On a Global Conformal Invariant of Initial Data Sets

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In the present paper a global conformal invariant \( Y \) of a closed initial data set is constructed. A spacelike hypersurface \( \Sigma \) in a Lorentzian spacetime naturally inherits from the spacetime metric a differentiation \( D_e \), the so-called real Sen connection, which turns out to be determined completely by the initial data \( h_{ab} \) and \( \chi_{ab} \) induced on \( \Sigma \), and coincides, in the case of vanishing second fundamental form \( \chi_{ab} \), with the Levi-Civita covariant derivation \( D_e \) of the induced metric \( h_{ab} \). \( Y \) is built from the real Sen connection \( D_e \) in the similar way as the standard Chern-Simons invariant is built from \( D_e \). The number \( Y \) is invariant with respect to changes of \( h_{ab} \) and \( \chi_{ab} \) corresponding to conformal rescalings of the spacetime metric. In contrast the quantity \( Y \) built from the complex Ashtekar connection is not invariant in this sense. The critical points of our \( Y \) are precisely the initial data sets which are locally imbeddable into conformal Minkowski space.

1. Introduction

In general relativity 3-manifolds play a distinguished role since in the initial value formulation of the Einstein theory the initial data, a metric \( h_{ab} \) and a symmetric tensor field \( \chi_{ab} \), are defined on connected orientable 3-manifolds \( \Sigma \). Because of the complexity of the initial value formulation, any invariant characterization of the initial data could provide a deeper understanding of the dynamics of general relativity. Mathematicians have extensively studied the geometry and the invariant characterization of three dimensional Riemannian manifolds. In particular, Chern and Simons [1] introduced a global conformal invariant of closed, orientable Riemannian 3-manifolds as the integral of the so-called Chern–Simons 3-form built from the Levi-Civita connection. The stationary points of this integral, viewed as a functional of the 3-metric, are precisely the conformally flat 3-geometries. Thus it might be interesting to generalize this result for initial data sets of general relativity, obtaining a global conformal invariant of the initial data; and it might also be interesting even from a pure mathematical point of view if there is a similar conformal invariant for connections on other trivial principal or vector bundles over \( \Sigma \).

In the present paper we show that the Chern–Simons functional, built from the real Sen connection on a four dimensional trivializable Lorentzian vector bundle over a closed orientable 3-manifold \( \Sigma \), is invariant with respect to rescalings of \( h_{ab} \) and \( \chi_{ab} \) corresponding to spacetime conformal rescalings; and the stationary points of this functional are precisely those triples \((\Sigma, h_{ab}, \chi_{ab})\) that can be locally imbedded into some conformally flat Lorentzian spacetime with first and second fundamental forms \( h_{ab} \) and \( \chi_{ab} \), respectively. For time symmetric initial data, i.e. when \( \chi_{ab} = 0 \), our invariant reduces to that of Chern and Simons, i.e. our invariant is a natural generalization of the latter.
The second section of this paper is a review of the most important properties of the Chern-Simons functional for a general gauge group and the specific conformal invariant of Chern and Simons for Riemannian 3-manifolds. The way in which we introduce their invariant, however, is slightly different from the original one, because it is this way that can be generalized to find Chern-Simons invariants for gauge groups larger than the rotation group.

The third section is devoted to the Chern-Simons invariant of a triple $(\Sigma, h_{ab}, \chi_{ab})$. We denote our invariant by $Y$. Although in this paper we will not use the Einstein equations (or any other field equations) for the sake of simplicity we call such a triple an initial data set. First we consider a trivializable Lorentzian vector bundle $V(\Sigma)$ over $\Sigma$, introduce the real Sen connection on it and then the Chern–Simons functional, built from the real Sen connection, will be introduced. In the third subsection we clarify some of its properties, in particular its conformal invariance, and we calculate its variational derivatives. For the sake of completeness in the fourth subsection we consider the Chern–Simons functional built from the complex Ashtekar connection. We consider this connection as a connection on the bundle of self-dual 2-forms on the Lorentzian vector bundle determined by the real Sen connection. It turns out, however, that this Ashtekar-Chern–Simons functional is not invariant with respect to conformal rescalings. Thus the conformal invariance depends on the representation in which the Chern–Simons functional is constructed. In fact, the stationary points of this functional are the initial data sets that can be locally isometrically imbedded into a flat spacetime.

The fourth section is devoted to the local isometric imbeddability of initial data sets into conformally flat geometries. More precisely, if $\Sigma$ is an $n$ dimensional manifold ($n \geq 3$), $h_{ab}$ a metric on $\Sigma$ with signature $(p, q)$, $p + q = n$, and $\chi_{ab}$ is a symmetric tensor field on $\Sigma$, then we are interested in the necessary and sufficient conditions for the triple $(\Sigma, h_{ab}, \chi_{ab})$ to be locally isometrically imbeddable into some conformally flat $(n+1)$ dimensional geometry $(M, g_{ab})$ with $g_{ab}$ of signature $(p+1, q)$ or $(p, q+1)$ and so that the induced metric and second fundamental form are $h_{ab}$ and $\chi_{ab}$, respectively. We find three tensor fields, built from $h_{ab}$ and $\chi_{ab}$, whose vanishing characterizes this local imbeddability. In three dimensions one of these tensor fields vanishes identically, and the remaining two are given by the variational derivatives of the Sen–Chern–Simons functional. Thus the stationary points of the Sen–Chern–Simons functional are precisely the initial data sets that can be imbedded, at least locally, into some conformally flat spacetime.

Ultimately, one wants to use the results of this paper in the study of solutions to the Einstein equations. An obvious question is that of the dependence of $Y$ on $(h_{ab}, \chi_{ab})$, when the latter is evolved via the Einstein vacuum equations. This topic will be addressed in work in progress.

Our conventions are mostly the same as those of [2]. In particular, the wedge product of forms is defined to be the anti-symmetric part of the tensor product, the signature of the spacetime and spatial metrics is $(+, - , -)$ and $(-, - , -)$, respectively. The curvature $F_{(\alpha)(\beta)}$ of a covariant derivation $\nabla_{\alpha}$ on a vector bundle is defined by $F_{(\alpha)(\beta)} := w_{\alpha} \nabla_{\alpha}(w_{\beta} \nabla_{\beta} X^{\gamma}) - w_{\beta} \nabla_{\beta}(w_{\alpha} \nabla_{\alpha} X^{\gamma}) + [w_{\alpha}, w_{\beta}] \nabla_{\alpha} X^{\gamma}$. Finally, the Ricci tensor is $R_{ab} := R^{c}_{abc}$ and the curvature scalar is the contraction of $R_{ab}$ with the metric. Although we mostly use the abstract index notation, sometimes the differential form notation will also be used. Every mapping, section, tensor field, etc. will be smooth. Our general differential geometric reference is [3].

2 The Chern–Simons functional

2.1 The general Chern–Simons functional

Let $G$ be any Lie group, $\mathcal{G}$ its Lie algebra and $\pi : P \to \Sigma$ a trivializable principal fibre bundle over $\Sigma$ with structure group $G$. Since $P$ is trivializable, it admits global cross sections $\sigma : \Sigma \to P$. Let $V$ be a
$k$ dimensional real vector space, $\rho : G \to \text{GL}(V)$ a linear representation of $G$ on $V$, $\rho_* : \mathcal{G} \to \text{gl}(V)$ the corresponding representation of its Lie algebra and let $E(\Sigma)$ be the vector bundle over $\Sigma$ associated to $P$ with the linear representation $\rho$ of $G$ on $V$. Because of the trivializability of $P$, $E(\Sigma)$ is also trivializable, and hence it admits $k$ pointwise linearly independent global sections $e_{\alpha}^a, a = 1, \ldots, k$. We call such a system of global sections a global frame field. Any global cross section of $P$ can be interpreted as such a global frame field, and the ‘gauge transformations’ as certain $k \times k$-matrix valued functions $A^a_{\alpha}$ on $\Sigma$.

Any connection on $P$ determines a connection on $E(\Sigma)$, whose connection coefficients with respect to a global frame field form a $\rho(\mathbb{G}) \subset \text{gl}(k, \mathbb{R})$-valued 1-form $A^a_{\alpha \mu}$ on $\Sigma$. Here $\mu$ is the abstract tensor index referring to the manifold $\Sigma$. If $F^a_{\alpha \mu \nu} := \partial_{\alpha} A^a_{\mu \beta} - \partial_{\beta} A^a_{\alpha \mu} + A^b_{\alpha \mu} A^a_{\beta \nu} - A^b_{\alpha \nu} A^a_{\beta \mu}$, the curvature of the connection on $\Sigma$, then the Chern-Simons functional of the connection is the integral

$$Y[A] := \int_{\Sigma} \text{Tr} \left( F_{\mu \nu} A_{\rho} + \frac{2}{3} A_{\mu \nu} A_{\rho} A_{\lambda} \right) := \int_{\Sigma} \left( F^a_{\alpha \mu \nu} A^b_{\beta \gamma} + \frac{2}{3} A^a_{\alpha \mu} A^b_{\beta \gamma} A^c_{\gamma \mu} \right) \frac{1}{3!} \delta^a_{\mu \nu \rho}.$$  \hspace{1cm} (2.1.1)

Obviously, $Y[A]$ is invariant with respect to orientation preserving diffeomorphisms of $\Sigma$ onto itself. Recalling that under a gauge transformation $A^a_{\alpha}$ the connection and the curvature transform as

$$A^a_{\alpha \mu} \mapsto A'^{a'}_{\alpha' \mu} := A^a_{\alpha \mu} \left( A^a_{\alpha} \Lambda_{a}^{a'} + \partial_{\alpha} \Lambda_{a}^{a'} \right),$$

$$F^a_{\alpha \mu \nu} \mapsto F'^{a'}_{\alpha' \mu' \nu'} := A^a_{\alpha \mu} F^a_{a \mu' \nu'},$$

where $\Lambda_{a}^{a'}$ is defined by $A^a_{\alpha} \Lambda_{a}^{a'} = \frac{\delta^{a'}_{a}}{2}$, the Chern-Simons functional transforms as

$$Y[A] - Y[A'] = 2 \int_{\Sigma} \partial_{\alpha} \left( A^a_{\beta \alpha} \Lambda^{a'}_{a} \left( \partial_{\beta} \Lambda_{a}^{a'} \right) \right) \frac{1}{3!} \delta^a_{\mu \nu \rho} +$$

$$+ \frac{2}{3} \int_{\Sigma} A^a_{\beta \alpha} \left( \partial_{\alpha} A^a_{\beta \mu} \right) A^{a'}_{\alpha \mu} \left( \partial_{\beta} \Lambda^{a'}_{a} \right) \Lambda^{a'}_{a} \left( \partial_{\gamma} \Lambda_{a}^{a'} \right) \frac{1}{3!} \delta^a_{\mu \nu \rho}.$$  \hspace{1cm} (2.1.2)

