\( \mu_\gamma - \mu_{\gamma \rightarrow} \) whenever \( \gamma, \gamma' \) are in different \( \hat{t} \)-trajectories.

(27)

It follows from (14) that every fragment \( \beta_{u_0} \beta_{u_1} \cdots \beta_{u_k} \), \( u = 0, 1, \ldots, 2^l - 1 \) of \( b^{(l)} \)

is of the form \( \beta_{u_0} u \beta_{u_1} \cdots \beta_{u_k} \), \( r' = rq_{k,u} - 1 \). Thus

\[
\bar{d}(b^{(l)}[t_x^r - 1, \lambda_x - 1], v(b^{(l)}[0, \lambda_x - l_x' - 1])) \leq \frac{2^l |\beta_{u_0}|}{|\beta_{u_0}|} \leq \frac{1}{p_x} \tag{28}
\]

Choose \( s \geq 0 \) such that \( \gamma \in \hat{G} \). The inequality (28) is valid for the blocks \( b^{(l)}_s \),

because \( v \circ \psi = v \circ \psi \). Thus the sequence \( (b^{(l)}_s)_{s=0}^{\infty} \) satisfies the conditions (a) and (b) of the Proposition 1. In this manner (26) is proved.

Now we prove (27). Let \( \gamma, \gamma' \) do not belong to the same \( \hat{t} \)-trajectory. Let \( \gamma, \gamma' \in \hat{G} \), and let \( g = g(\gamma, \gamma') \) satisfies (18). Then

\[
g = g_0 \epsilon_0^{(r)} + \cdots + g_{r-1} \epsilon_r^{(r)}.
\]
with \( g_0, \ldots, g_{r-1} = 0, 1, \ldots, n - 1 \).

Define
\[
a_t = g_0 + g_1 l + \cdots + l^{r-1}_t g_{r-1}.
\]

Then
\[
\frac{a_t}{l^r_t} \leq \frac{nr}{l_t} \to 0
\]
and
\[
\sum_{t=0}^{\infty} a_t \leq nr \to \infty.
\]

We will show that
\[
\lim_{t \to \infty} \left[ \int_X \gamma (\varphi_s^{(a)}(x)) \mu(dx) - \int_X \gamma' (\varphi_s^{(a)}(x)) \mu(dx) \right] \neq 0.
\]

Repeating the same calculations as in \([\text{GoKwLeLa}]\) and using (4) we get for \( t > s \)
\[
\int_X \gamma (\varphi_s^{(a)}(x)) \mu(dx) = \sum_{u \in G_t} \left\{ \frac{2^{l_t^r - 1}}{Q_t} \sum_{u} \frac{\hat{\gamma} (\hat{\varphi}_s^{(u)} (h))}{\mu_{2^r}} \bigg| \frac{a_t}{l^r_t} \bigg| \right\} + \rho_e (29)
\]
where
\[
\rho_e (29) = \frac{a_t}{l^r_t} \sum_{u \in G_t} \left\{ \frac{2^{l_t^r - 1}}{Q_t} \sum_{u} \frac{\hat{\gamma} (\hat{\varphi}_s^{(u)} (h))}{\mu_{2^r}} \right\}
\]
and
\[
\gamma = \gamma' \text{ or } \gamma' , \quad \rho_e \to \frac{a_t}{l^r_t} \to 0, \quad \beta_{u,0} = \beta_{u,0} (l^r_t)
\]

However
\[
\rho_e (29) = \rho_e (29)
\]
if \( u \equiv \pi (\mod 2^r) \).

