CLR-estimate for the Generators of Positivity Preserving and Positively Dominated Semigroups

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Supported by Federal Ministry of Science and Research, Austria
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CLR-ESTIMATE FOR THE GENERATORS OF POSITIVITY PRESERVING AND POSITIVELY DOMINATED SEMIGROUPS

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Dedicated to the memory of Mark Grigor’evich Krein

ABSTRACT. Let $B$ be a generator of positivity preserving (shortly, positive) semigroup in $L_2$ on a space with $\sigma$-finite measure. It is supposed that the semigroup $e^{-tB}$, $t > 0$, acts continuously from $L_2$ to $L_\infty$. For a measurable function $V \geq 0$ an estimate for the number of negative eigenvalues of the operator $B - V$ is obtained, generalizing the similar estimate for the the Schrödinger operator (the CLR-estimate) and reproducing, for that case, the best-known coefficient. A more general theorem replaces the positivity property of the semigroup by domination by a positive semigroup.

1. Introduction

1.1. The CLR inequality, in its original form, reads

\begin{equation}
N_\leq (-\Delta - V) \leq C(d) \int_{\mathbb{R}^d} V^{d/2} dx, \quad d \geq 3.
\end{equation}

Here $\Delta$ is the Laplacian on $\mathbb{R}^d$ and $V \geq 0$ is a measurable function (potential). By $N_\leq$ we denote the number of negative eigenvalues of a selfadjoint operator, provided its negative spectrum is discrete. The single assumption, under which (1.1) holds, is the finiteness of the integral on its right-hand side.

At least five different proofs of (1.1) are known up to present. In the chronological order, they were given by Rozenblum [23], Cwikel [11], Lieb [19], Li and Yau [18] and Conlon [9]. The proof of Rozenblum can be found also in [5] and [13], the one of Cwikel in [25]. The proof of Lieb, in two different versions, is presented in [22] and [26], and of Li and Yau in [13]. Various generalizations are also known: to higher order operators, such as $(-\Delta)^l$ instead of $-\Delta$; to a class of pseudodifferential operators; to various important operators of Mathematical Physics, such as Magnetic and Relativistic Schrödinger operators, Pauli operator, etc. - see [20] for a discussion of the applications of (1.1) and its analogs in Physics. A close topic is estimation of the number of eigenvalues of a perturbed operator, appearing in the gaps of the spectrum of the unperturbed one, see e.g. [4], where additional references can be found.

1991 Mathematics Subject Classification. 35P15 (47A10, 47B38, 47B65, 47D06).

Key words and phrases. Positivity preserving semigroups, CLR-estimate.
1.2. We are not going to describe here different approaches to the proof of (1.1). We only mention that the proofs given in [23], [11], and [9] use some specific tools of Real Analysis. As a result, they apply only to operators on \( \mathbb{R}^d \), and even the extension to manifolds needs an additional technique. In contrast to this, the approaches of [19] and [18] can be transplanted to a more general setting (though they do not work for differential operators of order higher than two). For Li-Yau’s approach this was done in [17]. It turned out that the appropriate class of operators \( B \), which could be substituted for \(-\Delta\) in (1.1), is the class of Markov (in a more rigorous terminology, sub-Markov) generators.

Here is the main result of [17].

**Theorem 1.1.** Let \( \Omega \) be a space with a \( \sigma \)-finite measure \( \mu \) and \( B > 0 \) be a self-adjoint operator in \( L_2(\Omega, \mu) \), generating a Markov semigroup. Suppose that the following embedding inequality is satisfied, with some \( q > 2 \):

\[
\left( \int_{\Omega} |u|^q d\mu \right)^{2/q} \leq K \| B^{1/2} u \|_{L^2}^2, \quad u \in \text{Dom}(B^{1/2}).
\]

Then for any \( 0 \leq V \in L_p(\Omega, \mu) \), \( p = (1 - \frac{2}{q})^{-1} \), one has

\[
N_-(B - V) \leq C(q)K \int_{\Omega} V^p d\mu.
\]

In the present paper we give an “abstract version” of Lieb’s approach. The results are even more general. For instance, in our Theorem 2.1 the semigroup \( e^{-iB} \) is supposed to be positivity preserving but not necessarily Markov. Another important property required in our approach, is \((L_2 \to L_\infty)\)-boundedness or, shortly, \((2, \infty)\)-boundedness of \( e^{-iB} \). The function \( M_B(t) := \| e^{-B} \|_{L_2 \to L_\infty}^2 \) is involved in the estimates in an explicit way. One more result, Theorem 2.4, concerns semigroups, which are dominated by a positivity preserving semigroup. In the most general version of our results, Theorem 2.5, the estimate is given in the terms of the integral kernel of \( e^{-iB} \), rather than in the ones of the corresponding function \( M_B(t) \).

For Markov semigroups, the property of \((2, \infty)\)-boundedness is called ultraretractive. Under the assumptions of Theorem 1.1, ultraretractivity of \( e^{-iB} \) follows from (1.2) due to the remarkable result of Varopoulos (see e.g. [28, Ch.II]), and \( M_B(t) \leq cKt^{-p} \) both at zero and at infinity. So, our theorems contain Theorem 1.1 as a special case. Their formulations are much more flexible. In particular we do not restrict ourselves to “purely powerlike” functions \( M_B(t) \): in some applications \( M_B(t) \) behaves as \( O(t^{-\alpha_0}) \) at zero and as \( O(t^{-\alpha_\infty}) \) at infinity, with different exponents \( \alpha_0 \) and \( \alpha_\infty \), see e.g. Subsections 3.3-3.5.

Lieb’s approach to the proof of (1.1) is based upon the path integrals technique. We do not use this formalism, though our main technical tool, the “suspended Trotter formula” (see Lemma 4.4), imitates path integrals. However, we make no
use of any probabilistic technique. It is the decisive point: exactly this allows us to prove the results in such a general setting. A simple trace class analysis replaces convergence properties in infinite-dimensional integration and the only structure we need for our approach is the one of measure space. Note that in the standard approach, new applications either require existence of some additional structure (for example, the one of Lie group, [10]), or involve more and more advanced theories of path integration, cf. [26] and [8], where further references can be found.

An important for Physics and still unsolved problem, related to (1.1), concerns the sharp value of the constant $C(d)$. The approaches of [23] and [11] lead to enormously excessive values, while the approaches of [18] and [9] give for $C(d)$ a reasonable estimate. The smallest known value of this constant was found by Lieb in [19]. We show that our approach allows one to recover Lieb’s constant without using path integrals. In doing this, we combine approximation arguments, used in two different expositions (in [22] and [26]) of Lieb’s proof.

In principle, possibility of direct application of Trotter’s formula instead of path integrals technique was understood long ago. Say, this was mentioned by Simon in the Introduction to his book [26]. However, it seems that so far there were no attempts to carry out this idea.