Its first term on the right is zero by $\partial \Sigma = \emptyset$. First suppose that the gauge transformation $\Lambda_{a}^{a'}$ is homotopic to the identity transformation, i.e. there is a 1 parameter family of global gauge transformations $\Lambda_{a}^{a'}(t), t \in [0, 1]$, such that $\Lambda_{a}^{a'}(0) = \delta^{a'}_{a}$ and $\Lambda_{a}^{a'}(1) = \Lambda_{a}^{a'}$ (‘small gauge transformations’). Then substituting $\Lambda_{a}^{a'}(t)$ into (2.1.2) and taking the derivative with respect to $t$ at $t = 0$ we obtain that the right hand side is vanishing; i.e. the Chern-Simons functional is invariant with respect to small gauge transformations. For general gauge transformations, however, the second term on the right of (2.1.2) is not zero. In fact, as a consequence of the integrality of the second Chern class, for the left hand side of (2.1.2) we have (see e.g. [1])

$$Y[A] - Y[A'] = 16 \pi^2 N,$$  \hspace{1cm} (2.1.3)

for some integer $N$ depending on the global gauge transformation $\Lambda_{a}^{a'}$. We will see that the geometric content of this formal result is connected with a certain homotopy invariant of the mapping $\Lambda : \Sigma \to G$. In particular, for $G = SO(3)$ or $SO(1, 3)$, the connected component of $SO(1, 3)$, $N$ is just the integer that can be interpreted as twice the winding number of $\Lambda$.

Finally, let us consider any smooth 1 parameter family $A^a_{\alpha \mu}(t)$ of connections on $E(\Sigma)$ and the corresponding Chern-Simons functional $Y[A(t)]$. Then

$$\delta Y[A] := \left. \frac{d}{dt} Y[A(t)] \right|_{t = 0} = 2 \int_{\Sigma} \left( \text{Tr} \left( F_{\mu \nu} \delta A_{\rho} \right) + \partial_{\mu} \left( \text{Tr} \left( A_{\nu} \delta A_{\rho} \right) \right) \right),$$  \hspace{1cm} (2.1.4)

where $\delta A^a_{\alpha \mu} := \left. \left( \frac{d}{dt} A^a_{\alpha \mu}(t) \right) \right|_{t = 0}$, the ‘variation’ of the connection 1-form. Thus $Y[A]$ is functionally differentiable and the derivative is essentially the curvature.
Let $R \to \Sigma$ be a trivializable principal bundle over $\Sigma$ with structure group $SO(3)$, $\rho$ the defining representation of $SO(3)$ and let $E(\Sigma)$ be the associated trivializable vector bundle. The global sections of $R$ can be interpreted as globally defined frame fields $E_i^a$ of $E(\Sigma)$, $i = 1, 2, 3$. Then one can introduce the negative definite fibre metric $h_{ab}$ for which $E_i^a$ and hence any frame field obtained from $E_i^a$ by the action of $SO(3)$ is orthonormal; i.e. if $\vartheta_i^a$ is the basis dual to $E_i^a$ and $\eta_{ij} := \text{diag}(-1,-1,-1)$, then $h_{ab} := \vartheta_i^a \vartheta_j^b \eta_{ij}$. (The dual basis can also be interpreted as a vector bundle isomorphism $E(\Sigma) \to \Sigma \times \mathbb{R}^3: (p, X^a) \mapsto (p, X^a)$ and the fibre metric $h_{ab}$ is the pull back of the constant metric $\eta_{ij}$ along $\vartheta_i^a$.) Any connection on $R$ determines a covariant derivation $D_a$ on $E(\Sigma)$, annihilating the fibre metric $h_{ab}$. If $E_i^a$, $\vartheta_i^a$ is a pair of dual global $h_{ab}$-orthonormal frame fields, then the connection can be characterized completely by its connection coefficients $\gamma_{ij}^a := \vartheta_i^a D_j E_i^a$.

Since for any orientable $3$-manifold the tangent bundle is trivializable, there is a vector bundle isomorphism, the so-called soldering form, between the tangent bundle and the abstract vector bundle $E(\Sigma)$. It is $\theta : T\Sigma \to E(\Sigma) : (p, v^a) \mapsto (p, v^a \theta_i^a)$. By means of the soldering form $T\Sigma$ and $E(\Sigma)$ can be identified (and there will not be any difference between the Greek and Latin indices) and $h_{ab}$ will be a metric on $T\Sigma$. Obviously, for any fixed soldering form, there is a one-to-one correspondence between the negative definite metrics on $T\Sigma$ and the global frame fields in $E(\Sigma)$ modulo the $SO(3)$ action. Furthermore, the connection on $E(\Sigma)$ determines a linear metric connection on $T\Sigma$. Requiring the vanishing of the torsion of this linear connection, the connection on $E(\Sigma)$ will be completely determined and the connection coefficients $\gamma_{ij}^a$ become the Ricci rotation coefficients of the Levi-Civita connection. Thus the Chern–Simons functional, built from that connection on $E(\Sigma)$ whose pull back to $T\Sigma$ is the Levi-Civita one, is completely determined by $E_i^a$. Consequently, for such connections $Y$ will be a second order functional of $E_i^a$, invariant with respect to homotopically trivial gauge transformations, but it will depend on the homotopy class of the global frame field on $\Sigma$. Therefore $h_{ab}$ determines $Y[E_i^a]$ modulo $16\pi^2$ only.

To understand the root of this obstruction, recall that $\Lambda : \Sigma \to SO(3)$ is a proper map (i.e. the inverse image of any compact subset of $SO(3)$ is compact, because $\Sigma$ itself is compact) and $\dim \Sigma = \dim SO(3)$. Thus there is an integer, $\text{deg}(\Lambda)$, the degree of $\Lambda$, such that for any $3$-form $\omega$ on $SO(3)$ $\int_{\Sigma} \Lambda^* (\omega) = \text{deg}(\Lambda) \int_{SO(3)} \omega$ [5]. In particular, for the normalized invariant volume element of $SO(3)$, $dv := \frac{1}{4\pi^2} \text{Tr}((\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda))$, by (2.1.2) and (2.1.3) we have

$$\text{deg}(\Lambda) = \frac{1}{4\pi^2} \int_{\Sigma} \Lambda^* (dv) = \frac{1}{4\pi^2} \int_{\Sigma} \text{Tr} \left( (\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \right) = \frac{1}{2} N.$$ (2.2.1)

But $\text{deg}(\Lambda)$ counts how many times $\Sigma$ covers the rotation group by the mapping $\Lambda$, and hence $N$ may be interpreted as twice the winding number of the map $\Lambda : \Sigma \to SO(3)$. In particular, for $\Sigma \simeq S^3$ the homotopy classes of the mapping $\Lambda$ are precisely the elements of $\pi_3(SO(3))$.

A direct calculation shows that $Y[E_i^a]$ is invariant with respect to the conformal rescalings $E_i^a \mapsto \Omega^{-1} E_i^a$, and hence $Y[E_i^a]$ modulo $16\pi^2$ is a conformal invariant of $(\Sigma, h_{ab})$. Any $1$-parameter family $E_i^a(t)$ of global frame fields yields a $1$-parameter family $\gamma_{ij}^a(t)$ of connection coefficients, i.e. any variation $\delta_{ij}^a$ determines a variation $\delta \gamma_{ij}^a$. Thus any variation of the metric $h_{ab}$ determines the variation of the connection coefficients, apart from an unspecified small gauge transformation. Then by (2.1.4) it is a straightforward calculation to show that the variational derivative of $Y[E_i^a]$ with respect to $h_{ab}$ is well defined and it is the Cotton–York tensor [1]. Thus the stationary points of the $SO(3)$ Chern–Simons functional are, in fact, the conformally flat Riemannian metrics. It is this picture that we generalize in finding our conformal invariant of initial data sets in the next section.
3 The Chern–Simons invariant of initial data sets

3.1 The Lorentzian vector bundle

Let \( L \to \Sigma \) be a trivializable principal bundle over \( \Sigma \) with the structure group \( SO(1, 3) \), \( \rho \) its defining representation and let \( V(\Sigma) \) be the associated vector bundle. \( V(\Sigma) \) is therefore a trivializable real vector bundle of rank 4 over \( \Sigma \). The global sections of \( L \) can be considered as globally defined frame fields \( e_a^\mu \), \( a = 0, \ldots, 3 \), with given ‘space’ and ‘time’ orientation; and one can define the Lorentzian fibre metric \( g_{ab} \) on \( V(\Sigma) \) for which \( e_a^\mu \) is orthonormal. Explicitly, if \( \xi^a \) is the basis dual to \( e_a^\mu \) and \( \eta_{ab} := \text{diag}(1, -1, -1, -1) \) then \( g_{ab} := \xi^a \xi^b \eta_{ab} \). \( \xi^a \) can also be interpreted as a vector bundle isomorphism \( V(\Sigma) \to \Sigma \times \mathbb{R}^4 : (p, X^a) \mapsto (p, X^a) \) and \( g_{ab} \) as the pull back of \( \eta_{ab} \) from \( \Sigma \times \mathbb{R}^4 \) to \( V(\Sigma) \).

