Late Time Behaviour of the Maximal Slicing of the Schwarzschild Black Hole

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Abstract

A time-symmetric Cauchy slice of the extended Schwarzschild spacetime can be
evolved into a foliation of the $r > 3m/2$-region of the spacetime by maximal surfaces
with the requirement that time runs equally fast at both spatial ends of the manifold.
This paper studies the behaviour of these slices in the limit as proper time-at-infinity
becomes arbitrarily large and gives an analytic expression for the collapse of the lapse.
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1 Introduction

In this work we study the time function $\tau$ on the Schwarzschild black hole spacetime having the following properties:

(i) The level sets of $\tau$ result from evolution of a time-symmetric Cauchy slice of Schwarzschild by maximal surfaces under the additional requirement that proper time for asymptotic observers at infinity, which are at rest relative to the slicing, runs equally fast at both spatial ends.

(ii) The time function $\tau$ is zero on the time-symmetric slice and coincides with the proper time of the infinite observers. (This means that $\alpha$, the lapse of the time function goes to one at both infinities along each slice.)

Note that (i) is really a property only of the slicing defined by $\tau$ rather than $\tau$ itself. This time function which has first been considered in [1, 2] has two key properties: The first property is that $\tau$ takes all real values or, in other words, the future singularity at $r = 0$ does not prevent $\tau$ from assuming arbitrarily large positive values (and similarly for the past). It is believed that this property holds on vacuum spacetimes more general than Schwarzschild. Here it is important to realize that such spacetimes are not “given” to us; rather they have to be generated by a Cauchy problem: One first constructs regular asymptotically flat initial data, satisfying the vacuum constraints, say maximal, and then tries to evolve these in time by analytical or numerical means. Doing this involves an a priori choice of gauge which in particular implies that the resultant globally hyperbolic spacetime comes already equipped with a specific time function. Suppose the initial data has a future-trapped surface. Then, by the Penrose singularity theorem [3], any Cauchy evolved spacetime is singular in the sense of having future-incomplete null geodesics. (Similar conclusions, but in both the future and the past direction, hold when the initial data has an outer-trapped surface [4] or when the topology is nontrivial e.g. in the sense that there is more than one asymptotic end [5].) Many maximal initial data sets having one of these properties exist (for trapped surfaces, see [6]). Now there is the conjecture, due to Moncrief and Eardley [7], that if one evolves the initial data in a gauge where the whole slicing is maximal and $\tau$ is proper time at infinity, the evolution should be extendable to arbitrarily large values of $\tau$, irrespective of whether singularities form or not. This global existence result, if true, would, in spirit at least, go a long way toward settling in the affirmative the Penrose Cosmic Censorship hypotheses [8] in the case of asymptotically flat vacuum data. The spacetime evolved in the way described, in the Schwarzschild case, has the second property that it is in fact extendable: there are no maximal spherically symmetric Cauchy slices of Schwarzschild reaching radii less than or equal to $r = 3m/2$. Thus maximal slices of Schwarzschild “avoid the singularity at $r = 0$”. It is this last property which numerical relativists expect to be true for evolutions of more general initial data and which is clearly desirable if numerical codes based on maximal slicings are used.

Take any observer at rest relative to the slicing defined by $\tau$ ("Eulerian observer"). Then $\int \alpha \, d\tau$ along the trajectory of that observer is her or his proper time. Since proper
time is finite as the slicing approaches the limiting maximal slice at \( r = 3m/2 \), we must have \( \int \alpha d\tau < \infty \), and thus \( \lim_{\tau \to \infty} \alpha(\tau) = 0 \) (“collapse of the lapse” [9]). Our main result is that, along the Eulerian observers going through the bifurcation 2-sphere,

\[
\alpha(\tau) \sim \frac{4}{3\sqrt{2}} \exp \left( \frac{4A}{3\sqrt{6}} \exp \left( -\frac{4\tau}{3\sqrt{6} m} \right) \right) \quad \text{as} \quad \tau \to \infty,
\]

where the constant \( A \) is given by Equ. (3.41). The exponent in (1.1) has been estimated before [1, 10] by a mixture of numerical and model calculations. The estimate in [10] of this exponent is 1.82 which agrees quite closely with our exact \( \frac{3\sqrt{2}}{4} \sim 1.83 \). We hope that this result will be useful for the numerists as an accurate test for codes based on maximal slicings. An extension of the work here to the late time behaviour of \( \alpha \) along the trajectories of arbitrary Eulerian observers will appear elsewhere [17].

Our plan is as follows: In § 2 we review some generalities on lapse functions and foliations. Then we give a precise definition of the time function under study. In § 3 we perform the asymptotic analysis leading to Equ. (1.1). In Appendix A we essentially rederive the Schwarzschild metric in terms of spherically symmetric, maximal Cauchy data. In Appendix B we prove a calculus lemma which is basic for our analysis.

### 2 Generalities

Let \((M, ds^2)\) be a globally hyperbolic spacetime and \( \tau : M \to \mathbb{R} \) a time function, i.e. a function the level sets of which form a foliation \( \mathcal{F}_\tau \) of \( M \) by Cauchy surfaces \( \mathcal{C} \). Then the function \( \alpha : M \to \mathbb{R} \) defined by

\[
\alpha := (-(\nabla \tau)^2)^{-1/2}
\]

is called the lapse of \( \mathcal{F}_\tau \). The reason for this name is that \( \alpha \) measures “lapse of proper time” along trajectories normal to the leaves of \( \mathcal{F}_\tau \) as a function of \( \tau \). To make this explicit, define the vector field \( \tau^\mu \) by

\[
\tau^\mu = -\alpha^2 \nabla^\mu \tau \Rightarrow \tau^\mu \nabla_\mu \tau = 1
\]

which is timelike and future (i.e. increasing \( \tau \)) pointing. The vector \( \tau^\mu \) yields an orthogonal decomposition of \( M \) as \( M = \mathbb{R} \times \mathcal{C} \), as follows: Construct a diffeomorphism \( \varphi : \mathbb{R} \times \mathcal{C} \), i.e. \( \varphi : (\lambda, y^i) \in \mathbb{R} \times \mathcal{C} \mapsto x^\mu = \varphi^\mu_\lambda(y^i) \in M \), by

\[
\varphi^\mu_\lambda(y) = \frac{d}{d\lambda} \varphi^\mu_\lambda(y^i) = \tau^\mu(\varphi_\lambda(y))
\]

\[
\varphi_0 = \tau^{-1}(0).
\]