Thus
\[
I_t = \sum_{u \in G_t} \left\{ \frac{2^{l_t^r - 1}}{Q_t} \sum_{u} \frac{\hat{\gamma} (\hat{\varphi}_s^{(u)} (h))}{\mu_{2^r}} \bigg| \frac{a_t}{l^r_t} \bigg| \right\}
\]
\[
= \sum_{u \in G_t} \left\{ \frac{2^{l_t^r - 1}}{Q_t} \sum_{u} \frac{\hat{\gamma} (\hat{\varphi}_s^{(u)} (h))}{\mu_{2^r}} \right\}
\]
\[
(30)
\]
Take \( j = 0, 1, \ldots, l^{r-1}_t - 1 \). We can represent it as
\[
j = j_0 + j_1 l + \cdots + m_{r-1} l^{r-1}_t,
\]
where \(j_0, j_1, \ldots, j_{r-1} = 0, 1, \ldots, l_2 - 1\).

Let

\[ K_t = \{0 \leq j \leq l'^r_t - 1; 0 \leq j_0, j_1, \ldots, j_{r-1} \leq l_2 - n - 1\}. \]

We have

\[
\frac{\#K_t}{l'^r_t} \geq (1 - \frac{n}{l_2})^r. \tag{31}
\]

If \(j \in K_t\) then if easy to check that

\[
\beta_{u, \vartheta}[j + a_1] - \beta_{u, \vartheta}[j] = g_u e^{r_1} + g_1 e^{r_2} + \cdots + g_{r_2 - 1} e^{r_2 - 1} = g^*_z, \quad z = 0, 1, \ldots, 2^r - 1, u \equiv z \pmod{2^r}. \tag{32}
\]

In particular \(g^*_z = g(\gamma, \gamma')\).

(31) and (32) imply

\[
o_{\vartheta, z} (g^*_z) \geq (1 - \frac{n}{l_2})^r. \tag{33}
\]

Using (8),(29),(30),(31),(32),(33) we obtain

\[
\int_{\mathcal{X}} \gamma(\varphi^{(a \beta)}(x)) \mu(dx) = \\
= \sum_{z=0}^{2^r-1} \omega_z \left[ \int_{\mathcal{X}} \gamma^*(\varphi^*(x)) \mu(dx) + \rho + \rho' \right], \quad \rho \to 0, \rho' \leq 1 \quad \text{and} \quad \lim_{\rho' \to 0} \frac{\rho'}{\rho} = 0.
\]

Now, if \(t \in N(\gamma, \gamma')\) then (18),(19),(21),(22) and (23) imply

\[
\lim_{t \to \infty} \left[ \int_{\mathcal{X}} \gamma(\varphi^{(a \beta)}(x)) \mu(dx) - \int_{\mathcal{X}} \gamma'(\varphi^{(a \beta')})(x) \mu(dx) \right] = \\
= c(\gamma, \gamma') \left[ A_\gamma(g) - A_{\gamma'}(g) \right] + b,
\]

and

\[
|b| \leq 2(1 - c(\gamma, \gamma')) < \frac{1}{2} c[A_\gamma(g) - A_{\gamma'}(g)].
\]

In this way

\[
\lim_{t \in N(\gamma, \gamma') \atop t \to \infty} \left[ \int_{\mathcal{X}} \gamma(\varphi^{(a \beta)}(x)) \mu(dx) - \int_{\mathcal{X}} \gamma'(\varphi^{(a \beta')})(x) \mu(dx) \right] \neq 0.
\]

We have shown \(\mu_{\gamma'} = \mu_{\gamma}\) by the Proposition 2. It follows from (5) and from the simplicity \(U_{\varphi'}\) on \(L_{\gamma, \gamma} \in \hat{G}\) that

\[
m(T_{\varphi'}) = \max\{\text{lengths of } \hat{\xi}\text{-trajectories of } \hat{G}\} = r. \tag{31}
\]

**Proposition 5** \(r(T_{\varphi'}) = \infty\)
The sequence of blocks and Set Propositions in [KwLal]. We get \( r(T_i) = r_s \). In this manner \( r(T_i) = \lim r_s = \infty \).

5 Example 2.

In this part we construct a \( M \)-cocycle \( \varphi \) such that \( T_\varphi \) has the properties announced in the second part of the Introduction.