Since both \( T\Sigma \) and \( V(\Sigma) \) are trivializable, there are imbeddings \( \Theta : T\Sigma \to V(\Sigma) : (p, r^a) \mapsto (p, r^a \Theta^a_\alpha) \) such that the vectors \( r^a \Theta^a_\alpha \) are all spacelike with respect to the fibre metric \( g_{ab} \). Or, in other words, the pull back of \( g_{ab} \) along \( \Theta \), \( h_{ab} := \Theta^a_\alpha \Theta^b_\beta g_{ab} \), is a negative definite metric on \( T\Sigma \). Thus \( \Theta(T_p \Sigma) \) is a spacelike subspace of the fibre \( V_p \) in \( V(\Sigma) \) over \( p \in \Sigma \), and hence, apart from a sign, there is a uniquely determined global section \( t^a \) of \( V(\Sigma) \) which has unit norm with respect to \( g_{ab} \), and is a normal of \( \Theta(T\Sigma) \): \( t^a \Theta^a_\alpha t_\alpha = 0 \) for all \( t^a \) tangent vector of \( \Sigma \). The orientation of \( t^a \) will be chosen to be compatible with the ‘time’ orientation above. Then \( P^a_b := \delta^a_b - t^a t_b \) is the projection of the fibre \( V_p \) onto \( \Theta(T_p \Sigma) \) at each point \( p \) of \( \Sigma \). Thus if \( X^a \) is any section of \( V(\Sigma) \) then it can be decomposed in a unique way as \( X^a = N t^a + N^a \), where \( N \) is a function and \( N^a \) is a section of \( V(\Sigma) \) such that \( P^a_b N^b = N^a \). \( N \) and \( N^a \) may be called the lapse and shift parts of \( X^a \), respectively. Obviously, this decomposition depends on the imbedding \( \Theta \). Any such decomposition of the sections of \( V(\Sigma) \) into its lapse and shift parts defines a vector bundle isomorphism \( \iota \) between \( V(\Sigma) \) and the Whitney sum of the trivial line bundle \( \Sigma \times \mathbb{R} \) and \( T\Sigma \). For fixed \( \Theta \) we can, and in fact we will, identify the tangent bundle \( T\Sigma \) with its \( \Theta \)-image in \( V(\Sigma) \). Then the Greek indices become \( P^a_b \)-projected Latin indices. In spite of this identification we use the Greek indices if we want to emphasize that they are indices tangential to \( \Sigma \). Obviously, the negative definite metric \( h_{ab} \) does not fix the Lorentzian fibre metric \( g_{ab} \) completely: \( g_{ab} \) and \( \tilde{g}_{ab} \), determined by the same spatial metric \( \tilde{g}_{ab} = g_{ab} + \tau t_\alpha t_\beta \), where \( \tau : \Sigma \to (-1, \infty) \) is an arbitrary function. If \( X^a \) is any section of \( V(\Sigma) \) then, under the transformation \( g_{ab} \to g_{ab} + \tau t_\alpha t_\beta \), its lapse part transforms as \( N \to \sqrt{1 + \tau} N \), and hence this freedom corresponds to the pure rescaling of the lapse and the changing of the vector bundle isomorphism \( \iota \) above. Thus the Lorentzian vector bundle \( V(\Sigma) \) is completely determined by \( h_{ab} \) and the knowledge of the lapse and shift parts of its sections. The vector bundle \( V(\Sigma) \) can be interpreted as the restriction of the spacetime tangent bundle \( TM \) to an imbedded spacelike hypersurface \( \Sigma \), and \( \Theta \) as the differential of the injection \( \Sigma \to M \).

A \( g_{ab} \)-orthonormal global frame field will be said to be compatible with the imbedding \( \Theta \) if it is of the form \( \{ t^a_i, E_i^\mu \} \), \( i = 1, 2, 3 \). Thus \( E_i^\mu \) is a triad of orthonormal vectors tangent to the distribution \( \Theta(T\Sigma) \) everywhere. The set of all such \( \Theta \)-compatible frame fields defines a reduction \( SO(1, 3) \to SO(3) \) of the gauge group (‘time gauge’). As the next lemma shows, there is no topological obstruction excluding the possibility of such a gauge reduction.

**Lemma 3.1.1:** For any global frame field \( e_a^\mu \) there exists a globally defined one parameter family of Lorentz transformations \( \Lambda(t) : \Sigma \to SO(1, 3) \), \( t \in [0, 1] \), such that \( \Lambda e_0^a(0) = \frac{e_0^a}{\sqrt{2}} \) and \( \Lambda e_0^a(1) \) takes \( e_a^\mu \) into a \( \Theta \)-compatible frame field.

**Proof:** Because of the trivializability of \( L \), there are globally defined Lorentz transformations taking \( e_a^\mu \) into a \( \Theta \)-compatible global frame. These transformations are unique only up to spatial rotations keeping the normal \( t^0 \) fixed. Or, in other words, we search for global Lorentz transformations modulo rotations, i.e. an element of the coset space \( SO(1, 3)/SO(3) \) being homotopic to the identity. But \( SO(1, 3)/SO(3) \) is homeomorphic to...
\( \mathbb{R}^3 \), which is a contractible topological space. Hence any two mappings \( \Sigma \to SO(1,3)/SO(3) \) are homotopic. In particular, there is a Lorentz transformation, taking \( e_2 \) into a \( \Theta \)-compatible frame, which is homotopic to the identity transformation.

This Lemma implies that there is a natural one-to-one correspondence between the homotopy classes of the global rotations \( \Sigma \to SO(3) \) and of the Lorentz transformations \( \Sigma \to SO_h(1,3) \).

### 3.2 The real Sen connection

Any connection on \( L \) determines a covariant derivation \( \mathcal{D} \) on \( V(\Sigma) \) which annihilates the Lorentzian fibre metric \( g_{ab} \). However, we would like to build up our connection from the tensor fields \( h_{ab}, \chi_{ab} \) of the initial data set. Thus we follow the philosophy of subsection 2.2 in tying the connection with the fields on \( \Sigma \), and we specify \( \mathcal{D} \) by imposing the following restrictions on its action on independent sections of \( V(\Sigma) \):

1. For the normal section \( t_a \) let us define \( \chi_{ab} := \mathcal{D}_a t_b \), and for which we require that \( \chi_{ab} = \chi_{(ab)} \).
2. For vector fields \( v^a \) on \( \Sigma \) we require that (\( \mathcal{D}_a v^b \) and \( \mathcal{D}_a \chi_{ab} \)) are homotopic.

Thus, for given \( \Theta \), the covariant derivation \( \mathcal{D} \) is completely determined by \( g_{ab} \) and \( \chi_{ab} \); i.e. for given \( t \), \( \mathcal{D} \) is completely determined by the initial data set. Suppose for a moment that \( \Sigma \) is a spacelike hypersurface in a Lorentzian spacetime \( (M, g_{ab}) \), \( \nabla_e \) is the four dimensional Levi-Civita covariant derivation and define \( \mathcal{D}_a := P^b_e \nabla_b \), the so-called 3-dimensional Sen operator [6]. Obviously \( \mathcal{D}_a \) is well defined on any tensor field defined on the submanifold \( \Sigma \), it annihilates the spacetime metric and satisfies the requirements i. and ii. above. It is easy to prove the converse of this statement, namely that the differential operator on the restriction to \( \Sigma \) of the spacetime tangent bundle satisfying i. and ii. and annihilating the spacetime metric is unique. Thus we call the connection satisfying i. and ii. the real Sen connection on \( V(\Sigma) \). The contraction of (3.2.1) with \( t_a \) and the projection of it to \( \Theta(T\Sigma) \), respectively, are

\[
(\mathcal{D}_a X^a) t_a = D_a N - \chi_{ab} N^a,
\]

(3.2.2)

\[
(\mathcal{D}_a X^a) P^b_a = D_a N^b + N^b \chi_{ab}.
\]

(3.2.3)

Thus \( \mathcal{D} \) can also be considered as a covariant derivation on the bundle of the pairs \( (N, N^a) \) on \( \Sigma \), the Whitney sum of the trivial real line bundle \( \Sigma \times \mathbb{R} \) and \( T\Sigma \).

Next calculate the action of the commutator of two \( \mathcal{D} \)'s on functions and on sections of \( V(\Sigma) \):

\[
(\mathcal{D}_e \mathcal{D}_f - \mathcal{D}_f \mathcal{D}_e) \phi = -2 \chi_{ab} t_f D_b \phi,
\]

(3.2.4)

\[
(\mathcal{D}_e \mathcal{D}_f - \mathcal{D}_f \mathcal{D}_e) X^a = -2 \chi_{ab} t_f D_b X^a - \left( R_{bef}^a + \chi_{ae} \chi_{bf} - \chi_{af} \chi_{be} \right) X^b - (t^a (D_e \chi_{fb} - D_f \chi_{eb}) - t_b (D_e \chi_{af} - D_f \chi_{ae})) X^b.
\]

(3.2.5)
where $R^a_{bcf}$ is the curvature tensor of the Levi-Civita connection of $(\Sigma, h_{ab})$. Then one can read off the curvature and the ‘torsion’ of the Sen connection:

$$F^a_{\ bcf} := R^a_{bcf} + \chi^a_{\ e} \chi_{bf} - \chi^a_{\ f} \chi_{be} + t^a_{\ e}(D_t \chi_{fb} - D_f \chi_{tb}) - t_b(D_t \chi_{bf} - D_f \chi_{tb}),$$

(3.2.6)

$$T^a_{\ eb} := 2\chi^a_{\ [eb]}.$$  

(3.2.7)

Thus $F^a_{\ bcf}$ represents the Gauss and Codazzi tensors, built from the initial data $h_{ab}$ and $\chi_{ab}$, appearing in the 3+1 decomposition of the curvature tensor of a Lorentzian spacetime. Namely, if $\Sigma$ is a spacelike hypersurface in $(M, g_{ab})$ and $M R^a_{bcdf}$ is the spacetime curvature tensor then $F^a_{\ bcf} = M R^a_{bcdf} P^e P^d$. Note that $F^a_{\ bcf}$ is the curvature in the strict sense of differential geometry [3]; i.e. it is a globally defined $so(1, 3)$ Lie algebra valued 2-form on $\Sigma$. On the other hand, $T^a_{\ eb}$ is not a torsion in the strict sense, because the torsion is defined only for connections on principal bundles that are reduced subbundles of the linear frame bundle of the base manifold; i.e. if there is a soldering form. The true torsion, the pull back to the base manifold of the covariant exterior derivative of the soldering form, is always a vector valued 2-form on the base manifold. Here $T^a_{\ eb}$ is not such a 2-form on $\Sigma$, its projection to $\Sigma$ is zero.