Thus \( \lambda \), viewed as a function on \( M \), coincides with \( \tau \). We will, by abuse of notation, use the same letter \( \tau \) for \( \lambda \) viewed in this way. Furthermore, since \( \tau^\mu \) is normal to the leaves of \( \mathcal{F}_\tau \), the lines of constant \( y^i \) are orthogonal trajectories, in other words

\[
\varphi^\mu_\tau(y) \varphi^\nu_{\tau,i}(y) g_{\mu\nu}(\varphi_\tau(y)) = 0.
\]
Consequently, in the \((\tau, y^i)\)-coordinates, the metric takes the form
\[
\varphi_{\tau}^* (ds^2) = \varphi_{\tau}^* \varphi_{\tau} \varphi_{\tau} g_{\mu \nu} d\tau^2 + \varphi_{\tau,i} \varphi_{\tau,j} g_{\mu \nu} dy^i dy^j
= g_{\tau \tau}(\tau, y) d\tau^2 + g_{ij}(\tau, y) dy^i dy^j,
\]
where \(g_{ij}\) is the induced metric on the leaves and
\[
g_{\tau \tau}(\tau, y) = -\alpha^2(\varphi_{\tau}(y)).
\]
Thus, along \(y^i = \text{constant}\), the proper time \(s\) is given by
\[
s = \int \alpha(\varphi_{\tau}(y)) d\tau'.
\]
Note that, when \(\tau'\) is another time function giving the same foliation, i.e. \(\tau' = \tau'(\tau)\), the lapse \(\alpha\) changes according to \(\alpha' = \left(\frac{d\tau'}{d\tau}\right)^{-1} \alpha\). Suppose now we are given another vector field \(\xi^\mu\) on \(M\). This can be uniquely decomposed
\[
\xi^\mu = N n^\mu + X^\mu, \quad X^\mu n_\mu = 0
\]
where \(n^\mu = -\alpha \nabla^\mu \tau\), is the future normal of \(\mathcal{F}_\tau\). To distinguish \(N\) from \(\alpha\), we call \(N\) the **boost function** of \(\xi^\mu\) relative to \(\mathcal{F}_\tau\). If \(N\) is non-zero on some leaf \(\Sigma_{\tau_0}\), it can be viewed as the restriction to \(\Sigma_{\tau_0}\) of the lapse of the time function \(t'\) obtained by \(\xi^\mu \nabla_\mu t' = 0\), \(t'|_{\Sigma_{\tau_0}} = \text{const.}\).

We have the relation
\[
N = \alpha \xi^\mu \nabla_\mu \tau,
\]
which is of course trivial in the present context, but will be extremely useful in our computation of the lapse \(\alpha\) of a maximal foliation of the extended Schwarzschild spacetime, where \(\xi^\mu\) can be chosen as the "static" Killing vector.

We now recall some features of Schwarzschild which are used in our construction. In the exterior region \(r > 2m > 0\) we have
\[
ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad -\infty < t < \infty.
\]
\(ds^2\) can be smoothly extended across \(r = 2m\) to the Kruskal spacetime \(M\) on which \(r\) is a globally defined function \(r : M \rightarrow \mathbb{R}^+\), which has saddle points at \(\mathcal{S}\), the bifurcation 2-sphere of the horizon. The Killing vector field \(\partial / \partial t\) extends to a global Killing vector field \(\xi^\mu\) on \(M\) which is spacelike in the interior, i.e. black and white hole, regions, null on the horizon and zero on \(\mathcal{S}\). Both the black hole region and the right exterior region can be written in the form (2.10) with the understanding that the functions \((\theta, \varphi)\) and \(r\) together with the retarded Eddington–Finkelstein coordinate
\[
u = t - r - 2m \ln |r - 2m|
\]
covers both regions and the horizon at \( r = 2m \). The function \( t \) goes to \( \infty \) at the right component (where “right” refers to the original unextended spacetime) and goes to \(-\infty\) at the left horizon. The set where \( t \) vanishes is the union of \( S \), the original \( t = 0 \)-spacelike hypersurface (extended in the obvious way to the left exterior region) and the timelike, totally geodesic cylinder \( \Gamma \), which is ruled by timelike radial geodesics through \( S \) which are orthogonal to \( S \), and which hit the singularity as \( r \to 0 \). Since \( r \) is constant along the trajectories of \( \xi^\mu \) and \( r \) is, by (2.10), an “areal radius”, it follows that every spherically symmetric spacelike slice has a spherical minimal surface (a “throat”) exactly where it is tangential to \( \xi^\mu \) (which of course can only happen in the interior and it necessarily has to happen there for slices leaving to the other (left) exterior region).

Consider the function \( h(r, C) \) given by

\[
h(r, C) = -\int_{r_C}^{r} \frac{C}{(1 - 2m/x)(x^4 - 2mx^3 + C^2)^{1/2}} dx,
\]

where the integral is to be understood in the Cauchy-principal-value sense for \( r > 2m \) and where \( 0 < C < 3\sqrt{\frac{3}{4}} m^2 \), \( r > r_C \) and \( r_C \) is the unique root of \( P(x) = x^4 - 2mx^3 + C^2 \) for this range of \( C \) in the interval \( 3m/2 < r_C < 2m \). For \( x > r_C \), we have \( P(x) > 0 \). Thus \( h(r, C) + r + 2m \ln |r - 2m| \) depends smoothly on \((r, r_C)\). We easily infer that

\[
t = h(r, C)
\]

defines, for each fixed \( C \), a spacelike slice \( \Sigma_C \) which smoothly extends to the black hole, where it intersects \( \Gamma \) at \( r = r_C \).

In order to see that this surface extends smoothly and symmetrically through \( \Gamma \), we use for \( r < 2m \) the parameter

\[
\ell(r) = \int_{r_C}^{r} \frac{x^2 dx}{[P(x)]^{1/2}},
\]

which is the proper distance along the slice, as can either be seen from (2.10,12,13) or from Appendix A. Then, from (2.12,14) we have the system of ODE’s

\[
\frac{dh}{d\ell} = -\frac{C}{r^2 - 2mr}, \quad \frac{dr}{d\ell} = \frac{m}{r^2 - \frac{2C^2}{r^5}},
\]

with \( h(0) = 0, r(0) = r_C, \frac{dr}{d\ell}(0) = 0 \), which is regular at \( \ell = 0 \). Thus the function \( r \) along the slice is symmetric with respect to \( \ell = 0 \) and smooth. This implies that \( dr/d\ell = (1 - 2m/r + C^2/r^4)^{1/2} \) is antisymmetric.

