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Vienna, Preprint ESI 430 (1997)  
March 17, 1997

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at
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March 18, 1997

Abstract

We extend ideas developed for the loop representation of quantum gravity to diffeomorphism-invariant gauge theories coupled to fermions. Let $P \to \Sigma$ be a principal $G$-bundle over space and let $F$ be a vector bundle associated to $P$ whose fiber is a sum of continuous unitary irreducible representations of the compact connected gauge group $G$, each representation appearing together with its dual. We consider theories whose classical configuration space is $\mathcal{A} \times \mathcal{F}$, where $\mathcal{A}$ is the space of connections on $P$ and $\mathcal{F}$ is the space of sections of $F$, regarded as a collection of Grassmann-valued fermionic fields. We construct the ‘quantum configuration space’ $\mathcal{A} \times \mathcal{F}$ as a completion of $\mathcal{A} \times \mathcal{F}$. Using this we construct a Hilbert space $L^2(\mathcal{A} \times \mathcal{F})$ for the quantum theory on which all automorphisms of $P$ act as unitary operators, and determine an explicit ‘spin network basis’ of the subspace $L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G})$ consisting of gauge-invariant states. We represent observables constructed from holonomies of the connection along paths together with fermionic fields and their conjugate momenta as operators on $L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G})$. We also construct a Hilbert space $\mathcal{H}_{\text{diff}}$ of diffeomorphism-invariant states using the group averaging procedure of Ashtekar, Lewandowski, Marolf, Mourão and Thiemann.
1 Introduction

In this paper we address the problem of the quantization of a wide class of diffeomorphism-invariant theories. We consider theories whose degrees of freedom are described in the Hamiltonian framework by a connection on some principal $G$-bundle on space together with a set of fermionic fields on space taking values in representations of the compact connected gauge group $G$. Theories of this class can be thought of as giving a diffeomorphism-invariant description of fermionic matter interacting by means of gauge fields. One of the main motivations for considering such theories is that general relativity can be reformulated as a diffeomorphism-invariant theory having as canonically conjugate variables a connection on space and a densitized triad field [1]. This means that the gravitational force coupled to Yang-Mills gauge fields and fermionic matter fits into our framework.

Our approach builds upon existing work on the loop representation of quantum gravity and other diffeomorphism-invariant theories of connections [3]. The loop quantization of theories of connections is based on an assumption that Wilson loop functionals of connection become well-defined quantum operators. Similarly, our approach is based on an assumption that certain fermionic analogs of Wilson observables become well-defined operators. An important feature of our scheme is that we do not introduce closed Wilson loops as fundamental observables, as typical in the loop representation of a pure gauge theory. Instead, we use open Wilson paths with fermions or their conjugate momenta at the endpoints, which we call ‘fermionic path observables’. Closed loops arise as secondary quantities.

The basic strategy of the paper can be summarized as follows. We introduce a set of path observables on the classical phase space of our theory that forms a closed algebra with respect to Poisson brackets. The main idea then is to construct a ‘kinematical’ Hilbert space for the theory and represent our algebra by operators on this Hilbert space. This is not the space of physical states, because the constraints — gauge-invariance, spatial diffeomorphism-invariance, and the Hamiltonian constraint — have not yet been implemented. The quantization procedure we adopt satisfies some natural requirements; for example, gauge-invariant classical quantities become operators that preserve the gauge-invariant subspace of the kinematical Hilbert space, ‘real’ quantities become self-adjoint operators, and so on. It turns out, however, that representation of our algebra is, in general, reducible even on the subspace of gauge-invariant states.

An important issue in quantizing diffeomorphism-invariant gauge theories coupled to fermions is that of respecting the classical reality conditions on the quantum level. In field theories on Minkowski spacetime the operator $\dagger$ of Hermitian conjugation of fermionic fields sends configuration fermionic fields into their canonically conjugate momenta. However, in diffeomorphism-invariant theories the $\dagger$ operation is more complicated. Consider for example a theory in
which the only gauge field is the gravitational field, described by the connection field $A_a$ canonically conjugate to the densitized triad field \cite{1}. The configuration and momentum fermionic fields have different density weights, and the action of the operator involves the square root of the determinant of the metric:

$$\hat{\sigma}^\dagger = -i\sigma \xi.$$  \hspace{1cm} (1)

Here $\sigma$ is the square root of the determinant of the metric, defined in terms of the densitized triad field.

In the quantum theory, one wants to impose this ‘reality condition’ in an appropriate form. However, in trying to impose the reality condition (1) in a naive way one encounters the following problem. In general relativity the triad field is a dynamical variable. Thus, in the quantum theory the quantity $\sigma$ constructed from the triad field becomes an operator. Naively one might require that the adjoint of the operator $\hat{\sigma}$ corresponding to the fermionic momentum be given by:

$$(\hat{\sigma})^\dagger = -i\sigma \xi.$$  

To illustrate the problems arising from this naive definition, consider the commutator

$$\left[ \int d^n x \hat{\sigma}(x) f(x), \int d^n x \text{Tr}(\hat{A}_a(x) g^a(x)) \right]$$

of operators $\hat{\sigma}$ and $\hat{A}_a$ smeared with test functions. For simplicity of the argument that follows let us consider the case of a real connection $\hat{A}_a$, corresponding to Riemannian general relativity. One might expect this commutator to vanish, since the corresponding Poisson bracket is zero in the classical theory. However, if we apply the $\dagger$ operation to this commutator, we obtain

$$\left[ \int d^n x (-i\hat{\sigma})(x) \xi(x), \int d^n x \text{Tr}(\hat{A}_a(x) g^a(x)) \right]$$

which appears to be nonzero, since the operator $\hat{\sigma}$ does not commute with $\hat{A}_a$. Thus, the $\dagger$ operation defined in this naive way seems to lead to inconsistencies.

Our strategy for dealing with this issue is as follows. One role of the $\dagger$ operation on functions on the classical phase space is to say which of these functions are real. Therefore, a possible way to impose the classical reality conditions on the quantum level is to require that classical real quantities are represented by self-adjoint operators at the quantum level. One then has a satisfactory quantum theory if a sufficient number of physically interesting real observables are represented by self-adjoint operators. The quantization scheme adopted in this paper respects the classical reality conditions in the sense that a certain set of real observables is represented by self-adjoint operators. Note, however, that in this paper only a rather small set of real observables is treated. For instance, the quantization of the Hamiltonian constraint, which is itself a
real functional on the phase space of the theory (in its ‘real’ formulation used in [13]), is not considered in this paper.

The organization of this paper is as follows. In Sec. 2 we start by reviewing the main results of the loop representation approach to quantizing diffeomorphism-invariant theories of connections. In Sec. 3 we specify the class of theories which are of interest for us here and give some examples of theories belonging to this class. In Sec. 4 we describe the path observables which are the basic building blocks of our quantum theory. In Sec. 5 we describe the classical configuration space and a certain completion of it which we call the ‘quantum configuration space’. We then construct the kinematical Hilbert space as a space of functions on the quantum configuration space. In Sec. 6 we construct the Hilbert space of gauge-invariant states, and describe a basis of this space given by ‘open spin networks’ with ends labelled by fermionic fields. In Sec. 7 we define operators representing the path observables. The issue of diffeomorphism-invariant states is discussed in Sec. 8. We conclude with a discussion of the results obtained, focusing on the issue of superselection sectors.

Despite its great importance, we do not discuss dynamics in this paper, as the correct treatment of the Hamiltonian constraint is still a matter of controversy in the loop representation of pure gravity, despite recent progress on this front [13]. We hope, however, that our work will serve as a basis for future work on dynamical aspects of the quantization of diffeomorphism-invariant gauge theories coupled to fermions.

2 Quantization of diffeomorphism-invariant theories of connections

This section reviews the main steps of quantization of diffeomorphism-invariant theories of connections. See references [2, 3, 4] for more details.

In the Hamiltonian formalism, the kinematical phase space of such a theory consists of pairs \((A, \tilde{E})\) satisfying suitable regularity conditions. Here \(A\) is a connection on a principal \(G\)-bundle \(P\) over a manifold \(\Sigma\) which represents ‘space’, and the field \(\tilde{E}\) is the canonically conjugate momentum (the tilde means that it is a densitized vector field). For technical reasons we assume that \(\Sigma\) is real-analytic and \(G\) is a compact connected Lie group. Actually the assumption that \(\Sigma\) is analytic is unnecessary [5]; much of what we do in this paper generalizes to the smooth case, but analyticity makes things a bit simpler.

The classical configuration space of our theory is the space of smooth connections on \(P\), which we denote as \(\mathcal{A}\). In the case of canonical quantization of a theory with a finite number of degrees of freedom one normally represents quantum states by square-integrable functions on the classical configuration space. The Hilbert space of states is then constructed as the space of such functions with the inner product being defined by the usual integral over the configuration space.
space, $\langle \psi | \phi \rangle = \int \overline{\psi} \phi \, d\mu$, where $d\mu$ is some measure on the configuration space. However, in the case of theories with an infinite number of degrees of freedom, which is of interest for us here, there often does not exist an appropriate measure on the classical configuration space. For example, there is no ‘Lebesgue measure’ on an infinite-dimensional vector space. However, one can often complete the classical configuration space in some topology and construct a suitable measure on the resulting ‘quantum configuration space’. In field theories, this larger space usually includes distributional fields in addition to smooth fields. Also, it is common in field theories for the classical configuration space to have zero measure with respect to the measure on the quantum configuration space. The quantum configuration space depends on a choice of a functions on the classical configuration space, which play the role of distinguished observables. In our case we work with (smooth) ‘cylinder functions’ on $A$, that is, functions dependin smoothly on the holonomy of the connection $A$ along finitely many analytic paths. In other words, $\Psi$ is a cylinder function if it is of the form

$$\Psi(A) = \psi(P \exp \int_{\epsilon_1} A, \ldots, P \exp \int_{\epsilon_n} A)$$

for some analytic paths $\epsilon_1, \ldots, \epsilon_n$ in $\Sigma$ and some smooth function $\psi$ on $G^n$. The most well-known functions of this form are the Wilson loops, introduced as observables for quantum gravity by Rovelli and Smolin in their original paper on the loop representation. Unlike the Wilson loops, the above cylinder functions are not necessarily gauge-invariant. This allows them to serve as a complete set of functions on $A$.