The main objective of [17] was an adaptation of Li - Yau’s approach to the widest possible setting. When discussing [17], E.Lieb, L.Saloff-Coste and H.Siedentop suggested to attempt to generalize, in a similar way, Lieb’s approach. This idea was supported by T.Hoffman-Ostenhof. We use this opportunity to express our gratitude to them. We also are indebted to V.Liskevich for useful consultations in the theory of positivity preserving semigroups.

We are also grateful to the Erwin Schrödinger Institute in Vienna. The main results of the paper were obtained in January - February 1997, when the authors enjoyed its hospitality.

2. Main results

2.1. Positivity preserving semigroups. Let $\Omega$ be a space with a $\sigma$-finite measure $\mu$. Generally, $L_p$ will denote the space $L_p(\Omega, \mu)$, $1 \leq p \leq \infty$. Any selfadjoint nonnegative operator $B$ in $L_2$ generates a strongly continuous contractive semigroup $Q(t) = Q_B(t) = e^{-tB}$, $0 \leq t < \infty$. We are interested in positivity preserving or, shortly, positive semigroups. This class is defined by the property $Q(t)u \geq 0$ a.e. for any nonnegative function $u \in L_2$. It is well known that generators of positive semigroups can be characterized in terms of their quadratic forms (the “first Beurling - Deny criterion”, see e.g. [22, Th.XIII.50]). We do not suppose in this paper $Q(t)$ to be a Markov, or sub-Markov semigroup. So, $Q(t)$ is contractive in $L_2$ but not necessarily in $L_p$ with $p \neq 2$.

Another notion we need is the one of $(2, \infty)$-bounded semigroup. We use this term for a selfadjoint contractive semigroup $Q(t) = e^{-tB}$ in $L_2$, which for each $t > 0$ is bounded as acting from $L_2$ to $L_\infty$. Such operators prove to be integral operators, see e.g. [16, Sec.XI.1] or [1, Th.1.3]. If we denote by $Q(t; x, y) = Q_B(t; x, y)$ the
The resulting function is well defined as an element of $L_{\infty}$ which does not depend on the particular choice of $t_1$ and $t_2$. Thus, (2.1) can be rewritten as

\[ M_B(t) := \text{ess sup}_x \int_{\Omega} |Q_B(\frac{t}{2}; x, y)|^2 \, dy = \|e^{-tB}\|_{L_2 \to L_{\infty}}^2 < \infty \]

(for brevity, we write $dy$ instead of $\mu(dy)$). In the theory of Markov semigroups the $(2, \infty)$-boundedness property is called ultracontractivity. In order to avoid any ambiguity, we do not apply this term to semigroups which are only positive.

The kernel $Q(t; x, y)$ is defined almost everywhere on $\Omega \times \Omega$ for any $t > 0$. One can re-define the kernel on a set of measure zero for each $t > 0$ in such a way that it becomes measurable in all variables (see [1, Lemma 2.2]) and symmetric: $Q(t; x, y) = Q(t; y, x)$ a.e. We always suppose this already done.

By duality, $Q(t)$ is also bounded as an operator from $L_1$ to $L_2$. The semigroup property $Q(t_1)Q(t_2) = Q(t_1 + t_2)$ shows that $Q(t)$ acts from $L_1$ to $L_{\infty}$ and is factorized through $L_2$. This makes it possible to define the value of the kernel on the diagonal $y = x$ for almost all $x$:

\[ Q(t; x, x) = \int_{\Omega} Q(t_1; x, y)Q(t_2; y, x) \, dy, \quad t_1, t_2 > 0 \text{ and } t_1 + t_2 = t. \]

The same semigroup property, together with contractivity in $L_2$, imply that $M_B(t)$ is nonincreasing on $(0, \infty)$. We usually require

\[ \int_a^\infty M_B(t) \, dt < \infty, \quad a > 0. \]

We will write $B \in \mathcal{P}$ if the selfadjoint operator $B$ generates the semigroup which is both positive and $(2, \infty)$-bounded. For such $B$, the kernel $Q_B(t; x, y)$ is nonnegative:

\[ Q_B(t; x, y) \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \Omega \times \Omega, \quad Q_B(t; x, x) \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \Omega. \]

If $B \in \mathcal{P}$ and $B \geq \gamma$ with some $\gamma \geq 0$, then for any $r \geq -\gamma$ the operator $B_r = B + r$ also belongs to $\mathcal{P}$. The corresponding semigroup is $Q_B(t) = e^{-rt}Q_B(t)$, thus $M_B(t) = e^{-rt}M_B(t)$. It follows that $M_B(t)$ with $r > -\gamma$ decays at infinity exponentially. Note also that the property of $Q_B(t)$ to be sub-Markov, provided $Q_B(t)$ is, may fail for $r < 0$. 

\[ \text{(2.4)} \]
2.2. The CLR-estimate for generators of positive semigroups. Let $B$ be a nonnegative selfadjoint operator in $L_2$. Suppose that a given measurable function $V \geq 0$ (more rigorously, the operator of multiplication by $V$) is form-bounded with respect to $B$, with a bound smaller than 1. Then the selfadjoint bounded from below operator $B - V$ is defined by the method of quadratic forms. Denote by $N_-(B - V)$ the number of its negative eigenvalues (counting multiplicities), with the usual convention $N_-(B - V) = \infty$ if there is some essential spectrum below zero.

Let $G(z)$ be a function on $[0, \infty)$, polynomially growing at infinity and such that $z^{-1}G(z)$ is integrable at zero. With any such $G$ we associate another function $g = L(G)$:

\begin{equation}
(2.5) \quad g(\lambda) = L(G)(\lambda) := \int_0^\infty z^{-1}G(z)e^{-\frac{\lambda}{z}}dz.
\end{equation}

So, $g(1/\lambda)$ is the Laplace transform of $z^{-1}G(z)$. Evidently, $G \geq 0$ implies that $g$ is positive and monotone.

**Theorem 2.1.** Let $B \in \mathcal{P}$ be such that $M_B(t)$ satisfies (2.4) and $M_B(t) = O(t^{-\alpha})$ at zero, with some $\alpha > 0$. Fix a nonnegative convex function $G$, polynomially growing at infinity and such that $G(z) = 0$ near $z = 0$. Put $g = L(G)$. Then

\begin{equation}
(2.6) \quad N_-(B - V) \leq \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_B(t)G(tV(x))dx,
\end{equation}

as long as the expression on the right-hand side is finite.

**REMARKS.** 1. The finiteness of the last expression guarantees that the operator $B - V$ is bounded from below.

2. It follows from convexity that $G(z)$ grows at infinity at least as $O(z)$. Therefore, the condition (2.4) is necessary in order that the estimate (2.6) be meaningful.

2.3. Positively dominated semigroups. For certain applications the assumption of positivity of $e^{-tB}$, required in Theorem 2.1, is too restrictive. A more general class of semigroups is singled out by the positive domination property.