To do this choose a prime number \( p > r \) and denote \( G_t = Z_{p^t}, t \geq 0 \) and by \( \pi : G_{t+1} \rightarrow G_t \) the natural homomorphisms. Next, let \( \pi : G_t \rightarrow G_{t+1} \) be defined by \( \pi(g) = g, g = 0, 1, \ldots, p^{t+1} - 1 \). The groups \( G_t \), the homomorphisms \( \pi_t \) and the mappings \( \pi_t \) satisfy the conditions described in 3.2. Take a probability vector \( \omega(i) > 0, i = 1, 2, \ldots, r; \omega(i) > 0 \). Select positive integers \( \lambda_1^{(t)}, \ldots, \lambda_r^{(t)} \) such that

\[
\lambda_t^{(i)} = t_t^{(i)} p_t^{1}, \quad t_t^{(i)} \not\equiv 0 \quad (34)
\]

\[
\omega_t(i) = \frac{\lambda_t^{(i)}}{\lambda_t} \rightarrow \omega(i), \quad i = 1, 2, \ldots, r, \quad \lambda_t = \lambda_1^{(t)} + \cdots + \lambda_r^{(t)}. \quad (35)
\]

Set

\[
\beta_t^{(l)} = \beta_t = 0(i)(2i) \cdots (l-1)i), \quad l = \lambda_t^{(i)}
\]

and

\[
b_t^{(l)} = \beta_t^{(i)} \beta_t^{(i)} \cdots \beta_t^{(i)}.
\]

The sequence of blocks \( \{b_t^{(l)}\} \) determine a \( M \)-cocycle \( \varphi \) over the group \( G = \lim(G_t, \pi_t) \) of \( p \)-adic integers and \( M \)-cocycles \( \varphi_t \) over \( G_t \) according to the definitions in 3.2.

Proposition 6 There is a probability vector \( \omega(i) > 0, i = 1, \ldots, r \) such that \( r(T_\varphi) = r, F^*(T_\varphi) = \omega(1) = \frac{1}{1}, \omega(1) \geq \omega(i) > 0, i = 2, \ldots, r, {C(T_\varphi)} = \infty \). \( T_\varphi \) has simple spectrum.

Proof. It is proved in [FiKw] that for every \( s \geq 0, T_\varphi \) is ergodic and \( r(T_\varphi) = r, F^*(T_\varphi) = \max(\omega(1), \ldots, \omega(r)) = \omega(1) \). Then \( r(T_\varphi) = \lim r(T_\varphi) \) and \( F^*(T_\varphi) = \lim F^*(T_\varphi) = \omega(1) \). To prove the next properties of \( T_\varphi \) let us remark that the set

\[
\bigcup_{s=0}^\infty T_s
\]

from 3.2 coincides with the set of all rational \( p \)-adic integers. For \( g \in G \) let \( \sigma_g : X \times G \rightarrow X \times G \) be defined by \( \sigma_g(x, h) = (x, g + h), \quad h \in G \). Of course \( \sigma_g \in C(T_\varphi) \) (\( G \) acts as a group of measure preserving transformations in \( X \times G \)).
Consider \( \sigma_g, g \in G_s \simeq \overline{\mathbb{R}}^s, s \geq 0 \). We will show that \( \sigma_g \not\in \text{wc}\{T_{\varphi}^n, n \in \mathbb{Z}\} \). Assume that \( (T_{\varphi})^{s+1} \rightarrow \sigma_g \) \( C(T_{\varphi}) \). Then \( (T_{\varphi})^{s+1} \rightarrow \sigma_g \) for every \( s \geq 0 \) what implies

\[
\mu \{ x \in X; \varphi_s^{(u)}(x) \neq g \} = \epsilon_s \rightarrow 0. \tag{36}
\]

Fix \( s \geq 0 \). Choose \( T(u) = T \) such that \( \frac{u}{p_T} < \frac{\epsilon_s}{2} \). It follows from (3) that

\[
\varphi_s^{(u)}(x) = B_T[i + u] - B_T[i],
\]

if \( x \in D_T, i = 0, 1, \ldots, p_T - u_0 - 1 \).

