Constrained Dynamics for Quantum Mechanics I.
Restricting a Particle to a Surface

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Restricting a Particle to a Surface.

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Abstract We analyze constrained quantum systems where the dynamics do not 
preserve the constraints. This is done in particular for the restriction of  
a quantum particle in \(\mathbb{R}^n\) to a curved submanifold, and we propose  
a method of constraining and dynamics adjustment which produces the  
right Hamiltonian on the submanifold when tested on known examples.  
This method will be the germ of a “Dirac algorithm for quantum con-
straints.” We generalise it to the situation where the constraint is a  
general selfadjoint operator with some additional structures.

Keywords: constraints, quantum mechanics, surface, Hamiltonian operator.
1. Introduction.

The problem of enforcing constraints in classical mechanics has had satisfactory solutions for some time now [Di, GNH, MW]. The same is not true in quantum mechanics, contrary to claims of "quantizations" of classical constrained systems after constraining. Quantization maps are rarely unique nor well-defined [VH, Wo, Ri, GGT]. Moreover, using noncommutativity, some constraints can be defined in quantum mechanics with zero classical limit. This means that the general problem of quantum constraints needs to be solved in the quantum arena without appealing to classical methods.

There is presently a wide variety of methods for imposing quantum constraints [Di, GH, HT, La, SW] but these methods deal with the kinematics only at the quantum level, and their interrelations are unclear. When the given quantum dynamics preserves the constrained subsystem that is fine, one only needs to restrict it to the constrained subsystem to obtain the constrained dynamics. However, when this is not the case the problem arises of how to appropriately adjust the constraint set and the dynamics into a stable system. (New constraints produced by such a method are generally called secondary constraints). The obvious strategy of fixing the dynamics and extending the constraint set to its orbit under the dynamics, produces the wrong physics in examples. It is our opinion that the question of secondary quantum constraints and dynamics adjustment is a physical problem, and cannot be decided by mathematical arguments alone. In this paper we set out to solve this problem of secondary quantum constraints in the limited class of systems consisting of the restriction of a quantum particle on $\mathbb{R}^n$ to a smooth (possibly curved) submanifold i.e. the case of holonomic constraints (cf. p75 [Ar]). The motivation for this choice is as follows:

(1) there are several examples of quantum mechanics on surfaces available (e.g. $S^2$) against which we can test the results of our analysis.

(2) in $\mathbb{R}^n$ there is ample geometry available to guide the intuition, e.g. a metric, hence normal vectors and Lebesgue measure,

(3) generalisations are easy from this class, in fact in Sect. 6 we propose a generalisation of it to the case of a general selfadjoint operator on a Hilbert space
(4) A constraint in this situation is of the form \( \varphi(q) = 0 \) which has an immediate obvious quantization in the Schrödinger representation, whereas a constraint of the form \( \varphi(q, p) = 0 \) need not have a unique quantization,

(5) physically these systems in \( \mathbb{R}^3 \) can be approximated by very thin films or wires and so are close to experimental verification,

(6) the exposition of constraint methods for these systems are more transparent than for more general ones,

The kinematics of quantum systems of this type has already been done concretely by Landsman [La] and abstractly at the C*-level by Grundling and Hurst [GH], so any new proposed method should be compatible with these. We will not consider BRST-methods [HT], as we believe these not to be equivalent at the quantum level to the other methods [Gr, LL, Mc]. So to summarize; we aim to solve for this class of systems the problem of secondary quantum constraints, i.e. given a constrained system where the dynamics do not preserve the constraints, to find a method of adjustment which agrees on known constrained systems with the right physics.

The architecture of the paper is as follows; in Sect. 2 we summarize the Dirac procedure for constraining a classical particle to a submanifold, both local and global, and we also do two examples. In Sect. 3 we consider the question of quantum constraints, we first summarize the usual Dirac procedure, then start to analyze the situation of restricting a quantum particle in \( \mathbb{R}^n \) to a lower dimensional submanifold \( \Gamma \). We obtain the constraining map \( \kappa \) from the Hilbert space \( L^2(\mathbb{R}^n) \) of the unconstrained particle to that of the constrained particle \( L^2(\Gamma) \), and we discuss generalisations to other types of constrained systems. In Sect. 4 we solve the problem of how to obtain from the Hamiltonian of the original particle a constrained Hamiltonian on \( L^2(\Gamma) \), and we apply it to two examples; restricting a free particle in \( \mathbb{R}^3 \) to a sphere and a cylinder, and we obtain in each case the correct Hamiltonian. In Sect. 5, we consider how to obtain observables on \( L^2(\Gamma) \) from the original observables on \( L^2(\mathbb{R}^n) \), i.e. we analyse when we obtain sensible liftings of operators through the map \( \kappa \), and how to obtain a suitable field algebra for the constrained particle. In Sect. 6 we suggest a
generalisation of the method of the preceding sections to the constraining situation where the constraint is a selfadjoint operator on an abstract Hilbert space, with zero in the continuous part of the spectrum. We find that we need to assume some additional structure to do the work which the geometry of $\mathbb{R}^n$ does in the method of the preceding sections. The reader in a hurry can start with Sect. 3.


We start by recalling the basic Dirac–Bergman method [Di, Su, SM] which is of course a local procedure. However to keep this brief, we only present it in the context of a particle in $\mathbb{R}^n$ constrained to an $(n \perp 1)$-dimensional submanifold $\Gamma$, which is the system at the focus of this paper.

The full phase space is $\mathbb{R}^{2n}$ with generic point $(q, p)$ and the usual Poisson algebra $\mathcal{P} = (C^\infty(\mathbb{R}^n), \{\cdot, \cdot\})$ is the setting for the problem. We specify a smooth $(n \perp 1)$-dimensional submanifold $\Gamma$ in $\mathbb{R}^n$ as the zero set of a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$, i.e., $\Gamma = \varphi^{-1}(0)$. We also assume $\nabla \varphi \neq 0$ on $\Gamma$ so we can use the gradient to define a normal on $\Gamma$. The Hamiltonian for a particle in a potential $V$ is:

$$H_c = \frac{1}{2m} p \cdot p + V(q).$$

So we have a constrained system with a single primary constraint $\varphi(q, p) := \varphi(q)$ and primary constraint manifold in phase space $\varphi^{-1}(0) = \Gamma \times \mathbb{R}^n =: \Gamma_p$. The Dirac procedure starts with the new Hamiltonian:

$$H_p := \frac{1}{2m} p \cdot p + V(q) + \mu \varphi,$$

where $\mu \in \mathcal{P}$ is a free multiplier, to be determined below. So the new time evolution for $A \in \mathcal{P}$ on $\Gamma_p$ is

$$\dot{A}|_{\Gamma_p} = \{A, H_p\}|_{\Gamma_p} = \left(\{A, H_c\} + \mu \cdot \{A, \varphi\}\right)|_{\Gamma_p}.$$

Since time evolution must preserve $\Gamma_p$ we require that

$$0 = \dot{\varphi}|_{\Gamma_p} = \{\varphi, H_p\}|_{\Gamma_p} = \frac{1}{m} p \cdot (\nabla \varphi)|_{\Gamma_p}.$$
Thus $\chi := \frac{1}{m} p \cdot (\nabla \varphi)$ is a secondary constraint determining a smaller submanifold $\Gamma_f := \varphi^{-1}(0) \cap \chi^{-1}(0)$, which again must be preserved by time evolution.

In the examples in the rest of this paper we will constrain free motion to a surface, so for this case \((V = 0)\):

\[
\dot{\xi}|_{\Gamma_f} = 0 = \left(\{\xi, H_\varphi\} + \mu \cdot \{\xi, \varphi\}\right)|_{\Gamma_f}, \quad \text{where} \quad \xi \quad \text{denotes} \quad \chi, \varphi
\]

so

\[
0 = \left(\{\frac{1}{m} p \cdot (\nabla \varphi), \frac{1}{2m} p \cdot p\} + \mu \{\frac{1}{m} p \cdot (\nabla \varphi), \varphi\}\right)|_{\Gamma_f}
\]

\[
= \left(\frac{1}{2m^2} p \cdot \{\nabla \varphi, p \cdot p\} + \frac{1}{m} \mu (\nabla \varphi) \cdot \{p, \varphi\}\right)|_{\Gamma_f}
\]

\[
= \left(\frac{1}{m^2} p \cdot (\nabla (p \cdot \nabla \varphi)) - \frac{1}{m} \mu (\nabla \varphi) \cdot (\nabla \varphi)\right)|_{\Gamma_f}.
\]

Since $\nabla \varphi \neq 0$ on $\Gamma$, we solve on $\Gamma_f$:

\[
\mu = \frac{p \cdot (\nabla (p \cdot \nabla \varphi))}{m \|\nabla \varphi\|^2} = \frac{(p \cdot \nabla)\varphi}{m \|\nabla \varphi\|^2}
\]

thus

\[
H_p = \frac{1}{2m} p \cdot p + \varphi \cdot \frac{(p \cdot \nabla)\varphi}{m \|\nabla \varphi\|^2}.
\]

Notice that the second term in $H_p$ when extended near $\Gamma_f$ acts like a “potential” to keep the motion on $\Gamma$, and that the secondary constraint

\[
\chi = \frac{1}{m} p \cdot \nabla \varphi)|_{\Gamma} = 0
\]

 guarantees that the motion is tangent to the surface and there are no more constraints. Moreover, since

\[
\{\varphi, \chi\} |_{\Gamma_f} = \frac{1}{m} \|\nabla \varphi\|^2 |_{\Gamma_f} \neq 0,
\]

the constraints are second class. The Dirac algorithm continues to construct a Dirac bracket, but we will not need this. For later use we specialize the above to a sphere and a cylinder.

Let $\Gamma$ be a sphere of radius $a$ in $\mathbb{R}^3$. The phase space is $\mathbb{R}^6$, $\Gamma_p = S^2 \times \mathbb{R}^3$ and we have

\[
\varphi(q) = q \cdot q \perp a^2, \quad \text{so} \quad \chi = \frac{1}{m} p \cdot \nabla \varphi = \frac{2}{m} p \cdot q
\]

and since $\nabla (p \cdot \nabla \varphi) = 2p$ here, we get

\[
H_p = \frac{1}{2m} p \cdot p \left(1 + \frac{q \cdot q \perp a^2}{\|q\|^2}\right)
\]

(2.3)
An explicit calculation produces

\[ H_p|_{\Gamma_f} = \left. \frac{1}{2m} p \cdot p \right|_{q_1 - a = 0} = \frac{|L|^2}{2ma^2} \tag{2.4} \]

where \( L = q \times p \bigg|_{r=a} \) is angular momentum on the sphere.

If we take \( \Gamma \) to be a cylinder of radius \( a \) around the \( z \)-axis in \( \mathbb{R}^3 \), we have

\[ \varphi(q) = q_1^2 + q_2^2 \perp a^2, \quad \text{so} \quad \chi = \frac{2}{m}(p_1 q_1 + p_2 q_2) \]

\[ H_p = \left. \frac{1}{2m} p \cdot p \right|_{q_1^2 + q_2^2 - a^2} = \frac{p_1^2 + p_2^2}{2m(q_1^2 + q_2^2)}. \tag{2.5} \]

And now:

\[ H_p|_{\Gamma_f} = \left. \frac{1}{2m} p \cdot p \right|_{q_1^2 + q_2^2 - a^2} = \frac{1}{2m} \left( \frac{L_3^2}{a^2} + p_3^2 \right)|_{\Gamma_f} \tag{2.6} \]

where \( L_3 := q_1 p_2 - q_2 p_1 \) is the usual angular momentum around the \( q_3 \)-axis.

Next we recall the global version of the Dirac–Bergman method of constraints as worked out by Gotay, Nester and Hinds [GNH], which is a considerable simplification. Start with a symplectic manifold \((M, \omega)\) (finite dimensional here) and a Hamiltonian \( H \in C^\infty(M) \). Let \( M_1 \subset M \) be a given primary constraint submanifold. Now the evolution equation \( \iota_X \omega = dH \) may not have solutions for the vector field \( X \) tangential to \( M_1 \), so let

\[ M_2 := \{ m \in M_1 \mid dH(m) = \iota_X \omega(m) \ \text{for some} \ X \in T_m M_1 \} . \]

However the time evolution must now preserve \( M_2 \), so iterate:

\[ M_{k+1} := \{ m \in M_k \mid dH(m) = \iota_X \omega(m) \ \text{for some} \ X \in T_m M_k \} . \]

When this iteration converges sensibly, we obtain a final constraint manifold \( M_c \) on which \( dH = \iota_X \omega \) has solutions \( X \in \Gamma^\infty(M_c) \equiv \) smooth vector fields on \( M_c \), which provide the desired time evolutions. Geometrically we can think of \( M_c \) as the largest submanifold which has a vector field \( X \) which is tangential at
each $m \in M_e$ to a trajectory of the time evolution of the original unconstrained manifold $M$. The completeness of $X$ is still an open question for the general analysis. (Below we will refer to the above as the GNH-algorithm).

We will take this geometric picture as the guiding principle for constraining the dynamics of a quantum system below in Sect. 4.

### 3. Constraining the representation space.

For reference we recall the usual Dirac prescription for first class quantum constraints. Given a field algebra $\mathcal{A}$ preserving a dense domain $\Delta$ in a Hilbert space $\mathcal{H}$, as well as a Hamiltonian $H \in \mathcal{A}_{sa}$ and a set of constraints $\{ \hat{\varphi}_i \in \mathcal{A}_{sa} \mid i \in I \}$ (where $I$ is an abstract index set), one selects the physical subspace by

$$\mathcal{H}_{\text{phys}} := \{ \psi \in \Delta \mid \hat{\varphi}_i \psi = 0 \ \forall i \}$$

and this is assumed to be nonzero (hence zero must be in the discrete spectrum of each $\hat{\varphi}_i$). Decompose $\mathcal{H} = \mathcal{H}_{\text{phys}} \oplus \mathcal{H}^{-}_{\text{phys}}$, so for each $A \in \mathcal{A}$ we have a decomposition $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Clearly the only $A \in \mathcal{A}$ which, together with $A^*$ can be restricted to $\mathcal{H}_{\text{phys}}$ are those of the form $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

Denote the $^*$-algebra of these by $\mathcal{O}$. Those which restrict trivially are of the form $A = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ and comprise an ideal $\mathcal{D}$ of $\mathcal{O}$. Then the $^*$-algebra of physical observables are

$$\mathcal{O} \big|_{\mathcal{H}_{\text{phys}}} \cong \mathcal{O}/\mathcal{D} =: \mathcal{R}.$$ 

If $\exp itH$ preserves $\mathcal{H}_{\text{phys}}$ then $\text{Ad} \exp(itH)$ lifts to automorphisms on $\mathcal{R}$. Otherwise we need to alter it (cf. Sect. 4). Let $P_{\text{phys}}$ be the projection on $\mathcal{H}_{\text{phys}}$ and define the sesquilinear form

$$(\psi_1, \psi_2)_D := (P_{\text{phys}} \psi_1, P_{\text{phys}} \psi_2) = (\psi_1, P_{\text{phys}} \psi_2)$$

then $\text{Ker}(\cdot, \cdot)_D = \mathcal{H}^{-}_{\text{phys}}$ and we write $P_{\text{phys}}$ as a factorization $P_{\text{phys}} : \mathcal{H} \to \mathcal{H}/\text{Ker}(\cdot, \cdot)_D \cong \mathcal{H}_{\text{phys}}$ (obvious identification), and we can think of $\mathcal{O}$ as all $A \in \mathcal{A}$ which together with $A^*$ can lift through the factorisation.

Next we introduce our main object of study. Consider a quantum particle in $\mathbb{R}^n$ in the Schrödinger representation. Its basic data consists of the Hilbert space
\[ \mathcal{H} = L^2(\mathbb{R}^n, \mu) \] (with \( \mu \) the Lebesgue measure), and on the dense subspace \( C^\infty_c(\mathbb{R}^n) \) of smooth functions of compact support, we have the position operator \( \hat{q}(\psi)(x) := x\psi(x) \), the momentum operator \( \hat{p}(\psi)(x) := (i\nabla\psi)(x) \) and a Hamiltonian \( H \) (which is \( \frac{1}{2m}\hat{p} \cdot \hat{p} \) when the particle is free). These operators are all essentially selfadjoint and preserve \( C^\infty_c(\mathbb{R}^n) \). We wish to constrain this particle to a given smooth \((n \perp 1)\)-dimensional submanifold \( \Gamma \subset \mathbb{R}^n \). Assume we have a bounded real-valued constraint function \( \varphi \in C^\infty(\mathbb{R}^n) \) with \( \nabla \varphi \) nonzero and bounded on a neighbourhood of \( \Gamma \), and

\[ \Gamma = \{ x \in \mathbb{R}^n \mid \varphi(x) = 0 \} = \varphi^{-1}(0). \]

(Notice that the level hypersurfaces of \( \varphi \) form a foliation of this neighbourhood [To].) Quantize \( \varphi \) by the multiplication operator \( \varphi(\hat{q}) \), i.e.

\[ (\hat{\varphi}(\psi))(x) = \varphi(x) \cdot \psi(x) \quad \forall \psi \in \mathcal{H} \]

which is selfadjoint since \( \varphi \) is real-valued, and bounded because \( \varphi \) is.

