$z$-graded Extensions of Poisson Brackets

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Abstract

A Z-graded Lie bracket $\{ , \}_P$ on the exterior algebra $\Omega(M)$ of differential forms, which is an extension of the Poisson bracket of functions on a Poisson manifold $(M,P)$, is found. This bracket is simultaneously graded skew-symmetric and satisfies the graded Jacobi identity. It is a kind of an ‘integral’ of the Koszul-Schouten bracket $[ , ]_P$ of differential forms in the sense that the exterior derivative is a bracket homomorphism: $[d\mu, dv]_P = d\{\mu, v\}_P$. A naturally defined generalized Hamiltonian map is proved to be a homomorphism between $\{ , \}_P$ and the Frölicher-Nijenhuis bracket of vector valued forms. Also relations of this graded Poisson bracket to the Schouten-Nijenhuis bracket and an extension of $\{ , \}_P$ to a graded bracket on certain multivector fields, being an ‘integral’ of the Schouten-Nijenhuis bracket, are studied. All these constructions are generalized to tensor fields associated with an arbitrary Lie algebroid.

1 Introduction

The classical Poisson bracket

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

was first introduced by Poisson in the early nineteenth century in his study of the equation of motion in celestial mechanics. About thirty years later, Jacobi discovered the famous ‘Jacobi identity’ and Hamilton, using the Poisson bracket,

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found that the equations of motion could be written in the form of what is now called 'Hamilton's equations'. Since then, Poisson brackets in more and more general form have been exploited for almost two hundred years in geometry and physics.

We found a way to extend the Poisson brackets to \( \mathbb{Z}_2 \)-graded Lie brackets on the exterior algebra of differential forms. The aim of this paper is to present this extension together with observations on its relations to other graded Lie brackets over a Poisson manifold.

The graded Lie brackets have become a topic of interest in physics in the context of 'supersymmetries' relating particles of differing statistics (cf. [C-N-S]). The growing interest to graded Lie algebras in mathematics started in the context of deformation theory (see the survey [Ro]) and the discovery of Schouten [Sc] and Nijenhuis [Ni], who observed that the standard Lie bracket of vector fields can be extended to a graded Lie bracket of multivector fields. This bracket – the Schouten-Nijenhuis bracket \([\cdot, \cdot]^{\mathbb{Z}_2-N}\) – satisfies the Leibniz rule and the whole algebraic structure is a prototype of what is now called a Gerstenhaber algebra (see [KS1]). The last notion goes back to Gerstenhaber's work on cohomology rings of algebras [Ge].

The Schouten-Nijenhuis bracket detects Poisson structures: \( P \in \Gamma(\Lambda^2 T M) \) is a Poisson tensor if and only if \([P, P]^{\mathbb{Z}_2-N} = 0\).

Similarly, the Frölicher-Nijenhuis bracket \([\cdot, \cdot]^{\mathbb{Z}_2-N}\) on the graded space \( \Omega(M; TM) \) of vector valued differential forms detects complex structures: a nearly complex structure \( J \in \Omega^1(M; TM) \) is complex (integrable) if and only if \([J, J]^{\mathbb{Z}_2-N} = 0\), as states the famous theorem of Newlander and Nirenberg. It is also used to define Nijenhuis operators – an important tool in the theory of integrable systems.

The Koszul-Schouten bracket \([\cdot, \cdot]_P\) defined on differential forms on a Poisson manifold \((M, P)\) plays, in turn, an important role in the theory of Poisson Lie groups, where it is used to define dressing actions.

Recent papers by Lian and Zuckerman [L-Z], Getzler [Gz] and several papers on string theory make also an extensive use of graded Lie brackets and their generalizations (cf. the appearance of Batalin-Vilkovisky algebras in the BRST cohomology of topological field theories).