We say that a semigroup $P(t) = e^{-tA}$ of selfadjoint contractions in $L_2$ is dominated by a positive semigroup $Q(t) = e^{-tB}$, if

\begin{equation}
(2.7) \quad |P(t)\psi| \leq Q(t)|\psi| \quad \text{a.e. on } \Omega, \quad \text{any } \psi \in L_2.
\end{equation}

We also say that $A$ is dominated by $B$ and write $A \in PD(B)$. When using this notation, we always assume (by default) that the semigroup $e^{-tB}$ is positive. We call any $B$ such that $A \in PD(B)$ a *dominator* for $A$. In cases when $B$ need not to be specified, we simply say that $A$ generates a *positively dominated semigroup*.

The following domination criterion was found in [27] and [15].
Proposition 2.2. Let $A$ and $B$ be nonnegative selfadjoint operators in $L_2$, and $b[\ldots]$ be the quadratic form of $B$. Suppose that the semigroup $e^{-tB}$ is positive. Then $A \in \mathcal{P} \mathcal{D}(B)$ if and only if the following two conditions are satisfied:

1. $\psi \in \text{Dom}(A) \implies |\psi| \in \text{Dom}(b) (= \text{Dom}(B^{1/2})).$

2. For all $\psi \in \text{Dom}(A)$ and $0 \leq \phi \in \text{Dom}(b)$ one has

$$\text{Re} \left( (\text{sign } \psi) \phi, A \psi \right)_{L_2} \geq b[\phi, |\psi|].$$

Here $\text{sign } \psi = \psi/|\psi|$ for $\psi \neq 0$ and $\text{sign } 0 = 0$.

In particular, we get (when $\phi = |\psi|$)

$$A \in \mathcal{P} \mathcal{D}(B) \implies \{(A \psi, \psi) \geq b[|\psi|, |\psi|], \text{ any } \psi \in \text{Dom}(A)\}.$$

(2.8)

It is clear that $A_r = A + r \in \mathcal{P} \mathcal{D}(B + r)$, as long as $A \in \mathcal{P} \mathcal{D}(B)$. There is also a less trivial class of transformations of generators, which preserve domination.

A nonnegative continuous function $f$ on $[0, \infty)$ is called absolute monotone if it is $C^\infty$ for $\lambda > 0$ and $(-1)^k f^{(k)}(\lambda) \geq 0$ for each $k \geq 1$. It was proved in [7] (see especially Section 6 there) that for any absolute monotone $f$, the positivity of $e^{-tB}$ implies the one of $e^{-tf(B)}$, and the inclusion $A \in \mathcal{P} \mathcal{D}(B)$ implies $f(A) \in \mathcal{P} \mathcal{D}(f(B))$. It also was shown in [7] that this fact is, in a sense, invertible.

For $0 < \alpha < 1$ the function $\lambda^\alpha$ is absolute monotone, so we have, in particular:

Proposition 2.3. Let the semigroup $e^{-tB}$ be positive, $A \in \mathcal{P} \mathcal{D}(B)$ and $0 < \alpha < 1$. Then $e^{-tB^\alpha}$ is also positive, and $A^\alpha \in \mathcal{P} \mathcal{D}(B^\alpha)$.

If $Q(t)$ is $(2, \infty)$-bounded, (2.7) implies that $P(t)$ is also $(2, \infty)$-bounded and, moreover, $A \in \mathcal{P} \mathcal{D}(B) \implies M_A(t) \leq M_B(t)$. Such semigroup $P(t)$ consists, therefore, of integral operators. Again, after changing on a set of measure zero for any $t$, we get for $P(t)$ a measurable kernel $P(t; x, y)$. The inequality (2.7), defining domination, is now equivalent to

$$|P(t; x, y)| \leq Q(t; x, y) \text{ a.e. on } \mathbb{R}_+ \times \Omega \times \Omega,$$

(2.9)

see [24, Sec.IV.8]. The similar inequality holds on the diagonal $x = y$.

2.4. The CLR estimate for generators of positively dominated semigroups. Let $A$ generate a positively dominated semigroup. Like in the positive case, suppose that multiplication by a given $V \geq 0$ is form-bounded with respect to $A$, with a bound smaller than 1. (Note that this is always the case, provided the above condition is satisfied for a dominator $B$; this is an easy consequence of (2.8).) Then we can define the selfadjoint operator $A - V$ and, as above, consider the quantity $N_-(A - V)$. 


Theorem 2.4. Let $B \in \mathcal{P}$ and $A \in \mathcal{PD}(B)$. Suppose that $M_B(t)$ satisfies (2.4) and $M_B(t) = O(t^{-\alpha})$ at zero, with some $\alpha > 0$. Let $G$ and $g$ be functions as in Theorem 2.1. Then
\[ N_-(A - V) \leq \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_B(t)G(tV(x))dx, \]
as long as the expression on the right-hand side is finite.

Comparing Theorems 2.1 and 2.4, we see that $N_-(A - V)$ and $N_-(B - V)$ have the same estimate. However, the natural conjecture $N_-(A - V) \leq N_-(B - V)$ is, generally, wrong.

Evidently, Theorem 2.1 is a particular case of Theorem 2.4, for $A = B$. In its turn, Theorem 2.4 is an easy consequence of the following, more general but a bit less transparent result.

Theorem 2.5. Let $B \in \mathcal{P}$ and $A \in \mathcal{PD}(B)$. Suppose that $M_B(t) = O(t^{-\alpha})$ at zero, with some $\alpha > 0$. Let $Q(t) = e^{-tB}$ and $G, g$ be functions as in Theorem 2.1. Then
\[ N_-(A - V) \leq \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \lim_{\varepsilon \to 0^+} \sup \int_\Omega Q_B(t + \varepsilon; x, x)G(tV(x))dx, \]
as long as the expression on the right-hand side is finite.

REMARK. The condition
\[ \int_0^\infty Q_B(t; x, x)dt < \infty \ a.e. \ on \ \Omega \]
is necessary in order that the estimate (2.10) to be meaningful. Of course, (2.11) follows from (2.4). However, here, we do not assume (2.4) to be satisfied.

The proof of Theorem 2.5 is given in Sections 4 and 5.