One can complete the algebra of cylinder functions with respect to the sup norm to obtain a commutative C*-algebra $\text{Fun}(A)$ for which the $*$-operation is just pointwise complex conjugation. The Gelfand-Naimark theorem then tells us that this C*-algebra is isomorphic to the algebra of all continuous functions on its spectrum. We take the spectrum of $\text{Fun}(A)$ to be the quantum configuration space of our theory. We denote this space by $\overline{A}$, because it contains the classical configuration space $A$ as a dense subset.

Elements of the quantum configuration space are called ‘generalized connections’, and can be described as follows. First, define a ‘transporter’ from the point $p$ to the point $q$ to be a map from $P_p$ to $P_q$ that commutes with the right action of $G$ on the bundle $P$. If we trivialize the bundle over $p$ and $q$, we can think of such a transporter simply as an element of $G$. A ‘generalized connection’ $A$ is a map assigning to each oriented analytic path $\epsilon$ in $\Sigma$ a parallel transporter $A_\epsilon: P_p \to P_q$, where $p$ is the initial point of the path $\epsilon$ and $q$ is the final point. We require that $A$ satisfy certain obvious consistency conditions: $A$ should assign the same transporter to two paths that differ only by an orientation-preserving reparametrization, it should assign to the inverse of any path the inverse transporter, and it should assign to the composite of two paths the composite transporter.
An ordinary smooth connection $A$ gives a generalized connection where the parallel transporter $A_e$ along any path $e$ is simply the holonomy of $A$ along this path, so $A \subseteq \mathcal{A}$. Also, any cylinder function extends to a continuous function on $\mathcal{A}$ by setting

$$\Psi(A) = \psi(A_{e_1}, \ldots, A_{e_n})$$

for any generalized connection $A$. This lets us think of cylinder functions as functions on the quantum configuration space.

There is a natural measure $\mu_0$ on $\mathcal{A}$, which comes from Haar measure on the group $G$. One can show that any reasonable measure on $\mathcal{A}$ is determined by the values of the integrals of all cylinder functions. Thus, to define the measure $\mu_0$ we specify the values of the integrals $\int \Psi d\mu_0$ for all cylinder functions $\Psi$ given as above. Such an integral is defined as the integral over the corresponding copies of the group $G$:

$$\int \Psi d\mu_0 = \int \psi(g_1, \ldots, g_n) \, dg_1 \cdots dg_n$$

where $dg$ denotes the normalized Haar measure on $G$. The measure $\mu_0$ has three important properties. First, it is gauge-invariant. More precisely, note that any gauge transformation $g$ acts on $A \in \mathcal{A}$ to give a generalized connection $A'$ with $A'_e = g(q)^{-1} \circ A_e \circ g(p)$, where $p, q$ are the final and the initial point of the path $e$, respectively. This gives an action of the group $G$ of gauge transformations of $P$ on the space $\mathcal{A}$, and this action preserves the measure $\mu_0$. Second, $\mu_0$ is diffeomorphism-invariant. More precisely, $\mu_0$ is invariant under all automorphisms of the bundle $P$, not necessarily acting as the identity on the base space $\Sigma$. Third, $\mu_0$ is strictly positive, meaning that $\int \Psi d\mu_0 > 0$ for any nonnegative integrable function on $\mathcal{A}$ except the function $0$.

We define the kinematical Hilbert space $L^2(\mathcal{A})$ to be the space of functions on $\mathcal{A}$ that are square-integrable with respect to the measure $d\mu_0$. This space is not the physical state space, since it contains states that are not invariant under the 'gauge' symmetries of our theory, that is, gauge transformations and diffeomorphisms of spacetime. One way to try to find physical states is to look for solutions of quantum constraints in $L^2(\mathcal{A})$. In general, the solutions may not live in $L^2(\mathcal{A})$, but in some completion thereof, but for the Gauss law constraint this problem does not occur: there is a large subspace of $L^2(\mathcal{A})$, the 'gauge-invariant Hilbert space', consisting of gauge-invariant square-integrable functions on $\mathcal{A}$. Alternatively, since the measure $\mu_0$ is gauge-invariant, it gives rise to a well-defined measure on the space $\mathcal{A}/G$ of generalized connections modulo gauge transformations. Then we may equivalently define the gauge-invariant Hilbert space to be the space $L^2(\mathcal{A}/G)$ of square-integrable functions on $\mathcal{A}/G$.

We can construct an explicit basis of $L^2(\mathcal{A}/G)$ using 'spin networks' [4, 12]. A spin network is a triple $(\gamma, \rho, i)$ consisting of:

1. a graph $\gamma$ analytically embedded in $\Sigma$,
(2) a labelling $\rho$ of each edge $e$ of $\gamma$ with an irreducible representation $\rho_e$ of $G$,
(3) a labelling $\iota$ of each vertex $v$ of $\gamma$ with an intertwining operator $\iota_v$.

Here the edges of the graph $\gamma$ are assumed to be unparametrized but oriented, and $\iota_v$ is an intertwining operator from the tensor product of the representations corresponding to the incoming edges at the vertex $v$ to the tensor product of the representations labeling the outgoing edges. (For more details see references [2, 4]. Here we allow graphs with isolated vertices, i.e., vertices that are not vertices of any edge. This is unimportant now but will be important when we come to fermions.) Without loss of generality, spin networks $\Gamma$ are always assumed to satisfy a fourth non-degeneracy condition:

(4) All representations $\rho_e$ are nontrivial and $\gamma$ is a ‘minimal’ graph, in the sense that it cannot be obtained from another graph $\gamma'$ by subdividing edges of $\gamma'$.

The ‘spin network state’ $\Psi_\Gamma$ is a gauge-invariant cylinder function on $\mathcal{A}$ constructed from the spin network $\Gamma$ as follows:

$$\Psi_\Gamma(A) = [\bigotimes_e \rho_e(A_e)] \cdot [\bigotimes_v \iota_v],$$

where ‘$\cdot$’ stands for contracting, at each vertex $v$ of $\gamma$, the upper indices of the matrices corresponding to the incoming edges, the lower indices of the matrices assigned to the outgoing edges, and the corresponding indices of the intertwining operator $\iota_v$. Being gauge invariant, such states lie in $L^2(\mathcal{A}/G)$. We obtain an orthonormal basis of $L^2(\mathcal{A}/G)$ if we use spin network states corresponding to all possible choices of $\gamma$ and $\rho$ and a choice of an orthonormal basis of intertwining operators $\iota_v$ for each vertex $v$ and each choice of representations labeling incident edges.

The states in $L^2(\mathcal{A}/G)$ are still not invariant under the action of diffeomorphisms of $\Sigma$. There is a natural unitary representation of $\text{Diff}(\Sigma)$ on $L^2(\mathcal{A}/G)$. For instance, the action of the operator $U(\phi)$ corresponding to a diffeomorphism $\phi$ of $\Sigma$ on any spin network state is given by

$$U(\phi) \Psi_{(\gamma, \rho, \iota)} = \Psi_{(\phi\gamma, \phi\rho, \phi\iota)},$$

Here $\phi\gamma$ is the image of the graph $\gamma$ under the diffeomorphism $\phi$, and $\phi\rho, \phi\iota$ are the corresponding representations and intertwining operators associated with the new graph $\phi\gamma$.

To find diffeomorphism-invariant states one has to impose the quantum diffeomorphism constraint. There are very few diffeomorphism-invariant states in $L^2(\mathcal{A}/G)$, so here we have to follow the strategy of looking for solutions in some larger space. One often needs to do this when solving quantized constraint equations, and the larger space is usually chosen to be a space of linear functionals on some dense subspace of the initial Hilbert space. That is, we should
choose some dense subspace $C$ of the gauge-invariant Hilbert space $L^2(\mathcal{A}/G)$, equip it with some topology in which it is complete, and look for solutions in the topological dual $C^\ast$. Note that $C \subseteq L^2(\mathcal{A}/G) \subseteq C^\ast$.

Since there is a simple geometrical action of diffeomorphisms on cylinder functions, it is natural to choose the subspace of gauge-invariant cylinder functions as $C$. This has a natural topology in which it is complete, the inductive limit topology. To solve the diffeomorphism constraint, we then seek diffeomorphism-invariant vectors in $C^\ast$.