Assume we are given a concrete field algebra \( \mathcal{F} \subset \mathcal{B}(\mathcal{H}) \), which should be a C*-algebra containing all relevant operators in bounded form, e.g. \( \hat{\varphi} \), \( \exp i\mathbf{a} \cdot \hat{q} \), \( \exp i\mathbf{a} \cdot \hat{p} \) (\( \mathbf{a} \in \mathbb{R}^n \)). The choice of field algebra will turn out to be important for producing a meaningful algebra of observables in the constrained system, but we will return to this matter in a later section. In summary, we are assuming the following data:

- the operators \( \hat{p} \), \( \hat{q} \) and \( H \) on \( C^\infty_c(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \),

- A bounded \( \varphi \in C^\infty(\mathbb{R}^n) \) with \( \Gamma = \varphi^{-1}(0) \) a smooth \((n \perp 1)\)-dimensional submanifold, and with \( \nabla \varphi \) nonzero and bounded in a neighbourhood of \( \Gamma \),

- A unital field algebra \( \mathcal{F} \) on \( L^2(\mathbb{R}^n) \).

We would like to find a way of imposing the constraint "\( \hat{\varphi} = 0 \)" on this system. However since \( \Gamma \) is a null-set of \( \mu \), we have \( \text{Ker} \hat{\varphi} = \{0\} \), and so Dirac's method of restriction to the kernel of the constraint fails. The fact that zero is in the continuous spectrum of \( \hat{\varphi} \) is physically due to the uncertainty principle; a particle confined to \( \Gamma \) will violate Heisenberg's uncertainty principle in a direction normal to \( \Gamma \).
At present there are two methods which can handle the above situation: the T-procedure of [GH], as well as Landsman’s application of Rieffel induction [La]. The T-procedure “ignores” the original representation of $\mathcal{F}$, considers $\mathcal{F}$ as an abstract C*-algebra, for which it then finds those representations $\pi$ for which zero is in the discrete part of the spectrum of $\pi(\hat{\phi})$ and works out the algebraic structures associated with factoring $\text{Ker} \pi(\hat{\phi})$ out of each of those representations. On the other hand, Landsman’s method assumes a locally compact group $H$ with a continuous proper action on $\mathbb{R}^n$ such that $\Gamma$ is precisely the set of points left invariant. Assuming some initial representation $\pi$ of $H$ on $L^2(\mathbb{R}^n)$, he makes $C_c(\mathbb{R}^n)$ into an $(A \perp B)$-bimodule where $B$ is the algebra $C_c(H)$ (with convolution for multiplication) and $A$ is the part of the commutant of $\pi(H)$ preserving $C_c(\mathbb{R}^n)$. There is then a natural rigging map, and this allows a Rieffel induction from the trivial representation of $B$ to $A$.

The T-procedure selects the set of all representations in which the Dirac algorithm makes sense, which necessarily excludes the original representation of the current system. This means that one loses the original physical interpretation of the state vectors, and one has the problem of which representation to choose for the constrained system. From the point of view of the physics, one may sometimes be more interested in constructing the constrained operators directly out of the original ones. Landsman’s method does this, but requires some additional group structure, and quantizes a fairly small algebra. Both methods do only the kinematics, and if the dynamics do not preserve $\Gamma$ it is hard to see from these structures what should be done.

We will propose a method for constraining the kinematics of a quantum particle to $\Gamma$ which generalises the usual Dirac prescription and in which the constrained operators are explicitly constructed from the original ones. It will then be fairly easy to see what to do with the dynamics.

In a sense the solution of the kinematics problem is already known. One takes the subspace $C_c(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and restricts it to $\Gamma$, thus obtaining a dense subset of the physical Hilbert space $L^2(\Gamma, \gamma)$ ($\gamma$ denotes the measure induced
on $\Gamma$ by $\mu$). The restriction is the same as factoring out the subspace

$$N := \{ f \in C_c(\mathbb{R}^n) \mid f \mid_{\Gamma} = 0 \}$$

from $C_c(\mathbb{R}^n)$. The observables are those operators which “restrict to $\Gamma$,” i.e., operators $A$ which preserve $C_c(\mathbb{R}^n)$ and $N$, in which case $A$ lifts through the restriction to define an operator on $C_c(\Gamma) \subset L^2(\Gamma)$. This is the procedure which we intend to refine. Note however that the restriction map $\rho : C_c(\mathbb{R}^n) \to C_c(\Gamma)$ is unbounded with respect to the $L^2$-norms and it is not closable as an operator. Let $\psi_k \in L^2(\mathbb{R}^n)$ be a sequence of continuous functions with compact support, all of which have the same restriction to a nonzero $\theta \in L^2(\Gamma)$, and which converge to zero w.r.t. the $L^2$-norm. Then $\psi_k \to 0$, $\rho(\psi_k) \to \theta \neq 0$, so $\lim \rho(\psi_k) \neq \rho(\lim \psi_k)$. So we expect some pathologies to arise in operator questions. We would also like to build the inner product of $L^2(\Gamma)$ explicitly out of the inner product of $L^2(\mathbb{R}^n)$. To do this, we adapt a well-known argument from statistical mechanics [Kh], and p80 of [Fa].

Since $\nabla \varphi(x) \neq 0$ on a neighbourhood of $\Gamma$, the critical points of $\varphi$ are a finite distance away from $\Gamma$. So we have the geometrically obvious:

**Lemma 3.2.** There is a $t > 0$ such that in $\varphi^{-1}(t, t]$ we have locally a smooth curvilinear orthogonal coordinate system $(y, \varphi)$ where $y = \text{constant}$ are curves normal to the level hypersurfaces of $\varphi$.

**Proof:** Consider the vector field $X(x) = n(x)/|\nabla \varphi(x)|$ on $\varphi^{-1}(t, t)$, where $n = \nabla \varphi/|\nabla \varphi|$. Then its integral curves are normal to all level surfaces of $\varphi$ in $\varphi^{-1}(t, t)$, and moreover for each $s \in (t, t)$ we have a diffeomorphism $\alpha_s : \varphi^{-1}(0) \to \varphi^{-1}(t, t)$ such that $\alpha_0 = \iota$ and $d/ds \alpha_s(x) = X(\alpha_s(x))$ [AM]. In fact $\varphi(\alpha_s(x)) = s$ for all $x \in \varphi^{-1}(0) = \Gamma$, which we see as follows. For small $\varepsilon$:

$$\alpha_\varepsilon(x) = x + \varepsilon \frac{n(x)}{|\nabla \varphi(x)|} + O(\varepsilon)$$

so

$$\varphi(\alpha_\varepsilon(x)) = \varphi(x) + (\varepsilon/|\nabla \varphi(x)|) \nabla \varphi(x) \cdot n(x) + O(\varepsilon)$$

$$= \varphi(x) + \varepsilon + O(\varepsilon) \quad \forall x \in \varphi^{-1}(t, t)$$
using a Taylor expansion and \( n = \nabla \varphi / |\nabla \varphi| \). So \( \frac{d}{ds} \varphi(\alpha_s(x)) = 1 \) with solution \( \varphi(\alpha_s(x)) = s + \varphi(x) \) and if \( x \in \Gamma \), the last term is zero. Thus we have diffeomorphisms \( \alpha_s : \varphi^{-1}(0) \to \varphi^{-1}(s) \). Now equip an open neighbourhood \( U \subset \varphi^{-1}(0) \) with local coordinates \( y \) by a chart, then \( \alpha_s \) will equip \( \alpha_s(U) \subset \varphi^{-1}(s) \) with the same coordinates. Thus we have the desired local coordinates \((y, \varphi)\) on the set \( V := \varphi^{-1}(\perp t, t) \cap \alpha_{(-t, t)}(U) \). (The incompleteness of \( X \) on \( \varphi^{-1}[\perp t, t] \) is not an obstacle to the argument: just attenuate \( X \) with appropriate smooth bump functions to make it integrable with speed 1 on \( \varphi \) where we want it).

For a function \( f \in L^1(\mathbb{R}^n) \) we have

\[
\int_V f(x) \, d\mu(x) = \int_V f(y, \varphi) \, J(y, \varphi) \, dy \, d\varphi
\]

where \( J \) is the Jacobian and \( V \) is as in the preceding proof. Observe that for a fixed \( \varphi = s \) that \( J(y, s) \, dy \) is not yet the surface measure \( d\gamma_s \) on \( \varphi^{-1}(s) \).

For that the orthogonal coordinate \( \varphi \) needs to be expressed in terms of the length of the curves \( y = \text{constant} \). Since \( d\varphi = |\nabla \varphi| \, d\ell \) with \( d\ell \) the length measure on a curve \( y = \text{constant} \), we conclude that \( d\gamma_s(y) = [J \cdot |\nabla \varphi|](y, s) \, dy \). Now

\[
\int_V |\nabla \varphi(x)| \, f(x) \, d\mu(x) = \int_V f(y, \varphi) \, [J \cdot |\nabla \varphi|] \, dy \, d\varphi = \int_{-t}^{t} \left( \int_U f \cdot J \cdot |\nabla \varphi| \, dy \right) \, d\varphi
\]

(3.3)

and this is the expression we wish to exploit. Let \( f \in C_c(\mathbb{R}^n) \), and let the thickness of the shell \( \varphi^{-1}[\perp t, t] \) around \( \Gamma \) approach zero, then

\[
\lim_{s \to 0} \frac{1}{2s} \int_{V \cap \varphi^{-1}[-s, s]} |\nabla \varphi| \cdot f \, d\mu = \lim_{s \to 0} \frac{1}{2s} \int_{-s}^{s} \left( \int_U f \cdot J \cdot |\nabla \varphi| \, dy \right) \, d\varphi = \int_U (f \cdot J \cdot |\nabla \varphi|)(y, 0) \, dy
\]

where the use of the fundamental theorem of calculus is justified because the function

\[
\varphi \to \int_U (f \cdot J \cdot |\nabla \varphi|)(y, \varphi) \, dy
\]
is continuous due to the uniform continuity of the integrand; a consequence of \(f \in C_c(\mathbb{R}^n)\) and \(J \cdot |\nabla \varphi| \in C^\infty(\mathbb{R}^n)\). Thus

\[
\lim_{s \to 0} \frac{1}{2s} \int_{\varphi^{-1}([-s, s])} |\nabla \varphi| \cdot f \, d\mu = \int_U (f | \Gamma) \, d\gamma.
\]

By doing this for all open patches \(U \subset \Gamma\) (equipped with charts), we conclude

\[
\lim_{s \to 0} \frac{1}{2s} \int_{\varphi^{-1}([-s, s])} |\nabla \varphi| \cdot f \, d\mu = \int_\Gamma (f | \Gamma) \, d\gamma \tag{3.4}
\]

for all \(f \in C_c(\mathbb{R}^n)\).

In particular, let \(\psi_1, \psi_2 \in C_c(\mathbb{R}^n)\), then we have

\[
(\psi_1 | \Gamma, \psi_2 | \Gamma)_{L^2(\Gamma)} = \lim_{t \to 0} \frac{1}{2t} \int_{\varphi^{-1}([-t, t])} \overline{\psi_1} \cdot \psi_2 \cdot |\nabla \varphi| \, d\mu
= \lim_{t \to 0} \frac{1}{2t} (\psi_1, |\nabla \varphi| \cdot \chi_{\varphi^{-1}([-t, t])} \psi_2)
= \lim_{t \to 0} \frac{1}{2t} (\psi_1, \hat{h} P_t \psi_2) \tag{3.5}
\]

where we used the notation \((\hat{h} \psi)(x) := |\nabla \varphi(x)| \psi(x)\) and \(P_t := \hat{\chi}_{\varphi^{-1}([-t, t])}\) and this is the desired relation between the inner products of the initial space \(L^2(\mathbb{R}^n)\) and of the physical space \(L^2(\Gamma)\). (Note that \(P_t\) is just the spectral projection of \(\hat{\varphi}\) on the interval \([-t, t]\), and that \(\hat{h}\) is a positive bounded operator on \(P_t H\)). Now the right hand side of this equation will exist for a much larger class of functions in \(L^2(\mathbb{R}^n)\) than \(C_c(\mathbb{R}^n)\), though for some of these the restrictions to \(\Gamma\) on the left hand side may not be defined. However the limit in (3.5) will definitely fail to exist for some elements of \(L^2(\mathbb{R}^n)\). Define

\[
(\psi_1, \psi_2)_\Gamma := \lim_{t \to 0} \frac{1}{2t} (\psi_1, \hat{h} P_t \psi_2) \tag{3.6}
\]

for all pairs \(\psi_1, \psi_2\) for which the rhs is defined and finite. To obtain a positive sesquilinear form from (3.6), we need to decide on a dense domain on which \((\cdot, \cdot)_\Gamma\) is defined and finite. Clearly \(C_c(\mathbb{R}^n)\) is one such domain, but it is not maximal. In fact, \((\cdot, \cdot)_\Gamma | C_c(\mathbb{R}^n)\) has no closed extensions as a sesquilinear form (cf. [RS] p278), so there is no canonical way of getting a maximal domain. The set

\[
S := \{ \psi \in L^2(\mathbb{R}^n) \mid (\psi, \psi)_\Gamma \text{ exists and is finite} \}\]
We want to see what the above structure looks like in the restriction theorem \(^3/\)\(^7/\). But now it implicitly involves a limiting process, which we will exploit below. This means the restriction map can be written as

\[
\ker (\cdot, \cdot) = \{(\psi_1, \psi_2) \in C_c(\mathbb{R}^n) \mid \psi_1 \perp \psi_2 \} = \{\psi \in C_c(\mathbb{R}^n) \mid \psi|_{\Gamma} = 0\}
\]

with the structures above, denote by \(\mathcal{C} : C_\varphi \to C_\varphi/\ker (\cdot, \cdot)\) the factor map, then

\[\mathcal{C} : C_\varphi \to C_\varphi/\ker (\cdot, \cdot),\]

and let \(\mathcal{C} \psi := \psi|_{\Gamma}\) which is in \(C_c(\Gamma)\) since \(\Gamma\) is a closed subset of \(\mathbb{R}^n\). To see that \(U\) is well defined on \(C_c(\mathbb{R}^n)/\ker (\cdot, \cdot)\), let \(\psi_1, \psi_2 \in C_c(\mathbb{R}^n)\) with \((\psi_1 \perp \psi_2)|_{\Gamma} = 0\). Then by \((3.4)\) we find \(\psi_1 \perp \psi_2 \in \ker (\cdot, \cdot)\), i.e., \(\mathcal{C}\psi_1 = \mathcal{C}\psi\) so \(U\) is well-defined. Moreover from \((1)\) we see that \(U\) is unitary on the dense subspace \(C[C_c(\mathbb{R}^n)]\), hence it extends to a unitary on \(\mathcal{H}\). Clearly \(C_c(\mathbb{R}^n)|_{\Gamma} = C_c(\Gamma)\), so since the image of \(U\) contains a dense subspace and \(U\) is unitary, we find \(U : \mathcal{H} \to L^2(\Gamma)\) onto.

This means the restriction map can be written as \(\kappa := U \circ \mathcal{C} : C_\varphi \to L^2(\Gamma)\), but now it implicitly involves a limiting process, which we will exploit below. Note that \(C_c(\mathbb{R}^n)\cap \ker (\cdot, \cdot) = \{\psi \in C_c(\mathbb{R}^n) \mid \psi|_{\Gamma} = 0\}\) and that \(\kappa\) is unbounded (w.r.t. the Hilbert space norms).

**Example.** We want to see what the above structure looks like in the restriction of a quantum particle in \(\mathbb{R}^3\) to a sphere. So take \(\mathcal{H} = L^2(\mathbb{R}^3)\), \(\hat{p}\psi = i\nabla\psi\), \(\hat{q}\psi(x) = x\psi(x)\) as usual and for the constraint \(\varphi(x) = \)
\[ |x|^2 \perp a^2, \, a > 0. \] Now \((\nabla \varphi)(x) = 2x \neq 0\) except if \(x = 0\), and so the critical point of \(\varphi\) is distance \(a\) away from \(\Gamma = \varphi^{-1}(0)\). We have \(\varphi^{-1}[s, t] = \{x \in \mathbb{R}^3 \mid x^2 \in [a^2 \perp s, a^2 + t]\}\), \(s, t > 0\). So

\[
(\psi_1, \psi_2)_r = \lim_{t \to 0} \frac{1}{2t} (\psi_1, \hat{P}_t \psi_2)
= \frac{1}{2} \left\{ \begin{array}{l}
\frac{d}{dt} \int_{\varphi^{-1}[-t, t]} 2|\vec{x}|(\vec{\psi}_1 \psi_2)(x) d^3x \bigg|_{t=0} \\
= \frac{d}{dt} \int_{\sqrt{a^2 - t}}^{\sqrt{a^2 + t}} r^2 dr \int d\Omega \, r(\vec{\psi}_1 \psi_2)(x) \bigg|_{t=0}
\end{array} \right.
\]

where we used polar coordinates and \(\Omega\) denotes the measure on the unit sphere \(S^2\). For functions of the form \(\psi_i(x) = f_i(r) \xi_i(\theta, \phi)\), \(\xi_i \in L^2(S^2)\), and \(f_i\) continuous on \((a \perp \varepsilon, a + \varepsilon)\) we have by the fundamental theorem of calculus

\[
(\psi_1, \psi_2)_r = \left( \frac{d}{dt} \int_{\sqrt{a^2 - t}}^{\sqrt{a^2 + t}} r^3 f_1 f_2(r) dr \bigg|_{t=0} \right) \int \xi_1 \xi_2(\theta, \phi) d\Omega
= a^2 f_1(a) f_2(a) \int \xi_1 \xi_2 d\Omega,
\]

as expected from (3.5). Moreover we have that \(\psi_i \in \text{Ker}(\cdot, \cdot)_r\) whenever \(f_i(a) = 0\).