Next we observe that the level sets of \( \sigma = t - h(r, C) \), for fixed \( C \) in the allowed range, give rise to maximal surfaces on the Kruskal manifold, i.e. they satisfy

\[
\nabla^\mu([-((\nabla \sigma)^2)^{-1/2}\nabla_\mu \sigma]) = 0.
\]

(2.16)
The function $\sigma$ is not the time function of interest to us (in fact: $\sigma$ being not differentiable at $r_C$ it does not define a global foliation). Rather this local foliation arises from moving a given maximal slice, say $\sigma = 0$, along the flow of $\xi^\mu = (\partial / \partial t)^\mu$. The function $N = [-(\nabla \sigma)^2]^{-1/2}$ is nothing but the boost function of $\partial / \partial t$ relative to $\sigma = 0$. In fact, in the explicit solution of (2.14) due to Reinhart [2], he first, essentially by guessing, finds $N$ to be

$$N = \left(1 - \frac{2m}{r} + \frac{C^2}{r^4}\right)^{1/2},$$

and from this (2.12) can be inferred. For a more illustrative derivation from the initial-value point of view see Appendix A. Note that $N$ as a function of $\ell$ is antisymmetric relative to $\sigma = 0$.

We now claim that the surface $t = h(r, C')$ lies everywhere in the future of $t = h(r, C)$ when $C' > C$; and that $t = h(r, C)$ lies to the future of $S = \Sigma_0$. Amusingly we are unable to see this from the explicit integral (2.12). Instead, one first computes $\frac{dr_C}{dC}$ from

$$r_C^4 - 2mr_C^3 + C^2 = 0,$$

(2.18)

Thus the claimed behaviour is true at least along the throat. Next observe that our slices are asymptotically flat at both spatial ends and that $t_{\infty}(C) = \lim_{r \to \infty} h(r, C)$ exists. Suppose that $h(R, C) = h(R, C')$ for some $R > r_C$ to the right of $\Gamma$. Then, by the symmetry w.r.t. $\Gamma$ this would have to happen also to the left of $\Gamma$. Thus we would have a lens-shaped region spanned by two maximal slices. But this, by an elegant argument due to Brill and Flaherty [11], is impossible, except if the two slices are identical, which they are not in our case. This argument continues to be valid for $R = \infty$. Thus $t(r, C)$ monotonically increases with $C$ for fixed $r$, and so does $t_{\infty}(C)$.

It follows that the equation $t = h(r, C)$ can be solved for $C$ to yield a smooth time function defined on the $r < 3m/2$-subset of the part of Kruskal lying in the future of the Cauchy slice $S$. $C$ gives the foliation we are interested in, but it is not yet the time function we want: rather this is obtained from eliminating $C$ in terms of $\tau$ in the relation

$$\tau = t_{\infty}(C),$$

(2.20)

Suppose we had started with the Cauchy slice $t = 0$ which, being time-symmetric, is in particular maximal and evolve it into a maximal slicing by a lapse function $\alpha$ going to 1 at both spatial ends. This is possible in a unique way (see [12]). Then the resultant time function is spherically symmetric and symmetric w.r.t. $\Gamma$, and so it has to coincide with the one obtained above. In particular it follows that our $\tau$ can be smoothly extended to negative values of $\tau$ which would have been very non-obvious from the explicit formula (2.12).
We next compute the lapse function $\alpha$ of $\tau$. Using (2.9) this involves computing

$$(\xi^\mu \nabla_\mu \tau)^{-1} = \frac{dC}{d\tau} \frac{\partial h}{\partial C} \bigg|_r.$$  

(2.21)

Note that (2.21) blows up at $r = r_C$ but in such a way that

$$\alpha = (\xi^\mu \nabla_\mu \tau)^{-1} N$$

(2.22)

has a smooth limit as $r \to r_C$, as it has to be. Using formula (B.12), there results

$$\alpha = \left( \frac{d\tau}{dC} \right)^{-1} \left[ \frac{1}{2} \frac{1}{r - 3m/2} - \int_{r_C}^{r} \frac{x(x - 3m)dx}{(x - 3m/2)^2 [x^4 - 2mx^3 + C^2]^{1/2}} \right].$$

(2.23)

with

$$\frac{d\tau}{dC} = \int_{r_C}^{\infty} \frac{x(x - 3m)dx}{(x - 3m/2)^2 [x^4 - 2mx^3 + C^2]^{1/2}}.$$  

(2.24)

Note that $N$ and $\alpha$ are linearly independent radial solutions of

$$\left( \Delta - K_{ij} K^{ij} \right) f = 0,$$

(2.25)

where $N$ goes to 1 at the right infinity and to $-1$ at the left one whereas $\alpha$ goes to 1 at both ends.

We are interested in studying $\alpha$ along the trajectories of Eulerian observers. This requires choosing a coordinate $\rho = \rho(r, \tau)$ the level surfaces of which are timelike cylinders orthogonal to our slicing. (One such timelike cylinder is already known, namely $\Gamma$ given by $r = r_C$.) Such a coordinate can be found without any calculation. Recall that maximal slicings preserve spatial volumes along Eulerian observers. Thus a suitable coordinate will be the “volume radius” on each slice, defined by

$$\rho^3(r, \tau) = 3 \int_{r_{C(t)}}^{r} \frac{x^4dx}{[x^4 - 2mx^3 + C^2]^{1/2}},$$

(2.26)

using that the spatial metric on each slice has the form (see Appendix A)

$$g_{ij} dx^i dx^j = \left( 1 - \frac{2m}{r} + \frac{C^2}{r^4} \right)^{-1} dr^2 + r^2 d\Omega^2.$$  

(2.27)

In the coordinates $(t, \rho, \theta, \varphi)$, the Schwarzschild metric for $r > 3m/2$ reads

$$ds^2 = -\alpha^2 d\tau^2 + \left( \frac{\rho}{r} \right)^4 d\rho^2 + r^2 d\Omega^2$$

(2.28)

where $r = r(\rho, \tau)$ is given implicitly by (2.26) and $\alpha$ by (2.23). (To check (2.28) explicitly one should first observe that $C \frac{\partial h}{\partial C} \bigg|_r = \rho^2 \frac{\partial \rho}{\partial C} \bigg|_r$.)
Note that, as $C$ approaches $\sqrt{27/16} \ m^2$, $r_C$ approaches the value $3m/2$, since

$$P(x) = x^4 - 2mx^3 + C^2 = \left(x - \frac{3m}{2}\right)^2 \left(x^2 + mx + \frac{3m^2}{4}\right) + O\left((x - \frac{3m}{2})^2\right). \quad (2.29)$$

Eqn. (2.29) also shows that $r_C$ approaches a double root of $P(x)$ as $C \to \sqrt{27/16} \ m^2$. Thus, as one lets $\tau$ tend to infinity for fixed $\rho$, the function $r$ approaches $3m/2$. In that sense the slices approach the limiting maximal slice at $r = 3m/2$. We are interested in estimating $\alpha$ in that limit. For simplicity we will confine ourselves to $\rho = 0$, i.e. the throat $\Gamma$.