We obtain these by averaging gauge-invariant cylinder functions over the action of the diffeomorphism group, following the procedure of Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [3]. It is easiest to average a special sort of gauge-invariant cylinder function, namely a spin network state $\Psi_\Gamma$, where $\Gamma = (\gamma, \rho, i)$ is some spin network. For technical reasons we assume $\gamma$ is 'type I', i.e., that for every edge of $\gamma$ there is an analytic function on $\Sigma$ that vanishes only on the maximal analytic curve extending that edge. Let $S(\gamma)$ be any set of diffeomorphisms of $\Sigma$ with the following property: for any graph $\gamma'$ which equals $\phi \gamma$ for some $\phi \in \text{Diff}(\Sigma)$, there is a unique diffeomorphism $\phi' \in S(\gamma)$ with $\gamma' = \phi' \gamma$. Also let $GS(\gamma)$ be the group of 'graph symmetries' of $\gamma$, that is, the group $\text{Iso}(\gamma)/\text{TA}(\gamma)$, where $\text{Iso}(\gamma)$ is the group of diffeomorphisms mapping $\gamma$ to itself, and $\text{TA}(\gamma)$ is the subgroup fixing each edge of $\gamma$. We may define an element $\overline{\Psi}_\Gamma \in C^\ast$ by:

\[ \overline{\Psi}_\Gamma(\Phi) = \sum_{\phi_1 \in S(\gamma)} \sum_{[\phi_2] \in GS(\gamma)} \langle \Psi_{(\phi_1 \circ \phi_2)\Gamma} | \Phi \rangle, \]

where $\Phi \in C$, and where we choose one representative $\phi_2$ for each equivalence class $[\phi_2] \in GS(\gamma)$. It is easy to check that $\overline{\Psi}_\Gamma$ is diffeomorphism-invariant. More generally, suppose $\Psi$ is any gauge-invariant cylinder function. We may write $\Psi$ as a (possibly infinite) linear combination of spin network states $\sum_\Gamma a_\Gamma \Psi_\Gamma$. Then we define $\overline{\Psi} \in C^\ast$ by:

\[ \overline{\Psi}(\Phi) = \sum_\Gamma a_\Gamma \overline{\Psi}_\Gamma(\Phi). \]

It is easy to see that the sum converges and defines a diffeomorphism-invariant element $\overline{\Psi} \in C^\ast$.

We may define the inner product of diffeomorphism-invariant vectors of this form by

\[ \langle \overline{\Psi} | \overline{\Phi} \rangle = \overline{\Psi}(\Phi). \]

Completing this space of such vectors in this inner product we thus obtain a Hilbert space, the 'diffeomorphism-invariant Hilbert space' $\mathcal{H}_{\text{dif}}$. The diffeomorphism-invariant spin network states $\overline{\Psi}_\Gamma$ form an orthogonal (but not orthonormal) basis of $\mathcal{H}_{\text{dif}}$ as we let $\Gamma$ range over a set of spin networks containing one from each diffeomorphism equivalence class. For any spin network $\Gamma$, the
diffeomorphism-invariant state \( \mathfrak{H}_F \) takes a zero value on spin networks that do not belong to the same equivalence class, and a nonzero value on spin networks from the same equivalence class.

In short, the main steps of the quantization procedure include: (i) construction of the kinematical Hilbert space of states \( L^2(\mathcal{A}) \), which requires the specification of a quantum configuration space together with an appropriate measure on this space; (ii) construction of the gauge-invariant Hilbert space \( L^2(\mathcal{A}/G) \) (and, conveniently, an explicit basis for this space), and (iii) construction of the diffeomorphism-invariant Hilbert space \( \mathcal{H}_{\text{diff}} \). In what follows we generalize all these steps to the case of theories involving fermions. Finally, there is the problem of imposing the Hamiltonian constraint and the problem of finding solutions to this constraint. In this paper we do not enter into this all-important problem.

In the next section we describe the class of theories which are considered in this paper and give several examples of physically interesting theories belonging to this class. Their quantization will be discussed in the following sections.

3 The class of theories

In this paper we consider a special class of theories with fermionic degrees of freedom. Each theory belonging to this class is specified, first, by: (i) a principal \( G \)-bundle \( P \to \Sigma \), where the gauge group \( G \) is a compact connected Lie group and \( \Sigma \) is a real-analytic manifold. We also need to specify (ii) a finite list \( I \) of irreducible continuous unitary representations of \( G \). Each representation in \( I \) corresponds to an elementary fermion appearing in the theory. Using any representation \( \rho \in I \) we may associate to \( P \) a vector bundle \( P \times_G \rho \) over \( \Sigma \). It will be convenient to lump these together by forming a single vector bundle

\[
F = P \times_G \left( \bigoplus_{\rho \in I} \rho \right),
\]

sections of which simultaneously describe all the fermionic fields in the theory.

From this data we can construct the classical configuration space and phase space of the theory. We have already described the classical configuration space for the gauge fields; this is the space \( \mathcal{A} \) of smooth connections on \( P \). The corresponding classical phase space is the space of pairs \( (A, E) \) where \( A \) is a smooth connection on \( P \) and \( E \) is a smooth Ad\(P\)-valued vector field of density weight 1. We denote this classical phase space as \( T^* \mathcal{A} \), since we can think of \( E \) as a cotangent vector to \( \mathcal{A} \) satisfying certain smoothness conditions. Both \( \mathcal{A} \) and \( T^* \mathcal{A} \) become infinite-dimensional smooth manifolds in a natural way.

Similarly, the classical configuration space for the fermionic fields is the space \( F \) of smooth sections of \( F \), and the classical phase space for the fermionic fields is space of pairs consisting of a smooth section of \( F \) together with a densitized smooth section of \( F^* \) with density weight 1. We denote this classical phase
space as $T^*\mathcal{F}$. Both $\mathcal{F}$ and $T^*\mathcal{F}$ become infinite-dimensional topological vector spaces in a natural way.

The classical configuration space for the whole theory is thus the product $\mathcal{A} \times \mathcal{F}$, while the classical phase space is $T^*\mathcal{A} \times T^*\mathcal{F}$. In down-to-earth terms, a point in the classical phase space is simply a list of pairs $(A, E), (\xi, \tilde{\xi}), \ldots, (\eta, \tilde{\eta})$. Here $A$ is the connection on $P$ and $E$ is its canonically conjugate momentum. Similarly, the fields $\xi, \ldots, \eta$ are fermionic configuration fields corresponding to the representations in $I$, while the fields $\tilde{\xi}, \ldots, \tilde{\eta}$ are their canonically conjugate momenta.

In what follows we shall treat the fermionic fields as Grassmann-valued, in order to guarantee that the Pauli principle holds for our fermions. This amounts to treating $\mathcal{F}$ as a supermanifold with all odd coordinates [8]. Thus we define the algebra $C^\infty(\mathcal{F})$ of ‘smooth functions’ on $\mathcal{F}$ to be the exterior algebra $\Lambda \mathcal{F}^*$ generated by the continuous linear functionals on $\mathcal{F}$. Note that these are not functions on $\mathcal{F}$ in the standard sense, but only in the sense of supermanifold theory. Similarly, we define the algebra $C^\infty(T^*\mathcal{F})$ of ‘smooth functions’ on the fermionic classical phase space $T^*\mathcal{F}$ to be the exterior algebra generated by the continuous linear functions on $T^*\mathcal{F}$. We also define ‘smooth functions’ on the configuration space and phase space of the whole theory as follows:

$$C^\infty(\mathcal{A} \times \mathcal{F}) = C^\infty(\mathcal{A}) \otimes C^\infty(\mathcal{F}),$$
$$C^\infty(T^*\mathcal{A} \times T^*\mathcal{F}) = C^\infty(T^*\mathcal{A}) \otimes C^\infty(T^*\mathcal{F}).$$

The latter algebra is the algebra of classical observables of the theory.

Theories may have the same phase space and differ only in the form of the Hamiltonian, so a theory is finally determined by: (iii) the Hamiltonian. Since we shall not actually treat dynamics in this paper, we shall not be very precise about the class of allowed Hamiltonians, but we are interested in those for which the action can be written as follows:

$$S = \int dt \int d^3x (\text{Tr} \ \hat{E}^a \mathcal{L}_t A_a + \tilde{\xi} \mathcal{L}_t \xi + \tilde{\omega} \mathcal{L}_t \eta + \cdots + \sum_{i=1}^{N} \hat{H}^{(i)} + N^a \tilde{\hat{H}}_a + \text{Tr} \ \mathcal{N} \tilde{\mathcal{G}}).$$  \quad (2)

Here $\mathcal{L}_t$ stands for the time derivative and the quantities $\mathcal{N}, N^a$, and $\mathcal{N}$ are Lagrange multipliers. The quantities $\hat{H}, \tilde{\hat{H}}_a,$ and $\tilde{\mathcal{G}}$ are functionals on the phase space corresponding to the Hamiltonian, diffeomorphism and Gauss law constraints. Here tildes over the momentum fields keep track of density weights: a single tilde over a symbol stands for a density of weight 1, a double tilde over a symbol stands for a density of weight 2, and a single tilde under a symbol denotes a density of weight $-1$.

Once the canonical variables are chosen (or in other words, once the phase space is chosen), a theory is determined solely by the Hamiltonian. The other two constraint functionals are determined by the requirement that they generate diffeomorphisms and gauge transformations on the phase space of the theory.

Let us give two examples.
Example A. An example of such a theory was considered in \cite{10}. It describes a massless fermionic field interacting with a gravitational field in four-dimensional (Riemannian) spacetime, with the gravitational field being described using a chiral spin connection. The gauge group is SU(2), and the gravitational degrees of freedom are described by the SU(2) connection $A_a$ and its canonically conjugate momentum field $\tilde{E}^a$. The only fermionic field of the theory, $\xi$, takes values in the spin-1/2 representation. The Hamiltonian constraint of the theory consists of two parts:

$$\tilde{H} = \frac{1}{2} \text{Tr}(\tilde{E}^a \tilde{E}^b F_{ab}) + \tilde{E}^a B^B D_a \xi B \tilde{\pi}^A,$$

where the first part is the Hamiltonian of the gauge field and the second part is responsible for the dynamics of the fermionic field. Here $F_{ab}$ is the curvature of the connection field $A_a$, and $D_a = \partial_a + A_a$ is the covariant derivative operator. This theory is the simplest theory of gravity coupled to matter.