3. Some applications

3.1. Schrödinger operator. Let $\Omega = \mathbb{R}^d$ with Lebesgue measure, $B = -\Delta$. The kernel of $e^{-tB}$ equals $Q(t; x, x) = (2\pi)^{-d/2}t^{-d/2}$ on the diagonal. So, (2.4) (or (2.11)) dictates $d \geq 3$. Following Lieb [19] (see also [26, p. 96]), select $G(z) = (z - a)_+$, with some positive constant $a$ to be chosen. Then (2.6) gives
\[ N_-(\Delta - V) \leq (2\pi)^{-d/2} g(1)^{-1} \int_0^\infty \int_{\mathbb{R}^d} t^{-d/2-1} G(tV(x))dxdt, \]
where, according to (2.5), $g(1) = \int_\mathbb{R}^d (1 - az^{-1})e^{-z}dz$. After the change of variables we come to
\[ N_-(\Delta - V) \leq C(G) \int_{\mathbb{R}^d} V(x)^{d/2}dx, \]
where $C(G) = (2\pi)^{-d/2} g(1)^{-1} \int_0^\infty (t - a)t^{-d/2-1}dt$. Then one should optimize in $a$. For $d = 3$, it turns out that the best constant in (3.1) is obtained when $a = .25$ which gives $C(G) = .1156$, the constant found by Lieb. It is explained in [6] that such choice of $G$ is optimal.
3.2. Magnetic Schrödinger operator. For a given magnetic vector potential \( a(x) = \{a_j(x)\}_{1 \leq j \leq d} \in L^2_{\text{loc}}(\mathbb{R}^d) \), consider the magnetic Schrödinger operator \( A = H_a = -(\nabla - i a)^2 \). This operator was studied, e.g. in [2] and in [26], and it was shown that \( H_a \in \mathcal{P} \mathcal{D}(-\Delta) \). Theorem 2.4 gives

\[
N_-(H_a - V) \leq C(G) \int_{\mathbb{R}^d} V(x)^{d/2} dx, \quad d \geq 3,
\]

with the same constant as in (3.1). The proof of the magnetic CLR-estimate (3.2), outlined in [26, p.168], uses the Itô stochastic integration. A more elementary proof in [21] gives a somewhat worse constant than in the non-magnetic case.

3.3. The relativistic Schrödinger operator. In the space \( L^2_{\text{loc}}(\mathbb{R}^d) \), \( d \geq 3 \), consider the operator \( B = H_R = (-\Delta + 1)^{1/2} - 1 \). This is a pseudodifferential operator of order 1, selfadjoint and nonnegative. It was studied, e.g., in [8], [20] and [29]. By Proposition 2.3, the semigroup \( Q(t) = e^{-tB} \) is positive. Its \((2, \infty)\)-boundedness is implied by the explicit representation of the kernel on the diagonal

\[
Q(t; x, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t((\xi^2 + 1)^{1/2} - 1)} d\xi.
\]

It follows from (3.3) that (2.2) and (2.3) hold, with \( M_B(t) \leq C(t^{-d/2} + t^{-d}) \). Thus, Theorem 2.1 applies, with the same \( G \) as above. This leads to the estimate, first stated in [12]:

\[
N_-(H_R - V) \leq C \left( C_{G,d} \int_{\mathbb{R}^d} V(x)^d dx + C_{G,d/2} \int_{\mathbb{R}^d} V(x)^{d/2} dx \right), \quad d \geq 3,
\]

where \( C_{G,a} = g(1)^{-1} \int_0^\infty t^{-a-1} G(t) dt \). We do not try to optimize \( G \) here.

3.4 The relativistic magnetic Schrödinger operator. Consider the operator \( A = H_{a,R} = (H_a + 1)^{1/2} - 1 \), where \( H_a \) is the magnetic Schrödinger operator as in Subsection 3.2. Due to Proposition 2.3, the semigroup \( e^{-tH_{a,R}} \) is dominated by \( e^{-tH_R} \). Now, Theorem 2.4 gives for \( N_-(H_{a,R} - V) \) the same estimate (3.4) as for the non-relativistic magnetic case. Such estimate seems to be new.

3.5. Sublaplacian on a nilpotent Lie group. The material on Lie groups we need here, including the estimate of the heat kernel, is contained in [28, Ch.IV].

Let \( \mathcal{X} = \{X_1, \ldots, X_k\} \) be a Hörmander system of left invariant vector fields on a nilpotent Lie group \( \mathcal{G} \). With any such a system, two integers, \( d \) and \( D \), are associated in a standard way: \( d \) is the local dimension of the couple \((\mathcal{X}, \mathcal{G})\) and \( D \) is the dimension of \( \mathcal{G} \) at infinity. The operator \( B_\mathcal{X} = -\sum_{j=1}^k X_j^2 \) is called Sublaplacian, associated with the system \( \mathcal{X} \). It generates a Markov ultracontractive semigroup, for which \( M_{B_\mathcal{X}}(t) = O(t^{-d/2}) \) at zero and \( M_{B_\mathcal{X}}(t) = O(t^{-D/2}) \) at infinity.
So Theorem 2.1 applies to the operator $B_X - V$, provided $D \geq 3$. It shows that

$$
(3.5) \quad N_-(B_X - V) \leq C \left( \int_{\mathbb{R}^d} V(x)^{d/2} \, dx + \int_{\mathbb{R}^d} V(x)^{D/2} \, dx \right)
$$

Some estimates of $N_-(B_X - V)$ were obtained in [10] and in [17], but only for $d \leq D$. In this case Theorem 1.1 applies for any $q \in [d, D]$, and the estimate (3.5) can be considerably improved, see [17, Th.3.8]. The methods of [10] and [17] do not apply if $d > D$.

4. Technical considerations

We preclude the proof of Theorem 2.5 by some assertions of a technical character. In this section we always suppose $0 \leq V \in L_1 \cap L_\infty$.

4.1. Perturbation of a generator.

**Lemma 4.1.** Let $A \in \mathcal{PD}(B)$. Then also $A + V \in \mathcal{PD}(B)$.

The fact directly follows from the criterion of domination (Proposition 2.2).

4.2. Estimates in Schatten classes. By $\mathfrak{S}_p$ we denote the Schatten classes of compact operators (see, e.g. [14, Ch.III] or [25, Ch.2]). In particular, $\mathfrak{S}_1$ is the trace class, $\mathfrak{S}_2$ is the Hilbert-Schmidt class.

**Lemma 4.2.** Let $A \in \mathcal{PD}(B)$. Then for any $p \geq 1$

$$
(4.1) \quad \left\| V^{1/2} e^{-tA} \right\|_{\mathfrak{S}_{2p}}, \left\| e^{-tA} V^{1/2} \right\|_{\mathfrak{S}_{2p}} \leq M_B(2t)^{1/p} \left\| V^{1/2} \right\|_{L_{2p}}
$$

and

$$
(4.2) \quad \left\| V^{1/2} e^{-tA} V^{1/2} \right\|_{\mathfrak{S}_p} \leq M_B(t)^{1/p} \left\| V \right\|_{L_p}.
$$

Similar estimates are valid with $e^{-t(A+V)}$ instead of $e^{-tA}$.

**Proof.** Clearly (4.2) is a direct consequence of (4.1). Further, (4.1) for $p = 1$ is a consequence of (2.1): one simply writes the expression for the norm in $\mathfrak{S}_2$. The estimate (4.1) in the operator norm (we take formally $p = \infty$) follows from $V \in L_\infty$ and contractivity of $e^{-tA}$ in $L_2$. The rest is just interpolation. For the perturbed semigroup the same estimates hold due to Lemma 4.1. 