### 3 The late time analysis

It is convenient to replace the parameter $C$ by $\delta$ defined by

$$\delta = r_C - \frac{3m}{2}, \quad r_C^4 - 2mr_C^3 + C^2 = 0. \quad (3.1)$$

As $C$ ranges between 0 and $3 \sqrt{3} \ m^2$, $\delta$ ranges monotonically from $m/2$ to 0. Using the rescaled quantities

$$\bar{C} = \frac{C}{m^2}, \quad \bar{\tau} = \frac{\tau}{m}, \quad \bar{\delta} = \frac{\delta}{m}, \quad (3.2)$$

we find that

$$\bar{\tau}(\bar{\delta}) = -\bar{C} \int_{3/2+\bar{\delta}}^{\infty} \frac{ydy}{(y-2)(y^4 - 2y^3 + \bar{C}^2)^{1/2}}, \quad (3.3)$$

where

$$\bar{C} = \left(\bar{\delta} + \frac{3}{2}\right)^{3/2} \left(\frac{1}{2} - \bar{\delta}\right)^{1/2}. \quad (3.4)$$

We have the

**Lemma:**

$$\bar{\tau}(\bar{\delta}) = -\frac{3\sqrt{6}}{4} \ln \bar{\delta} + \frac{3\sqrt{6}}{4} \ln \left|18(3\sqrt{2} - 4)\right| - 2 \ln \left|\frac{3\sqrt{5} - 5}{9\sqrt{6} - 22}\right| + O(\bar{\delta})$$

$$= -\frac{3\sqrt{6}}{4} \ln \bar{\delta} + A + O(\bar{\delta}) \quad \text{as } \bar{\delta} \to 0. \quad (3.5)$$

**Proof:** First note that

$$\frac{d}{d\bar{C}} \left[\frac{\bar{C}}{(y^4 - 2y^3 + \bar{C}^2)^{1/2}}\right] = \frac{y^3(y - 2)}{(y^4 - 2y^3 + \bar{C}^2)^{3/2}}. \quad (3.6)$$
Thus, from the mean value theorem,

\[
\left| \frac{y}{y - 2} \left( (y^4 - 2y^3 + C^2)^{1/2} - \frac{\sqrt{27/16}}{(y^4 - 2y^3 + 27/16)^{1/2}} \right) \right| \leq \frac{\sqrt{27} \tilde{\delta} y^4}{(y^4 - 2y^3 + C^2)^{3/2}} \tag{3.7}
\]

where we have used

\[
\sqrt{27/16} - C \leq \sqrt{27} \tilde{\delta}^2. \tag{3.8}
\]

(3.7) is valid for \( y \neq 2 \) but, by continuity, also for \( y = 2 \). We will find it convenient to sometimes express \( C \) in terms of \( \tilde{\delta} \), using (3.4). Writing

\[
Q(s) = s^2 \left( s^2 + 4s + \frac{9}{2} \right) - \tilde{\delta}^2 \left( \frac{\tilde{\delta}}{2} + 4 + \frac{9}{2} \right). \tag{3.9}
\]

Equ. (3.3) can, after substituting \( s = y - 3/2 \), be written as

\[
\tau = \left( \frac{3}{2} \right)^{3/2} \left( \frac{1}{2} - \tilde{\delta} \right)^{1/2} \int_{\tilde{\delta}}^{\infty} \frac{(s + 3/2)ds}{(1/2 - s)\sqrt{Q(s)}}. \tag{3.10}
\]

It is elementary to see that, for \( s \geq \tilde{\delta} \),

\[
0 \leq \frac{9}{2} (s^2 - \tilde{\delta}^2) \leq Q(s) \leq (s^2 - \tilde{\delta}^2) \left[ \frac{9}{2} + 2s(4 + s) \right] \tag{3.11}
\]

which, using \( \sqrt{1 + x} \leq 1 + x/2 \) for \( x \geq 0 \), implies

\[
\left| \frac{1}{[Q(s)]^{1/2}} - \frac{1}{[\frac{3}{2}(s^2 - \tilde{\delta}^2)]^{1/2}} \right| \leq \frac{2s(4 + s)}{[\frac{3}{2}(s^2 - \tilde{\delta}^2)]^{1/2}}. \tag{3.12}
\]

The estimate (3.7) now takes the form

\[
\left| \frac{s + 3/2}{1/2 - s} \left( (\frac{3}{2})^{3/2}(1/2 - \tilde{\delta})^{1/2} \right) - \frac{\sqrt{27/16}}{[s^2 + 4s + 9/2]^{1/2}} \right| \leq \frac{\sqrt{27} \tilde{\delta}^2 (s + 3/2)^4}{[Q(s)]^{3/2}} \tag{3.13}
\]

The inequalities (3.12,13) are the basic estimates we will be using. We now split the integration domain in (3.10) as follows:

\[
\tilde{\delta} \leq s \leq \sqrt{\tilde{\delta}/2}, \quad \sqrt{\tilde{\delta}/2} \leq s \leq \infty \tag{3.14}
\]

and write

\[
\tau = \tau_1 + \tau_2 \tag{3.15}
\]
Accordingly, we furthermore define \((0 < \tilde{\delta} < 1/2)\)

\[
\tilde{\pi}_1^0 = \sqrt{27/16} \int_\delta^{\sqrt{\delta^2}} \frac{s + 3/2}{(1/2 - s)[2(s^2 - \tilde{\delta}^2)]^{1/2}} ds
\]

and

\[
\tilde{\pi}_2^0 = \sqrt{27/16} \int_\delta^{\infty} \frac{s + 3/2}{\sqrt{\delta^2} (1/2 - s)[s^2(s^2 + 4s + 9/2)]^{1/2}} ds.
\]