Example B. Another example is the theory describing massive fermions interacting both gravitationally and electromagnetically, which was considered in \cite{9}. The gauge group here is SU(2) $\times$ U(1). The gravitational and electromagnetic degrees of freedom are described together by a SU(2) $\times$ U(1) connection field $A_a$ and canonically conjugate momentum field $\tilde{E}^a$. The fermionic degrees of freedom are described by two Grassmann-valued fields $\xi, \eta$ taking values in the representations of spin 1/2 and charge $\pm 1$. The Hamiltonian of the theory consists of the following three parts. The part generating the dynamics of the gravitational field is

$$\tilde{H}_{\text{grav}} = \frac{1}{2} \eta \text{Tr}(\tilde{E}^a \tilde{E}^b B^c),$$

where the magnetic field $\tilde{B}^a$ is the dual of the curvature $F_{ab} = \eta_{abc} \tilde{B}^c$, $\eta_{abc}$ being the totally antisymmetric tensor of weight $-1$. The part generating the dynamics of the electromagnetic field is

$$\tilde{H}_{\text{em}} =$$

$$\frac{1}{32} \sigma^{-2} \eta_{abc} \eta_{def} \text{Tr}(\tilde{E}^a \tilde{E}^c) \left[ \text{Tr}(\tilde{E}^b \tilde{E}^d) \text{Tr}(\tilde{B}^e) \text{Tr}(\tilde{B}^f) - \text{Tr}(\tilde{E}^b \tilde{E}^d) \text{Tr}(\tilde{E}^e) \text{Tr}(\tilde{E}^f) - \text{Tr}(\tilde{E}^a \tilde{E}^c) \text{Tr}(\tilde{E}^d) \text{Tr}(\tilde{B}^e) \text{Tr}(\tilde{B}^f) \right]$$

where $\sigma$ is the square root of the determinant of the metric, as defined using the $\tilde{E}$ field. Finally, the part of the Hamiltonian responsible for the fermionic fields is

$$\tilde{H}_{\text{ferm}} = [\tilde{E}^a B^B - \frac{1}{2} \text{Tr}(\tilde{E}^a B^B)] (D_a \xi B \tilde{\pi}^A + D_a \eta A \tilde{\omega} B) + \text{im} [\tilde{\pi}^A \tilde{\omega} A - (\sigma)^2 \eta A \xi A]$$

where $m$ is the fermion mass.
In both the theories above particles appear in the theory with their antiparticles. Mathematically this is manifested by the fact that each irreducible representation that appears in the theory appears with its dual. However, this comes about in very different ways for the two different theories above. The SU(2) theory in Example A contains only one fermionic field, but this field transforms under a representation that is isomorphic to its dual. It describes a particle that is its own antiparticle. In Example B we have two different fermionic fields transforming under dual representations of the group SU(2) × U(1). This theory describes a particle that is distinct from its antiparticle.

One can see that representations must appear with their duals in order to construct a gauge-invariant mass term in the Hamiltonian, or any other gauge-invariant expression bilinear in the configuration fermionic fields. It is sensible therefore to limit our attention to theories for which each irreducible representation of G appears in the list together with its dual. Actually, to streamline the exposition, we shall consider only the case where particles are distinct from their antiparticles. That is, we assume: (iia) the list I of representations of G is of the form (ρ1, ρ′1, ..., ρn, ρ′n). It is not difficult to extend our analysis to the case of particles that are their own antiparticles, and we mention a few of the changes that need to be made as we come to them.

Having described the class of theories which are the subject of this paper we are now ready to proceed with the quantization program. For this we need to construct fermionic path observables, which are the basic building blocks of the quantum theory.

4 Fermionic path observables

We now introduce ‘path observables’, which play in our theories a role similar to that of the standard Wilson loop observables in theories of connections. We start with by describing some path observables that are functions on the classical configuration space A × F. These observables are built from the holonomy of the connection along a path in space together with fermion fields at the endpoints of this path.

Note first that given two fermionic fields ξ, η transforming under dual representations ρ and ρ′ of the gauge group, one can construct a gauge-invariant quantity from these fields using the G-invariant bilinear pairing between ρ and ρ′. This quantity depends on the values ξp, ηp of these fields at a point p ∈ Σ and is given simply by ⟨ξp, ηp⟩. More generally, given points p and p′ and a path ε from p to p′, one can parallel transport ξp to p′ along the path ε using the connection A and then pair it with ηp′, obtaining the gauge-invariant quantity

\[ ⟨ξp_ε|ηp′⟩ = (ρ(P \exp \int_ε A)ξp, ηp′) \]

which we call a ‘configurational path observable’. It is also helpful to have a
graphical notation for a configurational path observable, in which we represent the oriented path $\epsilon$ by a line with an arrow on it, and draw the fermionic fields $\xi$ and $\gamma$ as dots at the endpoints:

$$\begin{align*}
(\xi_p | \epsilon | \eta_{p'}) & \\
\xi & \rightarrow \eta
\end{align*}$$

We may think of the dot labelled by $\xi$ as a particle and the the dot labelled by $\eta$ as its antiparticle. Configurational path observables can be introduced for any representation in the list $I$ together with its dual, or in other words, for any particle in the theory together with its antiparticle.

Note that the configurational path observables are even since they involve a product of two fermion fields. As a result they commute. We may also think of them as functions on the classical phase space, but since they involve no momenta they all Poisson-commute. Let us now introduce the basic path observables involving fermionic momentum fields and describe the Poisson algebra these observables generate.

Recall that the momentum field $\pi$ canonically conjugate to $\xi$ transforms under the dual representation. This means that one can apply the above construction to the $\xi, \pi$ fields. Namely, given two points $p, p'$ and a path $\epsilon$ connecting them, one can construct the quantity

$$(\xi_p | \epsilon | \pi_{p'}) = (\rho | \mathcal{P} \exp \int_{\epsilon} A | \xi_p, \pi_{p'}),$$

which is gauge invariant and depends only on the values of fields $\xi, \pi$ at the endpoints $p, p'$ of the path $\epsilon$ and on the holonomy of the connection field along $\epsilon$.

The quantity $(\xi | \gamma | \pi)$ is almost the one which we need. However, the Poisson bracket of this quantity with a configurational path observable $(\xi | \gamma | \eta)$ is a distribution. Following [9], let us introduce some 'averaged' momentum observables in such a way that the Poisson bracket of a configurational path observable with a momentum observable is again a configurational path observable. In other words, let us introduce momentum observables in such a way that the resulting Poisson algebra contains no distributions. To do this we choose an arbitrary rule which specifies a path $\epsilon_{pp'}$ from $p$ to $p'$ given points $p, p'$ in some region $\mathcal{R} \subset \Sigma$, an open set whose closure is compact. Having this rule at our disposal, we can integrate $(\xi | \epsilon_{pp'} | \pi)$ over $\mathcal{R}$ as a function of $p'$. We obtain the quantity

$$\int_{\mathcal{R}} d^n p' (\xi_p | \epsilon_{pp'} | \pi_{p'}),$$

Here in our graphical notation we use a dot as before to stand for the fermionic configuration field, but use a little circle to represent the averaged fermionic momentum field. The result is a gauge-invariant quantity, which depends on
values of the momentum field \( \pi \) and the connection field \( A \) over a region of the spatial manifold, on the value of \( \xi \) at the point \( p \), and on a rule \((p,p') \mapsto \epsilon_{p'p}\).

Similarly, one can introduce a quantity

\[
\int_{\mathcal{R}} d^n p \langle \tilde{\omega}_{p'} | \epsilon_{p'p} | \eta_p \rangle
\]

de pending on the momentum field \( \tilde{\omega} \) canonically conjugate to \( \eta \) (or, physically, the momentum field of the antiparticle). One can also construct quantities involving two momentum fields

\[
\int_{\mathcal{R}} d^n p \int_{\mathcal{R}'} d^n p' \langle \tilde{\omega}_{p'} | \epsilon_{p'p} | \tilde{\pi}_{p'} \rangle,
\]

Again we represent averaged momentum fields by little circles.

Note that were we to consider a theory in which \( \xi \) coincided with its antiparticle \( \eta \), there would be only one path observable linear in the momentum field, instead of two different quantities \((4)\) and \((5)\). Some of the Poisson brackets below would be different in this case, but the necessary modifications are not difficult.

The quantities we have described so far are not invariant under the Hermitian conjugation operation \( \dagger \). This operation is defined when the gauge fields of our theory include gravity, and the crucial property it satisfies is given by equation \((1)\). Let us now introduce a set of 'real' observables, by which we mean functions on the classical phase space that are preserved by this \( \dagger \) operation. Since Hermitian conjugation sends fermionic momentum fields into configuration fields, real observables should involve both fields. Let us introduce the following observables which are linear in the fermionic momenta:

\[
\int_{\mathcal{R}} d^n p \langle \xi_p, \tilde{\pi}_p \rangle,
\]

\[
\int_{\mathcal{R}} d^n p \langle \tilde{\omega}_p, \eta_p \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the bilinear pairing between the dual representations \( \rho \) and \( \rho^* \). As we shall see, when quantized these observables play the role of number operators measuring the number of particles of the appropriate sort in the region \( \mathcal{R} \). The graphical representation that we use for these observables is as shown above.