In the next statement we we estimate an integral, involving a special linear combination of the semigroups $e^{-t(A+\gamma V)}$, $\gamma_j \geq 0$. Such linear combinations (with $A$ replaced by $A_r$ with $r > 0$) will play the crucial role in the proof of Theorem 2.5. Here we require $M_B(t)$ (where $B$ is a dominator for $A$) to satisfy (2.4). Recall that this is always true for $M_B(t)$ with $r > 0$.
Lemma 4.3. Let operators $A$ and $B$ satisfy the assumptions of Theorem 2.4, and $J \geq N > \alpha - 1$. Suppose that the exponents $\gamma_j \geq 0$ and the coefficients $c_j$, $j = 1, \ldots, J$ are chosen so that

$$\sum_{j=1}^J c_j \gamma_j^k = 0 \quad k = 0, \ldots, N - 1.$$  \hfill (4.3)

Then

$$\int_0^\infty \left\| \sum_j c_j V \frac{1}{2} e^{-t(A+\gamma_j V)} V^{\frac{1}{2}} \right\|_{\mathcal{E}_1} \, dt < \infty. \hfill (4.4)$$

Proof. Termwise convergence at infinity follows from (2.4) and (4.2) for $p = 1$. Thus, we have to justify only convergence at $t = 0$.

Introduce the operator-valued functions

$$X_k(t) = \sum_j c_j \gamma_j^k e^{-t(A+\gamma_j V)}, \quad k \geq 0,$$

so $\|V^{\frac{1}{2}} X_0(t) V^{\frac{1}{2}}\|_{\mathcal{E}_1}$ is the integrand in (4.4). All $X_k$ are bounded uniformly in operator norm. They also are strongly differentiable and

$$X'_k(t) = -AX_k(t) - VX_{k+1}(t).$$

Besides, in view of (4.3), $X_k(0) = 0$ for $k = 0, \ldots, N - 1$. It follows

$$X_k(t) = -\int_0^t e^{-(t-s)A} V X_{k+1}(s) \, ds.$$ 

Iterating this equality, we get

$$X_0(t) = (-1)^N \int_0^t \int_0^{t_1} \cdots \int_0^{t_{N-1}} e^{-(t-t_1)A} V e^{-(t_1-t_2)A} V \cdots$$

$$\times V e^{-(t_{N-1}-t_N)A} V X_N(t_N) dt_1 \ldots dt_N.$$ 

Therefore,

$$\|V^{\frac{1}{2}} X_0(t) V^{\frac{1}{2}}\|_{\mathcal{E}_1} \leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{N-1}} \|V^{\frac{1}{2}} e^{-(t-t_1)A} V^{\frac{1}{2}}\|_{\mathcal{E}_N} \cdots$$

$$\times \|V^{\frac{1}{2}} e^{-(t_{N-1}-t_N)A} V^{\frac{1}{2}}\|_{\mathcal{E}_N} \|V^{\frac{1}{2}} X_N(t_N) V^{\frac{1}{2}}\|_{\mathcal{E}_N} dt_1 \ldots dt_N.$$ 

By (4.2), the $k$-th factor of the integrand does not exceed $(t_{k-1} - t_k)^{-\frac{N}{2}} \|V\|_{L_N}$ (we take $t_0 = t$ and $t_{N+1} = 0$). This gives $\|V^{\frac{1}{2}} X_0(t) V^{\frac{1}{2}}\|_{\mathcal{E}_1} \leq Ct^{-(\alpha - N)} \|V\|_{L_N}^N$ and the desired result follows. \hfill $\blacksquare$
4.3. On the Trotter formula. Let $A$ and $W$ be two nonnegative selfadjoint operators in a Hilbert space $\mathcal{H}$, $W$ being bounded. Given an element $\psi \in \mathcal{H}$, denote $\psi(s) = e^{-s(A+W)}\psi$. It follows from the $C_0$-property and contractivity of $e^{-s(A+W)}$ that $\psi(s)$ is uniformly continuous on $[0, \infty)$.

A short proof of the Trotter formula is presented in [26, p.5]. In particular, it is shown there that for any $t > 0$

$$
\|e^{-t(A+W)}\psi - \left( e^{-\frac{tA}{n}} e^{-\frac{tW}{n}} \right)^n \psi \| 
\leq n \sup_{0 \leq s \leq t} \| \left( e^{-\frac{t}{n}(A+W)} - e^{-\frac{tA}{n}} e^{-\frac{tW}{n}} \right) \psi(s) \|.
$$

(4.5)

It is then shown that the right-hand side of (4.5) tends to zero as $n \to \infty$. As a result, we see that (for $\psi$ fixed)

$$
\lim_{n \to \infty} \left( e^{-\frac{tA}{n}} e^{-\frac{tW}{n}} \right)^n \psi = e^{-t(A+W)} \psi
$$

uniformly in $t \in [0, t_0]$ for any finite $t_0$.

Consider now the sequence of piecewise-constant functions

$$
\psi_n(s) = \left( e^{-\frac{A}{n}} e^{-\frac{W}{n}} \right)^l \psi \quad \text{for} \quad \frac{l-1}{n} < s \leq \frac{l}{n}, \ l = 1, \ldots, n.
$$

Then $\psi_n \to \psi$ uniformly on $[\delta, 1]$ for any $\delta > 0$. Indeed, for $t = l/n$ one has

$$
\psi_n(l/n) - \psi(l/n) = \left( e^{-\frac{A}{n}} e^{-\frac{W}{n}} \right)^l \psi - \psi(t).
$$

(4.7)

Now, suppose that the difference between two terms of (4.6) is less than $\varepsilon$ for $0 \leq t \leq 1$ and $n > n_\varepsilon$. If $n > \delta^{-1} n_\varepsilon$ and $l/n > \delta$, then $l > n_\varepsilon$ and therefore the norm of the right-hand side of (4.7) is smaller than $\varepsilon$. The estimate (with, say, $2\varepsilon$ instead of $\varepsilon$) remains valid for any $s \in [\delta, 1]$, due to the uniform continuity of $\psi(s)$.

The same is true if in the definition of $\psi_n$ we take $e^{-\frac{W}{n}} e^{-\frac{A}{n}}$ instead of $e^{-\frac{A}{n}} e^{-\frac{W}{n}}$.

The next statement can be called “suspended version” of the classical Trotter formula. It is an immediate consequence of the above argument.