The importance of the Hamiltonian formalism and the belief that the propagation of higher-order geometric singularities can be described in terms of suitably extended Hamiltonian formalism provoke several attempts to extend the Poisson bracket defined on \( C^\infty(M) \) by a symplectic form, or, more generally, by a Poisson tensor \( P \), to the graded algebra \( \Omega(M) \) of differential forms (see [Mi1], [C-V]). It should be stressed that for this extension the bracket degree coincides with the degree of a form, whereas the degree of a \( k \)-form with respect to the Koszul-Schouten bracket is \((k-1)\). Michor in [Mi1] for symplectic, and Cabras and Vinogradov in [C-V] for arbitrary Poisson structures, proposed several graded brackets extending the Poisson bracket \([\cdot, \cdot]_P\) of functions. Their brackets, however, fail to be either skew-symmetric (and those may be viewed...
as prototypes of what is now called Leibniz or today brackets \([KS2]\), or to satisfy the Jacobi identity. The direct skew-symmetrization of the first ones is not the right solution, since it leads again to brackets not satisfying the Jacobi identity.

In this paper, we propose a true graded Lie bracket extending the Poisson bracket of functions. Our bracket fails to satisfy the Leibniz rule (in fact, we prove that Leibniz rule contradicts another natural property of the extended bracket), but it seems to be a right one, since it coincides on co-exact forms with the bracket of Michor, Cabras, and Vinogradov and it is nicely related to other graded Lie brackets on the manifold: the exterior derivative is a homomorphism into the Koszul-Schouten bracket, a generalized Hamiltonian map is a homomorphism into the Frölicher-Nijenhuis bracket and another Hamiltonian map is a homomorphism into the Schouten-Nijenhuis bracket. Moreover, our extension of the canonical Poisson bracket on the cotangent bundle \(T^*M\) contains the Frölicher-Nijenhuis and the symmetric Schouten bracket on \(M\) (cf. [DV-M]). We get also a graded analog of the well-known exact sequence of Lie algebra homomorphisms on a symplectic manifold \((\mathfrak{g},\omega)\)

\[
0 \rightarrow H^0(M) \rightarrow C^\infty(M) \xrightarrow{\mathcal{H}} \text{LHam}(\omega) \xrightarrow{\varepsilon} H^1(M) \rightarrow 0,
\]

where \(H^0(M)\) and \(H^1(M)\) are the De Rham cohomology spaces with trivial Lie brackets, \(C^\infty(M)\) is taken with the Poisson bracket, \(\text{LHam}(\omega)\) is the Lie algebra of locally Hamiltonian vector fields, \(\mathcal{H}\) is the Hamiltonian map, and \(\varepsilon\) assigns to \(X \in \text{LHam}(\omega)\) the cohomology class of \(i_X \omega\).

We are also able to extend the Poisson bracket to a graded bracket on the exterior algebra generated over \(C^\infty(M)\) by Hamiltonian vector fields, which is an ‘integral’ of the Schouten-Nijenhuis bracket. In the case of a symplectic manifold (i.e. nondegenerate \(P\)) it is defined on all multivector fields and on ‘co-exact’ multivector fields we get the same structure as Cabras and Vinogradov [C-V].

Our proofs are chosen in this way that they can be used immediately in a more general setting of Poisson tensors for an arbitrary Lie algebroid. This fact, together with the lack of the Leibniz rule, makes some of the proofs computationally complicated.

Recall that the notion of a Lie algebroid is a straightforward generalization of a Lie algebra and, what is more significant in our case, also a generalization of a tangent bundle and plays a significant role in Poisson geometry (see [C-D-W]). All generalizations are presented in the last section, so the readers, who are not familiar with the concept of a Lie algebroid, may simply concentrate on ‘classical’: vector fields, differential forms, Poisson structures, etc.

In this paper, we do not give paper any direct applications of the introduced bracket to physics, rather concentrating on its properties and making the paper rigorously mathematical. However, our belief is that, because of its naturality and nice relations to other significant structures, the presented extension of the
Poisson bracket will find its applications in geometry and physics, as the other mentioned brackets do.

The paper is organized as follows. In the next section, we briefly recall main properties of the Schouten-Nijenhuis, Nijenhuis-Richardson, and Frölicher-Nijenhuis brackets.