If $\epsilon^a_2 \equiv \zeta^a_2 = \epsilon^a_2$ is a pair of dual $g_{ab}$-orthonormal frame fields then we can define the connection coefficients of the Sen connection with respect to these frames by $\Gamma^a_{b\ k} := \zeta^a_2 D_a \epsilon^k_2$. These form a globally defined $so(1, 3)$ matrix Lie algebra valued 1-form on $\Sigma$, and the tetrad components of the curvature in its ‘internal indices’, $F^a_{\ bcf} := \epsilon^a_2 \epsilon^c_2 F^a_{\ bcf}$, are built up from the connection components $\Gamma^a_{b\ k}$ in the well known manner.

Finally, let us consider the behaviour of the various quantities under conformal rescalings. For any function $\Omega : \Sigma \to (0, \infty)$ the conformal rescaling of the fibre metric, $g_{ab} \mapsto \tilde{g}_{ab} := \Omega^2 g_{ab}$, determines the rescaling of the spatial metric: $h_{ab} \mapsto \tilde{h}_{ab} := \Omega^2 h_{ab}$, but it doesn’t determine the rescaling of $\chi_{ab}$. However, recalling how the extrinsic curvature of a spacetime hypersurface behaves under a conformal rescaling of the spacetime metric, the new $\chi_{ab}$ is expected to depend on an additional independent function $\Omega : \Sigma \to \mathbb{R}$ too, and we define the new $\chi_{ab}$ by $\tilde{\chi}_{ab} := \Omega \chi_{ab} + \tilde{\Omega} h_{ab}$. If, for the sake of later convenience, we define $\Upsilon := D_t (\ln \Omega)$ and $\omega := \Omega^{-1} \tilde{\Omega}$, then the behaviour of the Levi-Civita and Sen derivations, respectively, are

$$D_t X^a = D_t X^a + \left( P^a_{\ b} \Upsilon^b + P^a_{\ c} \Upsilon^c - h_{ab} h^{cd} \Upsilon^d \right) X^b,$$

(3.2.8)

$$D_t X^a = D_x X^a + \left( P^a_{\ b} \Upsilon^b + P^a_{\ c} \Upsilon^c - h_{ab} h^{cd} \Upsilon^d \right) X^b + \omega (P^a_{\ b} t^b - \Omega^2 h_{ab}) X^b.$$  

(3.2.9)

One can now calculate the conformal behaviour of the curvature of the Levi-Civita connection, of the ‘torsion’ and of the curvature of the Sen connection:

$$\Omega^2 \tilde{R}_{\ cde} = R^c_{\ cde} + 4P^a_{\ [e} (D_{[d} \Upsilon_{b]} - \Upsilon_{d} \Upsilon_{b]}),$$

(3.2.10)

$$\tilde{T}^a_{\ ceb} = T^a_{\ ceb} + 2\omega P^a_{\ [c} t_{b]},$$

(3.2.11)

$$\Omega^2 \tilde{F}_{\ cde} = F_{\ cde} + 4P^a_{\ [e} \left( (D_{[d} \Upsilon_{b]} - \Upsilon_{d} \Upsilon_{b]} + \chi_{d} \chi_{b]} \omega \right) + \Omega \left( D_{d} \omega - \Upsilon_{d} + \chi_{d} \Upsilon_{e} \right) \right) +$$

$$+ P^a_{\ [c} (\Upsilon_{e} \Upsilon_{f} + \Omega^2),$$

(3.2.12)

where $P^a_{\ cde} := P^a_{\ pde} - P^a_{\ ped}$. If $\epsilon^a_2 \equiv \zeta^a_2 = \epsilon^a_2$ is a pair of dual orthonormal bases, then, under the conformal rescaling, they must be rescaled as $\epsilon^a_2 \mapsto \epsilon^a_2 := \Omega^{-1} \epsilon^a_2 \equiv \zeta^a_2$ and $\zeta^a_2 \mapsto \zeta^a_2 := \Omega \zeta^a_2$. Thus the behaviour of the connection coefficients and the curvature components in such a basis are
\[
\begin{align*}
\tilde{\Gamma}_{cd}^e &= \Gamma_{cd}^e + \zeta^e_{ab} \left( P_c \gamma_f - h_{cf} \gamma_e \right) c_d^f + \omega \zeta^e_{ab} \left( P_c t_f - t^c h_{cf} \right) c_d^f, \\
\tilde{F}_{cd} &= \zeta^e_{ab} \tilde{F}_{bced},
\end{align*}
\] (3.2.13)

where \( \tilde{F}_{bced} \) is given by (3.2.12).

### 3.3 The Sen–Chern–Simons functional on \( V(\Sigma) \)

Following the general prescription of subsection 2.1, we can introduce the Chern–Simons functional \( Y[\Gamma] \), built from the real Sen connection on the trivializable vector bundle \( V(\Sigma) \). We call \( Y[\Gamma] \) the Sen–Chern–Simons functional. Using formulae (3.2.12-14) it is a lengthy but straightforward calculation to derive how \( Y[\Gamma] \) transforms under conformal rescalings:

\[
Y[\Gamma] - Y[\tilde{\Gamma}] = \int_{\Sigma} D \left( \varepsilon^{abc} (\delta \gamma_e) c^e_d \right) \left( P_c \gamma_f - h_{cf} \gamma_e \right) c_d^f, \quad (3.3.1)
\]

where \( d\Sigma := \frac{1}{6} \varepsilon_{\alpha\beta\gamma} \), the metric volume element determined by the 3-metric \( h_{\alpha\beta} \). Thus for compact \( \Sigma \) the Sen–Chern–Simons functional is invariant with respect to rescalings that correspond to spacetime conformal rescalings; i.e. \( Y[\Gamma] \) modulo \( 16\pi^2 \) is a conformal invariant of the initial data set. Since by Lemma 3.1 there is a one-to-one correspondence between the homotopy classes of the global rotations \( \Sigma \rightarrow SO(3) \) and the global Lorentz transformations \( \Sigma \rightarrow SO(1,3) \), the integer \( N \) in (2.2.1) can still be interpreted as twice the winding number of the global Lorentz transformation.

Since for fixed \( \Gamma \) the real Sen connection is completely determined by \( h_{\alpha\beta} \) and \( \chi_{\alpha\beta} \), \( Y[\Gamma] \) can also be considered as a second order functional of the frame field \( c^e_d \) and a first order functional of \( \chi_{\alpha\beta} \). Similarly to the Riemannian case, any variation \( \delta h_{\alpha\beta} \) of the 3-metric yields a variation \( \delta_1 \Gamma_{\alpha\beta\gamma}^e \) of the connection coefficients and an unspecified small gauge transformation, and any variation \( \delta \chi_{\alpha\beta} \) yields a variation \( \delta_2 \Gamma_{\alpha\beta\gamma}^e \). Thus the variational derivatives of \( Y[\Gamma] \) with respect to \( h_{\alpha\beta} \) and \( \chi_{\alpha\beta} \) are well defined, and, using the general formula (2.1.4), these derivatives can be calculated. Since by Lemma 3.1 the pure boost gauge transformations are all small, these calculations can be carried out in the time gauge, where the formulae are considerably simpler. The results are

\[
\begin{align*}
\frac{\delta Y}{\delta \chi_{ab}} &= -8 \sqrt{|h|} \varepsilon^{cd(a} D_c \chi^{b)} d = \\
&= :8 \sqrt{|h|} H_{ab}, \quad (3.3.2) \\
\frac{\delta Y}{\delta h_{ab}} &= -4 \sqrt{|h|} \left( Y_{ab} - \varepsilon^{cd(a} D_c \left( \chi^{b)} d - \chi^{b)} e \chi_{ed} \right) - \frac{1}{2} \chi^{b)} e \left( D_e \chi_{cd} - D_d \chi \right) \right) + H^{(a} \chi^{b)} e = \\
&= : -4 \sqrt{|h|} \left( B_{ab} + H^{(a} \chi^{b)} e \right). \quad (3.3.3)
\end{align*}
\]

Here \( Y_{ab} := -\varepsilon_{cd(a} D^c R^d_{b)} \), the Cotton–York tensor of the intrinsic 3-geometry; and \( H_{ab} \) would play the role of the magnetic part of the Weyl curvature of the spacetime \((M, g_{\alpha\beta})\) if \( \Sigma \) were a spacelike hypersurface in \( M \). Both \( H_{ab} \) and \( B_{ab} \) are symmetric and trace free. Although, by (3.3.1), \( Y[\Gamma] \) is invariant with respect to any finite conformal rescaling, by (3.3.2) and (3.3.3) it is easy to prove directly its invariance with respect to infinitesimal conformal rescalings: If \((\Omega(t), \Omega^t)\) is a 1-parameter family of conformal factors such that \( \Omega(0) = 1 \) and \( \Omega(0) = 0 \), then
\[
\delta Y[\Gamma] := \left( \frac{d}{dt} Y[\Gamma(t)] \right)|_{t=0} = \int_{\Sigma} \left( \frac{\delta Y}{\delta h_{ab}} 2\delta \Omega h_{ab} + \frac{\delta Y}{\delta \chi_{ab}} (\delta \Omega \chi_{ab} + \delta \tilde{\Omega} h_{ab}) \right) d\Sigma = 0, \tag{3.3.4}
\]

where \(\delta \Omega := (\frac{d}{dt} \Omega(t))|_{t=0}\) and \(\Delta \Omega := (\frac{d}{dt} \tilde{\Omega}(t))|_{t=0}\). We give a geometric characterization of the stationary points of the Sen-Chern-Simons functional, \(B_{ab} = 0\) and \(H_{ab} = 0\), in section four.

### 3.4 The Ashtekar–Chern–Simons functional on \(\pm \Lambda^2(\Sigma)\)

Next we are constructing another representation of the gauge group, \(SO(1, 3)\), and the associated vector bundle. This will be the self-dual/anti-self-dual representation. We will see that the Chern–Simons functional constructed in this vector bundle is not invariant with respect to the conformal behaviour introduced in the second subsection. Thus the conformal invariance depends on the actual representation too.