**Lemma 4.4.** For any $t > 0$ and any $\delta > 0$ one has

$$
\int_{\delta t}^{(1-\delta)t} e^{-s(A+W)}W e^{-(t-s)(A+W)} ds

= (w) \lim_{n \to \infty} \frac{t}{n} \sum_{\delta < \frac{l}{n} \leq 1-\delta} \left( e^{-\frac{tA}{n}} e^{-\frac{tW}{n}} \right)^l W \left( e^{-\frac{tA}{n}} e^{-\frac{tW}{n}} \right)^{n-l}.
$$
4.4. The Birman - Schwenig principle. Let $A$ satisfy the assumptions of Theorem 2.5 and $0 \leq V \in L_1 \cap L_\infty$. For $r > 0$, denote
\[ K_r(V) = V^{\frac{1}{2}} A r^{-1} V^{\frac{1}{2}} = V^{\frac{1}{2}} (A + r)^{-1} V^{\frac{1}{2}}. \]

Let us show that $K_r(V)$ is compact and, moreover, $K_r(V) \in \mathcal{B}_p$ for $p > \alpha$. Indeed, $K_r(V)$ can be represented as $K_r(V) = \int_0^\infty V^{\frac{1}{2}} e^{-tA} V^{\frac{1}{2}} dt$. According to (4.2),
\[ \|K_r(V)\|_{\mathcal{B}_p} \leq \int_0^\infty \|V^{\frac{1}{2}} e^{-tA} V^{\frac{1}{2}}\|_{\mathcal{B}_p} dt \leq \|V\|_{L_p} \int_0^\infty M_B(t)^{1/p} dt. \]

The last integral is finite, therefore $K_r(V) \in \mathcal{B}_p$.

For a compact selfadjoint operator $K \geq 0$ with the eigenvalues $\lambda_k$, and a given number $\lambda > 0$, we denote $n(\lambda, K) = \# \{k : \lambda_k > \lambda\}$. The Birman-Schwenig principle, in its simplest form, consists in the equality
\[ (4.8) \quad N_-(A_r - V) = n(1, K_r(V)), \quad r > 0. \]

In this paper we do not use the more subtle version of (4.8) for $r = 0$, see [3].

Letting in (4.8) $r \to 0$, we come to
\[ (4.9) \quad N_-(A - V) = \lim_{r \to 0+} n(1, K_r(V)). \]

To estimate the right-hand sides in (4.8) and (4.9), take a function $g(\lambda)$ defined on $\mathbb{R}_+$, continuous, nonnegative and nondecreasing. The inequality
\[ (4.10) \quad n(1, K_r(V)) \leq g(1)^{-1} \text{Tr} \, g(K_r(V)) \]
holds as long as the right-hand side of (4.10) is finite. The trace in (4.10) depends on $g$ monotonically. Moreover, if a sequence of monotone continuous functions $g_m \geq 0$ converges to $g$ pointwise and monotonically, then
\[ \lim_{m \to \infty} \text{Tr} \, g_m(K_r(V)) = \text{Tr} \, g(K_r(V)). \]

4.5. An algebra of functions on the semiaxis. Denote by $\mathcal{F}$ the set of all finite linear combinations of the exponents $e^{-\gamma z}$ with $\gamma \geq 0$:
\[ (4.11) \quad F(z) = \sum_{j=1}^J c_j e^{-\gamma_j z}, \quad \gamma_j \geq 0. \]

We always assume that in the representation (4.11) the exponents $\gamma_j$ are mutually different. Evidently $\mathcal{F}$ is an algebra with respect to the usual operations on functions. Given $F \in \mathcal{F}$, put $G(z) = z F(z)$ and (cf. (2.5))
\[ (4.12) \quad g(\lambda) = \mathcal{L}(G)(\lambda) = \int_0^\infty F(z) e^{-z/\lambda} dz, \quad \lambda \geq 0. \]
In other words,

\[(4.13) \quad g(\lambda) = \lambda \sum_{j=1}^{J} c_j (1 + \gamma_j \lambda)^{-1}, \lambda \geq 0.\]

For a given \(N\), denote by \(\mathcal{F}_N\) the ideal in \(\mathcal{F}\), generated by the function

\[(4.14) \quad F_N(z) = (1 - e^{-z})^N \in \mathcal{F}.\]

For \(F \in \mathcal{F}_N\), there are no less than \(N+1\) terms in (4.11) and, since \(F(z) = O(z^N)\) at \(z = 0\), the equalities (4.3) are satisfied for the coefficients in (4.11). It follows from (4.13) and (4.3) that \(F \in \mathcal{F}_N\) yields \(g(\lambda) = O(\lambda^{N+1})\) at \(\lambda = 0\). In particular, this applies to the function \(g_N := \mathcal{L}(z F_N(z))\). Note that \(g_N\) is positive and monotone on \(\mathbb{R}_+\). According to (4.12), for a function \(F \in \mathcal{F}_N\) the inequality

\[(4.15) \quad |F(z)| \leq K F_N(z)\]

implies

\[|g(\lambda)| \leq K g_N(\lambda), \lambda \geq 0, \quad g = \mathcal{L}(z F(z)).\]

The best constant \(K\) in (4.15) will be called the norm of \(F \in \mathcal{F}_N\). This notion will also be used for any function \(F\) on \(\mathbb{R}_+\), satisfying (4.15).

**Lemma 4.5.** Any function \(0 \leq F \in C[0, \infty)\), having a finite limit as \(z \to \infty\) and such that \(F(z) = o(z^N)\) at zero, can be approximated in \(F_N\)-norm by nonnegative functions from \(\mathcal{F}_N\).

**Proof.** We can apply the Stone-Weierstrass theorem (its version for locally compact spaces) to the algebra \(\mathcal{F}\) and find an approximation of \(F/F_N\), uniform on \([0, \infty)\), by nonnegative functions \(F_m \in \mathcal{F}\). Then the functions \(F_N F_m \in \mathcal{F}_N\) approximate \(F\) in \(\mathcal{F}_N\). ●

**4.6. Trace class convergence.** When proving Theorem 2.5, we will employ some regularization procedures in order to improve convergence. Here we need the following elementary statement, cf. [14, Th. III.6.3].

**Lemma 4.6.** 1°. Let \(T_n\) be a sequence of bounded operators in a Hilbert space, converging weakly to an operator \(T\). If \(R_1, R_2 \in \mathcal{S}_2\), then \(R_1 T_n R_2 \to R_1 T R_2\) in \(\mathcal{S}_1\).

2°. Let \(T \in \mathcal{S}_1\), and \(\{R_{1,n}\}, \{R_{2,n}\}\) be two sequences of bounded operators, converging to \(T\) strongly. Then \(R_{1,n} T R_{2,n} \to T\) in \(\mathcal{S}_1\).

**Proof.** 1°. If \(R_1, R_2\) have rank one, \(R_j = (., f_j) g_j\), then

\[R_1 T_n R_2 = (T_n g_2, f_1)(., f_2) g_1 \to (T g_2, f_1)(., f_2) g_1 = R_1 T R_2.\]

This coincides with the convergence in \(\mathcal{S}_1\) since the operators involved are of rank one. The result extends to \(R_j\) of any finite rank and then, by the standard approximation procedure, to the general case.
If \( T = (., f)g \), then \( R_{1,n}TR_{2,n}^* = (., R_{2,n}f)R_{1,n}g \to (., f)g = T \). Again, this coincides with the convergence in \( \mathcal{E}_1 \) since the operators involved are of rank one. The proof completes as in 1°. 

Note that the assertion 2° becomes wrong if we, instead of \( R_{2,n}^* \), multiply by \( R_{2,n} \).