Then (36) implies

\[
\frac{1}{p_T} \{ 0 \leq i \leq p_T - u_0 - 1; B_T[i + u] - B_T[i] = g \} \geq 1 - \epsilon_s.
\]

On the other hand from [FiKw] we can deduce that

\[
\frac{1}{p_T} \{ 0 \leq i \leq p_T - u - 1; B_T[i + u] - B_T[i] \neq g \} \geq \rho > 0
\]

whenever \( g \neq 0 \) and \( 0 \leq u < \frac{p_T}{2} \).

In this way \( \sigma_g \not\in \text{wc}\{T_{\varphi}^n, n \in \mathbb{Z}\} \) for every \( g \in \bigcup_{s=0}^{\infty} G_s \). To finish the proof it remains to select a probability vector \( < \omega(g) >, g = 1, \ldots, r \) for \( T_{\varphi} \) to have simple spectrum. It follows from [KwSi] that if numerus \( \omega(g) \) satisfy a condition

\[
\sum_{g=1}^{r} [\gamma(g) - \gamma'(g)] \omega(g) \neq 0 \tag{37}
\]

whenever \( \gamma \neq \gamma', \gamma, \gamma' \in \hat{G} \) then \( T_{\varphi} \) has simple spectrum.

Fix \( \omega(1), \frac{1}{r} < \omega(1) < 1 \). If \( r = 2 \) then \( F^*(T_{\varphi}) > \frac{1}{2} \) and it is known [Fe2] that \( T_{\varphi} \) has simple spectrum.

Let \( r \geq 3 \). Consider a set

\[
\Delta = \{ (\omega(2), \ldots, \omega(r); 0 \leq \omega(i) \leq \omega(1), \sum_{i=2}^{r} \omega(i) = 1 - \omega(1)) \}.
\]

The set \( \Delta \) can be regarded as a subset of the Eukliden space \( R^{r-2} \). For fixed \( \gamma, \gamma' \in \hat{G} = \bigcup_{s=0}^{\infty} \hat{G}_s, \gamma \neq \gamma' \) we have \( (r-3) \)-dimensional plane \( D(\gamma, \gamma') \) in \( R^{r-2} \) described by

\[
D(\gamma, \gamma') = \{ (\omega(2), \ldots, \omega(r); \sum_{g=2}^{r} [\gamma(g) - \gamma'(g)] \omega(g) = [\gamma'(1) - \gamma(1)] \omega(1)) \}.
\]

To set \( \Delta_0 = \bigcup_{\gamma \neq \gamma'} D(\gamma, \gamma') \) has Lebesgue measure 0 (in \( R^{r-2} \)) so that we can find \( < \omega(g) > \in \Delta - \Delta_0, g = 2, \ldots, r \). Then the condition (37) is satisfied and \( T_{\varphi} \).
has simple spectrum for $s \geq 0$. However, $m(T_\psi) = \sup_s m(T_\psi_s) = 1$.

The proposition is proved. ■
REFERENCES


[dJ] A. del Junco A transformation with simple spectrum which is not rank one, Canad. J. Math. 29, 1977 655-663


[FeKw] S. Ferenczi, J. Kwiatkowski Rank and spectral multiplicity, Studia Mathematica 102, 1992, 121-144


[Kin] J. King The commutant is the weak closure of the powers, for rank 1 transformations, Ergodic Theory and Dynamical Systems 6, 1986, 363-385


[Rt2] E.A. Robinson Mixing and spectral multiplicity, Ergodic Theory and Dynamical Systems 5, 1985, 617-624
J. Kwiatkowski,
Mathematics and Informatics Faculty,
Nicholas Copernicus University,
ul. Chopina 12/18,
87–100 Toruń, Poland.
jkwiat@mat.uni.torun.pl