One may ask how the method above should be generalised to bigger classes of constraints, so towards that a few remarks.

**Remarks.**

1. If \(\Gamma\) has corners (so \(\varphi\) is not smooth) we can develop an “approximation” to it by a sequence of smooth submanifolds \(\{\Gamma_k\}\), but need to choose a notion of convergence for \(\Gamma_k \to \Gamma\).

2. If \(\Gamma\) has edges we can combine the procedure above with a Dirac constraining. For example if we want to constrain a particle in \(\mathbb{R}^3\) to the upper hemisphere \(\Gamma^+\) of the sphere \(\Gamma\) of the last example,

\[
\Gamma^+ := \{x \in \mathbb{R}^3 \mid |x| = a, \, x_1 \geq 0\}
\]

then first apply the method above to the constraint \(\varphi(x) = |x|^2 \perp a^2\) to obtain a dense subspace of \(L^2(\Gamma)\), and follow this by applying a Dirac procedure via (3.1) to the constraint \(\zeta(\theta, \phi) = \chi_{[0, \pi/2]}(\theta)\) in \(L^2(\Gamma)\).
(3) To constrain a particle in $\mathbb{R}^n$ to a smooth $(n \perp k)$-dimensional submanifold $\Gamma$, we need $k$ independent constraints $\varphi_1, \ldots, \varphi_k$ such that $\Gamma = \varphi_1^{-1}(0) \cap \cdots \cap \varphi_k^{-1}(0)$ with critical points well away from $\Gamma$. Now it is clear how to adapt the method based on (3.2); we choose a curvilinear coordinate system $(y, \varphi_1, \ldots, \varphi_k)$ where $\varphi_i$ are the coordinates measured along the trajectories of the gradient fields $\nabla \varphi_i$, and $y$ are the coordinates along the manifolds $\varphi_1^{-1}(c_1) \cap \cdots \cap \varphi_k^{-1}(c_k)$, $c$ in some $\varepsilon$-neighbourhood of zero in $\mathbb{R}^k$. Then with $J$ the Jacobian, the measure on $\Gamma$ is $d\gamma = (J \cdot |\nabla \varphi_1| \cdots |\nabla \varphi_k|)(y, 0, \ldots, 0)dy$ so (3.3) becomes for $f \in C_c(\mathbb{R}^n)$:

$$\lim_{s \to 0} [2^k s_1 \cdots s_k]^{-1} \int_{\varphi^{-1}([-s,s]^k)} |\nabla \varphi_1| \cdots |\nabla \varphi_k| \cdot f \, d\mu = \int_{\Gamma} (f |\Gamma) \, d\gamma$$

where $\varphi^{-1}([-s, s]^k) := \varphi_1^{-1}([-s_1, s_1]) \cap \cdots \cap \varphi_k^{-1}([-s_k, s_k])$. Further adaptations are straightforward.

Alternatively, we can impose the constraints $\varphi_i$ one-by-one, but for this we need to generalise the procedure above for $\mathbb{R}^n$ to general Riemannian manifolds.

(4) For $\varphi$ which are positive, $\Gamma = \varphi^{-1}(0)$ consists only of critical points, though if other critical points are well away from $\Gamma$, there is still a neighbourhood of $\Gamma$ foliated by the level hypersurfaces of $\varphi$, and it is possible to adapt the method above to this situation.

(5) In the case where $\varphi^{-1}(0) = \Gamma \cup \Delta$ with $\mu(\Gamma) = 0 \neq \mu(\Delta)$, $\Gamma \cap \Delta = \emptyset$, neither the Dirac method (which will only produce $L^2(\Delta)$) nor the method above seem appropriate. So what we want is a method which will give the method above on $\Gamma$, and the Dirac procedure on $\Delta$. (We choose not to allow surface terms on $\Delta$). Assume the critical points of $\varphi$ are well away from $\Gamma$, and that $\Delta$ is the closure of an open set. That is, we want a densely defined map $\kappa : L^2(\mathbb{R}^n) \to L^2(\Gamma) \oplus L^2(\Delta)$. Recalling that $L^2(\Delta) = P_0 L^2(\mathbb{R}^n)$, this can be done by defining:

$$(\psi_1, \psi_2)_R := (\psi_1, P_0 \psi_2) + \lim_{t \to 0} \frac{1}{2t} (\psi_1^N, \hat{h} P_t \psi_2^N)$$
for all \( \psi_i \in D_R := \{ \psi \in H \mid \psi^N \in C_\varphi \} \) where \( \psi^N := x_N \cdot \psi \) and \( N \) is an open neighbourhood of \( \Gamma \) such that \( \overline{N} \cap \Delta = \emptyset \). Then proceed as before, taking \( D_R/\Ker(\cdot, \cdot)_R \) with \( \kappa \) the composition of the factor map and the unitary from \( D_R/\Ker(\cdot, \cdot)_R \) to \( L^2(\Gamma) \oplus L^2(\Delta) \).

(6) At the cost of complicating the current exposition, we can enlarge the domain \( C_\varphi \) of \( \kappa \) considerably. For instance, by splitting the limit (3.6) into the average of the limits from above and below in \( t \), and letting the domain consist of those \( \psi \in L^2(\mathbb{R}^n) \) for which \( \varphi \to \psi(y, \varphi) \) is continuous from above and below at zero (where we used the curvilinear coordinates of Lemma 3.2), we obtain another useful domain for \( \kappa \). We chose not to do this, because the additional analytic details would have obscured the simplicity of our central idea.

(5) The methods proposed here for dynamics reduction can be extended to other holonomic constraints, i.e. if we are given a constraint \( \varphi(q, p) \) which involves only one member of each canonical pair \( (q_i, p_i) \), then a partial Fourier transform can convert it to the type \( \varphi(q) \) considered here. So there is a unitary transformation of such a system to one of the present type.

4. Constraining the Dynamics.

In this section we continue the analysis of the problem of constraining a quantum particle in \( \mathbb{R}^n \) to a subset and address the problem of how to constrain the dynamics, i.e. how to construct out of the given time evolution on \( L^2(\mathbb{R}^n) \) an acceptable time evolution on the physical Hilbert space. We consider four cases:

(1) For an ordinary Dirac constraining we restrict for example a quantum particle in \( \mathbb{R}^n \) with Hamiltonian \( H \) to live on a set \( T \) which is the closure of an open set \( S \) (hence \( \mu(T) \neq 0 \)). Kinematically, one constrains via the projection \( P_{\text{phys}} : L^2(\mathbb{R}^n) \to L^2(T) \), as described at the start of Sect. 3. If the dynamics preserve \( T \), i.e. each \( U_t := \exp itH \) is of the form \( U_t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) with respect to the decomposition \( L^2(\mathbb{R}^n) = L^2(T) \oplus L^2(T)^{-} \), then there is no problem: one just restricts \( U_t \) to \( L^2(T) \) to obtain the constrained dynamics. In practice this is a rare occurrence.
(2) For the case of a Dirac constraining as in (1) where the dynamics does not preserve $T$, e.g. if the particle is free, $H = \hat{p}^2/2m$ and $T$ is compact, then $H\lfloor L^2(T)$ is not selfadjoint. To equip the constrained particle with a time evolution, we need to choose some selfadjoint extension of the symmetric operator $H \lfloor C_c^\infty(S)$ (for this to make sense we need to assume that $H$ preserves $C_c^\infty(S)$, which will be true if it is a differential operator). This amounts to the choice of boundary conditions, i.e. deciding how the particle should behave at the walls (e.g. reflection), and this is a physical choice which cannot be determined from mathematical considerations alone.

(3) For the problem of the last section where we constrain a particle in $\mathbb{R}^n$ with Hamiltonian $H$ to a lower dimensional submanifold $\Gamma$, we will need to assume that $H$ is "smooth enough" near $\Gamma$, i.e. $H(Dom H \cap C_\varphi) \subset C_\varphi$ and $\kappa(Dom H \cap C_\varphi)$ is dense in $L^2(\Gamma)$. In the case where $H$ "preserves" $\Gamma$, i.e. $H$ preserves $Dom H \cap Ker \kappa$, $H$ will clearly lift through $\kappa$ and we only need to define the constrained Hamiltonian $H_\Gamma$ on $L^2(\Gamma)$ by $H_\Gamma \kappa(\psi) := \kappa(H \psi)$ for all $\psi \in Dom H \cap C_\varphi$, and require that it is essentially selfadjoint.

(4) The case of constraining a particle as in (3) but where $H$ does not preserve $\Gamma$, will be our object of study for the rest of this section. By analogy with case (2), we expect that some physical choices will need to be made. When $H$ is a Schrödinger operator with smooth potential, physicists already know what to do: one lets the constrained Hamiltonian be $\frac{1}{2m} \Delta_{\Gamma} + V|\Gamma$ where $\Delta_{\Gamma}$ is the Laplacian on $\Gamma$.

Henceforth we assume that $H$ on $L^2(\mathbb{R}^n)$ does not preserve $\Gamma$ in the sense of (3) above. Recalling that $\kappa : C_\varphi \to L^2(\Gamma)$ involves a factorisation, it is natural to look for linear sections of $\kappa$ which one can use to construct liftings of operators on $C_\varphi$ to $L^2(\Gamma)$. The image of such a section deserves a name of its own:

**Def.** A **transverse space** is a linear space $T \subset C_\varphi$ such that $Ker \kappa \cap T = \{0\}$ and $\kappa(T)$ is dense in $L^2(\Gamma)$.

Transverse spaces are abundant, and below we will use physical arguments to make a choice of a transverse space but first we show how one can use a transverse space to constrain $H$ given a transverse space $T \subset Dom H \cap C_\varphi$ such that
\( HT \subset C_\varphi \), we have the options:

(i) Define the constrained Hamiltonian \( H_{(i)} \) with dense domain \( \kappa(\mathcal{T}) \) by

\[
H_{(i)} \kappa(\psi) := \kappa(H\psi), \quad \forall \psi \in \mathcal{T}.
\]

This is well-defined since by definition of transverse spaces, \( \kappa \) is injective on \( \mathcal{T} \).

(ii) Given that the choice of \( \mathcal{T} \) has some physical content (see below), one may object to (i) in that \( H \) is not forced to preserve \( \mathcal{T} \) in any sense. If we want to include such a restriction, since it is only the behaviour near \( \Gamma \) which should be important, we can restrict \( H \) to \( \mathcal{T} \) “in the limit,” i.e. we consider the projection \( P_t^\Gamma H \) as \( t \) goes to zero in the definition of \( \kappa \), where \( P_t^\mathcal{T} \) denotes the projection onto \( P_t^\mathcal{T} \). So we propose an alternative constrained Hamiltonian \( H_{(ii)} \) on \( \kappa(\mathcal{T}) \) by

\[
(\kappa(\psi), \kappa(\phi))_{L^2(\Gamma)} = \lim_{t \to 0} \frac{1}{2t} (P_t^\mathcal{T} H\psi, \hat{\hbar} P_t\phi)
\]

for all \( \psi, \phi \in \mathcal{T} \).

Below we will test both methods for a suitable choice of transverse space. This choice is the issue we now want to address. Recall that in the classical global method of Gotay, Nester and Hinds [GNH], the idea was to adjust the dynamics so that the motion is always tangential to \( \Gamma \), thus forcing the particle to remain on \( \Gamma \). We look for some quantum mechanical version of this.

First we want to give meaning to the concept of motion “tangential to \( \Gamma \)” for a quantum particle. Whilst the classical state of a particle is a point in phase space, quantum mechanically a state is here a vector \( \psi \) in \( L^2(\mathbb{R}^n) \). Classically, a particle with constant mass at position \( q \) moves tangentially to \( \Gamma \) if its momentum \( p = m \dot{q} \) is tangential to \( \Gamma \), i.e. \( p \in T_q \Gamma \). Inspired by this, in a quantum mechanical setting, we say a particle in a state \( \psi \in C^\infty_c(\mathbb{R}^n) \) moves tangentially to \( \Gamma \) if \( (\hat{p}\psi)(q) \in T_q \Gamma \) for all \( q \in \Gamma \). Now, given the normal vectors \( \nabla \varphi \) to \( \Gamma \), we have \( v \in T_q \Gamma \) iff \( \nabla \varphi(q) \cdot v = 0 \). So a particle in a state \( \psi \in C^\infty_c(\mathbb{R}^n) \) moves tangentially to \( \Gamma \) when \( (\nabla \varphi \cdot \hat{p}\psi)(x) = 0 \) for all \( x \in \Gamma \), i.e. iff the component of the momentum of \( \psi \) normal to \( \Gamma \) is zero. We would like to generalise this notion away from \( C^\infty_c(\mathbb{R}^n) \). Since \( \Gamma \) is of \( \mu \)-measure
zero, we cannot in general specify a property for an \( L^2 \)-function \( \psi \in L^2(\mathbb{R}^n) \) on \( \Gamma \). Recall that \( \kappa \) is a limiting procedure on the shells \( \varphi^{-1}[\perp t, t] \), in which eventually \( t \to 0 \). So it may be enough to define tangentiality on the level sets of \( \varphi \) in such a shell, and depend on \( \kappa \) to restrict the property to \( \Gamma \). Fix a shell \( S_{t_0} := \varphi^{-1}[\perp t_0, t_0] \) around \( \Gamma \), then we say a \( \psi \in \text{Dom}(\hat{p}|L^2(S_{t_0})) \) has momentum tangential to the level sets of \( \varphi \) iff \( \nabla \varphi \cdot \hat{p}\psi = 0 \). Since in terms of the local coordinate system \((y, \varphi)\) of Sect. 3 this means \( \frac{\partial \psi}{\partial \varphi} = 0 \) on \( S_{t_0} \), we see that \( \psi|_{S_{t_0}} \) must be constant in the normal direction (i.e. along the trajectories of the vector field \( \nabla \varphi \)). So \( \psi|_{S_{t_0}} \) is uniquely determined by its restriction to \( \Gamma \). This defines the notion of “states with momentum tangential to the level sets of \( \varphi \) in \( S_{t_0} \)”.

Conversely, given any \( \psi \in L^2(\Gamma) \), we can make out of it a state \( \tilde{\psi} \in L^2(\mathbb{R}^n) \) by extending it constantly along the normals in \( S_{t_0} \) and set it equal to zero outside \( S_{t_0} \), i.e.

\[
\tilde{\psi}(x) = \psi(\varphi^{-1}(x)) \quad \text{for} \quad x \in S_{t_0} \quad \text{and zero for} \quad x \not\in S_{t_0}
\]

(cf. proof of lemma 3.2 for \( \alpha \)). Denote the space of these by \( H^T_{t_0} \), and the projection onto \( H^T_{t_0} \) by \( P^T_{t_0} \). Then \( H^T_{t_0} \) is thought of as a “thickening” of \( L^2(\Gamma) \) in \( L^2(\mathbb{R}^n) \), and note that \( H^T_{t_0} \cap C_\varphi \) is a transverse space. However, when we deal with differential operators, the discontinuity at the boundary of the shell \( S_{t_0} \) is a problem, so we prefer the following smooth version. Let \( \zeta_{t_0} \in C^\infty(\mathbb{R}) \) be a bump function which is one on \( [\perp t_0, t_0] \) and zero outside \( [\perp t_0 \perp \varepsilon, t_0 + \varepsilon] \) for a given \( \varepsilon \). Then define for a \( \psi \in L^2(\Gamma) \) the new thickening

\[
\tilde{\psi}(x) = \zeta_{t_0}(\varphi(x)) \cdot \psi(\varphi^{-1}(x)) \quad .
\]

Denote the space spanned by these by \( H^\zeta_{t_0} \subset L^2(\mathbb{R}^n) \), then clearly \( P_{t_0}H^\zeta_{t_0} = H^T_{t_0} \).

We now want to constrain the Hamiltonian in such a way that it can be thought of as “projecting the force down to its tangential component,” where a Hamiltonian tangential to \( \Gamma \) will be one which keeps tangential motion to \( \Gamma \) tangential, i.e. preserves the tangential states on some shell \( S_{t_0} \). Recall that in the GNH-algorithm [GNH] one restricted the Hamiltonian to all states with
motion tangential to \( \Gamma \). To do the same here, we can now use either of the two proposed methods \((i)\) or \((ii)\) above with the choice of transverse space as

\[
\mathcal{T}_{t_0} := \mathcal{H}_{t_0}^\perp \cap C_\varphi \cap \text{Dom} \, H.
\]

As long as \( C_c^\infty(\mathbb{R}^n) \subset \text{Dom} \, H \), we have that \( \kappa(\mathcal{T}_{t_0}) \) is dense, in which case this is indeed a transverse space. So assuming the latter, and that \( H \mathcal{T}_{t_0} \subset C_\varphi \), we define

\[
H_{(i)} : \kappa(\psi) := \kappa(H \psi) \quad \forall \psi \in \mathcal{T}_{t_0}.
\]

In fact, when \( \psi \) is smooth and \( H \) is a differential operator, this becomes \( H_{(i)}(\psi|\Gamma) = (H \psi)|\Gamma \), and then the method is nothing but the one used by S. Helgason pp 251–252 [He] for the restriction of a differential operator to a submanifold. This is the first reasonable method to consider.