Eqn. (3.17) is in the principal-value sense at \(s = 1/2\). These integrals and one following later (in Eqn.(3.31)) can be explicitly computed using the formulas (see e.g. [14])

\[
\int \frac{dx}{x^2 \sqrt{x^2 - \delta^2}} = \frac{\sqrt{x^2 - \delta^2}}{x \delta^2}, \quad x > \delta > 0,
\]

\[
\int \frac{dx}{\sqrt{x^2 - \delta^2}} = \ln \left| x + \sqrt{x^2 - \delta^2} \right|, \quad x > \delta > 0,
\]

\[
\int \frac{dx}{x \sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{c}} \ln \left| -2c(ax^2 + bx + c) + 2c + bx \right|, \quad c > 0.
\]

Using \(\frac{s + 3/2}{s(1/2 - s)} = \frac{3}{s} - \frac{4}{s - 1/2}\), there results after straightforward manipulations

\[
\tilde{\pi}_1^0 = -\frac{3\sqrt{6}}{4} \ln \delta + \frac{3\sqrt{6}}{4} \ln \sqrt{\delta/2} + o(1) \quad \text{as} \quad \tilde{\delta} \to 0
\]

\[
\tilde{\pi}_2^0 = -\frac{3\sqrt{6}}{4} \ln \sqrt{\delta/2} + \frac{3\sqrt{6}}{4} \ln 2 + \frac{3\sqrt{6}}{4} \ln \left| \frac{18}{4 + 3\sqrt{2}} \right| - 2\ln \left| \frac{3\sqrt{3} - 5}{9\sqrt{6} - 22} \right| + o(1)
\]

as \(\tilde{\delta} \to 0\).

Next we have to estimate the remainders. We have

\[
\Delta \tilde{\tau}_1 = \int_\delta^{\sqrt{\delta^2}} \frac{s + 3/2}{1/2 - s} \left[ \frac{C(\tilde{\delta})}{\sqrt{Q(s)}} \right]^{1/2} ds.
\]

Using \(C(\tilde{\delta}) = \sqrt{27/16} + O(\tilde{\delta}^2)\) and (3.11,12), this has

\[
|\Delta \tilde{\tau}_1| \leq \text{const} \int_\delta^{\sqrt{\delta^2}} \frac{sd\delta}{s \delta^2 - \delta^2} = O(\delta^{3/2}).
\]

Next

\[
\Delta \tilde{\tau}_2 = \int_\sqrt{\delta^2}^{\infty} \frac{s + 3/2}{1/2 - s} \left[ \frac{C(\tilde{\delta})}{\sqrt{Q(s)}} \right]^{1/2} \left| \sqrt{27/16} - \right| \frac{1}{\left[ \sqrt{27/16} \right]^{1/2}} ds.
\]

By (3.13), this has a bound of the form

\[
|\Delta \tilde{\tau}_2| \leq \text{const} \tilde{\delta}^2 \int_\sqrt{\delta^2}^{\infty} \frac{(s + 3/2)^4}{s^2(s^2 + 4s + 9/2)^{3/2}} ds = \text{const} \tilde{\delta}^2 I.
\]
The integral $I$ in (3.26) can be further split as $I = I_2 + I'_2$, where
\[ I_2 = \int_1^{\delta/2} \frac{(s + 3/2)^4}{s^2(s^2 + 4s + 9/2)^{3/2}} ds \leq \text{const } \int_1^{\delta/2} \frac{ds}{(s^2 - \delta^2)^{3/2}} = \frac{1}{\delta^2} \int_1^{\delta/2} \frac{df}{f^2 - 1} \right)^{3/2} = O \left( \frac{1}{\delta} \right). \] (3.27)

Now
\[ I'_2 = \int_1^{\infty} \frac{(s + 3/2)^4}{s^2(s^2 + 4s + 9/2)^{3/2}} ds \leq \text{const } \int_1^{\infty} \frac{ds}{s^2} < \infty. \] (3.28)

Thus $\Delta \tilde{\tau}_2 = O(\tilde{\delta})$. Putting all this together implies
\[ \tilde{\tau}(\tilde{\delta}) = -\frac{3\sqrt{6}}{4} \ln \tilde{\delta} + A + o(1) \text{ as } \tilde{\delta} \to 0, \] (3.29)

which is not quite good enough. From Equ. (B.12) in the limit that $r$ goes to infinity and
\[ 2\tilde{C} \frac{d\tilde{C}}{d\tilde{\delta}} = -4\tilde{\delta} \left( \tilde{\delta} + \frac{3}{2} \right)^2, \] (3.30)

we see that
\[ \frac{d\tilde{\tau}}{d\tilde{\delta}} = \frac{5(\tilde{\delta} + 3/2)^{1/2}}{(1/2 - \tilde{\delta})^{1/2}} \int_1^{\infty} \frac{(s + 3/2)(s - 3/2)}{s^{(Q(s))}^{1/2}} ds \]
\[ = \sqrt{3/5} \int_1^{\delta/2} \frac{s^2 - 9/4}{s^2} \frac{ds}{[Q(s)]^{1/2}} + O(\tilde{\delta}) = \sqrt{3/5} \tilde{\delta} J + O(\tilde{\delta}). \] (3.31)

$J$ can in turn be split as $J = J^0 + \Delta J$, where
\[ J^0 = \int_1^{\delta/2} \frac{s^2 - 9/4}{s^2} \frac{ds}{\sqrt{s^2 - \delta^2}^{1/2}} = \]
\[ = \sqrt{2/9} \int_1^{\delta/2} \frac{df}{f^2 - 1} - \frac{9/4}{\sqrt{2/9}} \frac{1}{\delta^2} \int_1^{\infty} \frac{df}{f^2 \sqrt{f^2 - 1}} + O \left( \frac{1}{\delta} \right) \]
\[ = O(\ln \tilde{\delta}) - \sqrt{9/8} \frac{1}{\delta^2} \cdot 1. \] (3.32)

Finally,
\[ \Delta J = \int_1^{\delta/2} \frac{s^2 - 9/4}{s^2} \left[ \frac{1}{[Q(s)]^{1/2}} - \frac{1}{\sqrt{[Q(s)]^{1/2}}} \right] ds. \] (3.33)

Thus, using (3.12)
\[ |\Delta J| \leq \text{const } \int_1^{\delta/2} \frac{s^2}{\sqrt{s^2 - \delta^2}} = \frac{\text{const}}{\delta}. \] (3.34)

Putting (3.31,32,33) together, there results
\[ \frac{d\tilde{\tau}}{d\tilde{\delta}} = -\frac{3\sqrt{6}}{4} \frac{1}{\tilde{\delta}} + O(1) \text{ as } \tilde{\delta} \to 0. \] (3.35)
Integrating (3.35), we obtain
\[ \tilde{\tau}(\tilde{\delta}) = -\frac{3\sqrt{6}}{4} \ln \tilde{\delta} + A' + O(\tilde{\delta}), \] (3.36)
for some constant \( A' \). Comparing with (3.29) we infer \( A = A' \) and the proof of the estimate (3.5) is complete.