The quantities we have introduced form a set of functions on the phase space of the theory. Let us now describe the Poisson algebra which these quantities generate (we also give a graphical description). First, the Poisson brackets of the observables linear in the momentum \((4-5)\) with the configurational path observables \((3)\) are again configurational path observables. The following identities
can be verified \[9\]:

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \xi | e_{p'p} | \tilde{\pi} | \pi \right) , \left( \xi | f | \eta_{\pi} \right) \right\} = \left( \xi | e_{p'q} \circ f | \eta_{\eta} \right),
\]

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \xi | e_{p'p} | \tilde{\pi} | \pi \right) , \left( \xi | f | \eta_{\pi} \right) \right\} = \left( \xi | e \circ f | \eta_{\pi} \right) \quad (9)
\]

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e_{p'p} | \pi \right) , \left( \xi | f | \eta_{\pi} \right) \right\} = \left( \xi | f \circ e_{p'p} | \eta_{\eta} \right),
\]

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e_{p'p} | \pi \right) , \left( \xi | f | \eta_{\pi} \right) \right\} = \left( \xi | f \circ e | \eta_{\pi} \right) \quad (10)
\]

Here we use \( \circ \) to denote composition of paths. Also, it is assumed in \((9)\) that \( q \in \mathcal{R} \). If the point \( q \) lies outside of the region \( \mathcal{R} \) the Poisson bracket is zero. Similarly, in \((10)\) we assume \( q' \in \mathcal{R} \); otherwise the Poisson bracket is zero. Note that if the result is not zero, in both cases it is given by a configurational path observable with the path being the composite of the paths \( e, f \) compatible with their orientation.

The Poisson bracket of a momentum observable \((4)\) with another observable of this sort gives

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \xi | e_{pp'} | \tilde{\pi} | \pi \right) , \int_{\mathcal{R}'} d^3 q' \left( \xi | f_{qq'} | \tilde{\pi} | \pi \right) \right\} =
\int_{\mathcal{R}} d^3 p' \left( \xi | e_{pp'} \circ f_{qq'} | \tilde{\pi} | \pi \right) + \int_{\mathcal{R}'} d^3 q' \left( \xi | f_{qq'} \circ e_{pp'} | \tilde{\pi} | \pi \right),
\]

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e_{pp'} | \pi \right) , \int_{\mathcal{R}'} d^3 q' \left( \tilde{\omega} | f_{qq'} | \eta_{\eta} \right) \right\} =
\int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e_{pp'} \circ f_{qq'} | \pi \right) + \int_{\mathcal{R}'} d^3 q' \left( \tilde{\omega} | f_{qq'} \circ e_{pp'} | \eta_{\eta} \right),
\]

\[
\left\{ \int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e | \pi \right) , \int_{\mathcal{R}'} d^3 q' \left( \tilde{\omega} | f | \eta_{\eta} \right) \right\} =
\int_{\mathcal{R}} d^3 p' \left( \tilde{\omega} | e \circ f | \pi \right) + \int_{\mathcal{R}'} d^3 q' \left( \tilde{\omega} | f \circ e | \eta_{\eta} \right).\]

where we assume that the point where the configuration field in each of the observables is evaluated lies inside the region over which the momentum field
in the other is smeared; otherwise one or both terms of the Poisson bracket vanishes.

The Poisson bracket of (6) and a configuration observable also consists of two terms:

\[
\left\{ \int_{\mathcal{R}} d^n p \int_{\mathcal{R}'} d^n p' (\tilde{\omega}_{p} | e_{p'p'}^{*} | \tilde{\pi}_{p'}) , \ (\xi_{q} | f \eta_{q'}) \right\} = \\
\int_{\mathcal{R}} d^n p (\tilde{\omega}_{p} | e_{pq}^{*} c f | \eta_{q'}) + \int_{\mathcal{R}'} d^n p' (\xi_{q} | f c e_{p'q'}^{*} | \tilde{\pi}_{p'}),
\]

(13)

\[
\left\{ \omega \circ \pi, \xi \pi \eta \right\} = \omega \pi \eta + \xi \pi \eta
\]

where again we assume \( q' \in \mathcal{R}' \) and \( q \in \mathcal{R} \). For the Poisson brackets between (6) and observables linear in momentum we have

\[
\left\{ \int_{\mathcal{R}} d^n p \int_{\mathcal{R}'} d^n p' (\tilde{\omega}_{p} | e_{p'p'}^{*} | \tilde{\pi}_{p'}) , \int_{\mathcal{R}'} d^n q' (\xi_{q'} | f_{q'q'} | \tilde{\pi}_{q'}) \right\} = \\
\int_{\mathcal{R}} d^n p \int_{\mathcal{R}'} d^n p' (\tilde{\omega}_{p} | e_{pq}^{*} c f_{q'q'} | \tilde{\pi}_{p'}),
\]

(14)

\[
\left\{ \omega \circ \pi, \xi \pi \eta \right\} = \omega \pi \eta + \xi \pi \eta
\]

\[
\left\{ \int_{\mathcal{R}} d^n p \int_{\mathcal{R}'} d^n p' (\tilde{\omega}_{p} | e_{p'p'}^{*} | \tilde{\pi}_{p'}) , \int_{\mathcal{R}'} d^n q' (\tilde{\omega}_{q'} | e_{q'q} | \eta_{q}) \right\} = \\
\int_{\mathcal{R}} d^n q ' \int_{\mathcal{R}'} d^n p' (\tilde{\omega}_{q'} | e_{pq'}^{*} c f_{q'q} | \tilde{\pi}_{p'}).
\]

(15)

\[
\left\{ \omega \circ \pi, \xi \pi \eta \right\} = \omega \pi \eta + \xi \pi \eta
\]

where in (14) we assume \( q' \in \mathcal{R}' \), and in (15) we assume \( q \in \mathcal{R} \); otherwise the brackets vanish.

One can also write down Poisson brackets between (3-6) and the real quantities (7-8). For example, the Poisson brackets between configurational path observables and these real quantities are given by

\[
\left\{ \int_{\mathcal{R}} d^n p (\xi_{q} , \tilde{\pi}_{p}) , \ (\xi_{q} | e | \eta_{q'}) \right\} = \ (\xi_{q} | e | \eta_{q'}),
\]

\[
\left\{ \xi \circ \pi , \xi \pi \eta \right\} = \xi \pi \eta
\]

\[
\left\{ \int_{\mathcal{R}} d^n p (\tilde{\omega}_{p} , \eta_{p}) , \ (\xi_{q} | e | \eta_{q'}) \right\} = \ (\xi_{q} | e | \eta_{q'}).
\]

(16)
Here in (16) we assume \( q \) is contained in \( \mathcal{R} \), and in (17) we assume \( q' \) is contained in \( \mathcal{R} \); otherwise the Poisson brackets vanish.

A nice feature of the graphical notation is that it gives a simple mnemonic for the Poisson bracket formulas (9-17) above. Note that in every case, the terms appearing in the Poisson bracket of two of our observables correspond to the ways of attaching them together by filling little empty circles (i.e., momentum fields) of one observable with dots (i.e. configurational fields) of the other, in a manner compatible with the orientations of the paths. (In theories for which particles coincided with their antiparticles, we would not include orientation arrows on the paths in our graphical notation, and not require compatibility of orientations when attaching paths together. This would give precisely the extra terms in the Poisson bracket relations that actually appear in such theories.)

We see that the observables we have introduced are closed with respect to Poisson brackets: a Poisson bracket of any two our observables is a linear combination of these observables. Note, however, that the path observables involving fermionic momenta depend on a rule assigning to any two points \( p, p' \in \Sigma \) a path \( \epsilon_{pp'} \). The rules that appear on the right hand side of (9)-(15) are of the form \( \epsilon_{pq} \circ f_{pq} \), i.e., they are formed by composition from the rules we started with, so if we fix a rule we do not obtain a Poisson algebra.

Instead, we work with the 'big' algebra generated by all observables of the form (3-8), allowing all possible choices of a rule \( \epsilon_{pp'} \) and all possible regions \( \mathcal{R} \) (open sets whose closure is compact). We denote this algebra as \( \mathcal{B} \).

We may summarize by saying that we have introduced a Poisson algebra \( \mathcal{B} \) of smooth functions on the classical phase space \( T^* \mathcal{A} \times T^* \mathcal{F} \). This algebra is not closed under the Hermitian conjugation operator \( \dagger \). However, it has a distinguished subalgebra of observables preserved by the \( \dagger \) operation, namely the real-linear combinations of products of the observables (7-8).

Our aim now is to quantize the observables (3-8), finding operators on some Hilbert space whose commutators mimic the classical Poisson bracket relations. We proceed in several steps. We first construct a kinematical Hilbert space. Then we describe a subspace of gauge invariant states in this space, which can be thought of as a Hilbert space of solution of Gauss law constraint, and represent the observables in \( \mathcal{B} \) as operators on this space.

5 Quantum configuration space and kinematical Hilbert space

Recall that for diffeomorphism-invariant theories of connections, the quantum configuration space was most efficiently obtained by completing the algebra of
cylinder functions to obtain a commutative C*-algebra, and taking the spectrum of this C*-algebra. We would like to follow a similar strategy for our theories with fermions. However, in our case the Gelfand-Naimark theorem is not applicable because of the presence of Grassmann-valued fields. Another strategy is needed.

Recall that in the case of theories of connections, we could also describe the quantum configuration space as a certain completion of the classical configuration space. We can implement this idea in our case and construct the fermionic quantum configuration space $\mathcal{F}$ as a completion of the classical configuration space $\mathcal{F}$. Namely, we define $\mathcal{F}$ to be the space of all (not necessarily smooth) sections of the vector bundle $F$ over $\Sigma$. The space $\mathcal{F}$ is dense in $\mathcal{F}$ in the topology of pointwise convergence.

Alternatively, one can describe $\mathcal{F}$ as the projective limit of configuration spaces of certain fermionic systems with finitely many degrees of freedom. Namely, given any finite subset of points $V \subset \Sigma$ we may consider the configuration space of a system of fermions living at the points of $V$; this is the product

$$\mathcal{F}_V = \prod_{p \in V} F_p$$

of copies of fibers of $F$, one for each point in $V$. Given finite subsets $V \subseteq V'$ of points in $\Sigma$ there is a natural projection from $\mathcal{F}_V'$ to $\mathcal{F}_V$, and the projective limit of all these spaces $\mathcal{F}_V$ is $\mathcal{F}$.