5. **Proof of Theorem 2.5**

5.1. **Trace estimate.** The next proposition gives a spectral estimate in the terms imitating path integrals. However, the latter do not appear formally. The proposition will serve as a basis for obtaining estimates in “finite terms”. Below \( Q_r \) and \( P_r \) denote the integral kernels of the semigroups \( e^{-tB_r} \) and \( e^{-tA_r} \).

**Proposition 5.1.** Let \( A \) and \( B \) meet the conditions of Theorem 2.5. Let \( N > \alpha - 1 \). Fix a function \( 0 \leq F \in \mathcal{F}_N \), put \( G(z) = zF(z) \) and define \( g \) by (4.12). For a given \( 0 \leq V \in L_1 \cap L_\infty \) and \( r > 0 \), introduce the quantity

\[
L_r = L_r(B, V, F) = \int_0^\infty \frac{dt}{t} \limsup_{\varepsilon \to 0^+} \limsup_{n \to \infty} \int_{\Omega \times \cdots \times \Omega} Q_r(\varepsilon; x_n, x_0) \\
\times \prod_{k=1}^{n} Q_r\left(\frac{t}{n}; x_{k-1}, x_k\right) G\left(\frac{t}{n} \sum_{\nu=1}^{n} V(x_{\nu})\right) dx_0 \cdots dx_n.
\]

Then

\[
\text{Tr } g(K_r(V)) \leq L_r(B, V, F).
\]

The proof of Proposition 5.1 will be given in the end of this section. Before this, we show how to derive Theorem 2.5 from it. This will be done in several steps.

5.2. **A rough CLR estimate.**

**Theorem 5.2.** Let \( A \) and \( B \) satisfy the conditions of Theorem 2.5. Let \( N > \alpha - 1 \), \( F_N \) be the function (4.14), \( G_N(z) = zF_N(z) \) and \( g_N = L(G_N) \). Take any convex function \( H(z) \) on \( \mathbb{R}_+ \), coinciding with \( G_N(z) \) for small \( z \) and majorizing \( G_N \). Then

\[
\text{Tr } g_N(K_r) \leq \int_0^{\infty} \frac{dt}{t} \limsup_{\varepsilon \to 0^+} \int_{\Omega} Q_r(t + \varepsilon; x, x) H(tV(x))dx.
\]

**Remark.** Theorem 5.2 already gives spectral estimates, but with the constants which are far from being optimal. The reason is that in (5.3) \( g_N \) is constructed, starting from the function \( G_N \) rather than from its convex majorant \( H \). This, of course, roughens the estimate. Nevertheless, this theorem serves as an important step in the proof of the sharp estimate.
Proof. For \( n, \varepsilon \) and \( t \) fixed, consider the multiple integral in (5.1), with \( G = G_N \). Since \( H(z) \geq G_N(z) \) and \( H \) is convex, we can majorize this integral by

\[
\frac{1}{n} \sum_{\nu=1}^{n} \int_{\Omega} Q_{\nu}(\varepsilon; x_{\nu}, x_0) \prod_{k=1}^{n} Q_{\nu}(\frac{t}{n}; x_{k-1}, x_k) H(tV(x_\nu)) dx_0 \ldots dx_n.
\]

Now, for \( \nu \) fixed, we integrate over all \( x_k \) with \( k \neq \nu \), using the semigroup property. All the resulting integrals are identical and thus (5.4) equals

\[
\int_{\Omega} Q_{\nu}(t + \varepsilon; x, x) H(tV(x)) dx
\]

which does not depend on \( n \). Integration in \( t \) gives (5.3). ●

**Corollary 5.3.** Under the assumptions of Theorem 2.5, both parts in (5.2) depend on \( F \in \mathcal{F}_N \) continuously.

**Proof.** If

\[
|G(z)| \leq \delta G_N(z), \quad \delta > 0,
\]

then the multiple integral in (5.1) is majorized by \( \delta \) times (5.4), which proves the assertion for the right-hand side of (5.2). As for the left-hand side, (5.5) implies \( |g(\lambda)| \leq \delta g_N(\lambda) \), and therefore,

\[
|\text{Tr} g(K_r(V))| \leq \| g(K_r(V)) \| \frac{\text{Tr} g_N(K_r(V))}{\text{Tr} g_N(K_r(V))} \leq \delta \text{Tr} g_N(K_r(V)).
\]

Since, according to Theorem 5.2, \( \text{Tr} g_N(K_r(V)) \) is finite, the last part of (5.6) tends to zero as \( \delta \to 0 \). ●

**REMARK.** The statement for the left-hand side does not follow directly from (5.1) since here we do not suppose positivity of \( F \).

### 5.3. The sharp CLR-estimate.

**Proposition 5.4.** Let \( F(z) \) be a nonnegative continuous function on \( \mathbb{R}_+ \), polynomially growing at infinity and such that \( F(z) = 0 \) near zero. Let \( G(z) = zF(z) \) and \( g \) be defined by (4.12). Then the inequality (5.2) holds (it is possible that both parts are infinite).

**Proof.** Corollary 5.3 implies (5.2) for any \( F \) satisfying the conditions of Lemma 4.5, in particular for \( F \in C_0^{\infty}(\mathbb{R}_+) \). Having a general \( F \), we approximate it by a monotonically growing, pointwise convergent sequence of \( C_0^{\infty} \)-functions \( F_m \geq 0 \). The corresponding functions \( g_m \) also monotonically converge to \( g \). Since both parts of (5.2) respect monotone convergence, we get (5.2) for the general case. ●
Proof of Theorem 2.5. Let first $V \in L_1 \cap L_\infty$. Then we repeat the reasoning of Subsection 5.2, starting this time from (5.2) for the chosen $G$ (and putting $F(z) = z^{-1}G(z)$). This gives

\begin{equation}
(5.7) \quad \text{Tr } g(K_r(V)) \leq \mathcal{R}(V) =: \int_0^\infty \frac{dt}{t} \limsup_{\varepsilon \to 0^+} \int_\Omega Q_r(t + \varepsilon; x, x)G(tV(x))dx.
\end{equation}

After this, everything is standard. On the right-hand side of (5.7) we replace $Q_r$ by $Q$, which is bigger, and thus, due to (4.10), $n(1, K_r(V)) \leq g(1)^{-1}\mathcal{R}(V)$. Then we approximate a given $V$ by a growing sequence $V_m \in L_1 \cap L_\infty$, converging to $V$ a.e. and pass to the limit in (5.7) written for $V_m$. Finally, we let $r \to +0$. •

5.4. Proof of Proposition 5.1. Our function $g$ is of the form (4.13), with the coefficients satisfying (4.3). Since $K_r(V)(1 + gK_r(V))^{-1} = V^{1/2}(A_r + gV)^{-1}V^{1/2}$, we have

\begin{equation}
(5.8) \quad g(K_r(V)) = \sum_{j=1}^J c_j V^{1/2} (A_r + \gamma_j V)^{-1} V^{1/2} = \int_0^\infty \sum_{j=1}^J c_j V^{1/2} e^{-t(A_r + \gamma_j V)} V^{1/2} dt.
\end{equation}