To motivate the use of method \((ii)\), consider how a physicist might object to (4.1). Since \( H \) need not preserve \( \mathcal{T}_{t_0} \), it is possible that \( H \psi \) has momentum which is not tangential to \( \Gamma \), in which case it seems there is a force acting on the particle, forcing it off \( \Gamma \). To project this force out of the total force acting on the particle the restriction procedure above may not be appropriate. Instead, we intend to use the Hilbert space projections to first project out of \( H \) the parts which affect the momentum component normal to the level surfaces of \( \varphi \) in the shell \( \varphi^{-1}[t_0, t_0] \), then in the limit of \( \kappa \) when the thickness of the shell goes to zero we will be left with the appropriate Hamiltonian on \( L^2(\Gamma) \) (having maintained tangentiality of momentum to \( \Gamma \) during the limiting process). In the light of this we define \( H_{(ii)} \) by:

\[
(H_{(ii)} : \kappa(\psi_1), \kappa(\psi_2))_{L^2(\Gamma)} := \lim_{t_0 \to 0} \frac{1}{2t} \left( P_{t_0}^T H \psi_1, \dot{h} P_{t_0} \psi_2 \right)
\]

for all \( \psi_1 \in \mathcal{T}_{t_0} \), for which this exists for all \( \psi_2 \in \mathcal{T}_{t_0} \), (so that (4.2) defines a vector \( H_{(ii)} : \kappa(\psi_1) \in L^2(\Gamma) \)). Since \( P_{t_0} \mathcal{T}_{t_0} \subset \mathcal{H}_t^\perp \), (4.2) coincides with method \((ii)\). This is the second reasonable method.

One can easily conceive of other methods apart from \((i)\) and \((ii)\) for constraining dynamics, for instance forcing the particle onto \( \Gamma \) by infinite potential walls:
Given a system \( L^2(\mathbb{R}^n), \varphi, H \) as before, restrict the particle to a box around \( \Gamma \), say the shell \( S_t \), but for it to be well-defined we need to assume \( HC_c(\mathbb{R}) \subseteq C_c(\mathbb{R}) \) and that we have a selfadjoint extension \( H_t \) on \( L^2(S_t) \) of the symmetric operator \( H \) (\( \{ \psi \in C_c(S_t) \mid \psi(\varphi^{-1}(\pm t)) = 0 \} \)).

In the case when \( H \) is a differential operator, this means one needs to decide how it behaves at the walls. Henceforth we assume \( H \) is a differential operator. If we choose ordinary reflection at the walls (quantum billiards), then a \( \psi \in \text{Dom} \ H_t \) must satisfy

\[
\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{when} \quad \varphi = \pm t,
\]

but without more detailed knowledge of \( H \) we do not know what additional boundary conditions \( \psi \) must satisfy to make \( H_t \) selfadjoint. So \( \text{Dom} \ H_t \) is a subspace of

\[
\mathcal{D}_t := \left\{ \psi \in P_t C_c(\mathbb{R}^n) \mid \psi \in L^2(\mathbb{R}^n), \frac{\partial \psi}{\partial \varphi} = 0 \text{ when } \varphi(x) = \pm t \right\} \subseteq C_c.
\]

Then in the limit when \( t \) goes to zero, the boxes force the particle (hence \( H_t \)) onto \( \Gamma \). The appropriate way to take this limit, is through \( \kappa \), i.e.

\[
(H_{(iii)} \cdot \kappa(\psi_1), \kappa(\psi_2))_{L^2(\Gamma)} := \lim_{t \to 0} \frac{1}{2t} \left( H_t P_t \psi_1, \hat{h} P_t \psi_2 \right)
\]

where \( \psi_i | S_t \in \text{Dom} \ H_t \) for all \( t \in (0, t_0) \) and some fixed \( t_0 \). Note that the possible \( \psi_i \)'s are restricted by the behaviour at the boundary. So here for reflection, we get that \( (\partial \psi_i / \partial \varphi)(y, t) = 0 \) for all \( t \in [\pm t_0, 0] \cup (0, t_0] \).

We recognise that \( P_t \psi \) is in our transverse space of earlier on. When the space of \( \kappa(\psi) \) for such \( \psi \) is dense in \( L^2(\Gamma) \), \( H_{(iii)} \) is well-defined.

Other versions of this is possible if we change the behaviour at the walls, e.g.

introduce a phase with the reflection.

We will not discuss any other methods, but now the surprise is that we have the following equivalences:

**Theorem 4.4.** (1) With the assumptions above, \( H_{(i)} = H_{(iii)} = H_\Gamma \).

(2) moreover if \( H \) is a differential operator, then \( H_{(i)} = H_{(iii)} \) on \( \text{Dom} \ H_{(iii)} \).
Proof: (1) We first show $H_{(i)} = H_{(ii)}$. Recall that $C_c^\infty(\mathbb{R}^n) \subset \text{Dom } H$, and that $\kappa(\mathcal{T}_0 \cap C_c^\infty(\mathbb{R}^n)) = C_c^\infty(\Gamma)$ hence it suffices to show that

$$((H_{(i)} \perp H_{(ii)}) \kappa(\psi_1), \kappa(\psi_2))_{L^2(\Gamma)} = 0$$

for all $\psi \in \mathcal{T}_0 \cap C_c^\infty(\mathbb{R}^n)$. By (4.1) and (4.2), the left hand side is:

$$\lim_{t \to 0} \frac{1}{2t} \int_{\Omega(t)} (\|(\mathbb{I} \perp P_{L_T}) H \psi_1, \hat{H} P_T \psi_2\|) = \lim_{t \to 0} \frac{1}{2t} \int_{\Omega(t)} (\|(\mathbb{I} \perp P_{L_T}) H \psi_1 \psi_2 \|) \nu \, d\mu$$

Partition $\Gamma$ into patches $U$ on which local coordinates exist as in Lemma 3.2, then the right hand side becomes a finite sum of terms:

$$\lim_{t \to 0} \frac{1}{2t} \int_{\Omega(t)} d\varphi \int_U d\gamma \|(\mathbb{I} \perp P_{L_T}) H \psi_1 \psi_2 \| \nu \, d\mu$$

(alternatively, take $\psi_2$ with support in $U$ and let $U$ vary) Now $(\mathbb{I} \perp P_{L_T}) H \psi_1 =: \rho_t \perp \mathcal{H}^T_{t}$, so for all $\phi \in \mathcal{H}^T_{t} \cap C_c^\infty(S_t)$ we have

$$0 = (\rho_t, \phi) = \int_U d\gamma \phi(\gamma) \int_{\Omega(t)} d\varphi \, \overline{\rho_t}$$

where we used the fact that on $S_t$, $\phi$ is independent of $\varphi$, so since this holds for all $\phi$ (which will span a dense subspace of $L^2(U)$), we have that $\int_{\Omega(t)} \rho_t(\gamma, \varphi) \, d\varphi = 0$ a.e. in $\gamma$. Now using the continuity of $\psi_2 \|\varphi\|$, (*) becomes:

$$\lim_{t \to 0} \frac{1}{2t} \int_U d\gamma \, (\psi_2 \|\varphi\|)(\gamma, 0) \int_{\Omega(t)} d\varphi \, \overline{\rho_t} = 0$$

Since this holds for all $U$, we find that $H_{(i)} = H_{(ii)}$.

(2) Next we prove that $H_{(i)} = H_{(iii)}$ on $\text{Dom } H_{(iii)}$. Clearly $\bigcap_{t \in [0, t_0]} D_t |S_{t_0} = H^T_{t_0}$, so it suffices to show that

$$((H_{(i)} \perp H_{(iii)}) \kappa(\psi_1), \kappa(\psi_2))_{L^2(\Gamma)} = 0 \quad \forall \psi \in C_c^\infty(S_{t_0}) \cap \bigcap_{t \in [0, t_0]} \text{Dom } H_t$$

i.e.

$$\lim_{t \to 0} \frac{1}{2t} \int_{\Omega(t)} (\|(H \perp H_t P_t) \psi_1, \hat{H} P_T \psi_2\|) = 0.$$

Now $((H \perp H_t P_t) \psi_1, \hat{H} P_T \psi_2) = ((P_t H \perp H_t P_t) \psi_1, \hat{H} P_T \psi_2)$, and let $\mathcal{Q}_t = \{ \phi \in C_c^\infty(S_t) \mid \phi(\varphi^{-1}(\pm t)) = 0 \}$
which is dense in $L^2(S_t)$, and $H = H_t$ on $Q_t$. Now given an open precompact set $X \subset S_t$, we can write any $\phi \in C_c^\infty(\mathbb{R}^n)$ as $\phi = \phi_X + \tilde{\phi}_X$ where $\phi_X \in Q_t$ and $\phi|X = \tilde{\phi}_X|X$. Let $\psi|S_t \in \operatorname{Dom} H_t \cap C_c^\infty(S_t)$ for all $t \in (0, t_0]$, let $\phi \in Q_t$ and $X$ be the interior of $\operatorname{supp}\phi$. Then

\[
((P_t H \perp H_t P_t)\psi, \phi) = ((P_t H \perp H_t P_t)(\psi_X + \tilde{\psi}_X), \phi) = ((H \perp H_t)\psi_X + (P_t H \perp H_t P_t)\tilde{\psi}_X, \phi) = ((P_t H \perp H_t P_t)\tilde{\psi}_X, \phi) \quad (\dagger)
\]

Since $H$ and $H_t$ are differential operators, they preserve supports, hence $\operatorname{supp}((P_t H \perp H_t P_t)\tilde{\psi}_X) \subseteq \operatorname{supp} \tilde{\psi}_X$ which is disjoint from the support of $\phi$, hence since the inner product in $(\dagger)$ is an integral, we conclude that $(\dagger)$ is zero, i.e. $(P_t H \perp H_t P_t)\psi \perp Q_t$ hence $(P_t H \perp H_t P_t)\psi = 0$.

We remark that any reasonable method of dynamics reduction should produce a constrained Hamiltonian $H_T$ on $L^2(\Gamma)$ which is essentially selfadjoint and the time evolution it generates, $\exp i\mathcal{T}H_T$, must preserve the algebra of observables $\mathcal{R}_S$ when we have obtained this (see next section). We now test the above methods on two examples.

**Example 4.5.** Recall the previous example of the sphere of radius $a$ in $\mathbb{R}^3$:

$\mathcal{H} = L^2(\mathbb{R}^3) = L^2(\mathbb{R}_+, r^2 dr) \otimes L^2(\Omega)$ with Hamiltonian $H = \hat{p}^2/2m + \frac{1}{2m} \Delta$, having domain of essential selfadjointness $C_c^\infty(\mathbb{R}^3)$ and which acts on the decomposable functions $\psi(x) = f(r) \cdot \xi(\theta, \phi) \in C_c^\infty(\mathbb{R}^3)$ by

\[
(\Delta \psi)(x) = \left(\frac{d^2}{dr^2} - \frac{2}{r} \cdot \frac{d}{dr}\right) f(r) \cdot \xi(\theta, \phi) + \frac{1}{r^2} f(r) \cdot B \xi(\theta, \phi)
\]

where $B = |L|^2$ is the Laplace–Beltrami operator on $L^2(\Omega)$ ($\Omega$ denotes the unit sphere). Now for $t \in (0, a)$ we have

$\mathcal{H}^T_t := \{ \psi \in \mathcal{H} \mid \psi(x) = \chi_{[-a^2-t, a^2+t]}(r) \cdot \xi(\theta, \phi), \quad \xi \in L^2(\Gamma) \}.$

To obtain the smoothed space, choose a $\zeta_t \in C_c^\infty(\mathbb{R})$ which is one on $[\perp t, t]$ and zero on $[\perp t \perp \varepsilon, t + \varepsilon]$ and set

$\mathcal{H}^c_t := \{ \psi \in \mathcal{H} \mid \psi(x) = \zeta_t(r^2 \perp a^2) \cdot \xi(\theta, \phi), \quad \xi \in L^2(\Gamma) \}$
and note that if $\psi|\Gamma = \xi \in C^\infty(\Gamma)$, then $\psi \in C^\infty(\mathbb{R}^n) \subset \text{Dom } H$. So we choose the transverse space $T_t = \left\{ \psi \in \mathcal{H}_t \mid \psi|\Gamma \in C^\infty(\Gamma) \right\}$.

Now for method $(i)$ for constraining $H$, let $\psi = (\zeta_t \circ \varphi) \cdot \xi \in T_t$. Then from the explicit formula for $H$, we see

$$ (H \psi)(x) = \frac{1}{2mr^2} (B\xi)(\theta, \phi) \cdot \chi_{(\sqrt{a^2-t}, \sqrt{a^2+t})}(r) + \rho(x) $$

where $\rho$ is a function with support disjoint from $\varphi^{-1}(\perp t, t)$. So, since $H\psi$ is continuous near $\Gamma$, we have $\kappa(H\psi) = (H\psi)|\Gamma$, i.e. $\kappa(H\psi)(\theta, \phi) = (B\xi)(\theta, \phi)/2ma^2$.

$$ H(i) \kappa(\psi) = \kappa(H\psi) = \frac{1}{2ma^2}B\kappa(\psi) $$

for all $\psi \in T_t$, so, since $\kappa(T_t) = C^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, $H(i)\xi = \frac{1}{2ma^2}B\xi$ for all $\xi \in C^\infty(\Gamma)$ is densely defined, agrees with the classical Hamiltonian $L^2/2ma^2$ obtained in Sect. 2 (eq. (2.4)), and is essentially selfadjoint. A good result.

Next, as an exercise in method $(ii)$, we calculate

$$ \left( P_t^T H \psi_1, \hat{h} P_t \psi_2 \right), \quad \psi_i \in T_{t_0}, $$

where $\psi_i = (\zeta_t \circ \varphi) \cdot \xi_i$ as above. Now

$$ (P_t^T H \psi_1)(x) = \left( P_t^T \frac{B}{2mr^2} \right) \xi(\theta, \phi) $$

if $r \in [\sqrt{a^2-t}, \sqrt{a^2+t}] =: I_t$ and zero otherwise. To work out the projection, we only need to consider the radial coordinate. The one-dimensional space $N$ generated by $\chi_{I_t} \in L^2(\mathbb{R}_+, r^2 dr)$ corresponds to $\mathcal{H}_t^T$. So we need to decompose the function $\chi_{I_t}(r)/2mr^2$ according to $N \oplus N^-$, i.e. $\frac{1}{2mr^2} \chi_{I_t}(r) = \lambda_t \cdot \chi_{I_t}(r) + h(r)$ where $\lambda_t$ is a constant and $h \in N^-$, i.e. $\int_{I_t} r^2 h(r) dr = 0$. Now $\left( \chi_{I_t}, \frac{1}{2mr^2} \chi_{I_t} \right) = \frac{1}{2m} \int_{I_t} dr = \frac{\lambda_t}{2m} \int_{I_t} r^2 dr$. i.e.

$$ \lambda_t = \frac{3(\sqrt{a^2+t} \perp \sqrt{a^2+t})}{2m[(a^2+t)^3/2 \perp (a^2 \perp t)^3/2]} $$

and so:
Thus\[ (P^T_{t} H \psi_1, \hat{h} P_{t} \psi_2) = \int \lambda_t \cdot \chi_{I_t}(r) (B \xi_1)(\theta, \phi) \cdot 2r \psi_2(x)r^2 dr d\Omega\]
\[ = 2\lambda_t \int \frac{\sqrt{a^2 + t}}{\sqrt{a^2 - t}} r^3 dr \int (B \xi_1) \cdot \xi_2 d\Omega\]
\[ = 2\lambda_t \frac{1}{4a^2} ((a^2 + t)^2 \perp (a^2 \perp t)^2) \cdot (B \xi_1, \xi_2)_{L^2(\Gamma)}\]
\[ = 2\lambda_t \cdot (B \xi_1, \xi_2)_{L^2(\Gamma)}\]

where we used the fact that the measure on $\Gamma$ is $a^2 d\Omega$ with $d\Omega$ the usual measure on the unit sphere. Now
\[
\lim_{t \to 0} \lambda_t = \frac{3}{2m} \lim_{t \to 0} \frac{\sqrt{a^2 + t} \perp \sqrt{a^2 \perp t}}{(a^2 + t)^{3/2} \perp (a^2 \perp t)^{3/2}} = \frac{1}{2ma^2}, \quad \text{hence}\]
\[
\lim_{t \to 0} \frac{1}{2t} (P^T_{t} H \psi_1, \hat{h} P_{t} \psi_2) = \lim_{t \to 0} \lambda_t \cdot (B \xi_1, \xi_2)_{L^2(\Gamma)}\]
\[ = \left( \frac{B}{2ma^2} \xi_1, \xi_2 \right)_{L^2(\Gamma)} = (H_{(ii)} \xi_1, \xi_2) \quad \text{for all } \xi_i \in C^\infty(\Gamma).\]

Thus $H_{(ii)} = B/2ma^2 = H_{(i)} = H_{\Gamma}$ on $C^\infty(\Gamma)$, in agreement with theorem 4.4.