We also wish to have a kinematical Hilbert space for the fermion fields. Although we shall denote this Hilbert space by $L^2(\mathcal{F})$, it is not a space of square-integrable functions on $\mathcal{F}$ in the standard sense. We use the notation $L^2(\mathcal{F})$ just to emphasize the similarity in notations between the fermionic and the gauge fields degrees of freedom.

The space $L^2(\mathcal{F})$ can be constructed in a variety of equivalent ways. Perhaps the easiest is to use the fact that the fibers $F_p$ are finite-dimensional Hilbert spaces and let $L^2(\mathcal{F})$ be the fermionic Fock space over the Hilbert space direct sum $\bigoplus_{p \in \Sigma} F_p$. This is the natural Hilbert space completion of the exterior algebra

$$\Lambda\left(\bigoplus_{p \in \Sigma} F_p\right).$$

Alternatively, we can think of $L^2(\mathcal{F})$ as the infinite 'grounded' tensor product

$$\bigotimes_{p \in \Sigma} \Lambda F_p$$

of the fermionic Fock spaces $\Lambda F_p$, one for each point $p$ in space. (For an introduction to the mathematics of fermionic Fock spaces and infinite grounded tensor products, see [6].)

In the case of connection fields we can define the kinematical Hilbert space as a completion of an algebra of cylinder functions. An analogous fact holds for
fermion fields if we define a ‘cylinder function’ on $\mathcal{F}$ to be a smooth function on $\mathcal{F}$ depending only on the value of the fermion fields at finitely many points of $\Sigma$. The space of cylinder functions is thus

$$\Lambda\left(\bigoplus_{p \in \Sigma} F^*_p\right) \subseteq \Lambda \mathcal{F}^*,$$

the exterior algebra over the algebraic direct sum $\bigoplus_{p \in \Sigma} F^*_p$. Since each $F_p$ is a Hilbert space, this space is naturally isomorphic to

$$\Lambda\left(\bigoplus_{p \in \Sigma} F_p\right),$$

and by the above remarks we see that it can be completed to obtain $L^2(\mathcal{F})$.

Finally, we may also describe $L^2(\mathcal{F})$ in the following physically appealing way. For each finite set $V$ of points of $\Sigma$ we may start with the theory of fermions living at these points. The configuration space for this theory is the finite-dimensional space $\mathcal{F}_V$ described above. Quantizing this system we obtain the fermionic Fock space over $\mathcal{F}_V$, which is just the exterior algebra $\Lambda \mathcal{F}_V$ equipped with its standard inner product. Given finite subsets $V \subseteq V'$ of points in $\Sigma$ there is a natural projection from $\Lambda \mathcal{F}_V$ to $\Lambda \mathcal{F}_{V'}$, and the projective limit of these Hilbert spaces is the kinematical Hilbert space $L^2(\mathcal{F})$.

Putting together the quantum configuration spaces for the gauge field and fermionic degrees of freedom, we obtain the quantum configuration space $\mathcal{A} \times \mathcal{F}$ for the whole theory. Similarly, the kinematical Hilbert space for the full theory is $L^2(\mathcal{A}) \otimes L^2(\mathcal{F})$, which we denote by $L^2(\mathcal{A} \times \mathcal{F})$. Note that cylinder functions on $\mathcal{A}$ are dense in $L^2(\mathcal{A})$, and cylinder functions on $\mathcal{F}$ are dense in $L^2(\mathcal{F})$. Thus if we define the algebra of ‘cylinder functions’ on $\mathcal{A} \times \mathcal{F}$ to be the tensor product of the algebras of cylinder functions on $\mathcal{A}$ and cylinder functions on $\mathcal{F}$, it follows immediately that cylinder functions on $\mathcal{A} \times \mathcal{F}$ are dense in $L^2(\mathcal{A} \times \mathcal{F})$.

In simple terms, a cylinder function on $\mathcal{A} \times \mathcal{F}$ is a wavefunction depending on the connection only via its holonomies along finitely many paths in $\Sigma$ and on the fermionic fields only via their values at finitely many points of $\Sigma$. As we shall see in the next section, cylinder functions may be thought of as states of gauge theories coupled to fermions living on graphs in $\Sigma$ with finitely many edges and vertices. Cylinder functions play an important role in the quantum theory, for it is natural to define quantum operators first on the dense subspace of cylinder functions, and then to extend them to the entire Hilbert space $L^2(\mathcal{A} \times \mathcal{F})$.

The Hilbert space $L^2(\mathcal{A} \times \mathcal{F})$ serves only as an auxiliary Hilbert space of the theory, since its states are typically not invariant under gauge transformations, diffeomorphisms of space, or time evolution. To find physical states we must impose constraints corresponding to invariance under these symmetries. In the next section we describe a basis of solutions to the Gauss law constraint, that is, gauge-invariant elements of $L^2(\mathcal{A} \times \mathcal{F})$. 
6 Gauge-invariant states

There is a natural unitary representation of the group \( G \) of smooth gauge transformations on the Hilbert space of quantum states \( L^2(\mathcal{A} \times \mathcal{F}) \). In this section we describe the subspace of \( L^2(\mathcal{A} \times \mathcal{F}) \) consisting of that are invariant under these gauge transformations. We denote this space by \( L^2(\mathcal{A} \times \mathcal{F})/G \). We also describe an orthonormal basis for \( L^2(\mathcal{A} \times \mathcal{F})/G \) that is a generalization of the spin network basis of \( L^2(\mathcal{A})/G \).

Given any graph \( \gamma \) analytically embedded in \( \Sigma \), let \( E \) be its set of edges and let \( V \) be its set of vertices. Any edge \( e \in E \) begins at some vertex \( s(e) \) called its `source' and ends at some vertex \( t(e) \) called its `target'. We define the space of connections on \( \gamma \) to be

\[
\mathcal{A}_\gamma = \prod_{e \in E} \mathcal{A}_e
\]

where \( \mathcal{A}_e \) is the space of transporters from \( s(e) \) to \( t(e) \). If we trivialize \( P \) at all the vertices of \( \gamma \) we can think of \( \mathcal{A}_e \) as a product of copies of the gauge group \( G \), one for each edge. Similarly, we define the classical configuration space for fermion fields on \( \gamma \) to be

\[
\mathcal{F}_\gamma = \prod_{p \in V} \mathcal{F}_p.
\]

We define \( L^2(\mathcal{A}_\gamma) \) using normalized Haar measure on \( G \), define \( L^2(\mathcal{F}_\gamma) \) to be the fermionic Fock space over \( \mathcal{F}_\gamma \), and set

\[
L^2(\mathcal{A}_\gamma \times \mathcal{F}_\gamma) = L^2(\mathcal{A}_\gamma) \otimes L^2(\mathcal{F}_\gamma).
\]

We may think of this as a subspace of \( L^2(\mathcal{A} \times \mathcal{F}) \).

For any graph \( \gamma \) there is a unitary representation of \( G \) on \( L^2(\mathcal{A}_\gamma \times \mathcal{F}_\gamma) \), and we denote the subspace consisting of gauge-invariant elements by \( L^2((A_\gamma \times F_\gamma)/G) \). As in the case of pure gauge fields \([2, 4]\), the union of these spaces as \( \gamma \) ranges over all graphs is dense in \( L^2(\mathcal{A} \times \mathcal{F})/G \). Thus to describe \( L^2(\mathcal{A} \times \mathcal{F})/G \) it suffices to describe these space for arbitrary graphs \( \gamma \).

Trivializing \( P \) at all the vertices of \( \gamma \) we have

\[
L^2(\mathcal{A}_\gamma \times \mathcal{F}_\gamma) = \bigotimes_{e \in E} L^2(\mathcal{A}_\gamma) \otimes \bigotimes_{v \in V} \Lambda F_v. \tag{18}
\]

Let us decompose the Grassmann algebra \( \Lambda F_v \) at each vertex as an orthogonal direct sum of irreducible representations of \( G \): \( \Lambda F_v = \bigoplus_{\rho \in S} \rho \), where we have denoted by \( S \) the list of irreducible representations appearing in \( \Lambda F_v \). Similarly, the Peter-Weyl theorem says that \( L^2(\mathcal{A}_\gamma) \cong \bigoplus_{\rho \in \text{Rep}(G)} \rho \otimes \rho^* \), where the set \( \text{Rep}(G) \) contains one irreducible continuous unitary representation of \( G \) from each equivalence class, so that (18) implies

\[
L^2(\mathcal{A}_\gamma \times \mathcal{F}_\gamma) = \bigotimes_{e \in E} \left( \bigoplus_{\rho \in \text{Rep}(G)} \rho \otimes \rho^* \right) \otimes \bigotimes_{v \in V} \left( \bigoplus_{\rho \in S} \rho \right). \tag{19}
\]
The right hand side of (19) can be rewritten as follows:

\[
L^2(\mathcal{A}_\gamma \times \mathcal{F}_\gamma) = \bigoplus_{\rho_e \in \text{Rep}(G)} \bigotimes_{v \in V} \left( \bigotimes_{t(e) \in v} \rho_e^* \otimes \bigotimes_{s(e) \in v} \rho_v \right) \tag{20}
\]