From \( A = A' \) and (3.36) it is elementary to infer that
\[ \tilde{\delta} = \exp \left( -\frac{4}{3\sqrt{6}}(\tilde{\tau} - A) \right) + O \left[ \exp \left( -\frac{8}{3\sqrt{6}} \tilde{\tau} \right) \right] \text{ as } \tilde{\tau} \to \infty. \] (3.37)

Using (3.30), (3.35) can be written as
\[ \frac{d\tilde{\tau}}{dC} = \frac{3}{4\sqrt{2}} \frac{1}{\tilde{\delta}^2} + O \left( \frac{1}{\tilde{\delta}} \right). \] (3.38)

We want to evaluate the lapse \( \alpha \) of the time function \( \tau = m \tilde{\tau} \) along the central throat \( r = r_C \). This, using (2.23) and (2.24), is given by
\[ \alpha(\tau) = \frac{1}{2m} \left( \frac{d\tilde{\tau}}{dC} \right)^{-1}. \] (3.39)

Using (3.36) this finally leads to
\[
\alpha(\tau) = \frac{4}{3\sqrt{2}} \tilde{\delta} + O(\tilde{\delta}^2) \\
= \frac{4}{3\sqrt{2}} \exp \left( \frac{4A}{3\sqrt{6}} \right) \exp \left( -\frac{4\tau}{3\sqrt{6} m} \right) + O \left[ \exp \left( -\frac{8\tau}{3\sqrt{6} m} \right) \right]\text{ for } \tau \to \infty.
\] (3.40)

We sum up our results in the

**Theorem:** For the chosen maximal foliation, with the time function \( \tau \) coinciding with proper time at infinity and being zero on the time symmetric leaf \( S \), the lapse along the central geodesics orthogonal to the leaves behaves, as a function of \( \tau \), according to (3.40) with \( A \) given by
\[ A = \frac{3\sqrt{6}}{4} \ln |8(3\sqrt{2} - 4)| - 2 \ln \left| \frac{3\sqrt{3} - 5}{9\sqrt{6} - 22} \right| = -0.2181. \] (3.41)

It would be interesting to estimate the lapse for large \( \tau \) along arbitrary Eulerian observers rather than just the ones along \( \Gamma \). In terms of the coordinate \( \rho \) introduced in § 2 we conjecture that
\[
\alpha(\rho, \tau) = B(\rho) \exp \left( -\frac{4}{3\sqrt{6}} \frac{\tau}{m} \right) + O \left[ B^2(\rho) \exp \left( -\frac{8}{3\sqrt{6}} \frac{\tau}{m} \right) \right],
\] (3.42)
where $B(\rho)$ behaves for large $\rho$ as

$$B(\rho) \sim \text{const} \cdot \cosh \frac{4}{3\sqrt{6}} \left( \frac{\rho}{m} \right)^3.$$  \hfill (3.43)

The form of $B(\rho)$ in Equ. (3.43) is motivated by the solution to the lapse equation (2.25) on the limiting slice at $r = 3m/2$, which is symmetrical with respect to the throat.

**Appendix A**

The following discussion is similar in spirit to [15]. Let $\Sigma$ be the manifold $\mathbb{R} \times S^2$ with a Riemannian, spherically symmetric metric, which we write in the “radial” gauge, i.e.

$$g = d\ell^2 + r^2(\ell)d\Omega^2, \quad r \in (0, \infty).$$ \hfill (A.1)

The unit vector $\ell^i = (\partial / \partial \ell)^i$ is geodesic and satisfies ($r' = dr/d\ell$)

$$D_i \ell_j = \frac{r'}{r} q_{ij}, \hfill (A.2)$$

where $q_{ij} = g_{ij} - \ell_i \ell_j$ and prime means derivative w.r.t. $\ell$. After a calculation, which most easily follows the lines of Besse [13], we find for the Riemann tensor

$$R_{ijkl} \ell^i \ell^j = \frac{r''}{r} q_{kl}$$ \hfill (A.3)

and

$$q'_{ij} q'^k_{ij} q'^k_{ij} R_{ijkl} \ell^i \ell^j = \frac{2}{r^2} (1 - r'^2) q_{ij} q_{ij}.$$ \hfill (A.4)

Identities (A.3,4) imply that

$$R_{ij} = -r'' \frac{(2 \ell_i \ell_j + q_{ij})}{r} + \frac{1 - r'^2}{r^2} q_{ij}, \hfill (A.5)$$

$$R = -4 \frac{r''}{r} + 2 \frac{1 - r'^2}{r^2}.$$ \hfill (A.6)

The extrinsic curvature on $\Sigma$, in order to be spherically symmetric, has to be of the form

$$K_{ij} = v \ell_i \ell_j + w q_{ij}. \hfill (A.7)$$

The condition $K_{ij} g^{ij} = 0$ implies that $v + 2w = 0$. Using (A.2) we have

$$D^i K_{ij} = \left( v' + 3 \frac{r'}{r} v \right) \ell_j.$$ \hfill (A.8)
Thus the maximal momentum constraint implies $v = 2C/r^3$ for some constant $C$. Consequently,