Without much extra effort, we can also show that Schrödinger operators $H = \frac{1}{2m} \hat{p}^2 + V(\hat{q})$ for $V$ smooth near $\Gamma$ produce for $\xi \in C^\infty(\Gamma)$:

\[
H_{(i)} \xi(\theta, \phi) = H_{(ii)} \xi(\theta, \phi) = \left( \frac{B}{2ma^2} \xi \right)(\theta, \phi) + V(a, \theta, \phi) \cdot \xi(\theta, \phi).\]

**Example 4.6.** We would like to restrict a free quantum particle in $\mathbb{R}^3$ to a cylinder $\Gamma$ of radius $a$ around the $z$-axis. We do this in exact analogy with the sphere. Let the constraint be $\varphi(x) = x^2 + y^2 \perp a^2$, then $\nabla \varphi(x) = (2x, 2y, 0)$ hence the critical points are well away from $\Gamma$. So

\[
\varphi^{-1}([s, t]) = \{ x \in \mathbb{R}^3 \mid r^2 \in [a^2 \perp s, a^2 + t] \}\]

where we henceforth use cylindrical coordinates $(z, r, \theta)$. Then

\[
(\psi_1, \psi_2)_r = \lim_{t \to 0} \frac{1}{2t} (\psi_1, \hat{h} P_t \psi_2)\]
\[
= \frac{1}{2} \frac{d}{dt} \left. \int \frac{\sqrt{a^2 + t}}{\sqrt{a^2 - t}} 2r^2 dr \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta (\overline{\psi}_1 \psi_2)(x) \right|_{t=0}.\]
Let \( \psi_i \) be decomposable \( \psi_i(x) = f_i(r)\xi_i(z, \theta) \) where \( \xi_i \in L^2(\Gamma) \) and \( f_i \) is continuous at \( r = a \). Then

\[
(\psi_1, \psi_2)_\Gamma = \int \xi_1 \xi_2 \, dz \, d\theta \cdot \frac{d}{dt} \int r^2 f_1 f_2(r) \, dr \bigg|_{t=0} \\
= a f_i(a) f_2(a) \int \xi_1 \xi_2 \, d\theta \, dz.
\]

The free Hamiltonian \( H = \frac{1}{2m} \Delta / \) in cylindrical coordinates acts by:

\[
\Delta \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}.
\]

For \( t \in (0, a) \) we have:

\[
\mathcal{H}^T_t = \{ \psi \in \mathcal{H} \mid \psi(x) = \chi_{I_t}(r) \cdot \xi(z, \theta), \ \xi \in L^2(\Gamma) \}
\]

where \( I_t := [\sqrt{a^2 - t}, \sqrt{a^2 + t}] \). Then with a bump function \( \zeta_i \in C^\infty(\mathbb{R}) \) which is one on \( [\pm t, t] \) and zero outside \( [\pm t \pm \varepsilon, t + \varepsilon] \), we have

\[
\mathcal{H}^\zeta_t := \{ \psi \in \mathcal{H} \mid \psi(x) = \zeta_i(r^2 \pm a^2) \cdot \xi(z, \theta), \ \xi \in L^2(\Gamma) \}.
\]

Since the smooth functions of compact support are in the domain of \( H \), we choose

\[
\mathcal{T}_t = \left\{ \psi \in \mathcal{H}^\zeta_t \mid \psi|\Gamma \in C_c^\infty(\Gamma) \right\}.
\]

Now to obtain \( H(i) \), let \( \psi = (\zeta \circ \varphi) \cdot \xi \in \mathcal{T} \), then

\[
(H\psi)(x) = \left(\frac{-1}{2m}\right) \chi_{J_t}(r) \left( \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \xi(z, \theta) + \rho(x)
\]

where \( J_t := (\sqrt{a^2 - t}, \sqrt{a^2 + t}) \) and \( \rho \) has support disjoint from \( J_t \).

Now since \( \kappa(H\psi) = (H\psi)|\Gamma \) , we get

\[
H(i) \xi(z, \theta) = \left(\frac{-1}{2m}\right) \left( \frac{1}{a^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \xi(z, \theta)
\]

for all \( \xi \in C_c^\infty(\Gamma) \). Thus via theorem 4.4

\[
H|\Gamma = \left(\frac{-1}{2m}\right) \left( \frac{1}{a^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right)
\]
on $C_c^\infty(\Gamma)$. This is precisely the quantization one would expect of the constrained classical Hamiltonian (2.6).

**Remarks.** (1) We regard the choice of a transverse space $\mathcal{T}$ as a decision on the direction from which $H$ should be reduced to $\Gamma$. Above we chose the normal direction. In quantum systems with less geometry (e.g. no metric on an underlying manifold), it may be difficult to decide on an appropriate $\mathcal{T}$. This choice has physical content, and seems analogous to the choice of a selfadjoint extension for a Hamiltonian in the Dirac approach sketched early this section. An alternative way of expressing the choice of the transverse space is to observe that on $L^2(S_{t_0})$ for $t_0$ small enough, the pair of operators $\hat{\varphi}$ and $P_\varphi := i\partial/\partial\varphi$ form a canonical pair, i.e. $[P_\varphi, \hat{\varphi}]\psi = i\psi$ for all smooth $\psi$ with compact support in the interior of $S_{t_0}$. Then $\mathcal{T}|S_{t_0} = \text{Ker} P_\varphi \cap \text{Dom}(H|L^2(S_{t_0}))$. So in a general quantum system one can look for a given "local canonical conjugate" to the constraint to obtain a transverse space as its kernel. In fact, this observation also gives a clue on how to enforce second-class constraints, in the sense that if we are given a canonical pair $\varphi, P_\varphi$ to enforce, then we do it as above, by using $P_\varphi$ to locally select a transverse space to $\varphi^{-1}(0)$.

(2) Recall that in Sect. 2, the classical secondary constraint we obtained for $\varphi$ was $(\nabla\varphi) \cdot \mathbf{p}|\Gamma$. The selection of the normal transverse space above, can be thought of as the enforcement of the constraint $\nabla\varphi \cdot \mathbf{p}\psi = 0$ near $\Gamma$, which looks very much like what one would expect a "quantization" of the classical secondary constraint to be. This leads one to ask whether we can quantize the infinitesimal Dirac procedure which produced this secondary constraint as a whole. Unfortunately this does not seem to work; we sketch how it goes awry. Given the primary constraint $\hat{\varphi}$ and the Hamiltonian $H = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q})$, the secondary constraint should be $[H, \hat{\varphi}]$ "restricted" to $\Gamma$. Now

$$[H, \hat{\varphi}]\psi = \perp_{1/2m}(\Delta\varphi)\psi \perp \perp_{1/m}(\nabla\varphi) \cdot (\nabla\psi)$$

for all $\psi \in C_c^\infty(\mathbb{R}^n)$. Restriction to $\Gamma$ is done by $\kappa$. However, whilst $\perp_{1/m}(\nabla\varphi) \cdot (\nabla\psi)$ is proportional to the desired secondary constraint $\nabla\varphi \cdot \mathbf{p}\psi$, the term $\perp_{1/2m}(\Delta\varphi)\psi$ need not vanish on $\Gamma$. Moreover if we use $[H, \hat{\varphi}]$
as a new secondary constraint to select the transverse space instead of $\nabla \varphi \cdot \hat{p}$, we obtain the wrong result on the sphere. Nevertheless, given the close parallel which the selection of the transverse space has to the classical secondary constraint, we regard the use of a transverse space here as the imposition of a secondary quantum constraint.

(3) We remark that one needs to reduce the dynamics infinitesimally, i.e., through the Hamiltonian, not directly on the time evolution unitaries $\exp(itH)$.

(4) In general, there is no guarantee that the constraining of a selfadjoint Hamiltonian $H$ to $\Gamma$ produces an essentially selfadjoint operator on $L^2(\Gamma)$. This is a difficult question that needs further investigation, and its classical equivalent in the GNH-algorithm also is unsolved, that is, we do not know whether reducing a given complete Hamiltonian vector field to a submanifold $\Gamma$ produces a complete vector field on $\Gamma$. When $\Gamma$ has edges, it is very easy to get examples of complete vector fields which will not be complete when constrained to $\Gamma$.

5. Constrained Observables.

In this section we continue the analysis of the problem of the previous sections specifically in regard to the observables. That is, given the unbounded map $\kappa : C_\varphi \to L^2(\Gamma)$ above, we wish to examine how selfadjoint operators and unitaries on $L^2(\mathbb{R}^n)$ lift through $\kappa$ to produce operators on $L^2(\Gamma)$. Due to the unboundedness and nonclosability of $\kappa$ there will be some pathology, even for bounded operators on $L^2(\mathbb{R}^n)$.

The choice of field algebra $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H})$ will turn out to be important for obtaining a nontrivial constrained field algebra on $L^2(\Gamma)$. In fact, the CCR-algebra $\overline{\Delta(\mathbb{R}^2)} = C^* \{ \exp(i \hat{q} \cdot a), \exp(i \hat{p} \cdot a) \mid a \in \mathbb{R}^n \}$ will be too small if $\Gamma$ is a curved manifold (see later in this section). A more suitable field algebra for curved $\Gamma$ is $C_b(\mathbb{R}^n) \rtimes \text{Diff} \mathbb{R}^n$, which is here concretely the C*-algebra generated in $\mathcal{B}(L^2(\mathbb{R}^n))$ by the multiplication operators $\{ T_f \mid f \in C_b(\mathbb{R}^n) \}$, 

$$(T_f \psi)(x) := f(x)\psi(x), \quad \psi \in L^2(\mathbb{R}^n) = \mathcal{H}$$

and the set of unitaries $\{ V_\beta \mid \beta \in \text{Diff} \mathbb{R}^n \}$, where

$$(V_\beta \psi)(x) := J_\beta(x)^{-1/2} \psi(\beta x), \quad \psi \in L^2(\mathbb{R}^n)$$
with \( J_\beta \) the Jacobian of \( \beta \in \text{Diff } \mathbb{R}^n \). We will concentrate on these two classes of operators below.

Now an operator \( A \in \mathcal{B}(\mathcal{H}) \) will lift through \( \kappa \) to a densely defined operator \( \Lambda(A) \) on \( L^2(\Gamma) \) if:

- there is a space \( S \subset C_\varphi \) such that \( AS \subseteq C_\varphi \) and \( \kappa(S) \) is dense in \( L^2(\Gamma) \),
- \( A\psi \in \text{Ker } (\cdot, \cdot)_{\Gamma} = \text{Ker } \kappa \) for all \( \psi \in S \cap \text{Ker } \kappa \).

In this case

\[
\Lambda(A)\kappa(\psi) := \kappa(A\psi) \quad \forall \psi \in S, \tag{5.1}
\]

or equivalently

\[
(\Lambda(A)\kappa(\phi), \kappa(\psi))_{\Gamma} = \lim_{t \to 0} \frac{1}{2t} (A\phi, \hat{P}_t \psi) \quad \forall \psi, \phi \in S.
\]

Since we are interested in \(*\)-algebras of operators, we will concentrate on situations where the dense subspace \( S \) is invariant under the class of operators under consideration. Define the \(*\)-algebra:

\[
\mathcal{F}_S := \{ A \in \mathcal{F} \mid A\psi \in S \implies A^*\psi \quad \forall \psi \in S \}.
\tag{5.2}
\]

There are three obvious dense subspaces \( S \) we can ask to be preserved, \( C_\varphi \), \( C_c(\mathbb{R}^n) \) and the transverse space \( \mathcal{T}_t = \mathcal{H}_t^\xi \cap C_\varphi \cap \text{dom } H \) of the last section (or better still, the space \( \mathcal{T}^{(t_0)} \) spanned by all \( \mathcal{T}_t \) for \( t \in (0, t_0] \) and all possible smoothings \( \zeta \)), and we will consider all these in due course. A useful way of characterising \( \mathcal{T}^{(t_0)} \) is as

\[
\mathcal{T}^{(t_0)} = \left\{ \psi \in C_c(\mathbb{R}^n) \mid \frac{\partial \psi}{\partial \varphi}(x) = 0 \quad \forall x \in S_{\psi}, \quad 0 < t_\psi < t_0 \right\} \cap \text{dom } H.
\]

The choices \( \mathcal{T}_t \) or \( \mathcal{T}^{(t_0)} \) can be thought of as enforcing secondary quantum constraints on the observables, cf. remark 2 of Sect. 4. However, we will see that in the quantum picture there is no compelling reason to do this.

Unless we specify what \( S \) is below, we will assume some choice has been made. The elements of \( \mathcal{F}_S \) which will lift via (5.1) are those which preserve \( S \cap \text{Ker } \kappa \), and for these we have: \( \Lambda(A)\Lambda(B)\kappa(\psi) = \kappa(AB\psi) = \Lambda(AB)\kappa(\psi) \).
for all $\psi \in S$, i.e. $\Lambda$ is an algebra homomorphism. However, there are two pathologies associated with $\Lambda$:

(i) given an $A \in \mathcal{B}(\mathcal{H})$ for which $\Lambda(A)$ exists, then $\Lambda(A)$ need not be bounded,

(ii) given an $A \in \mathcal{B}(\mathcal{H})$ for which both $\Lambda(A)$ and $\Lambda(A^*)$ exist, we need not have that $\Lambda(A^*) \subseteq \Lambda(A)^*$.

5.3. Example of (i):

Let $\mathcal{H} = L^2(\mathbb{R}^2)$, $\varphi(x) = x_2$ so $\Gamma = \varphi^{-1}(0)$ is the $x_1$-axis. Choose $S = C_c(\mathbb{R}^n)$. Clearly all $V_\beta$ preserve $S = C_c(\mathbb{R}^2)$ for all $\beta \in \text{Diff} \mathbb{R}^2$, so since $(V_\beta^* \psi)(x) = J_\beta^{-1/2}(\beta^{-1}x) \cdot \psi(\beta^{-1}x)$ we have $V_\beta \in \mathcal{F}_S$. If $\beta \Gamma \subseteq \Gamma$, we get that $V_\beta$ preserves $C_c(\mathbb{R}^2 \setminus \Gamma) = \text{Ker}(\cdot, \cdot)_\Gamma \cap C_c(\mathbb{R}^2)$, and so $\Lambda(V_\beta)$ exists. Consider now the $\beta \in \text{Diff} \mathbb{R}^2$ given by $\beta(x_1, x_2) := (ax_1, x_2e^{x_1})$, $a > 0$, so clearly $\beta \Gamma = \Gamma$ and $J_\beta(x) = ae^{x_1} \neq 0$. (That $\beta$ is a diffeomorphism is clear since its inverse is $\beta^{-1}(x_1, x_2) = (x_1/a, x_2e^{-x_1/a})$ which is also differentiable).

Now for $\psi \in C_c(\mathbb{R}^2)$ we have

$$(\Lambda(V_\beta) \kappa(\psi), \Lambda(V_\beta) \kappa(\psi))_{L^2(\Gamma)} = \langle V_\beta \psi, V_\beta \psi \rangle_{\Gamma} = \langle V_\beta \psi | \Gamma, V_\beta \psi | \Gamma \rangle_{L^2(\Gamma)}$$

$$= \int |(V_\beta \psi | \Gamma)|^2 dx_1$$

$$= \int J_\beta(x_1, 0) |\psi(ax_1, 0)|^2 dx$$

$$= \int e^{x_1/a} |\psi(x, 0)|^2 dx =: I_\psi$$

and now we can choose a $\psi \in C_c(\mathbb{R}^2)$ with $\|\kappa(\psi)\| = \|\psi | \Gamma\|_{L^2(\Gamma)} = 1$ which can make $I_\psi$ arbitrary large, e.g. $\psi = \chi_{[n, n+1] \times [0, 1]}$ and consider $I_\psi$ as $n \to \infty$. Thus $\Lambda(V_\beta)$ is unbounded.

We remark that if we chose a more restrictive space $S$, e.g. $\mathcal{T}^{(\epsilon)}$, then the diffeomorphism in the last example will preserve $\mathcal{T}^{(\epsilon)}$, so this pathology cannot be removed by enforcing “secondary quantum constraints.”

Example of (ii):

Continue the previous example, noting that $\Lambda(V_\beta^*)$ exists because $\beta^{-1} \Gamma = \Gamma$.

However, if $\Lambda(V_\beta^*) \subseteq \Lambda(V_\beta)^*$ we have for all $\psi \in C_c(\mathbb{R}^2)$ that

$$(\Lambda(V_\beta) \kappa(\psi), \Lambda(V_\beta) \kappa(\psi))_{L^2(\Gamma)} = \langle \kappa(\psi), \Lambda(V_\beta)^* \kappa(V_\beta \psi) \rangle_{L^2(\Gamma)}$$
which makes $\Lambda(V_\beta)$ unitary, hence bounded, in contradiction with example (i), so it is false that $\Lambda(V_\beta^*) \subseteq \Lambda(V_\beta)^*$.

Now, it appears reasonable to the authors that in a constraining method, boundedness and the adjoint operation should be preserved, at least on the physical variables. So we want to restrict the set of operators under consideration to those satisfying these two requirements. Observe that an $A \in \mathcal{F}_S$ will lift to a bounded operator $\Lambda(A)$ on $L^2(\Gamma)$ iff

$$(A\psi, A\psi)_\Gamma \leq M \cdot (\psi, \psi)_\Gamma \quad \forall \psi \in S \quad \text{and a fixed} \quad M < \infty.$$

(Clearly if $\psi \in S \cap \ker \kappa$ then the last inequality implies $(A\psi, A\psi)_\Gamma = 0$, i.e. $A\psi \in \ker \kappa$, so $\Lambda(A)$ exists).