The integral in (5.8) converges strongly. Moreover, for the dominator $B_r$ of $A_r$, the condition (2.4) is always satisfied and therefore, Lemma 4.3 applies. So, the integral in (5.8) converges also in $\mathfrak{S}_1$, provided $N > \alpha - 1$. This implies $g(K_r(V)) \in \mathfrak{S}_1$ and therefore (5.8) leads to

\begin{equation}
(5.9) \quad \text{Tr } g(K_r(V)) = \int_0^\infty \Gamma(t) dt
\end{equation}

where

$$
\Gamma(t) = \sum_{j=1}^J c_j \text{Tr} \left( V^{1/2} e^{-t(A_r + \gamma_j V)} V^{1/2} \right).
$$

We will estimate $\Gamma(t)$ for a fixed $t > 0$. To this end, fix some integer $p > 2$. Then, using the cyclicity of the trace, we get

\begin{equation}
(5.10) \quad \Gamma(t) = \frac{p}{t(p-2)} \sum_{j=0}^p \frac{t(1-p^{-1})}{t_p^{-1}} \int_{t_p^{-1}} \text{Tr} \left( e^{-s(A_r + \gamma_j V)} V^{-t(s)(A_r + \gamma_j V)} \right) ds.
\end{equation}

Here, Lemma 4.2 guarantees that for $t > 0$ fixed, the operators under the trace sign in (5.10) have the $\mathfrak{S}_1$-norms, bounded uniformly in $s$. So we can interchange trace and integration again to get

$$
\Gamma(t) = \text{Tr } W_p(t), \quad W_p(t) \in \mathfrak{S}_1,
$$

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where

\[
W_p(t) = \frac{p}{t(p-2)} \sum_{j=0}^{J} c_j \int_{t^{p-1}}^{t^{(1-p)^{-1}}} e^{-s(A_r + \gamma_j V)} V e^{-(t-s)(A_r + \gamma_j V)} ds.
\]

Although the operator \( W_p(t) \) depends on the choice of \( p \), its trace \( \Gamma(t) \) does not.

Now, choose an expanding (as \( \varepsilon \to 0 \)) family of subsets \( \Omega_\varepsilon \subset \Omega \) of finite measure, such that their union is \( \Omega \). Introduce a family of regularizers \( R_\varepsilon = \chi_\varepsilon e^{-\varepsilon B_r/2} \) where \( \chi_\varepsilon \) is the characteristic function of \( \Omega_\varepsilon \). By Lemma 4.2, \( R_\varepsilon \in \mathcal{S}_2 \) and besides, \( R_\varepsilon \to I \) strongly as \( \varepsilon \to 0 \). By Lemma 4.6(2°) \( R_\varepsilon W_p(t) R_\varepsilon^* \to W_p(t) \) in \( \mathcal{S}_1 \) and therefore,

\[
\Gamma(t) = \text{Tr} W_p(t) = \lim_{\varepsilon \to 0} \text{Tr} R_\varepsilon W_p(t) R_\varepsilon^*.
\]

To evaluate \( \text{Tr} R_\varepsilon W_p(t) R_\varepsilon^* \), we apply Lemma 4.4, setting \( A = A_r \), \( W = \gamma_j V \), and \( \delta = p^{-1} \). It is convenient to take \( n = mp, m \in \mathbb{N} \). The limit in this Lemma exists in weak sense, however, according to Lemma 4.6(1°) after multiplying by \( \mathcal{S}_2 \)-operators from both sides, we get trace class convergence for products. Therefore,

\[
\frac{p-2}{p} \text{Tr} R_\varepsilon W_p(t) R_\varepsilon^* = \lim_{n \to \infty} \text{Tr} \left( \frac{1}{n} \sum_{j=1}^{J} c_j \sum_{l=m+1}^{n-m} R_\varepsilon \left( e^{-\frac{t}{n} A_r} e^{-\frac{t}{n} \gamma_j V} \right)^l V \left( e^{-\frac{t}{n} A_r} e^{-\frac{t}{n} \gamma_j V} \right)^{n-l} R_\varepsilon^* \right).
\]

Write down the trace in the last expression as a multiple integral:

\[
\frac{1}{n} \sum_{j=1}^{J} c_j \sum_{l=m+1}^{n-m} \int_{1_{\Omega} \times \cdots \times \Omega} Q_r \left( \varepsilon \frac{t}{n}; y, x_0 \right) P_r \left( \frac{t}{n}; x_0, x_1 \right) e^{\frac{t}{n} \gamma_j V(x_1)} \cdots \left( \frac{t}{n}; x_{l-1}, x_l \right)
\times e^{\frac{t}{n} \gamma_j V(x_l)} V(x_l) \cdots P_r \left( \frac{t}{n}; x_{n-1}, x_n \right) e^{-\frac{t}{n} \gamma_j V(x_n)} Q_r \left( \frac{\varepsilon}{2}; x_n, y \right) \chi_\varepsilon(y) dydx_0 \cdots dx_n.
\]

Rearranging the terms, we come to

\[
\frac{1}{n} \int_{\Omega \times \cdots \times \Omega} \chi_\varepsilon(y) Q_r \left( \varepsilon \frac{t}{n}; y, x_0 \right) P_r \left( \frac{t}{n}; x_{k-1}, x_k \right)
\times \sum_{j=1}^{J} c_j e^{-\frac{t}{n} \gamma_j \sum_{v=1}^{n} V(x_v)} \sum_{l=m+1}^{n-m} V(x_l) \left( Q_r \left( \frac{\varepsilon}{2}; x_n, y \right) dydx_0 \cdots dx_n.\right.
\]

The sum over \( j \) is equal to \( F \left( \frac{t}{n} \sum_{v=1}^{n} V(x_v) \right) \) and therefore nonnegative. Thus we can majorize the last integral by replacing \( P_r \) by \( Q_r \). After this, all the terms in (5.12) become nonnegative; we make them even bigger by replacing \( \chi_\varepsilon \) by 1 and...
extending summation over \( l \) to all \( l, 1 \leq l \leq n \). Then we integrate over \( y \), using the semigroup property. According to (5.11), we arrive to

\[
\Gamma(t) \leq \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} p \frac{1}{p - 2} \int_{\Omega \times \cdots \times \Omega} Q_t(\varepsilon; x_n, x_0) \prod_{k=1}^{n} Q_t(\frac{t}{n}; x_{k-1}, x_k) \times G \left( \frac{t}{n} \sum_{\nu=1}^{n} V(x_\nu) \right) dx_0 \ldots dx_n,
\]

Finally, since \( p \) was taken arbitrarily and \( \Gamma(t) \) does not depend on \( p \), we can drop the factor \( \frac{p}{p - 2} \) in the last inequality. This, together with (5.9), gives (5.2). ●

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