$$K_{ij} = \frac{2C}{r^3} \ell_i \ell_j - \frac{C}{r^3} q_{ij} \quad (A.9)$$

$$K_{ij} K^{ij} = 6 \frac{C^2}{r^6}. \quad (A.10)$$

Inserting (A.10) and (A.6) into the Hamiltonian constraint, there results

$$-4 \frac{r''}{r} + 2 \frac{1 - r'^2}{r^2} = 6 \frac{C^2}{r^6}. \quad (A.11)$$

Next we define $m(r)$ by

$$m(r) := \frac{r}{2}(1 - r'^2) + \frac{C^2}{2r^3}. \quad (A.12)$$

Now (A.11) implies that $dm/dr$ is zero. Thus

$$r' = \left(1 - \frac{2m}{r} + \frac{C^2}{r^4}\right)^{1/2}. \quad (A.13)$$

Assuming $m > 0$ and $0 \leq |C| < 3 \frac{\sqrt{3}}{4} m^2$, there are two initial-data sets consistent with (A.9) and (A.13). One starts at $r = 0$, expands to an $r_{\text{max}} < 3m/2$ and collapses back to $r = 0$. The other is an asymptotically flat complete metric on $\mathbb{R} \times S^2$ with mass $m$ at both ends which is symmetric with respect to the throat at $r = r_C > 3m/2$ with

$$1 - \frac{2m}{r_C} + \frac{C^2}{r_C^4} = 0. \quad (A.14)$$

Here we restrict ourselves to asymptotically flat data. These constitute a 2-parameter family of solutions to the spherically symmetric, maximal vacuum constraints. Of course, we know from the Birkhoff theorem that members of this family with different $C$ but the same $m$ have all to lie in the same spacetime, namely the extended Schwarzschild spacetime. “Discovering” this fact in the present context amounts to finding the “height function” written down in Sect. 2. The trick is to try to find the remaining Killing vector and to seek the $\Sigma_C$’s as graphs over the surfaces orthogonal to this Killing vector. If $(g_{ij}, K_{ij})$ evolve to a spacetime having another Killing vector, there must be a function $N$, not identically zero, and a vector field $X^i$ so that

$$2NK_{ij} + 2D_{[i}X_{j]} = 0. \quad (A.15)$$

Assuming $X^i$ to be again spherical, i.e.

$$X_i = \mu \ell_i, \quad \mu = \mu(r), \quad (A.16)$$
and again using (A.2) and (A.9) we infer that
\[
-2 \frac{NC}{r^3} + 2r' \frac{\mu}{r} = 0 \tag{A.17}
\]
\[
4 \frac{NC}{r^3} + 2r' \frac{d\mu}{dr} = 0. \tag{A.18}
\]
After combining (A.17) and (A.18), there results
\[
\mu(r) = \frac{D}{r^2}, \quad D = \text{const} \tag{A.19}
\]
\[
N = \frac{D}{C} r', \tag{A.20}
\]
where we have assumed \( C \neq 0 \). We assume without loss that \( D = C \). The existence of \((N, X^i)\) solving Equ.’s (A.15) does not necessarily imply that the vacuum evolution of the initial data set has a static Killing vector. There also has to be satisfied
\[
\mathcal{L}_X K_{ij} + D_i D_j N = N(R_{ij} - 2 K_{il} K^l_j). \tag{A.21}
\]
It is straightforward to check that (A.19,20) do satisfy (A.21). (In the case where \( C \) is zero, \( X^i = 0 \) and Eq.(A.21) implies that \( N \sim r' \).)

We remark in passing that the function \( N \), by virtue of (A.15) and (A.21), satisfies
\[
\Delta N = N K_{ij} K^{ij}. \tag{A.22}
\]
(Of the two linearly independent spherical solutions of (A.22) \( N \) is that combination which vanishes on the throat.)

It now follows that for \( r > r_C \) the metrics
\[
d s^2 = -(N^2 - g_{ij} X^i X^j) d\sigma^2 + 2g_{ij} X^j dX^i d\sigma + g_{ij} dX^i dX^j, \tag{A.23}
\]
with \( N_i, X^i, g_{ij} \) extended in a \( \sigma \)-independent way to \( \mathbb{R} \times \Sigma \), are vacuum solutions evolving from the above initial data sets. They have \( \xi^\mu = (\partial / \partial \sigma)^\mu \) as a Killing vector. More explicitly, since
\[
N^2 - X_i X^i = 1 - \frac{2m}{r}, \tag{A.24}
\]
we have
\[
d s^2 = -\left(1 - \frac{2m}{r}\right) d\sigma^2 + 2 \frac{C}{r^2} dl d\sigma + dl^2 + r^2 d\Omega^2, \tag{A.25}
\]
where \( r(\ell) \) is given implicitly by
\[
\ell(r) = \int_{r_c}^r \frac{dx}{\sqrt{1 - 2m/x + C^2/x^4}}. \tag{A.26}
\]
(For \( C = 0 \), \( \ell(r) \) can be written as \( \ell(r) = r \sqrt{1 - 2m/r} + m \ln \left| \frac{1 + \sqrt{1 - 2m/r}}{1 - \sqrt{1 - 2m/r}} \right| \).)
Note that for $C \neq 0$ the above metrics extend smoothly across $r = 2m$. We now seek a function $t$ with level surfaces orthogonal to $\partial / \partial \sigma$. Writing this function as

$$t = F(r) + \sigma,$$

we obtain from

$$g_{\mu\nu} \xi^\mu dx^\nu = -(N^2 - X_i X^i) d\sigma + X_i dx^i = \omega(dF + d\sigma),$$

for some function $\omega$, the equation

$$-D_i F = \frac{X_i}{N^2 - X_j X^j}$$

which makes only sense off the horizon. Using (A.16,19) this leads to

$$\frac{dF}{dr} = -\frac{C}{r^2 - 2mr} \frac{1}{\sqrt{1 - 2m/r + C^2/r^4}}.$$  \hspace{1cm} (A.30)

Now consider the coordinate transformation

$$\sigma = t - F.$$

Then

$$ds^2 = -(N^2 - X_j X^j) dt^2 + \tilde{g}_{ij} dx^i dx^j,$$  \hspace{1cm} (A.32)

with

$$\tilde{g}_{ij} = g_{ij} + 2X_i F_j - (N^2 - X_k X^r) F_i F_j$$

$$= g_{ij} + (N^2 - X_k X^r)^{-1} X_i X_j,$$  \hspace{1cm} (A.33)

where $X_i := \tilde{g}_{ij} X^j$. Using (A.29) and (A.30),

$$\tilde{g}_{ij} dx^i dx^j = \left[ 1 + \left(1 - \frac{2m}{r}\right)^{-1} \frac{C^2}{r^4} \right] dt^2 + r^2 d\Omega^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (A.35)