To start with, let \(\Lambda^2(\Sigma)\) be the vector bundle of 2-forms on the fibres of \(V(\Sigma)\); i.e. the fibre of \(\Lambda^2(\Sigma)\) over a point \(p \in \Sigma\) is \(V_p^* \wedge V_p^*\). \(\Lambda^2(\Sigma)\) is a trivializable, real vector bundle over \(\Sigma\). The fibre metric \(g_{ab}\) on \(V(\Sigma)\) defines a fibre metric on \(\Lambda^2(\Sigma)\) by \(\langle W, Z \rangle := 2g^{ac}g^{bd}W_{ab}Z_{cd}\), for any \(W_{ab} = W[Z_{ab}]\) and \(Z_{ab} = Z_{[ab]}\). If \(\xi, \eta, \zeta, \nu = 0, \ldots, 3, \) is a basis in \(V_p^*\) (or a global frame field for \(V^*(\Sigma)\)), then \(\xi^a \xi^b, a < b, \) form a basis for \(V_p^*\) (or in \(\Lambda^2(\Sigma)\)), and \(\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = g^{ac}g^{bd} - g^{ad}g^{bc}\). Thus if \(\xi^a\) is \(g_{ab}\)-orthonormal, then \(\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = \left\{ \zeta^a \wedge \zeta^b, \zeta^c \wedge \zeta^d \right\}\), \(a, j, k, \ldots = 1, 2, 3, \) is \(-1\) orthornormal and \(\langle \zeta^a \wedge \zeta^b, \zeta^c \wedge \zeta^d \rangle = 1\); i.e. the signature of \(\langle \cdot \rangle\) is \((- - - - + + + )\).

Let \(\varepsilon_{abcd}\) be the \(g_{ab}\)-volume form on the fibres of \(V(\Sigma)\), and introduce the duality mapping in the standard way: \(* : \Lambda^2(\Sigma) \rightarrow \Lambda^2(\Sigma) : W_{ab} \mapsto *W_{ab} := \frac{1}{2} \varepsilon_{abcd} W_{cd}\). Then \(\langle *W, Z \rangle = \langle W, *Z \rangle\) and \(* * = -I_{\Lambda^2(\Sigma)}\). The dual eigenvalues of the linear mapping \(*\) are \(\pm i\), and hence its eigenvectors belong to \(\Lambda^2(\Sigma) \oplus C\), the complexification of \(\Lambda^2(\Sigma)\). \(* W_{ab} := \frac{1}{2}(W_{ab} + i * W_{ab})\) are called the self-dual/anti-self-dual part of the (real) 2-form \(W_{ab}\). Thus the complexification of \(\Lambda^2(\Sigma)\) can be decomposed in a natural way as the Wirtinger sum of two of its subbundles: \(\Lambda^2(\Sigma) \oplus C = \Lambda^2(\Sigma) \oplus -\Lambda^2(\Sigma)\) \(\Lambda^2(\Sigma)\) are the bundle of self-dual/anti-self-dual 2-forms, respectively, over \(\Sigma\). They are trivializable complex vector bundles of rank 3 over \(\Sigma\).

If \(\xi^a\) is any orthonormal dual global frame field then \(\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = -\varepsilon^{abcd}\), where \(\varepsilon^{abcd}\) is the anti-symmetric Levi–Civita symbol, by means of which it is easy to calculate the self-dual/anti-self-dual part of the basis 2-forms. One has \(\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = \pm i \varepsilon^{abcd}\). Thus \(\pm \xi_{ab} := 4\varepsilon_{abcd}\), \(i = 1, 2, 3,\) form a basis in \(\Lambda^2(\Sigma)\) and \(\pm \xi^a \wedge \xi^b = 8\varepsilon^{abcd}\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = 0\). Therefore the self-dual and the anti-self-dual 2-forms are orthogonal to each other and, by \(\pm \xi^a = -\xi^a\), they are also complex conjugate of each other. In the time gauge, i.e. if the pair of orthonormal global dual frame fields is \(\{\tilde{t}^a, \tilde{F}^a\}, \{\tilde{t}^a, \tilde{v}_a^b\}\), the contraction of the normal section of \(V(\Sigma)\) and the basis vectors of \(\pm \Lambda^2(\Sigma)\) is \(\varepsilon^{abcd}\langle \xi^a \wedge \xi^b, \xi^c \wedge \xi^d \rangle = \tilde{v}_a^b\). Therefore, in the time gauge, \(\pm \Lambda^2(\Sigma)\) can be identified with the complexified tangent bundle \(T\Sigma \oplus C\) and its complex conjugate bundle, respectively, and \(\tilde{v}_a^b\) can be chosen as a basis both in \(\Lambda^2(\Sigma)\) and in \(-\Lambda^2(\Sigma)\).

The real Sen connection on \(V(\Sigma)\) defines a unique connection on the vector bundles \(\pm \Lambda^2(\Sigma)\) by

\[
D_x \pm \Lambda^2 = \frac{1}{2} \left( D_x W_{ab} + \frac{1}{2} \varepsilon_{abcd} (D_x W_{cd}) \right). \tag{3.4.1}
\]

Thus if \(\{\varepsilon^a\}, \{c^a\}\) is a pair of dual \(g_{ab}\)-orthonormal global frame fields in \(V(\Sigma)\) and the corresponding connection coefficients of the real Sen connection are \(\Gamma^b_{ac} := \xi^b \varepsilon^a\), then the \(D_x\)-derivative of the basis fields are \(D_x \pm \xi^a = -(\Gamma^b_{ac} \pm i \Gamma^b_{ak} \varepsilon^{kb} \xi^b) \pm \varepsilon^a\); i.e. the connection coefficients of the connection (3.4.1) in the basis \(\pm \xi^a\) are

\[
\pm A^a_{bc} := \Gamma^a_{bc} \pm i \Gamma^a_{ak} \varepsilon^{kb} \xi^b, \tag{3.4.2}
\]
where $\varepsilon_{ijk} := \varepsilon_{bijk}$. In the time gauge, when $\Gamma^i_{kj}$ reduces to the Ricci rotation coefficients $\gamma^i_{kj}$ of the spatial metric $h_{ab}$ in the spatial basis $\{E^a_i\}$ and $\Gamma^0_{k0} = -\chi_{k0} E^0_k$, $\pm A^i_{kj}$ become Ashtekar’s connection coefficients [7]:

$$
\pm A^i_{kj} = \gamma^i_{kj} \mp i \chi_{k0} E^0_k \varepsilon^{kij}.
$$

(3.4.3)

Next let us consider the Chern–Simons functional built from the connection \( \pm A^i_{kj} \) given by (3.4.2). \( Y[\pm A] \) can also be considered as a second order functional of \( \varepsilon^{a} \) and a first order functional of \( \chi_{ab} \). Before calculating their variational derivatives, it seems useful to introduce the following notation:

$$
V_{abcd} := \chi a \chi_{bd} - \chi_{ad} \chi_{bc}, \quad V_{ab} := V^r_{a} e_{b} = \chi a \chi_{b} - \chi_{ae} \chi_{eb}, \quad V := V^r e = \chi^2 - \chi_{ab} \chi^{ab},
$$

(3.4.4)

$$
J_a := D_a \chi^b - D_a \chi.
$$

(3.4.5)

The algebraic symmetries of \( V_{abcd} \) and \( V_{ab} \) are the same those of the Riemann and Ricci tensors, respectively. Then the tensors \( B_{ab} \) and \( H_{ab} \) of the previous subsection take the form:

$$
B_{ab} = -\varepsilon_{cd(a} D^c (R^{d)}_{b}) + V^{d)}_{b}, \quad H_{ab} = -\varepsilon_{cd(a} D^c \chi^{d)}_{b}.
$$

(3.4.6)

(3.4.7)

Then the variational derivatives of \( Y[\pm A] \) with respect to \( h_{ab} \) and \( \chi_{ab} \), calculated most easily in the time gauge, are

$$
\frac{\delta Y[\pm A]}{\delta \chi_{cb}} = 2 \sqrt{|h|} \left( H^{cb} \mp i \left( R^{cb} - \frac{1}{2} R h^{cb} + V^{cb} - \frac{1}{2} V h^{cb} \right) \right),
$$

(3.4.8)

$$
\frac{\delta Y[\pm A]}{\delta h_{ab}} = - \sqrt{|h|} \left( B^{ab} + \chi^{(a} H^{b) e} \right) \mp
$$

$$
\mp i \sqrt{|h|} \left( \varepsilon^{(a} \varepsilon^{b)} D_c D_d \chi f + \chi^{(a} \left( (R^{b)e} - \frac{1}{2} R h^{b)(e} R + V^{b)e} - \frac{1}{2} h^{b)(e} V \right) \right).
$$

(3.4.9)

Using these formulae the variation of the Ashtekar–Chern–Simons functional under the infinitesimal conformal rescaling of the previous subsection can be given easily:

$$
\delta Y[\pm A] = \pm \frac{i}{2} \int_{\Sigma} \left\{ \delta \hat{\Omega} (R + V) + 4 \delta \hat{\Omega} \left( D_c D^c \chi - D_c D_b \chi^{ab} - \chi_{ab} \left( R^{cb} - \frac{1}{2} R h^{cb} + V^{cb} - \frac{1}{2} V h^{cb} \right) \right) \right\} d \Sigma.
$$

(3.4.10)

Thus \( Y[\pm A] \) is not invariant even with respect to infinitesimal conformal rescalings. Thus the invariance of the functional depends not only on the connection on the principal bundle, but the actual representation \( \rho \) of the structure group; i.e. the associated vector bundle too.