The sum here runs over all labelings of edges \(\epsilon\) by irreducible representations \(\rho_e\) of the group \(G\), and by all labelings of vertices by irreducible representations from the list \(S\). It follows that

\[
L^2((\mathcal{A}_\gamma \times \mathcal{F}_\gamma)/G) = \bigoplus_{\rho_e \in \text{Rep}(G)} \bigotimes_{v \in V} \text{Inv}\left(\left( \bigotimes_{t(e) \in v} \rho_e^* \otimes \bigotimes_{s(e) \in v} \rho_v \right) \right) \tag{21}
\]

where \(\text{Inv}\) denotes the \(G\)-invariant subspace of the given representation. Note that \(\text{Inv}\left(\bigotimes_{t(e) \in v} \rho_e^* \otimes \bigotimes_{s(e) \in v} \rho_v \right)\) has a natural inner product, and is isomorphic to the space of intertwining operators from \(\bigotimes_{t(e) \in v} \rho_e\) to \(\bigotimes_{s(e) \in v} \rho_v \otimes \rho_v\). We denote this space of intertwining operators by

\[
\text{Hom}\left( \bigotimes_{t(e) \in v} \rho_e, \left( \bigotimes_{s(e) \in v} \rho_v \right) \right).
\]

As in the case of pure gauge fields, the above discussion also gives an orthonormal basis of spin network states for \(L^2((\mathcal{A}_\gamma \times \mathcal{F}_\gamma)/G)\). Unlike spin networks for theories describing only gauge fields, the spin networks for theories with fermions can contain edges with ‘open ends’. Such open ends represent fermionic degrees of freedom. We describe the spin network basis for \(L^2((\mathcal{A}_\gamma \times \mathcal{F}_\gamma)/G)\) in the following theorem summarizing the results of this section:

**Theorem 1** The space \(L^2((\mathcal{A}_\gamma \times \mathcal{F}_\gamma)/G)\) has an orthonormal basis of ‘fermionic spin network states’, each such state being specified by a choice of:

1. a graph \(\gamma\) analytically embedded in \(\Sigma\),
2. a labelling of each edge \(\epsilon\) of \(\gamma\) by an irreducible representation \(\rho_e \in \text{Rep}(G)\),
3. a labelling of each vertex \(v\) of \(\gamma\) by an irreducible representation \(\rho_v \in S\),
4. a labelling of each vertex \(v\) of \(\gamma\) by an intertwining operator \(t_v \in \text{Hom}\left( \bigotimes_{t(e) \in v} \rho_e, \left( \bigotimes_{s(e) \in v} \rho_v \right) \right)\) chosen from an orthonormal basis of such intertwining operators.

We call the data \(\Gamma = (\gamma, \rho, t)\) a ‘fermionic spin network’, and denote the fermionic spin network state corresponding to \(\Gamma\) as above by \(\Psi_{\Gamma}\).
7 Quantization of fermionic path observables

We have now constructed the kinematical Hilbert space $L^2(A \times F)$ as a completion of the space of cylinder functions on $A \times F$, and found a spin network basis for the gauge-invariant subspace $L^2((A \times F)/G)$. Our construction so far has nothing directly to do with the fermionic path observables introduced in Sec. 4. Thus, we have to make sure that there is a way to quantize these observables and obtain operators on $L^2((A \times F)/G)$. In fact, we shall obtain operators on $L^2(A \times F)$ that commute with gauge transformations, and thus map $L^2((A \times F)/G)$ to itself.

There is an obvious way to promote the configurational path observables into operators. We represent the quantities $(\xi \zeta \eta)$ by multiplication operators $(\xi \zeta \eta)$ on the space of cylinder functions on $A \times F$ as follows:

$$(\xi \zeta \eta) \Phi = (\xi \zeta \eta) \Phi$$

where $(\xi \zeta \eta)$ on the right hand side is considered as a cylinder function on $A \times F$. Note that if $\Phi$ is a cylindrical function so is $(\xi \zeta \eta) \Phi$. One has, therefore, a well-defined action of $(\xi \zeta \eta)$ on the space of cylinder functions.

The observables (4-8) involving fermionic momenta can also be first promoted to operators on the space of cylinder functions. The natural way to do this is to replace the fermionic momentum field $\pi$ in such observables by the corresponding Berezin derivative $\delta/\delta \xi$. Berezin derivatives $[7]$ are usually used in the context of field theories in Minkowski spacetime. However, we can also make sense of this notion in the context of diffeomorphism-invariant theories.

We introduce Berezin derivatives as operators on the space of cylinder functions as follows. Recall that the algebra of cylinder functions on $F$ is the exterior algebra

$$A \left( \bigoplus_{p \in \Sigma} F^*_p \right).$$

For any point $p$ choose an orthonormal basis $\xi^a(p)$ of the subspace of $F^*_p$ corresponding to the representation $p \in I$ under which the field $\xi$ transforms. Note that $\xi^a(p)$ is a cylinder function on $F$, and the algebra of cylinder functions on $F$ is generated by functions of this form. Thus, formally speaking, we may define the Berezin derivative operator to be the superderivation (i.e., graded derivation) of the algebra of cylinder functions such that

$$\frac{\delta}{\delta \xi^a(p)} \xi^b(p') = \delta^b_a \delta^{\alpha}(p, p'),$$

where $\delta^a_b$ is the Kronecker delta. However, the right hand side is distributional, so strictly speaking we should smear this equation with a ‘test function’. In other words, for any section $f$ of $P \times_G \rho$, we define

$$\int_{\Sigma} d^np f^b(p) \frac{\delta}{\delta \xi^a(p)}$$
to be the unique superderivation on the algebra of cylinder functions such that
\[
\left( \int_{\Sigma} d^\ast p \, \tilde{f}^\beta(p) \frac{\delta}{\delta \xi^\beta(p)} \right) \xi^\alpha(p') = f^\alpha(p').
\]

To describe the action of this operator on an arbitrary cylinder function, let us introduce the interior product operator \( i[\xi]_\alpha(p) \), namely the adjoint of exterior multiplication by \( \xi^\alpha(p) \). We then have
\[
\left( \int_{\Sigma} d^\alpha p \, \tilde{f}^\beta(p) \frac{\delta}{\delta \xi^\beta(p)} \right) \Psi = \sum_{p \in \Sigma} f^\alpha(p) \, i[\xi]_\alpha(p) \Psi,
\]
where the right hand side, for any cylinder function \( \Psi \), contains only a finite number of non-zero terms.

Using this definition of Berezin derivative one can proceed with the quantization of observables (4-6). However, there is an operator ordering ambiguity that needs to be resolved in this procedure. Since the observables (4-6) involve both fermion fields and their canonically conjugate momenta, which do not anticommute, we must specify which acts first. Note that one of our classical quantities, namely \( \int_{\Sigma} d^\ast p \xi(p, \tilde{\xi}_p) \), has a natural interpretation as the number of particles of type \( \xi \) within the region \( \Sigma \). To obtain this number operator when we quantize, we need to choose the operator ordering in which the Berezin derivative corresponding to the momentum field \( \tilde{\xi}_p \) acts before the multiplication operator corresponding to the configuration field \( \xi_p \). Let us therefore demand that derivatives act before multiplication operators for all our observables, which amounts to a ‘normal ordering’ prescription.

Following this choice of operator ordering we can now quantize all our fermionic observables, obtaining operators on the space of cylinder functions. More precisely, we define the quantum analogs of the observables (4-6) as follows:
\[
\int_{\Sigma} d^\alpha p' \, (\xi_\beta | \tilde{\xi}_{p'} | \tilde{\phi}_p) = \sum_{p' \in \Sigma} \xi^\alpha(p) \, (\mathcal{P} \int_{\Sigma} A)^\beta_\alpha \, i[\xi]_\beta(p') \tag{24}
\]
\[
\int_{\Sigma} d^\alpha p' \, (\tilde{\omega}_{p'} | \tilde{\xi}_{p'} | \eta_\beta) = \sum_{p' \in \Sigma} \eta_\beta(p) \, (\mathcal{P} \int_{\Sigma} A)^\beta_\beta \, i[\eta]_\beta(p') \tag{25}
\]
\[
\int_{\Sigma} d^\alpha p \int_{\Sigma} d^\ast p' (\tilde{\omega}_p \, \tilde{\xi}_{p'} | \tilde{\phi}_p) = \sum_{p \in \Sigma} \sum_{p' \in \Sigma} \frac{i[\eta]}{\alpha} (p) \, (\mathcal{P} \int_{\Sigma} A)^\beta_\alpha \, i[\xi]_\beta(p') \tag{26}
\]
\[
\int_{\Sigma} d^\alpha p \, (\xi_p, \tilde{\xi}_p) = \sum_{p \in \Sigma} \xi^\alpha(p) \, i[\xi]_\alpha(p) \tag{27}
\]
\begin{equation}
\int \mathcal{R} d^n p \left( \hat{\omega}_p, \eta_p \right) = \sum_{p \in \mathcal{R}} \eta_\alpha(p) i[\eta^\alpha(p)]
\end{equation}

The operators (22-28) map cylinder functions to cylinder functions, and extend uniquely to bounded linear operators on $L^2(\mathcal{A} \times \mathcal{F})$. Since they commute with gauge transformations they may also be thought of as operators on $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G})$. Finite linear combinations of fermionic spin network states are dense in $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G}$, so the actions of these operators on $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G}$ are determined by their actions on fermionic spin network states. On such states the action is simply described in our graphical notation, which is very nice for explicit computations. Let give several simple examples; generalization to arbitrary spin network states is straightforward. We consider the action of our operators on a fermionic spin network state containing a single path, that is, a cylinder function of the form $(\xi|\eta)$. First, the fermionic path observables linear in momentum, (25-26), act to compose the path $\epsilon$ with a path $f$. That is,

\begin{align}
\xi \c o \epsilon \eta & = \xi \c o \epsilon \eta \\
\omega \c o \epsilon \eta & = \omega \c o \epsilon \eta
\end{align}

if the relevant endpoint of $\epsilon$ lies within the region $\mathcal{R}$; otherwise we get zero. The operators (27-28) on the same state are given as follows:

\begin{align}
\xi \c o \epsilon \eta & = \xi \c o \epsilon \eta \\
\omega \c o \epsilon \eta & = \omega \c o \epsilon \eta
\end{align}

if the relevant endpoint of $\epsilon$ lies within the region $\mathcal{R}$; otherwise we get zero. In general, these operators count the number of the fermions (resp. antifermions) in the region $\mathcal{R}$. One can easily see that all fermionic spin network states are eigenstates of these operators.