**Definition:**

$$\mathcal{O}_S := \{ A \in \mathcal{F}_S \mid (A\psi, A\psi)_\Gamma \leq M_A \cdot (\psi, \psi)_\Gamma \geq (A^*\psi, A^*\psi)_\Gamma \quad \forall \psi \in S \quad \text{and some} \quad M_A < \infty, \quad \text{and} \quad \Lambda(A^*) \subseteq \Lambda(A)^* \}.$$

$$\mathcal{D}_S := \{ A \in \mathcal{O}_S \mid AS \subseteq \ker \kappa \supseteq A^*S \}.$$

Note that if $A \in \mathcal{O}_S$, then $\Lambda(A)$ is bounded, so we have in fact that $\Lambda(A)^* = \overline{\Lambda(A^*)}$, so there are no problems with selfadjointness if $A$ is selfadjoint. These will occur however if we consider $\Lambda$ on unbounded operators.

**Lemma 5.4.** $\mathcal{O}_S$ and $\mathcal{D}_S$ are *-algebras.

**Proof:** Both sets are linear spaces. To see that $\mathcal{O}_S$ is closed under taking of adjoints, take the adjoint of $\Lambda(A^*) \subseteq \Lambda(A)^*$ to get $\Lambda(A^{**}) = \Lambda(A) \subseteq \overline{\Lambda(A)} = \Lambda(A)^{**} \subseteq \Lambda(A^*)^*$, i.e. $A^*$ is in $\mathcal{O}_S$ if $A$ is. Then clearly $\mathcal{D}_S$ is also closed under taking of adjoints. To see that $\mathcal{O}_S$ is an algebra, let $A, B \in \mathcal{O}_S$, then

$$(AB\psi, AB\psi)_\Gamma \leq M_A \cdot (B\psi, B\psi)_\Gamma \leq M_A M_B (\psi, \psi)_\Gamma$$
for all $\psi \in S$, and similarly for $B^* A^*$. Moreover,

$$\Lambda(AB)^* = [\Lambda(A)\Lambda(B)]^* \supseteq \Lambda(B)^* \Lambda(A)^* \supseteq \Lambda(B^*)\Lambda(A^*) = \Lambda((AB)^*)$$

and thus $AB \in \mathcal{O}_S$. That $\mathcal{D}_S$ is an algebra is obvious.

Now $\mathcal{D}_S = \text{Ker}(\Lambda|\mathcal{O}_S)$ is a $*$-ideal of $\mathcal{O}_S$, and $\mathcal{R}_S := \mathcal{O}_S/\mathcal{D}_S \cong \Lambda(\mathcal{O}_S) \subset \mathcal{B}(L^2(\Gamma))$. We think of $\mathcal{R}_S$ as the “physical observables” obtained from enforcing the constraint on $\mathcal{O}_S$, in analogy to the algebra $\mathcal{R}$ of the T-procedure for Dirac constraining. $\mathcal{R}_S$ cannot be zero because the identity operator $\mathbf{1} \in \mathcal{O}_S$ and $\Lambda(\mathbf{1}) = \mathbf{1}$. We do not expect $\mathcal{R}_S$ to be a C*-algebra, but it can easily generate a C*-algebra since $\mathcal{R}_S \cong \Lambda(\mathcal{O}_S) \subset \mathcal{B}(L^2(\Gamma))$.

Given that the commutant $\hat{\mathcal{F}}'$ is the traditional observables, we find here:

**Theorem 5.5.** $\mathcal{F}_S \cap \hat{\mathcal{F}}' \cap \hat{\mathcal{h}}' \subset \mathcal{O}_S$.

**Proof:** Let $A \in \mathcal{F}_S \cap \hat{\mathcal{F}}' \cap \hat{\mathcal{h}}'$, which is a $*$-algebra, so it also contains $A^*$. Then

$$(A\psi, \hat{h}P_t A\psi) = \|\hat{h}^{1/2}P_t A\psi\|^2 = \|Ah^{1/2}P_t\psi\|^2$$

$$\leq \|A\|^2 \cdot \|\hat{h}^{1/2}P_t\psi\|^2 = \|A\|^2 \cdot (\psi, \hat{h}P_t\psi)$$

for all $\psi \in S$. Thus

$$(A\psi, A\psi)_\Gamma = \lim_{t \to 0} \frac{1}{2t}(A\psi, \hat{h}P_t A\psi)$$

$$\leq \|A\|^2 \lim_{t \to 0} \frac{1}{2t}(\psi, \hat{h}P_t\psi) = \|A\|^2 \cdot (\psi, \psi)_\Gamma.$$

The same is true for $A^*$. Now for all $\psi, \phi \in S$:

$$\left(\Lambda(A)\kappa(\psi), \kappa(\phi)\right)_{L^2(\Gamma)} = \left(\kappa(A\psi), \kappa(\phi)\right)_{L^2(\Gamma)}$$

$$= (A\psi, \phi)_\Gamma = \lim_{t \to 0} \frac{1}{2t}(A\psi, \hat{h}P_t\phi)$$

$$= \lim_{t \to 0} \frac{1}{2t}(\psi, A^*\hat{h}P_t\phi) = \lim_{t \to 0} \frac{1}{2t}(\psi, \hat{h}P_tA^*\phi)$$

$$= (\psi, A^*\phi)_\Gamma = \left(\kappa(\psi), \Lambda(A^*)\kappa(\phi)\right)_{L^2(\Gamma)}$$

so $\Lambda(A^*) \subseteq \Lambda(A)^*$. Thus $A \in \mathcal{O}_S$.

**Remarks.** (1) Note that due to the limit in $(\cdot, \cdot)_\Gamma$, it is only the behaviour near $\Gamma$ which contributes in the constraining of an operator. In fact the proof actually
shows that an \( A \in \mathcal{F}_S \) will be in \( \mathcal{O}_S \) if
\[
\lim_{t \to 0} \frac{1}{t} \| \tilde{h}^{1/2} P_t A \psi \|^2 = \lim_{t \to 0} \frac{1}{t} \| \tilde{A} h^{1/2} P_t \psi \|^2
\]
and
\[
\lim_{t \to 0} \frac{1}{t} (\psi, [A^*, \hat{h} P_t] \phi) = 0
\]
for all \( \psi, \phi \in S \) and the same for \( A^* \). So if \( A \) commutes with \( \hat{\varphi} \) and \( \hat{h} \) on a small neighbourhood of \( \Gamma \), it will also be in \( \mathcal{O}_S \). Behaviour of \( A \) away from \( \Gamma \) is irrelevant.

(2) We note by 5.5 that all multiplication operators by bounded Borel functions which preserve \( S \) will be in \( \mathcal{O}_S \). In particular, for a reasonable physical system we expect \( \hat{\varphi} \) to preserve \( S \), so the constraint \( \hat{\varphi} \) is in \( \mathcal{O}_S \). In this case \( \hat{\varphi} \) is in \( \mathcal{D}_S \) by the following argument:
\[
\lim_{t \to 0} \| \tilde{h}^{1/2} P_t \hat{\varphi} \psi \|^2 = \lim_{t \to 0} \frac{1}{t} \| \hat{\varphi} \tilde{h}^{1/2} P_t \psi \|^2
\]
\[
\leq \lim_{t \to 0} \frac{1}{t} \sup \{ |\varphi(x)|^2 \mid x \in \varphi^{-1}([-t, t]) \} \cdot \| \tilde{h}^{1/2} P_t \psi \|^2
\]
\[
= \left( \lim_{t \to 0} \sup \{ |\varphi(x)|^2 \mid x \in \varphi^{-1}([-t, t]) \} \right) \cdot \lim_{t \to 0} \frac{1}{t} \| \tilde{h}^{1/2} P_t \psi \|^2
\]
\[
= 0 \quad \forall \psi \in S
\]
using the facts that the limit \( \lim_{t \to 0} \frac{1}{t} \| \tilde{h}^{1/2} P_t \psi \|^2 = \lim_{t \to 0} \frac{1}{t} (\psi, \hat{h} P_t \psi) \) exists due to \( \psi \in S \subset C_\varphi \), and that the limit of the supremum is zero. Hence
\[
(\hat{\varphi} \psi, \hat{\varphi} \psi)_\Gamma = 0 \; , \; \text{i.e.} \; \hat{\varphi} \psi \in \ker \kappa \; \text{for all} \; \psi \in S \; , \; \text{so} \; \Lambda(\hat{\varphi}) = 0 \; \text{as expected.}
\]

**Theorem 5.6.** An \( A \in \mathcal{F}_S \) preserves \( \ker \kappa \cap S \) iff
\[
\lim_{t \to 0} \frac{1}{t} \| P_t A (I \perp P_t) \psi \|^2 = 0
\]
for all \( \psi \in \ker \kappa \cap S \).

**Proof:** We need the following lemma:

**Lemma:** With assumptions and notation as above, we have
\[
\psi \in \ker (\cdot, \cdot)_\Gamma \; \text{iff} \; \lim_{t \to 0} \frac{1}{t} \| P_t \psi \|^2 = 0 \; .
\]

**Proof:** \( \psi \in \ker (\cdot, \cdot)_\Gamma \; \text{iff} \; 0 = (\psi, \psi)_\Gamma = \lim_{t \to 0} \frac{1}{2t} (\psi, \hat{h} P_t \psi) \; . \) Now
\[
(\psi, \hat{h} P_t \psi) = \int_{\varphi^{-1}([-t, t])} |\nabla \varphi(x)| \cdot |\psi(x)|^2 d\mu(x)
\]
\[
\geq \inf \{ |\nabla \varphi(x)| \mid x \in \varphi^{-1}([-t, t]) \} \int_{\varphi^{-1}([-t, t])} |\psi(x)|^2 d\mu(x) \; .
\]
Since \(|\nabla \varphi(x)|\) is assumed to be bounded, continuous and nonzero on a neighbourhood of \(\Gamma\) we have

\[
\liminf_{t \to 0} \left\{ |\nabla \varphi(x)| \mid x \in \varphi^{-1}([-t, t]) \right\} = M > 0.
\]

Thus

\[
\lim_{t \to 0} \frac{1}{t} \langle \psi, \hat{h}P_t \psi \rangle \geq M \lim_{t \to 0} \frac{1}{t} \int_{\varphi^{-1}([-t, t])} |\psi(x)|^2 d\mu(x)
= M \lim_{t \to 0} \frac{1}{t} \|P_t \psi\|^2
\]

so \(\psi \in \text{Ker}(\cdot, \cdot)_\Gamma\) implies \(0 = \lim_{t \to 0} \frac{1}{t} \langle \psi, \hat{h}P_t \psi \rangle \geq M \lim_{t \to 0} \frac{1}{t} \|P_t \psi\|^2\)

and so: \(\lim_{t \to 0} \frac{1}{t} \|P_t \psi\|^2 = 0\)

On the other hand, given a \(\psi \in \mathcal{H}\) satisfying the last equation, we have

\[
\langle \psi, \hat{h}P_t \psi \rangle = \int_{\varphi^{-1}([-t, t])} |\nabla \varphi(x)| \cdot |\psi(x)|^2 d\mu(x)
\leq \sup \left\{ |\nabla \varphi(x)| \mid x \in \varphi^{-1}([-t, t]) \right\} \int_{\varphi^{-1}([-t, t])} |\psi(x)|^2 d\mu(x)
\]

and the the limit of the supremum (denoted \(N\)) exists by the boundedness assumption, and is strictly positive. So

\[
\lim_{t \to 0} \frac{1}{t} \langle \psi, \hat{h}P_t \psi \rangle \leq N \cdot \lim_{t \to 0} \frac{1}{t} \|P_t \psi\|^2 = 0
\]

i.e. \(\lim_{t \to 0} \frac{1}{t} \langle \psi, \hat{h}P_t \psi \rangle = 0\), i.e. \(\psi \in \text{Ker}(\cdot, \cdot)_\Gamma\) . ▼

So by this lemma, \(A \in \mathcal{F}_S\) preserves \(\text{Ker} \cap S\) iff \(\lim_{t \to 0} \frac{1}{t} \|P_t A \psi\|^2 = 0\) for all \(\psi \in \text{Ker} \cap S\) . Now

\[
\lim_{t \to 0} \frac{1}{t} \|P_t A P_t \psi\|^2 \leq \lim_{t \to 0} \frac{1}{t} \|P_t A P_t\|^2 \|P_t \psi\|^2
\leq \|A\|^2 \cdot \lim_{t \to 0} \frac{1}{t} \|P_t \psi\|^2 = 0
\]

whenever \(\psi \in \text{Ker} \kappa\) . So by the triangle inequality:

\[
\|P_t A P_t \psi\| + \|P_t A(\mathbb{I} \perp P_t) \psi\| \geq \|P_t A \psi\|
\geq \left\|\|P_t A P_t \psi\| \perp \|P_t A(\mathbb{I} \perp P_t) \psi\|\right\|
\]

we obtain:

\[
\lim_{t \to 0} \frac{1}{\sqrt{t}} \|P_t A \psi\| = \lim_{t \to 0} \frac{1}{\sqrt{t}} \|P_t A(\mathbb{I} \perp P_t) \psi\|
\]
for all $\psi \in \text{Ker} \kappa \cap S$.

An immediate consequence is that $\Lambda(A)$ exists for all $A \in \hat{\varphi}' \cap \mathcal{F}_S$ (though $\Lambda(A)$ need not be bounded in general), and as before, we only need commutativity close to $\Gamma$ to get this. Also note the similarity with the condition $P_{\text{phys}} A(1 \perp P_{\text{phys}}) = 0$ for an observable in Dirac constraining. Moreover the proof of 5.6 used only the existence of the limit (3.6) for $\psi$, so it will work for larger domains than $C_\varphi$.

In the particular cases where $S$ is either $C_\varphi$ or $C_c(\mathbb{R}^n)$, we shorten the notation to $A_\varphi := A_{C_\varphi}$ and $A_c := A_{C_c(\mathbb{R}^n)}$ where $A$ can be $\mathcal{F}$, $\mathcal{O}$, $\mathcal{D}$ or $\mathcal{R}$. Note that

$$C_c(\mathbb{R}^n) \cap \text{Ker} \kappa = \{ \psi \in C_c(\mathbb{R}^n) \mid \psi|\Gamma = 0 \} = C_c(\mathbb{R}^n \setminus \Gamma)$$

$$\mathcal{D}_c = \{ A \in \mathcal{O}_c \mid AC_c(\mathbb{R}^n) \subseteq C_c(\mathbb{R}^n \setminus \Gamma) \supseteq A^* C_c(\mathbb{R}^n) \}$$

and for an $A \in \mathcal{O}_c$ we have $\Lambda(A)\kappa(\psi) = \kappa(A\psi) = (A\psi)|\Gamma$ for all $\psi \in C_c(\mathbb{R}^n)$. Analogous statements hold if we replace $C_c(\mathbb{R}^n)$ by $C_\varphi$.

On comparing the method above to that of Landsman [La], we note that the algebra which Landsman selects to impose the constraint on is the subalgebra of the commutant $\hat{\varphi}'$ which preserves $C_c(\mathbb{R}^n)$, i.e. $\hat{\varphi}' \cap \mathcal{F}_c$. In our case the algebra which we constrain, $\mathcal{O}_c$ can be considerably larger than that, with consequently larger algebra of observables $\mathcal{R}_c$, but on the other hand there are also nonzero $A \in \hat{\varphi}' \cap \mathcal{F}_c \setminus \mathcal{O}_c$ (see below).

Next we wish to examine whether particular classes of operators are in $\mathcal{O}_S$.

- Consider the multiplication operator $(T_f \psi)(x) := f(x) \cdot \psi(x)$ $\forall \psi \in \mathcal{H}$ where $f$ is bounded and Borel. It is not automatic that $T_f \in \mathcal{F}_S$. In fact if $S = C_c(\mathbb{R}^n)$, then $T_f C_c(\mathbb{R}^n) \subseteq C_c(\mathbb{R}^n)$ iff $f$ is continuous (in which case $T_f \in \mathcal{O}_c$ by 5.5). For the choice $S = C_\varphi$ we see that if a Borel function $f$ is discontinuous on $\Gamma$, then restriction to $\Gamma$ may not be defined, i.e. we may have $T_f \not\in \mathcal{F}_\varphi$. (This also shows $\hat{\varphi}' \cap \mathcal{F}_\varphi \neq \emptyset$ hence that $\hat{\varphi}' \not\subseteq \mathcal{O}_\varphi \cup \mathcal{O}_c$). On the other hand, if $f$ is continuous and bounded on some shell $S_t$, we have $T_f \in \mathcal{F}_\varphi$. So $\mathcal{F}_\varphi$ contains a larger class of multiplication operators than $\mathcal{F}_c$, and these are all in $\mathcal{O}_\varphi$. In particular,
if \( \exp iq \cdot a \in \mathcal{F} \) then \( \exp iq \cdot a \in \mathcal{O}_c \cap \mathcal{O}_\varphi \). However a \( T_f \) is only in \( \mathcal{F}_{T(f)} \) if \( f \in T^{(l)} \).