The first term of the imaginary part on the right hand side of (3.4.9) can also be rewritten as

$$
\varepsilon^{(a} \varepsilon^{b)} D_c D_d \chi f = -2 \varepsilon^{(a} D_c H^{b) d} - D_c (\varepsilon^{(a} H^{b) e}) + \frac{1}{2} h^{a b} D_c J^e - \frac{1}{2} D^{(a} J^b).
$$

(3.4.11)

Thus for the stationary points of \( Y[\pm A] \) we have
\[ H_{ab} = 0, \quad (a.) \]
\[ R_{ab} - \frac{1}{2} R h_{ab} + V_{ab} - \frac{1}{2} V h_{ab} = 0, \quad (b.) \]
\[ \varepsilon_{a(d(a)D^e (R^b_d + V^d b))} = \frac{1}{2} \chi^e_{(a \varepsilon_b)cd} J^d, \quad (c.) \]
\[ D_{(a} J_{b)} = h_{ab} D_{e} J^e. \quad (d.) \]

Now b. implies \( R_{ab} + V_{ab} = 0 \) and d. implies that \( D_{(a} J_{b)} = 0 \). We will show that these two, together with \( H_{ab} = 0 \), imply the vanishing of \( J_a \). (\( B_{ab} = 0 \), i.e. c., will not be used in what follows.) First we show that \( J_a \) is constant. By \( H_{ab} = 0 \) we have \( D_{[d} \chi_{a]} = \frac{1}{2} h_{[d} \chi_{a]} \) and, using \( R_{ab} + V_{ab} = 0 \), a straightforward calculation shows that \( D_{[d} J_{a]} = \frac{1}{2} D_{[d} J_{a]} \), i.e. \( J_a \) is, in fact, constant. Then taking the divergence of b., we get \( \chi^b_{[d} J_{b]} = 0 \). Taking the divergence again and using \( D_{a} J_{b} = 0 \) we finally get \( J_a J^a = 0 \), i.e. by the definiteness of \( h_{ab} \), that \( J_a = 0 \). But \( R_{ab} + V_{ab} = 0 \) and \( D_{b} \chi_{[a} = 0 \) together is just the Gauss-Codazzi condition for the local isometric imbeddability of \( (\Sigma, h_{ab}, \chi_{ab}) \) in a flat spacetime with first and second fundamental forms \( h_{ab} \) and \( \chi_{ab} \), respectively.

4. The criterion of non-contortedness of the initial data sets

Let \( \Sigma \) be an \( n \) dimensional manifold, \( n \geq 3 \), \( h_{ab} \) a pseudo-Riemannian metric with signature \((p, q), p + q = n \), and \( \chi_{ab} \) a symmetric tensor field on \( \Sigma \). The triple \((\Sigma, h_{ab}, \chi_{ab})\) will be said to be \emph{locally imbeddable} into the \( n + 1 \) dimensional pseudo-Riemannian manifold \((M, g_{ab})\) as a non-null hypersurface if each point \( p \) of \( \Sigma \) has an open neighbourhood \( U \) and there is an imbedding \( \phi : U \to M \) such that \( h_{ab} = \phi^* g_{ab} \) and \( \chi_{ab} = \phi^* K_{ab} \), where \( K_{ab} \) is the extrinsic curvature of \( \phi(\Sigma) \) in \( M \): \( K_{ab} := P^e_p P^f_q \nabla_a t_f \). Here \( t_a \) is the unit normal of \( \phi(\Sigma) \), \( g^{cb} t_d t_b = \pm 1 \), and \( P^a_b := \delta^a_b + t^a t_b \), the projection to \( \Sigma \) (the \( n \) dimensional, or hypersurface, Kronecker delta). The triple will be called \emph{non-contorted} \cite{8} if it is locally imbeddable as a non-null hypersurface into some \emph{conformally flat geometry} \((M, g_{ab})\). As is well known \cite{8}, for \( n = 3 \) \((\Sigma, h_{ab}, \chi_{ab})\) is non-contorted if and only if the hypersurface twistor equation is completely integrable, i.e. it admits four linearly independent solutions.

In the present section we give an equivalent characterization of the non-contortedness in any dimensions greater than two by the vanishing of three tensor fields. In three dimensions one of these vanishes identically, while the others are precisely \( B_{ab} \) and \( H_{ab} \). Thus the stationary points of our conformal invariant are precisely the non-contorted initial data sets. In addition to the characterization of these stationary points, \( B_{ab} = 0 \) and \( H_{ab} = 0 \) provide a new criterion for the complete integrability of the hypersurface twistor equation. The main result of this section is the following statement:

**Proposition 4.1** The initial data set \((\Sigma, h_{ab}, \chi_{ab})\) is non-contorted if and only if the following tensor fields vanish:

\[ E^{cd}_{ab} := C^{cd}_{eb} + \left( V^{cd}_{eb} - \frac{4}{(n - 2)} p^e_p V^d_b q d + \frac{2}{(n - 1)(n - 2)} p^e_p p^3 d V \right) = 0, \quad (4.1.i) \]
\[ H^{ijk}_{ab} := p^e_p p^d_a p^k_b \chi^e_{ab} = 0, \quad (4.1.ii) \]
\[ B^{ijk}_{ab} := \frac{1}{(n - 2)} \left( D_{[d} T^d_{ab]} + D_{[c} (V^d_{ab} - \frac{1}{2(n - 1)} V P^d_b q d) \pm \frac{n - 2}{(n - 1)} \chi^e_{(ab} (D^e \chi_{b)} - D_{b]} \chi) \right) = 0. \quad (4.1.iii) \]
Here \( L_{ab} := -(\mathcal{R}_{ab} - \frac{1}{2(g^{-1})} R h_{ab}) \), \( C^{ab}_{cd} := R^{ab}_{cd} + \frac{4}{(g^{-1})} p^{[a}_{[c} P^{b]}_{d]} \) is the Weyl tensor of the metric \( h_{ab} \), and \( V_{abc} \) and \( V_{a} \) are defined by (3.4.4). The sign \pm \) corresponds to the sign of the length of the normal of \( \Sigma \) in the imbedding: \( g_{ab}^m \Omega^m = \pm 1 \).

**Proof:** First suppose that \( (\Sigma, h_{ab}, \chi_{ab}) \) is locally imbedded into the conformally flat \( (M, g_{ab}) \) and for the sake of simplicity we identify \( \Sigma \) with its \( \phi \)-image in \( M \). Let \( \tilde{g}_{ab} \) be a flat metric on \( M \) such that \( g_{ab} = \Omega^2 \tilde{g}_{ab} \) for some positive function \( \Omega \) on \( M \), and let \( \tilde{\nabla}_a \) be the corresponding flat Levi-Civita covariant derivation. Since \( (M, \tilde{g}_{ab}) \) is flat, there exist \((n + 1)\) linearly independent 1-form fields \( K_a \) which are constant with respect to the flat connection: \( \tilde{\nabla}_a K_b = 0 \). Let \( \nabla_a \) be the covariant derivation associated with the conformally flat metric \( g_{ab} \). If \( C^{ab}_{cd} \tilde{\nabla} a := (\nabla_a - \tilde{\nabla}_a) X^a \) then

\[
C^{ab}_{cd} = 2 \delta^{a}_{[e} \tilde{\nabla}_b \Omega - \tilde{g}_{eb} \tilde{g}^{cf} \nabla_f \Omega = 2 \delta^{a}_{[e} \tilde{\nabla}_b \Omega - g_{eb} g^{cf} \nabla_f \Omega,
\]

and the Riemann tensor \( R^{ab}_{cd} \) of the connection \( \nabla_a \) takes the form

\[
\Omega^2 R^{ab}_{cd} = 4 \delta^{a}_{[e} \tilde{\nabla}_b \Omega \Omega - 4 \delta^{a}_{[e} \tilde{\nabla}_d \Omega \Omega + 2 \delta^{a}_{[e} \delta^{b]} \nabla_f \Omega \nabla_f \Omega.
\]

Here the raising and lowering of indices on the right hand side is defined by the flat metric, while \( M R^{ab}_{cd} = \tilde{g}^{eb} M R^{eab}_{cd} \). In what follows we rewrite every quantity using only the conformally flat metric \( g_{ab} \). In particular, in terms of \( \nabla_a \), eq. (2) takes the form

\[
M R^{ab}_{cd} = 4 \delta^{a}_{[e} \tilde{\nabla}_d \Omega \Omega + 4 \delta^{a}_{[e} \tilde{\nabla}_d \Omega \tilde{\nabla} \Omega = 2 \delta^{a}_{[e} \delta^{b]} \nabla_f \Omega \nabla_f \Omega,
\]

and the \( \tilde{\nabla} \)-constant 1-form fields satisfy

\[
\nabla_a K_b = -2 K_e \nabla_b \Omega - g_{eb} K_b \nabla_e \Omega.
\]

Let us define \( \tilde{k}_a := P_a \kappa_a \) and \( \tilde{\xi} := t^a K_a \), by means of which \( K_a = \tilde{k}_a \pm \tilde{\xi} t_a \). From eq. (4) we have

\[
D_a \tilde{k}_b \pm \tilde{\xi} \chi_{ab} = -2 \tilde{k}_a \Upsilon_a \kappa_b + h_{ab} \left( \tilde{k} \xi \Upsilon_a \pm \tilde{\xi} \Omega^{-1} \Omega \right)
\]

\[
D_a \tilde{\xi} - \chi_{ab} \tilde{k}^b = -\tilde{k}_a \Omega^{-1} \Omega - \tilde{\xi} \Upsilon_a.
\]

Here \( D_a \) is the Levi-Civita covariant derivation on \( \Sigma \), \( \Upsilon_a := t^a \tilde{\nabla} \Omega \) and \( \Upsilon_a := D_a \Omega \). Then by (5) and (6) \( D_a (\tilde{k}_a \kappa \pm \tilde{\xi}^2) = -2 \Upsilon_a (\tilde{k}_a \kappa \pm \tilde{\xi}^2) \), which implies that \( \Omega^2 (\tilde{k}_a \kappa \pm \tilde{\xi}^2) = \text{const.} \) Thus it seems natural to introduce the following notations:

\[
k_a := \Omega \tilde{k}_a, \quad \xi := \Omega \tilde{\xi}, \quad \omega := \Omega^{-1} \Omega.
\]

Then by (5)-(7) and the definition of \( \Upsilon_a \) we have

\[
D_a k_b \pm \xi \chi_{ab} = -k_e \Upsilon_b + h_{ab} \left( \tilde{k} \xi \Upsilon_a \pm \omega \xi \right)
\]

\[
D_a \xi - \chi_{ab} k^b = \omega k_a,
\]

\[
D_a \Upsilon_b = D_b \Upsilon_a.
\]