The operator corresponding to a configurational path observable acts to create a path:

\begin{align}
\xi \c o \epsilon \eta & = \xi \c o \epsilon \eta \\
\omega \c o \epsilon \eta & = \omega \c o \epsilon \eta
\end{align}

The result is a spin network state containing two paths. The operator involving two momentum fields acts to close off a path:

\begin{align}
\xi \c o \epsilon \eta & = \xi \c o \epsilon \eta
\end{align}

The operators (22-28) map cylinder functions to cylinder functions, and extend uniquely to bounded linear operators on $L^2(\mathcal{A} \times \mathcal{F})$. Since they commute with gauge transformations they may also be thought of as operators on $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G})$. Finite linear combinations of fermionic spin network states are dense in $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G}$, so the actions of these operators on $L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G}$ are determined by their actions on fermionic spin network states. On such states the action is simply described in our graphical notation, which is very nice for explicit computations. Let give several simple examples; generalization to arbitrary spin network states is straightforward. We consider the action of our operators on a fermionic spin network state containing a single path, that is, a cylinder function of the form $(\xi|\eta)$. First, the fermionic path observables linear in momentum, (25-26), act to compose the path $\epsilon$ with a path $f$. That is,

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The result is a spin network state containing two paths. The operator involving two momentum fields acts to close off a path:

\begin{align}
\xi \c o \epsilon \eta & = \xi \c o \epsilon \eta
\end{align}
if the endpoints of \( \epsilon \) lie within the corresponding regions of integration in the operator \( \int_{\mathcal{R}} d^3 p \int_{\mathcal{R'}} d^3 p' \langle \hat{\omega}_p | \hat{\epsilon}_{p'} | \hat{\epsilon}_p \rangle \); otherwise we get zero. Note that the result here is the simple loop state so familiar from the loop representation of pure gravity.

In short, we can quantize our fermionic path observables, implementing them as operators on \( L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G}) \). As usual in quantization, we do not obtain a true representation of the Poisson algebra \( \mathcal{B} \), in which Poisson brackets of arbitrary observables in \( \mathcal{B} \) are mapped to commutators of the corresponding operators. However, if we restore the factors of \( \hbar \) which we have suppressed above, and work with \( \hbar \delta / \delta \xi^a(p) \) where above we use \( \delta / \delta \xi^a(p) \), we find that the commutators of operators (22-28) match the Poisson brackets of the observables (3-8) up to terms of order \( \hbar^2 \).

\section{Diffeomorphism-invariant states}

The Hilbert space \( L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G}) \) of gauge-invariant states is still too large to be the space of physical states of the theory. The group of diffeomorphisms \( \text{Diff}(\Sigma) \) acts non-trivially on the elements of \( L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G}) \). To find diffeomorphism-invariant states one should adopt the strategy described in Sec. 2, namely the 'group averaging' technique of Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [3]. Since the details of the construction are very similar to those in the case of theories with pure connection degrees of freedom, we only sketch the main steps here.

We look for diffeomorphism-invariant elements of the space \( \mathcal{C}^* \), the topological dual of the space \( \mathcal{C} \) of gauge-invariant cylindrical functionals. We construct such solutions by averaging cylindrical states from \( L^2((\mathcal{A} \times \mathcal{F})/\mathcal{G}) \), just as described in Sec. 2. Each cylindrical state \( \Psi \in \mathcal{C} \) defines in this way a diffeomorphism-invariant element \( \Psi \) of \( \mathcal{C}^* \). There is an inner product on the space of such diffeomorphism-invariant states given by

\[ \langle \Psi | \Phi \rangle = \Psi(\Phi). \]

Completing this space in this inner product, we obtain the Hilbert space \( \mathcal{H}_{\text{diff}} \) of diffeomorphism-invariant states. This space has an orthogonal basis given by states of the form \( \Psi_{\Gamma} \), where \( \Gamma \) ranges over all diffeomorphism equivalence class of fermionic spin networks. (These states are not necessarily unit vectors, but they can be normalized to obtain an orthonormal basis.)

\section{Conclusions}

Having set up a kinematical framework for diffeomorphism-invariant gauge theories coupled to fermions, and dealt with the Gauss law and diffeomorphism constraints, there are still some interesting questions concerning superselection
sectors to deal with before moving on to the much deeper problem of the Hamiltonian constraint. In general one can obtain superselection sectors (invariant subspaces) for the action of an algebra on a Hilbert space by studying the orbits of various states under the action of this algebra. Let \( \mathcal{B} \) denote the algebra of operators generated by the quantized fermionic path observables (22-28) of Sec. (7). It is especially interesting to consider the orbit of the ‘empty state’ under this algebra. By the ‘empty state’ we mean the vector in \( L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G} \) given by

\[
1 \otimes 1 \in L^2(A_e) \otimes \Lambda(\bigoplus_{\gamma \in \Sigma} F_\gamma).
\]

This is the fermionic spin network state corresponding to a graph with no vertices and no edges. In general the empty state is not a cyclic vector for \( \mathcal{B} \); i.e., the orbit \( \mathcal{B}(1 \otimes 1) \) is not dense in \( L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G} \). Thus \( L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G} \) has nontrivial superselection sectors.

There are various reasons why the empty state may not be a cyclic vector. First, there may be a nontrivial subgroup \( H \subseteq G \) that acts trivially on all the representations in our list \( I \) corresponding to the fermions in our theory. If this occurs, the empty state will never be a cyclic vector for \( \mathcal{B} \). To see this, note that \( H \) is a closed normal subgroup of \( G \) so \( G' = G/\bar{H} \) is a compact connected Lie group. We may naturally construct from \( P \) a principal \( G' \)-bundle \( P' \) and think of \( \mathcal{F} \) as associated to \( P' \). Letting \( G' \) denote the group of gauge transformations of \( P' \), and letting \( A' \) denote the space of smooth connections on \( P' \), one may check that \( L^2(A' \times \mathcal{F})/\mathcal{G}' \) is in a natural way a proper closed subspace of \( L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G} \). One may also check that the orbit of the empty state lies in \( L^2(A' \times \mathcal{F})/\mathcal{G}' \), so the empty state is not a cyclic vector.

It is easy to get around this problem, though, because in this situation there is no harm in replacing the gauge group \( G \) of our theory by \( G' \) and taking \( L^2(A' \times \mathcal{F})/\mathcal{G}' \) as our space of gauge-invariant states. For example, consider the gravitational and electromagnetic fields coupled to a charged fermionic field as in Example B of Sec. 3. The particle in this theory transform according to the spin 1/2, charge 1 representation of \( G = \text{SU}(2) \times \text{U}(1) \), while its antiparticle transforms according to the spin 1/2, charge -1 representation. The group \( H \) thus consists of the elements \( \pm (1,1) \), so there is no harm in replacing \( G \) by the group \( G/H = \text{U}(2) \).

Suppose therefore that this problem does not occur: only the identity element of \( G \) acts trivially on all the representations in the list \( I \). The empty state may still fail to be a cyclic vector. In fact, this always occurs if \( G = \text{SU}(n) \) for odd values of \( n > 2 \) when the list \( I \) consists only of the fundamental representation and its dual. The reason is that the \( n \)th exterior power of the fundamental representation is the trivial representation, so one may construct a gauge-invariant quantity of order \( n \) in the fermion field. Applying this to the empty state gives a fermionic spin network state living on the graph with one vertex and no edges. This element \( L^2(\mathcal{A} \times \mathcal{F})/\mathcal{G} \) cannot be in the orbit of
the empty state because the operators (22-28) always change the total fermion number by an even integer.

However, the question of whether the empty state is a cyclic vector remains open for the interesting cases of \( G = SU(2) \) and \( G = SU(2) \times U(1) \). Another interesting open question is as follows. At least for \( G = U(n) \) and \( G = SU(n) \) one knows that products of Wilson loop operators labelled by the fundamental representation span \( L^2(\mathcal{A}/\mathcal{G}) \). As noted in Sec. 7, such Wilson loops can be expressed as products of fermionic path observables. Therefore in theories with these gauge groups and fermions transforming under the fundamental representation one knows that the closure of the orbit of the empty state at least contains \( L^2(\mathcal{A}/\mathcal{G}) \subseteq L^2(\mathcal{A} \times \mathcal{F}/\mathcal{G}) \). For what other groups and choices of representations does this result hold?

10 Acknowledgements

We would like to thank H. Morales-Técotl and Y. Shtanov for many stimulating discussions when the basic ideas behind this work were being developed. K. K. also thanks A. Ashtekar for a useful discussion. We are grateful to the Erwin Schrödinger Institute for Mathematical Physics in Vienna for their hospitality while much of this paper was written, and especially to Peter Aichelburg and Abhay Ashtekar for running the Workshop on Mathematical Problems of Quantum Gravity held there. J. B. also thanks the Center for Gravitational Physics and Geometry for their hospitality while this paper was being completed. K. K. was supported, in part, by the International Soros Science Education Program (ISSEP) through grant No. PSU062052, and this research was also supported in part by NSF grant PHY95-14240.

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