- Now consider the unitaries \( V_\beta \), \( \beta \in \text{Diff } \mathbb{R}^n \). For the choice \( S = C_c(\mathbb{R}^n) \), we have that both \( V_\beta \) and \( V_\beta^* \) preserve \( C_c(\mathbb{R}^n) \), hence \( V_\beta \in \mathcal{F}_c \) for all \( \beta \in \text{Diff } \mathbb{R}^n \). This is not however true for the choice \( S = C_\varphi \). In fact, let \( \beta \) be a fixed translation \( \beta x = x + a \) not preserving \( \Gamma \), and let \( \psi \in C_\varphi \) be continuous on a neighbourhood of \( \Gamma \) but so discontinuous on \( \beta \Gamma \) that restriction to \( \beta \Gamma \) is not defined. Then \( V_{\beta^{-1}} \psi \notin C_\varphi \), and so \( V_{\beta^{-1}} \notin \mathcal{F}_\varphi \). Thus \( \mathcal{F}_\varphi \) contains a smaller set of the unitaries \( V_\beta \) than \( \mathcal{F}_c \). Below we will examine when these are in \( \mathcal{O}_\varphi \).

- Let \( S = C_c(\mathbb{R}^n) \ni \psi \), then

\[
\Lambda(V_\beta) \kappa(\psi) = \kappa(V_\beta \psi) = (V_\beta \psi)|\Gamma = (J_\beta^{1/2} |\Gamma) \cdot (\psi \circ \beta)|\Gamma.
\]

So \( \Lambda(V_\beta) = I \) iff \( \beta x = x \quad \forall x \in \Gamma \) and \( J_\beta |\Gamma = 1 \), i.e. \( V_\beta \perp I \in \mathcal{D}_c \) iff \( \beta x = x \quad \forall x \in \Gamma \) and \( J_\beta |\Gamma = 1 \).

Since \( \kappa(\psi) = \psi |\Gamma \) for \( \psi \in C_c(\mathbb{R}^n) \), one might surmise that \( V_\beta |\Gamma \subseteq \Gamma \) iff \( V_\beta \in \mathcal{O}_c \). However, the example (5.3) shows that it is not true that \( \beta |\Gamma \subseteq \Gamma \Rightarrow V_\beta \in \mathcal{O}_c \). For the converse, we have:

**Lemma 5.7.** If \( V_\beta \in \mathcal{O}_c \cup \mathcal{O}_\varphi \), then \( \beta |\Gamma \subseteq \Gamma \).

**Proof:** Let \( V_\beta \in \mathcal{O}_c \), hence \( V_\beta \) preserves \( \text{Ker } (\cdot, \cdot)_\Gamma \cap C_c(\mathbb{R}^n) = C_c(\mathbb{R}^n \setminus \Gamma) \).

This is equivalent to \( \beta(\mathbb{R}^n \setminus \Gamma) \subseteq \mathbb{R}^n \setminus \Gamma \), which is equivalent to \( \beta |\Gamma \subseteq \Gamma \). This argument directly adapts to \( C_\varphi \).

Note that if \( \Gamma \) is curved and nonperiodic, then the translations will not preserve it, so \( \exp(iq \cdot a) \notin \mathcal{O}_c \cup \mathcal{O}_\varphi \) hence of the generating unitaries of \( \Lambda(\mathbb{R}^{2n}) \), only the commutative set \( \{ \exp(iq \cdot a) \mid a \in \mathbb{R}^n \} \) is in \( \mathcal{O}_c \cup \mathcal{O}_\varphi \). This is why we consider the CCR–algebra as too small a choice of field algebra for this type of constraining.

**Theorem 5.8.** Given notation above, we have:

\( V_\beta \in \mathcal{O}_c \cup \mathcal{O}_\varphi \) iff \( \beta |\Gamma \subseteq \Gamma \) and \( J_\beta(x) = J_\beta^r(x) \) for all \( x \in \Gamma \) where \( J_\beta \) (resp. \( J_\beta^r \)) is the Jacobian of \( \beta \) (resp. \( \beta |\Gamma \)) with respect to \( \mu \) (resp. \( \gamma \)).
Proof: Let $V_\beta \in \mathcal{O}_c \cup \mathcal{O}_\varphi$, so by 5.7 $\beta \Gamma \subseteq \Gamma$. Now since $\overline{\Lambda(V_\beta^*)} = \Lambda(V_\beta)^*$ we get

$$\Lambda(V_\beta)^* \Lambda(V_\beta) \kappa(\psi) = \Lambda(V_\beta)^* \kappa(V_\beta \psi)$$

$$= \Lambda(V_\beta^*) \kappa(V_\beta \psi) = \kappa(V_\beta^* V_\beta \psi)$$

$$= \kappa(\psi) = \Lambda(V_\beta) \Lambda(V_\beta)^* \kappa(\psi) \quad \forall \psi \in C_c(\mathbb{R}^n) \text{ or } C_\varphi.$$ 

Thus $\overline{\Lambda(V_\beta)}$ is unitary, so

$$(\psi, \psi)_\Gamma = (V_\beta \psi, V_\beta \psi)_\Gamma = (V_\beta \psi | \Gamma, V_\beta \psi | \Gamma)_{L^2(\Gamma)}$$

$$= \int_\Gamma |(V_\beta \psi)(x)|^2 d\gamma(x) = \int_\Gamma J_\beta(x) \cdot |\psi(\beta x)|^2 d\gamma(x)$$

$$= \int_\Gamma J_\beta(x) \cdot \mathbb{J}_\beta^{-1}(x) \cdot |\psi(\beta x)|^2 d\gamma(\beta x)$$

$$= \int_\Gamma J_\beta(\beta^{-1}x) \cdot \mathbb{J}_\beta^{-1}(\beta^{-1}x) \cdot |\psi(x)|^2 d\gamma(x)$$

$$= \int_\Gamma |\psi(x)|^2 d\gamma(x) \quad \forall \psi \in C_c(\mathbb{R}^n) \text{ or } C_\varphi$$

iff $J_\beta(\beta^{-1}x) \cdot \mathbb{J}_\beta^{-1}(\beta^{-1}x) = 1$ for all $x \in \Gamma$, i.e. $J_\beta(x) = \mathbb{J}_\beta(x)$ for all $x \in \Gamma$.

Conversely, observe that $\beta \Gamma \subseteq \Gamma$ guarantees that both $\Lambda(V_\beta)$ and $\Lambda(V_\beta^*)$ exist, so by the reversibility of the previous calculation when $J_\beta = \mathbb{J}_\beta$ on $\Gamma$, we see that $\overline{\Lambda(V_\beta)}$ is unitary, hence bounded, and similarly the same is true for $\overline{\Lambda(V_\beta^*)}$. Since $\Lambda$ is a homomorphism and inverses are unique, we get that $\overline{\Lambda(V_\beta^*)} = \Lambda(V_\beta)^*$, i.e. $V_\beta \in \mathcal{O}_c \cup \mathcal{O}_\varphi$.

Remarks: (1) Note that if $\beta \in \text{Diff } \mathbb{R}^n$ preserves all the level sets of $\varphi$ we have $\varphi \circ \beta = \varphi$, so $[V_\beta, \varphi] = 0$. However since we need not have $\mathbb{J}_\beta = J_\beta$ on $\Gamma$, there are certainly such $\beta$ for which $V_\beta \notin \mathcal{O}_c$. Thus $\mathcal{F}_c \cap \varphi \setminus \mathcal{O}_c \neq \emptyset$, i.e. Landsman [La] quantizes some operators which we have excluded from our observables. (Some of these can still be taken through $\Lambda$ using 5.6, but they need not preserve the adjoint or boundedness). Nevertheless, we conclude that the field algebra $C_b(\mathbb{R}^n) \times \text{Diff } \mathbb{R}^n$ has ample elements in its physical algebras $\mathcal{R}_c$ or $\mathcal{R}_\varphi$. 

\[\square\]
(2) There is no reason in general to expect that the time evolutions will preserve $\mathcal{R}_S$, so we will need to extend it by taking the constrained field algebra as the $\text{C}^*$-algebra generated in $\mathcal{B}(L^2(\Gamma))$ by $\mathcal{R}_S$ and the unitaries $\exp it \mathcal{H}_\Gamma$, $t \in \mathbb{R}$.

(3) Recall that we consider the enforcement of secondary quantum constraints as the selection of the set $S = \mathcal{T}^{(t_0)}$ which the observables should preserve. Since this is an analogy to the classical procedure, it is natural to look for reasons to justify such a choice, and a first attempt may be to ask whether the choice of transverse states for $S$ will make the Jacobian condition in 5.8 obsolete, i.e. whether for a $V_\beta \in \mathcal{F}_{\mathcal{T}^{(t)}(t)}$ we have $V_\beta \in \mathcal{O}_{\mathcal{T}^{(t)}(t)}$ iff $\beta \Gamma \subset \Gamma$. This is not true, as we can see by an easy counterexample. Continue the example (5.3) with the additional assumption that the Hamiltonian has $\text{Dom} \: H = C_c^\infty(\mathbb{R}^2)$ and that $\beta \in \text{Diff} \: \mathbb{R}^2$ is not the given one in (5.3), but $\beta(x_1, x_2) = (x_1, 2x_2)$. Then both $V_\beta$ and $V_\beta^*$ preserve

$$\mathcal{T}^{(t_0)} = \left\{ \psi \in C_c^\infty(\mathbb{R}^n) \mid \frac{\partial \psi}{\partial x_2}(x) = 0 \quad \forall \: x \in S_{t_0}, \: t_\psi < t_0 \right\}$$

hence $V_\beta \in \mathcal{F}_{\mathcal{T}^{(t_0)}(t)}$, and $\beta \Gamma = \Gamma$, $\beta | \Gamma = \mathbb{I}$, $J_\beta = 1$, $J_\beta = 2$, hence

$$\Lambda(V_\beta)^* = \sqrt{2} \mathbb{I} \neq \Lambda(V_\beta^*) = \frac{1}{\sqrt{2} \mathbb{I}},$$

so $V_\beta \notin \mathcal{O}_{\mathcal{T}^{(t_0)}(t)}$.

Another possible reason one may want to use to justify the enforcement of secondary quantum constraints, is to produce a common dense invariant domain $\kappa(S)$ for the observables $\mathcal{R}_S$. However, for the choice $S = \mathcal{T}^{(t)}$ there is no reason why the Hamiltonian $H_{\Gamma}$ should preserve $\kappa(\mathcal{T}^{(t)})$, so it will be unreasonable to require the observables to do so.

So at this stage we fail to see why secondary quantum constraints should be imposed on the observables.

(4) It is interesting to observe that for any Hilbert space operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ with dense range, we can obtain the structure above. That is, given a dense space $S \subseteq \text{dom} \: K$ for which $K(S)$ is dense in $\mathcal{H}_2$, we can define $\mathcal{F}_S$, $\mathcal{O}_S$, $\mathcal{D}_S$, $\mathcal{R}_S$ exactly as before by replacing $\mathcal{F}$ with $\mathcal{B}(\mathcal{H}_1)$, and $\kappa$
by \( K \). So every such operator \( K \) and space \( S \) defines a lifting problem, hence a short exact sequence of \(*\)-algebras

\[
0 \to \mathcal{D}_S \to \mathcal{O}_S \to \mathcal{R}_S \to 0.
\]

This then produces a similar short exact sequence for the automorphism groups and their lifting through the factorisation. In particular, for ordinary Dirac constraining the operator \( K \) is the projection \( P_{\text{phys}} : \mathcal{H} \to \mathcal{H}_{\text{phys}} \), the space \( S \) is \( \mathcal{H} \) and \( A_S \cap \mathcal{F}_S \) is just \( A \) where \( A \) denotes either \( O \) or \( D \). This way of thinking nicely unifies the current construction (where \( K = \kappa \)) with that of the Dirac approach. The main difference is that \( P_{\text{phys}} \) is bounded whilst \( \kappa \) is unbounded and nonclosable.

6. Constraining by a general selfadjoint operator.

For later reference we start by summarizing the constraining algorithm developed in the preceding sections. Given the following data: operators \( \hat{p}, \hat{q} \) and \( H \) on \( C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \), a bounded \( \varphi \in C^\infty(\mathbb{R}^n) \) such that \( \Gamma = \varphi^{-1}(0) \) is a smooth \((n \perp 1)\)-dimensional submanifold and moreover \( \nabla \varphi \) is bounded and nonzero on a neighbourhood of \( \Gamma \) and a unital field algebra \( \mathcal{F} \subset \mathcal{B}(L^2(\mathbb{R}^n)) \) containing \( \varphi \), \( \exp i \hat{p} \cdot \mathbf{a} \), \( \exp i \hat{q} \cdot \mathbf{a} \).

(1) Define \( (\psi_1, \psi_2)_r := \lim_{t \to 0} \frac{1}{2t} (\psi_1, \hat{h} P_t \psi_2) \) \( \forall \psi_i \in C_\varphi(\mathbb{R}^n) \).

\[
L^2(\Gamma) = [C_\varphi(\mathbb{R}^n) / \text{Ker}(\cdot, \cdot)_{\Gamma}]^{\perp} \subset L^2(\Gamma), \quad \kappa : C_\varphi \to L^2(\Gamma) \text{ is the factorisation map and } L^2(\Gamma) \text{ is the constrained space.}
\]

(2) Let \( \mathcal{T}_{t_0} := \mathcal{H}^c_{t_0} \cap C_\varphi \cap \text{Dom} H \), check that \( H \mathcal{T}_{t_0} \subset C_\varphi \) for \( t_0 \) small enough, and if so, define the constrained Hamiltonian \( H_{\Gamma} \) by \( H_{\Gamma} \cdot \kappa(\psi) := \kappa(H \psi) \) for all \( \psi \in \mathcal{T}_{t_0} \).

(3) Choose a space \( S \) in \( \mathcal{H} \) for which \( \kappa(S) \) is dense (e.g. \( C_\varphi \)), and define

\[
\mathcal{F}_S = \{ A \in \mathcal{F} \mid AS \subseteq S \supseteq A^* S \}
\]

\[
\mathcal{O}_S = \{ A \in \mathcal{F}_S \mid (A \psi, A \psi)_\Gamma \leq M_A(\psi, \psi)_\Gamma \geq (A^* \psi, A^* \psi)_\Gamma, \Lambda(A^*) \subseteq \Lambda(A) \}
\]

where \( \Lambda(A) \) denotes the closure of the operator defined by

\[
\Lambda(A) \kappa(\psi) = \kappa(A \psi) \quad \forall \psi \in S.
\]
Define \( D_S := \{ A \in \mathcal{O}_S \mid AS \subseteq \text{Ker} \kappa \supseteq A^*S \} \), then

\[
\Lambda(\mathcal{O}_S) \cong \mathcal{O}_S / D_S =: \mathcal{R}_S ,
\]

and the algebra of the constrained observables on \( L^2(\Gamma) \) is the C*-algebra generated by \( \mathcal{R}_S \) and \( \exp(i\mathbb{R}H_\Gamma) \).

In this section we want to generalise the method above to impose a constraint " \( C\psi = 0 \) " where \( C \) is a general bounded selfadjoint operator on a Hilbert space \( \mathcal{H} \) with zero in its continuous spectrum. (If we start with an unbounded selfadjoint operator \( C \) we can, without loss of generality convert the problem to a bounded one by replacing \( C \) with \( f(C) \) where \( f \) is a continuous bounded real-valued function with \( f(x) = x \) on a neighbourhood of zero). As before, we still assume that there is a unital field algebra \( \mathcal{F} \) acting on \( \mathcal{H} \), containing \( C \), and that there is also a (possibly unbounded) Hamiltonian \( H \) given on \( \mathcal{H} \), \( \exp(itH) \in \mathcal{F} \) for all \( t \). We will be concerned with the construction of three objects: the constrained Hilbert space (and the constraining map to it from the original space), the constrained Hamiltonian, and the algebra of constrained observables.

At the abstract C*-level, the kinematics part has already been solved [GH], but one obtained the set of all representations in which the constraint can be imposed as an eigenvalue condition. Here we want to build a particular concrete constrained system out of the given unconstrained one.

Now geometry was paramount in the analysis of the previous sections, in that we needed a metric (on an underlying space) to define gradients, norms of vectors, normals to surfaces and the Lebesgue measure. This vital piece of information is missing in the problem under consideration, and we somehow need to augment the given data \( \{ \mathcal{F}, \mathcal{H}, H, C \} \) in order to adapt the method previously found to this problem. The extra information we will assume is:

(i) a maximally commutative C*-algebra \( \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) containing \( C \).

(Call \( \mathcal{A} \) a polarisation, and we know that it always has a cyclic and separating vector [BR 2.5.3])

(ii) A choice of a cyclic and separating vector \( \Omega \) for \( \mathcal{A} \). (Call \( \Omega \) the vacuum).

(iii) A choice of scaling operator \( K \in \mathcal{A}_+ \), \( \text{Ker} \, K = \{0\} \).

(iv) A selfadjoint operator \( \mathcal{P}_C \) such that \( [\mathcal{P}_C, C]\psi = i\psi \) for all \( \psi \in \mathcal{D} \equiv \) a dense invariant subspace of \( \mathcal{P}_{ie}\mathcal{H} \) where \( \mathcal{P}_i \) denotes the spectral projection.
of $C$ of the interval $[\perp, t]$.