Equations (8-10) form a system of partial differential equations for \( k_a \) and \( \xi \), whose conditions of integrability are
\[ 0 = \left( D_a D_b - D_b D_a \right) \xi = 2 \left( D_{[a} \chi_{b]} + h_{c[a} \left( D_{b]} \omega + \gamma_{b]} - \chi_{b]} \gamma \right) \right) k^c, \]

\[ R^{c}{}_{ab} k_d = - \left( D_a D_b - D_b D_a \right) k^e = \pm 2 \left( D_{[a} \chi_{b]} + P_{[a}^e \left( D_{b]} \omega + \omega \gamma_{b]} - \chi_{b]} \gamma \right) \right) \xi + \]

\[ + 2 \left( [\chi_{[a} \chi^d_{b]} + 2 P_{[a}^{[e} D_{b]} \gamma^{d]} + P_{[a}^{[e} \gamma_{b]} \gamma^{d]} \pm \right. \]

\[ \pm \omega P_{[a}^{[e} \chi_{b]}^{d]} \xi - \frac{1}{2} P_{[a}^{[e} P_{b]}^{d]} \left( \gamma_{e} \gamma^{e} \pm \omega^2 \right) \right) \right) k_d. \]

Applying \( P^a_b \) to eq. (3) we obtain

\[ M R_{ijkl} P^i_b P^j_c P^k_d P^l_a = 2 \left( h_{c[a} D_{b]} \gamma_{c]} - h_{d[a} D_{b]} \gamma_{c]} + h_{c[a} \gamma_{d]} \gamma_{c]} - h_{d[a} \gamma_{d]} \gamma_{c]} \right) \]

\[ \pm \omega \left( h_{c[a} \chi_{d]} - h_{d[a} \chi_{d]} \right) - h_{c[a} \chi_{d]} \gamma_{e} \gamma^{e} \pm \omega^2 \right) \right). \]

On the other hand the \((n+1)\) dimensional curvature tensor can be expressed in terms of the \(n\) dimensional curvature tensor and the extrinsic curvature, and hence we finally have

\[ R^{ab}_{cd} \pm 2 \chi^{a}_{[c} \chi^{b}_{d]} = 4 P_{[a}^{[e} D_{b]} \gamma^{d]} + 4 P_{[c}^{[e} \gamma_{d]} \gamma^{b]} \pm 4 \omega P_{[a}^{[e} \gamma^{d]} \gamma^{b]} - 2 P_{[a}^{[e} P^{b]}_{d]} \left( \gamma_{e} \gamma^{e} \pm \omega^2 \right), \]

\[ D_{e} \chi_{ab} - D_{a} \chi_{eb} = -2 h_{b[a} \left( D_{d} \omega + \omega \gamma_{d} - \chi_{d]} \gamma \right). \]

Thus by (15), (16) the integrability conditions (11,12) of the system (8,9) are satisfied identically. Equations (15,16) contain two kinds of information: One is already in the form of conditions on \((h_{ab}, \chi_{ab})\). The other kind is a system of partial differential equations on \((\omega, \gamma_{a})\), which we obtain by contraction eqs.(15,16), namely eqs. (28,29) below, and which is again overdetermined. By writing down the integrability conditions to this latter system, we will finally arrive at the complete characterization of non-contortedness.

The contractions of (15,16) are

\[ R_{ab} \pm \left( \chi \chi_{ab} - \chi_{ab} \chi \right) = \left( n - 2 \right) \left( D_b \gamma_d + \gamma_b \gamma_d \pm \omega \chi_{ad} - h_{ab} \gamma_{e} \right) \]

\[ + h_{a[b} D_{c} \gamma^{e} \pm \omega \chi \pm (n - 1) \omega^{2} \right), \]

\[ R \pm \left( \chi^2 - \chi \chi_{ab} \chi^{ab} \right) = \left( n - 1 \right) \left( 2 D_{e} \gamma^{e} - \left( n - 2 \right) \gamma_{e} \gamma^{e} \pm 2 \omega \chi \pm n \omega^{2} \right) \]

\[ D_{c} \chi_{d} - D_{a} \chi_{eb} = \left( n - 1 \right) \left( D_{d} \omega + \omega \gamma_{d} - \chi_{de} \gamma \right). \]

Then by (17,18)

\[ L_{bd} = \pm \left( \left( \chi \chi_{bd} - \chi_{bd} \chi \right) - \frac{1}{2(n - 1)} h_{bd} \left( \chi^2 - \chi \chi \right) \right) - \]

\[ - \left( n - 2 \right) \left( D_b \gamma_d + \gamma_b \gamma_d \pm \omega \chi_{bd} - \frac{1}{2} h_{bd} \left( \gamma_e \gamma^e \pm \omega^2 \right) \right). \]

Then substituting (20) back into eq.(15) and using the definition of the Weyl tensor we obtain

\[ E^{ab}_{cd} := C^{ab}_{cd} \pm \left( V^{ab}_{cd} - \frac{4}{(n - 2)} P_{[c}^{[e} V^{d]}_{a]} + \frac{2}{(n - 1)(n - 2)} P_{[a}^{[e} P^{d]}_{b]} \right) = 0. \]
$E_{abcd}$ plays the role of the Weyl tensor for the initial data sets. If $n = 3$ then $C_{abcd}$ and the term involving $V^a_{\,\,cd}$ in the expression for $E_{abcd}$ are separately zero identically. Also, in this case, equations (15) and (20) are equivalent. Next consider equation (16) and its contraction, eq. (19). By means of (19) eq. (16) can be rewritten as

$$D^c \chi^d_{\ b} - D^d \chi^c_{\ b} = \frac{2}{(n-1)} P^c_{\ b} \left( D_c \chi^{df} - D^f \chi \right).$$

(22)

Contracting this equation with $P^{ijk}_{\ cde} := 3! p^{ij} \ P^{k}_{\ de}$ we obtain

$$\frac{1}{(n-1)} P^{ijk}_{\ ab} \left( D_c \chi^{cd} - D^d \chi \right) = P^{ijk}_{\ cde} P^{k}_{\ d} \chi^d_{\ b} = P^{ijk}_{\ cde} D^c \chi^d_{\ b}.$$  

(23)

Since its left hand side is antisymmetric in $ab$ and its right hand side is symmetric in $ab$, they must vanish separately:

$$A^{ijk}_{\ ab} := \left( \frac{1}{(n-1)} P^{ij}_{\ c} P^{k}_{\ de} + \frac{1}{2} P^{ij}_{\ ab} P^{k}_{\ cd} \right) D^k \chi^d_{\ c} = 0,$$

$$H^{ijk}_{\ ab} := P^{ijk}_{\ cde} D^c \chi^d_{\ b} = 0.$$ 

(24) \hspace{1cm} (25)

The possible independent contractions of $H^{ijk}_{\ ab}$ are

$$h^{ab} H^{ijk}_{\ ab} = 0$$

$$H^{ijk}_{\ ab} = (n-1) \left( D^k \chi^d_{\ c} - \frac{1}{(n-1)} P^k_{\ b} \left( D_c \chi^{df} - D^f \chi \right) \right).$$

(26) \hspace{1cm} (27)

Thus by (27) $H^{ijk}_{\ ab} = 0$ is equivalent to (22), and hence implies $A^{ijk}_{\ ab} = 0$. Thus eq.(16) is equivalent to eq.(19) together with eq.(25).

Next let us consider the contracted equations (19) and (20):

$$(n-1) D_b \omega = - \left( D_c \chi^b_{\ c} - D_b \chi \right) - (n-1) \left( \omega Y_{ab} - \chi_{bc} Y^c \right),$$

$$(n-2) D_b Y_{ab} = - L_{ab} \pm \left( V_{ab} - \frac{1}{2(n-1)} h_{ab} V \right) - (n-2) \left( Y_{ab} \pm \omega \chi_{ab} - \frac{1}{2} h_{ab} \left( Y^c Y_c \pm \omega^2 \right) \right).$$

(28) \hspace{1cm} (29)

These equations can be considered as a system of partial differential equations for $\chi$ and $\omega$. Their integrability conditions are

$$0 = (D_c D_b - D_b D_c) \omega = \frac{1}{12} (n-1) h_{ai} h_{bj} \left( R^{efij}_{\ cde} Y^{f} - \frac{1}{(n-2)} D^f H^{efij} \right),$$

$$R^{cd}_{\ ab} Y_c = (D_c D_b - D_b D_c) Y^d = \left( - \frac{4}{(n-2)} P^{ef}_{\ ai} L_{dj} + (V_{cd} - \frac{4}{(n-2)} P^c_{\ e} P^d_{\ e}) \right) Y^d$$

$$+ \frac{2}{(n-1)(n-2)} V^c_{\ e} P^d_{\ e} \right) \omega \ h_{ai} h_{bj} H^{fij}_{\ cde} h_{ef} h_{cd} -$$

$$- \frac{1}{2(n-1)} \left( D^c Y_{b} \pm D_{b} \left( V_{ca} - \frac{1}{2(n-1)} V P^d_{\ e} \right) \right) \pm$$

$$\pm \frac{1}{n-1} \chi^d_{\ e} \left( D^e \chi_{b} - D_b \chi \right).$$

(30) \hspace{1cm} (31)
Thus by (22) and (27) the first condition is satisfied, while, using (21, 22, 27), the second can be rewritten as

$$B_{ab} = \frac{1}{(n-2)} \left( D_{[a}L_{b]} + D_{[a}V_{b]} - \frac{1}{2(n-1)} V P_{ab}^d \right) + \frac{(n-2)}{(n-1)} \chi^{[d} \left( D^c \chi_{bd} - D_b \chi \right) \right) =$$

$$= -\frac{1}{2} E_{ab} \Psi \gamma^c + \frac{1}{(n-1)} \omega h_{ab} R fck h_{cd} = 0.$$  (32)

Obviously, $B_{ab}b = B_{[ab]}b = 0$. Thus, to summarize, if $(\Sigma, h_{ab}, \chi_{ab})$ is non-contorted then $E_{ab} = 0$, $R fck = 0$ and $B_{ab} = 0$.