We have thus recovered the Schwarzschild metric. In particular this calculation shows that the parameter $C$ in our initial-data sets is “pure gauge”; initial data with the same $m$ lie in the same spacetime, namely as level sets of the function $\sigma$. They can also be written as

$$t = F_C(r)$$

and its translates under $\xi^\mu = (\partial / \partial t)^\mu$, where

$$F_C(r) = -C \int_{r_0}^r \frac{dx}{(1 - m/x)(1 - 2m/x + C^2/x^4)^{1/2}},$$

for some $r_0$. Taking $r_0 = r_C$ we have, with $h(r, C) = F_C(r)$, recovered (2.12).
It is shown in Sect. 2 that
\[ t = h(r, C) \]  
(A.38)
implicitly defines a smooth time function on the \( r > 3m/2 \)-subset of the future half of Kruskal. The boost function \( N \) obtained in this Appendix satisfies the same equation, on each leaf \( \Sigma_C \) as the lapse function of \( C \), namely (A.22). The reason for this is that, for fixed \( \Sigma_C \), \( \xi^\mu \) defines another local foliation, which is again maximal since \( \xi^\mu \) is a Killing vector.

## Appendix B

Consider
\[ F(x, E) = \int_{x_E=V^{-1}(E)}^{x} \frac{W(y)}{[E - V(y)]^{1/2}} dy, \]  
(B.1)
where \( V \) is a smooth function \( V: [x_0, \infty) \rightarrow \mathbb{R} \) with
\[ 0 < V(x_0), \quad V'(x) < 0 \text{ for } x > x_0, \quad V(\bar{x}) = 0 \]  
(B.2)
and
\[ 0 < E < V(x_0). \]  
(B.3)
The function \( W \) is smooth except perhaps at \( x = \bar{x} \), where it may have a simple pole. (Thus the pole of \( \sqrt{E} W(y)/[E - V(y)]^{1/2} \) is independent of \( E \).) In the latter case (B.1) is to be understood in the principal-value sense and the following operations valid for \( x \neq \bar{x} \). Next define (we follow [16] in spirit)
\[ J(x, E) = \int_{x_E}^{x} [E - V(y)]^{1/2} V(y) W(y) dy. \]  
(B.4)
Note that \( V \cdot W \) is smooth. Equ. (B.4) can be rewritten as follows
\[ J(x, E) = -\frac{2}{3} \int_{x_E}^{x} \frac{d}{dy} \left[ E - V(y) \right]^{3/2} \frac{V(y) W(y)}{V'(y)} dy, \]  
(B.5)
\[ J(x, E) = -\frac{2}{3} \int_{x_E}^{x} \left[ E - V(y) \right]^{3/2} \frac{V(x) W(x)}{V'(x)} + \frac{2}{3} \int_{x_E}^{x} \left[ E - V(y) \right]^{3/2} \frac{d}{dy} \left[ \frac{V(y) W(y)}{V'(y)} \right] dy. \]  
(B.6)
Differentiating (B.6) w.r.t. \( E \) twice, we obtain
\[ \frac{d^2}{dE^2} J(x, E) = -\frac{1}{2} \int_{x_E}^{x} \frac{1}{[E - V(x)]^{1/2}} \frac{V(x) W(x)}{V'(x)} + \frac{1}{2} \int_{x_E}^{x} \frac{1}{[E - V(y)]^{1/2}} \frac{d}{dy} \left[ \frac{V(y) W(y)}{V'(y)} \right] dy. \]  
(B.7)
On the other hand, differentiating (B.4) once w.r.t. \( E \) it follows that
\[ \frac{d}{dE} J(x, E) = \frac{1}{2} \int_{x_E}^{x} \frac{V(y)}{[E - V(y)]^{1/2}} W(y) dy \]
\[ = \frac{1}{2} \int_{x_E}^{x} \frac{V(y) - E + E}{[E - V(y)]^{1/2}} W(y) dy \]
\[ = \frac{1}{2} \int_{x_E}^{x} [E - V(y)]^{1/2} W(y) dy + \frac{1}{2} EF(x, E). \]  
(B.8)
Differentiating (B.8) once more w.r.t. $E$ and comparing with (B.7) we finally find

$$\frac{1}{4} F(x, E) + \frac{1}{2} E \frac{d}{dE} F(x, E) =$$

$$= -\frac{1}{2} \frac{1}{[E - V(x)]^{1/2}} \frac{V(x) W(x)}{V'(x)} + \frac{1}{2} \int_x^r \frac{1}{[E - V(y)]^{1/2}} \frac{d}{dy} \left[ \frac{V(y) W(y)}{V'(y)} \right] dy, \quad (B.9)$$

In our case we will have that $V'(x_0) = 0$ and we study the blow-up of $F_{\infty}(E) = \lim_{x \to \infty} F(x, E)$ as $E$ tends to $E_0 = V(x_0)$. As for a mechanical analogue, we could think of a particle on a half-line in a repulsive potential $V(x)$ and imagine $F(x, E)$ to be the time it takes a particle of energy $E$ to travel from $x_0$ to $x$. (If it were not for the presence of $W(y)$ in (B.1), this interpretation would be literally true.) The force on the particle grows so fast for large $x$, that the particle reaches infinity in finite time $F_{\infty}(E)$. There is an unstable equilibrium point at $x = x_0$. We ask for the way in which $F_{\infty}(E)$ blows up as $E$ approaches $V(x_0)$. If the energy $E$ is further increased, the orbits reach $x = 0$: this corresponds to maximal slices hitting the singularity.

To make contact with our function $h(r, C)$, set

$$V(x) = -x^4 + 2mx^3, \quad E = C^2, \quad W(x) = -\frac{1}{1 - 2m / x},$$

$$h(r, C) = CF(r, C^2),$$

$$x_0 = \frac{3m}{2}, \quad \bar{x} = 2m. \quad \tag{B.10}$$

Thus

$$\frac{d}{dC} h(r, C) = 2E \left. \frac{d}{dE} F(r, E) \right|_{E=C^2} + F(r, C^2) \quad \tag{B.11}$$

which, combined with (B.9), gives

$$\frac{d}{dC} h(r, C) = \frac{1}{2(r - 3m/2) \sqrt{1 - 2m / r + C^2 / r^4}} - \frac{1}{2} \int_r^x \frac{x(x - 3m)dx}{(x - 3m/2)^2 (x^4 - 2mx^3 + C^2)^{1/2}}, \quad \tag{B.12}$$

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**References**