This is thought of as a “local” canonical momentum for $C$, which will define the normal direction to the constrained system.

Now define the space $\mathcal{L} := \{ A \in \mathcal{A} \mid \omega_0(A) \text{ exists} \}$ where

$$
\omega_0(A) := \lim_{t \to 0} \frac{(P_t \Omega, A P_t \Omega)}{\| P_t \Omega \|^2}.
$$

Since obviously $\mathbb{1}$ and $P_t \in \mathcal{L}$, this space is not zero. Moreover, the space is selfadjoint, and we also have that $|\omega_0(A)| \leq \| A \|$ for all $A \in \mathcal{L}$ because $\| (P_t \Omega, A P_t \Omega) / \| P_t \Omega \|^2 \| \leq \| A \|$ and so $\omega_0$ extends to the closure of $\mathcal{L}$. Now since $\omega_0$ is obviously positive on the positive elements of $\mathcal{L}$, we can apply [KR] 4.3.13 to conclude that $\omega_0$ extends to a state on $\mathcal{A}$. Henceforth we fix a choice of extension and still denote it by $\omega_0$.

Denote the GNS-representation of $\omega_0$ by $(\pi_0, \Omega_0, \mathcal{H}_0)$. Noting that $\omega_0(K) > 0$, we define a state $\omega_K$ on $\mathcal{A}$ by

$$
\omega_K(A) := \omega_0(K A) / \omega_0(K), \quad A \in \mathcal{A}.
$$

Then, inspired by the following lemma and subsequent example, we identify the constrained Hilbert space with $\mathcal{H}_\omega_K$.

**Lemma 6.1.** $\omega_K$ is a “Dirac state” in the sense that $\omega_K(AC) = 0$ for all $A \in \mathcal{A}$, i.e. $\pi_{\omega_K}(C) \Omega_{\omega_K} = 0$.

**Proof:** It suffices to show that $\omega_K(C^2) = 0$, since $C$ is selfadjoint. So

$$
\frac{\omega_0(K C^2)}{\omega_0(K)} = \frac{(A^* \Omega, K P_t C \Omega)}{\| P_t \Omega \|^2 \omega_0(K)}.
$$

Now

$$
\| (K P_t \Omega, P_t C^2 \Omega) \| \leq \| K C \| \cdot \| P_t \Omega \| \cdot \| P_t C \Omega \|
$$

$$
= \| K C \| \cdot \| P_t \Omega \| \cdot \int_{-t}^{t} \lambda dP(\lambda) \| \Omega
$$

$$
\leq \| K C \| \cdot \| P_t \Omega \| \cdot t \int_{-t}^{t} dP(\lambda) \| \Omega
$$

$$
= t \| K C \| \cdot \| P_t \Omega \|^2.
$$

Hence

$$
\| \omega_K(C^2) \| \leq \frac{1}{\omega_0(K)} \lim_{t \to 0} t \| K C \| = 0.
$$
Remark: The proof above is easily adapted to show that $\omega_0$ is also a Dirac state on $A$. More precisely, we have for the left kernels that $N_{\omega_0} = N_{\omega_K}$ due to $\text{Ker} K = \{0\}$ and the fact that $A$ is commutative. Hence we have an identification map $\theta: A/N_{\omega_0} \to A/N_{\omega_K}$ by $\theta(A + N_{\omega_0}) = A + N_{\omega_K}$ and $\theta$ extends to a map $\theta: \mathcal{H}_{\omega_0} \to \mathcal{H}_{\omega_K}$ because
\[
\|\theta(A + N_{\omega_0})\|_{\mathcal{H}_{\omega_K}} = \omega_K(\theta(A + N_{\omega_0})^* \theta(A + N_{\omega_0}))^{1/2}
= \omega_K(A^* A)^{1/2} = \omega_0(K A^* A)^{1/2}
\leq \|K\| \cdot \omega_0(A^* A)^{1/2} = \|K\| \cdot \|A + N_{\omega_0}\|_{\mathcal{H}_0}.
\]

To define a constraining map $\kappa: \mathcal{H} \to \mathcal{H}_{\omega_K}$ from the unconstrained to the constrained space, recall that $\Omega$ separates $A$, hence the map
\[
\kappa(A\Omega) := \pi_{\omega_K}(A)\Omega_{\omega_K}, \quad A \in A
\]
is well-defined on the dense subspace $A\Omega$ with dense range. This will be our choice of constraining map for this context. Since $\|\kappa(A\Omega)\|^2 = \omega_K(A^* A)$, we see that $\text{Ker} \kappa = N_{\omega_K}$, $\Omega = N_{\omega_0}$, $\Omega$.

Example. To motivate the preceding structures, and to see what is involved in the choices of $A, \Omega, K$, we now reconsider the constraint situation $\hat{\varphi}, L^2(\mathbb{R}^n)$ of the preceding sections.

Starting with $\hat{\varphi} = \varphi(\hat{q})$, the natural choice of polariisation is $A := \{ f(\hat{q}) \mid f \in L^\infty(\mathbb{R}^n) \} = \{ \exp(i\hat{q} \cdot a) \mid a \in \mathbb{R}^n \}''$, and for $\Omega$ we can then choose any positive nowhere vanishing $L^2$-function. Choose the Gaussian $\Omega(x) = \exp(-a|x|^2)$, $a > 0$ and note that $C_c(\mathbb{R}^n) = C_c(\mathbb{R}^n)\Omega$, hence $C_c(\mathbb{R}^n) \subset A\Omega$. Now for $K$ we must have $K \in A_+$, so $K = k(\hat{x})$ for some $k \in L^\infty(\mathbb{R}^n)$, $k^{-1}(0) = \emptyset$. For the moment we will choose $k$ to be continuous, and then below deduce the precise choice which will correspond with the previous results for this situation. In particular, what we want to show is that
\[
(\kappa(A\Omega), \kappa(B\Omega))_{\mathcal{H}_{\omega_K}} = b((A\Omega|\Gamma), (B\Omega|\Gamma))_{L^2(\Gamma)}
\]
for all $A, B \in C_c(\mathbb{R}^n) = \mathcal{L}$ for the right choice of $K$, where $\kappa$ is the map defined in this section and $b$ is a normalising constant. Now
\[
(\kappa(A\Omega), \kappa(B\Omega))_{\mathcal{H}_{\omega_K}} = \omega_K(A^* B) = \lim_{t \to 0} \frac{\langle A\Omega, K P_t B\Omega \rangle}{\|P_t\Omega\|^2 \omega_0(K)}
\]
whenever \( A^* B \in \mathcal{L} \) (which we will see below is all \( C_c(\mathbb{R}^n) \)) so if we write 
\( A = f_A(\hat{q}) \), \( B = f_B(\hat{q}) \) with \( f_A, f_B \in C_c(\mathbb{R}^n) \) and remember that 
\[
(P_t \psi) (x) = \chi_{\frac{1}{2} [-t, t]} (x) \cdot \psi (x),
\]
we have 
\[
\frac{(A \Omega, K P_t B \Omega)}{\| P_t \Omega \|^2} = \int_{\varphi^{-1} [-t, t]} \bar{f}_A (x) (x) f_B (x) \exp (\pm 2a |x|^2) \, d\mu (x) 
\]
Now recall that for \( f \in C_c(\mathbb{R}^n) \) and small \( t \) we have 
\[
\int_{\varphi^{-1} [-t, t]} f (x) \, d\mu (x) = \int_t (\int \frac{f (y, \varphi)}{\nabla \varphi (y, \varphi)} \, d\gamma (y)) \, d\varphi = 2t \int \frac{f (y, 0)}{\nabla \varphi (y, 0)} \, d\gamma (y) + O (t)
\]
\[
= 2t \int \frac{f (y, 0)}{\nabla \varphi (y, 0)} \, d\gamma (y) + O (t).
\]
thus: 
\[
(\kappa (A \Omega), \kappa (B \Omega))_{\mathcal{H}_{\omega \kappa}} = \lim_{t \to 0} \frac{(A \Omega, K P_t B \Omega)}{\| P_t \Omega \|^2 \omega_0 (K)} = \frac{\int_\Gamma \left( \overline{f}_A k f_B e / |\nabla \varphi| \right) |\Gamma \, d\gamma}{\omega_0 (K) \int_\Gamma (e / |\nabla \varphi|) \, |\Gamma \, d\gamma}
\]
where \( e \) denotes the function \( \exp (\pm 2a |x|^2) \). Now since by a similar argument 
\[
\omega_0 (K) = \lim_{t \to 0} \frac{(P_t \Omega, K P_t \Omega)}{\| P_t \Omega \|^2} = \frac{\int_\Gamma (k e / |\nabla \varphi|) \, |\Gamma \, d\gamma}{\int_\Gamma (e / |\nabla \varphi|) \, |\Gamma \, d\gamma}
\]
(the existence of this limit shows \( \mathcal{L} = C_c(\mathbb{R}^n) \)) we have 
\[
(\kappa (A \Omega), \kappa (B \Omega))_{\mathcal{H}_{\omega \kappa}} = \frac{\int_\Gamma \left( \overline{f}_A k f_B e / |\nabla \varphi| \right) |\Gamma \, d\gamma}{\int_\Gamma (k e / |\nabla \varphi|) \, |\Gamma \, d\gamma}
\]
\[
= \frac{\int_\Gamma (f_A f_B e) \, |\Gamma \, d\gamma}{\int_\Gamma e \, |\Gamma \, d\gamma} = \frac{((A \Omega | \Gamma), (B \Omega | \Gamma))}{\| \Omega | \Gamma \|^2_{L^2 (\Gamma)}},
\]
where we made the choice \( k = |\nabla \varphi| \), which is just the normalised inner product of \( L^2 (\Gamma) \), and this will produce the same \( \kappa \) than we had before. If there is also a group of symmetries which needs to be unitarily implemented on the constrained Hilbert space \( \mathcal{H}_{\omega \kappa} \), we can use this unitarity to determine \( K \).
So now that we have the constraining map \( \kappa : \mathcal{H} \to \mathcal{H}_{\omega_K} \) as above, we can proceed to constrain the dynamics. This is where we will use the assumed operator \( \mathcal{P}_C \) in the role of the “normal derivative \( i \frac{\partial}{\partial \varphi} \) near \( \Gamma \).” Thus start by selecting a space of transverse states by

\[
\mathcal{H}^T_t := \left\{ P_t \psi \mid \psi \in \text{Dom} \mathcal{P}_C, \ P_t \mathcal{P}_C \psi = 0 \right\}
\]

and

\[
\mathcal{T}_t := \left\{ \psi \in \text{Dom} \mathcal{H} \mid P_t \psi \in \mathcal{H}^T_t \right\}
\]

and note that \( \mathcal{T}_t \subset \mathcal{T}_s \) if \( t > s \). For a consistent constraining we need to assume there is a \( t_0 > 0 \) such that the set \( S_{t_0} := \left\{ \psi \in \mathcal{T}_{t_0} \mid H \psi \in \mathcal{A} \Omega \right\} \) satisfies \( \text{Ker} \ \kappa \cap S_{t_0} = \{0\} \) and \( \kappa(S_{t_0}) \) is dense in \( \mathcal{H}_{\omega_K} \). Given this, we define the constrained Hamiltonian \( H_C \) on domain \( \kappa(S_{t_0}) \) to be

\[
H_C \cdot \kappa(\psi) := \kappa(H \psi) \ \forall \ \psi \in S_{t_0}.
\]

**Example.** Continue the previous example with \( C = \dot{\varphi} \) and choice \( K = \lvert \nabla \varphi \rvert \). Choose

\[
\mathcal{P}_\varphi \psi := f(\varphi) \frac{\partial}{\partial \varphi} \psi \ \forall \psi \in C^\infty_c(\mathbb{R}^n)
\]

where \( f \) is a smooth bump function which is one on \([\bot t_0, t_0]\) and zero outside \([\bot t_0, \varepsilon, t_0 + \varepsilon]\) for some \( \varepsilon > 0 \). Then

\[
\mathcal{H}^T_{t_0} = \left\{ \psi \in \mathcal{H} \mid \text{supp} \psi \subset \varphi^{-1}(\bot t_0, t_0), \ \psi(y, \varphi) \text{ independent of } \varphi \right\}
\]

on \( \varphi^{-1}(\bot t_0, t_0) \)

If \( H \) is a differential operator with domain \( C^\infty_c(\mathbb{R}^n) \), we have

\[
\mathcal{T}_{t_0} = \left\{ \psi \in C^\infty_c(\mathbb{R}^n) \mid \psi \lvert_{\varphi^{-1}(\bot t_0, t_0)} \right\}
\]

is constant in the direction \( \nabla \varphi \).

Since \( \mathcal{H}^T_{t_0} \cap C^\infty_c(\mathbb{R}^n) \subset \mathcal{T}_{t_0} \), we conclude that

\[
H_{\dot{\varphi}} \kappa(\psi) = \kappa(H \psi) \ \forall \psi \in \mathcal{T}_{t_0}
\]

defines the same operator as before.

Finally, to obtain the constrained kinematics as in Sect. 4, let \( S \subseteq \mathcal{A} \Omega \) be a subspace such that \( \kappa(S) \) is dense in \( \mathcal{H}_{\omega_K} \). Define in complete analogy with
Sect. 4,

\[ \mathcal{F}_S := \{ A \in \mathcal{F} \mid A \psi \in S \exists A^* \psi \ \forall \psi \in S \} \]

\[ \mathcal{O}_S := \{ A \in \mathcal{F}_S \mid (A \psi, A \psi)_K \leq M_A \cdot (\psi, \psi)_K \geq (A^* \psi, A^* \psi)_K \ \forall \psi \in S, \]

\[ M_A < \infty \ \text{and} \ \Lambda(A^*) \subseteq \Lambda(A) \star \}

\[ \mathcal{D}_S := \{ A \in \mathcal{O}_S \mid A S \subseteq \text{Ker } \kappa \supseteq A^* S \} \]

where \( \Lambda(A) \kappa(\psi) := \kappa(A \psi) \ \forall \psi \in S, \ A \in \mathcal{O}_S. \)

The previous proofs carry over verbatim, so \( \mathcal{D}_S = \text{Ker } (\Lambda \mid \mathcal{O}_S) \) and the algebra of observables acting on \( \mathcal{H}_{\omega_K} \) is the C*-algebra generated by \( \exp i \mathbb{R} H_C \) and \( \mathcal{R}_S := \mathcal{O}_S / \mathcal{D}_S \cong \Lambda(\mathcal{O}_S) \subset B(\mathcal{H}_{\omega_K}). \)

**Example.** In continuation of the last example, it is obvious that if we take \( \mathcal{F} = C_c(\mathbb{R}^n) \times \text{Diff } \mathbb{R}^n \) and \( S = C_c(\mathbb{R}^n), \) then we will get the same observable algebra than before.

**Remark** Another way to proceed could have been to extend the Dirac state \( \omega_K \) on \( \mathcal{A} \) to \( \mathcal{F}. \) However, this would have enlarged the GNS-space (producing a wrong result for the example of this paper), and in addition there is the problem that the extension need not be unique.
7. Conclusions and Discussion.

To summarize, we took the problem of restricting a quantum particle in $\mathbb{R}^n$ to a lower dimensional submanifold not conserved by the dynamics. We found a method to do the restriction, and this came in three parts: a map $\kappa$ from a dense subspace of $L^2(\mathbb{R}^n)$ to the constrained Hilbert space $L^2(\Gamma)$, a method for constructing the constrained Hamiltonian from the unconstrained one, and a method for the construction of the algebra of observables $\mathcal{R}_\Gamma$ on $L^2(\mathbb{R}^n)$. For each of these three parts there were choices to be made: for the first part a domain for $\kappa$ (e.g., $C_c(\mathbb{R}^n)$ and $C_\varphi$ seem the canonical choices), for the second part a choice of transverse space (here tangentiality of momentum to the level sets of $\varphi$ near $\Gamma$ seems to be the physically justified criterion), and for the third part a choice of space $S$ on which to reduce the observables (likely choices here are $C_c(\mathbb{R}^n)$, $C_\varphi$ or the transverse space used). We regard these choices as made as physical ones, and in the two examples, the choices made were evident, and produced results in agreement with the known solutions. The necessity of these choices should be compared with the choice of boundary conditions required in the constraining of a free quantum particle to a box.

There are mathematical pathologies associated with our proposed method, of which the central one is that the map $\kappa$ is neither bounded nor even closable. As a consequence, the constraining of observables need not preserve boundedness or involution, so we had to restrict the method to those observables for which these pathologies do not occur. The method is also sensitive with respect to the original choice of field algebra, and may not provide enough observables for some choices.

Whilst this method was developed in analogy with the classical constraint method, the analogy became quite vague at certain points, for instance, one can interpret the necessity for a choice of transverse space as an imposition of a secondary constraint, but this constraint could not be interpreted as obtained from a similar method as the classical one. Moreover secondary constraints do not seem to be necessary for the constraining of the observables.

We expect from our work in [GH] that the pathologies which occurred here can be circumvented by a suitable generalisation to a $C^*$-algebra framework, inso-
far the T-procedure solves already without pathology the kinematics part of the
question there, but this would involve losing the original representation which may
contain some physical information. The advantage of the current method is that it
constructs the constrained system concretely from the given representation of the
original system. Nevertheless, we intend to further pursue this line of thought.

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