Conversely, let the initial data set $(\Sigma, h_{ab}, \chi_{ab})$ satisfy the conditions i.-iii. of the proposition. We show that this data set can be imbedded locally into a conformally flat geometry. First let us consider the system of partial differential equations (28), (29) for $\omega$ and $\gamma_e$. Its integrability conditions are the equations (30) and (31), which, by the conditions i.-iii., are satisfied independently of $\omega$ and $\gamma_e$. Thus by the Darboux theorem the system (28), (29) is completely integrable: for any 1-form $\gamma_e(p_0)$ at a given point $p_0 \in \Sigma$ and real number $\omega(p_0)$ there is a uniquely determined solution of the system (28), (29) whose value at $p_0$ is just the pair $\gamma_e(p_0), \omega(p_0)$. Then, by i. and ii. the pair $(\gamma_e, \omega)$ is also a solution of the system of equations (15), (16). Next, for a given pair $(\gamma_e, \omega)$, let us consider the system of partial differential equations (8), (9) for $k_a$ and $\xi$. Its integrability conditions are (11) and (12), which, by (15) and (16), are identically satisfied independently of $k_a$ and $\xi$. Thus the system (8), (9) is completely integrable, and it has $n+1$ linearly independent solutions $(k^a, \xi^a)$, $a = 0, 1, \ldots, n$, specified in the following way. Let $\{x^a\}, a = 1, \ldots, n$, be a local coordinate system around $p_0 \in \Sigma$ in which $h_{ab}(p_0) = \eta_{ab} := \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. (The number of +1’s is $p$ and the number of -1’s is $q$) Then the components of the solution 1-forms $k^a$ and $\xi^a$ in this coordinate system at $p_0$ and the value of the $\xi^a$’s at $p_0$ are chosen to satisfy $k^a_0 = 0, \xi^a = 1$ and $k^a = \delta^a, \xi^a = 0$.

In a sufficiently small neighbourhood $U''$ of $p_0$ the 1-form $\gamma_e$ is not only closed (by (29)), but exact. Thus there exists a strictly positive smooth function $\Omega: U'' \to (0, \infty)$ such that $\gamma_e = D_e \ln \Omega$. Then let us define the following rescaling: $\tilde{k}^a_0 := \Omega^{-1} k^a_0, \tilde{\xi}^a := \Omega^{-1} \xi^a$ and define $\tilde{\Omega} := \Omega$. Then $\tilde{k}^a_0$ and $\tilde{\xi}^a_0$ defined only on $U''$, satisfy

$$D_a \tilde{k}^a_0 + \tilde{\xi}^a \chi_{ab} = -2\tilde{k}^0 \gamma_0 + h_{ab} \left( \tilde{k}^0 \gamma_e + \tilde{\xi} \Omega^{-1} \tilde{\Omega} \right) \right) \right)$$

$$D_a \tilde{\xi} - \chi_{ab} \tilde{k}^b = -\tilde{k}^0 \Omega^{-1} \tilde{\Omega} - \tilde{\xi} \gamma_e.$$  (34)

By (33) $\tilde{k}^a_0$ are closed 1-forms on $U''$. Thus in a sufficiently small open neighbourhood $U' \subset U''$ of $p_0$ they are exact too, and hence there exist smooth functions $\phi^a: U' \to \mathbb{R}$ such that $\tilde{k}^a_0 = D_a \phi^a$. Because of the special choice of the $k^a_0$ at $p_0$ there is an open neighbourhood $U \subset U'$ of $p_0$ on which the rank of the mapping $\phi := \{\phi^a\}: U' \to \mathbb{R}^n$ is $n$, i.e. $\phi$ is an imbedding of $U$ into the n-dimensional manifold $\mathbb{R}^n$ with the natural Descartes coordinates $x^a$. At the points of $\phi(U) \subset \mathbb{R}^n_+$ let us define the functions $\tilde{\gamma}^{ab}(\phi(p)) := \Omega^2(\phi)(\pm \tilde{\xi}^{a}(\phi(p))\tilde{\xi}^{b}(\phi(p)) + \phi^a(\phi(p))\phi^b(\phi(p)) \eta^{ab}(\phi(p))) \forall p \in U$. By (33) and (34) these are constant on $\phi(U): D_p \tilde{\gamma}^{ab} = 0$, and, because of the special choice of the independent solution 1-forms and functions $(k^a, \xi^a)$ at $p_0$, $\tilde{\gamma}^{ab}(\phi(p_0)) = \eta^{ab} := \text{diag}(1, 1, \ldots, 1, -1, \ldots, -1)$. Then extend $\tilde{\gamma}^{ab}$ to $\mathbb{R}^n_+ + 1$ in a constant way. Thus $\mathbb{R}^n_+ + 1$ together with $\tilde{\gamma}^{ab}$, the inverse of $\tilde{\gamma}^{ab}$, is a (flat) pseudo-Euclidean geometry. Since by $\phi^a \tilde{\gamma}^{ab} \tilde{\xi}^b = 0$ the 1-form $\tilde{\xi}^{a} \gamma_{ab}$ annihilates every vector tangent to $\phi(U)$, this 1-form is a normal of $\phi(U)$ in $\mathbb{R}^n_+ + 1$; and its norm with respect to $\gamma_{ab}$ is $\tilde{\xi}^{a} \tilde{\gamma}^{b} \gamma_{ab} = (\tilde{\xi}^{a} \gamma_{ac}) (\tilde{\xi}^{b} \gamma_{bd}) \tilde{\gamma}^{cd} = \pm \Omega^2 (\tilde{\xi}^{a} \gamma_{ac})^2$, i.e. $\tilde{\xi}^{a} \gamma_{ac} = \pm 1$. Let us extend the function $\Omega$ from $\phi(U)$ onto $\mathbb{R}^n_+ + 1$ to be positive everywhere and satisfying $\tilde{\xi}^{a} \delta_{b} \Omega = \tilde{\Omega}$, where $\delta_a$ is the partial derivative with respect to $x^a$. Then $\gamma_{ab} := \Omega^2 \tilde{\gamma}_{ab}$ is a conformally flat metric on $\mathbb{R}^n_+ + 1$ with respect to which $\tilde{\xi}^{a}$ is a unit normal of $\phi(U)$. The pull back to $U$ of this metric is $\phi^a \phi^b \gamma_{ab} = \Omega^2 \phi^a \phi^b \tilde{\gamma}^{cd} \gamma_{ab} \phi_{bd} = (\phi^a \phi^b \gamma_{ac}) (\phi^d \phi^b \phi_{bd}) \tilde{\gamma}^{cd} (\phi^d \phi_{bd} \phi_{bd} \phi_{bd})$, implying that $\phi^a \phi^b \gamma_{ab} = k_{ab}$.
Finally, let us calculate the pull back to $U$ of the extrinsic curvature of $\phi(U)$. The Christoffel symbols of $(\mathbb{R}^{n+1}, g_{ab})$ in the coordinates $x^a$ are $\Gamma^a_{\mu \nu} = \frac{1}{2} g^{a \gamma} \partial_\mu g_{\nu \gamma}$, thus the pullback to $U$ of the extrinsic curvature is $\tilde{E}_{\alpha \beta}^e = \chi_{\alpha \beta} - \Omega^{-1} \hat{\Omega} h_{\alpha \beta}$, where $\chi_{\alpha \beta}$ is the intrinsic curvature of $U$. The results are:

\begin{align}
\hat{E}_{\alpha \beta \gamma} &= E_{\alpha \beta \gamma}, \\
\hat{H}^{ij} &= \Omega^{-3} H^{ij}, \\
\hat{B}_{abc} &= B_{abc} + \frac{1}{2} E_{abc} d \ln \Omega + \frac{1}{(n-1)} \Omega^{-1} \hat{\Omega} h_{ij} h_{kl} H^{jk}_{f}. \tag{4.2}
\end{align}

Thus the conditions i - iii. are, in fact, conformally invariant.

Next let us consider the physically important special case of $n = 3$. As we mentioned in the proof above, in three dimensions $E_{abcd} = 0$ identically. Furthermore $A_{ij}^{jk}$ is also zero identically and ii. is equivalent to $H_{ij} = \frac{1}{36}(-\partial^2 \varepsilon_{ijk} H_{ij}^{jk} = (-\partial^2 \varepsilon_{cd(a} D_{bc}^e \chi^d_{b)} = 0$, the vanishing of the conformal magnetic curvature. (Here $\omega$ is the number of -1’s in the pseudo-euclidean form of $h_{ab}$.) Finally,

\begin{align}
\varepsilon^{cd}_{\alpha} B_{\alpha \beta} &= Y_{\beta} + \varepsilon_{cd(i} (D^j (\chi^d_{b)} \varepsilon D_{j(e} \chi^e_{d)} - \frac{1}{2} \chi_{b}^e (D_{e}^j \chi^d - D^d \chi)) \\
&= \Omega^{-1} \hat{\Omega} h_{ij} h_{kl} H^{jk}_{f}.
\end{align}

and therefore ii., iii. are equivalent to

\begin{align}
H_{ij} &= \frac{1}{36}(-\partial^2 \varepsilon_{ijk} H_{ij}^{jk} = (-\partial^2 \varepsilon_{cd(a} D_{bc}^e \chi^d_{b)} = 0, \tag{ii'}. \\
B_{ij} &= Y_{ij} + \varepsilon_{cd(i} (D^j (\chi^d_{b)} \varepsilon D_{j(e} \chi^e_{d)} - \frac{1}{2} \chi_{b}^e (D_{e}^j \chi^d - D^d \chi)) = 0. \tag{ii'}'.
\end{align}

Both $H_{ij}$ and $B_{ij}$ are traceless and symmetric, for negative definite $h_{ab}$ they are the tensors $H_{ij}$ and $B_{ij}$ introduced in subsection 3.3, and if $\chi_{ab} = 0$ (i.e. the initial data set is ‘time symmetric’) then $H_{ij}$ vanishes and $B_{ij}$ reduces to the Cotton-York tensor. Thus we have proven the following corollary:

**Corollary** The three dimensional initial data set is non-contorted if and only if $B_{ij} = 0$ and $H_{ij} = 0$.

The conformal behaviour of the symmetric traceless tensors $B_{ij}$ and $H_{ij}$ are:

\begin{align}
\hat{B}_{ij} &= \Omega^{-1} (B_{ij} + (\partial^a \Omega^{-1} \hat{\Omega} H_{ij}), \tag{4.6}
\hat{H}_{ij} &= H_{ij}. \tag{4.7}
\end{align}

Thus, as is well known, $H_{ij}$ is a conformal invariant of the initial data set; and for ‘internal’ conformal rescalings (i.e. when $\hat{\Omega} = 0$) $B_{ij}$ transforms covariantly, i.e. it has definite conformal weight, namely -1.

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