In Section 3, we deal with a Poisson manifold \((M, P)\), defining the Poisson bracket \(\{\cdot, \cdot\}_P\) of functions and the Koszul-Schouten bracket. We also define a generalized Hamiltonian and related maps.

Section 4 is devoted to the definition of the graded extension of \(\{\cdot, \cdot\}_P\) to the proof that it is a graded Lie bracket and to its main properties and relations to other graded Lie brackets. Extensions of the Poisson bracket \(\{\cdot, \cdot\}_P\) to multivector fields are studied in Section 5.

In the last section we consider these structures in a more general setting of an arbitrary Lie algebroid.

## 2 Graded Lie brackets on a manifold

A **graded Lie bracket** on a graded vector space \(V = \bigoplus_{n \in \mathbb{Z}} V_n\) ("graded" means always \(\mathbb{Z}\)-graded throughout this paper) is a bilinear operation \(\{\cdot, \cdot\}: V \times V \to V\), being graded

\[
[V_n, V_m] \subset V_{n+m},
\]

(2.1)

graded skew-symmetric

\[
[x, y] = -(-1)^{|x||y|}[y, x],
\]

(2.2)

and satisfying the graded Jacobi identity

\[
[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]],
\]

(2.3)

where \(|x|\) is the degree of \(x\), i.e. \(x \in V_{|x|}\).

Let us fix the convention that we write simply \(x, \mu\), etc., for the degrees \(|x|\) (or \(|X|\), \(|\mu|\), etc., when no confusion is possible.

One sometimes writes the graded Jacobi identity in the form

\[
(-1)^{|x||y|}[[x, y], z] + (-1)^{|y||z|}[[y, z], x] + (-1)^{|z||x|}[[z, x], y],
\]

(2.4)

which is equivalent to 2.3 for graded skew-symmetric brackets. However, for non-skew-symmetric brackets the formula 2.5 seems to be better, since it means that the adjoint map \(x \mapsto \text{ad}_x \overset{\text{def}}{=} [x, \cdot]\) is a representation of the bracket, i.e. \(\text{ad}_x\) is equal to the graded commutator

\[
[\text{ad}_x, \text{ad}_y] \overset{\text{def}}{=} \text{ad}_x \circ \text{ad}_y - (-1)^{|y|\text{ad}_y \circ \text{ad}_x} = \text{ad}_{[x,y]},
\]

(2.5)

whereas 2.4 has no clear direct meaning.
With a given smooth \((C^\infty)\) manifold \(M\) there are associated several natural graded Lie brackets of tensor fields. Historically the first was probably the famous Schouten-Nijenhuis bracket \([\ , \ ]^{s-n}\) defined on multivector fields (see [Sc] for the original and [Mi2] for a modern version). It is the unique graded extension of the usual bracket \([\ , \ ]\) on the space \(\mathcal{X}(M)\) of vector fields to the exterior algebra \(A(M) = \bigoplus_{n \in \mathbb{Z}} A^n(M)\) of multivector fields (where \(A^n(M) = \Gamma(\Lambda^n TM)\) is the space of \(n\)-vector fields for \(n \geq 0\) and \(A^n(M) = \{0\}\) for \(n < 0\)) such that

1. the degree of \(X \in A^n(M)\) with respect to the bracket is \((n - 1)\),

\[
\text{(2) } [X, f]^{s-n} = \mathcal{L}(X)f \quad \text{for } X \in A^1(M), \ f \in A^0(M) = C^\infty(M) \tag{2.6}
\]

and \(\mathcal{L}(X)\) being the Lie derivative along \(X\),

\[
\text{(3) } [X, Y \wedge Z]^{s-n} = [X, Y]^{s-n} \wedge Z + (-1)^{(s-1)p} Y \wedge [X, Z]^{s-n} \tag{2.7}
\]

for \(X \in A^k(M), Y \in A^l(M)\), i.e. \(\text{ad}\) is a representation of the Schouten-Nijenhuis bracket in graded derivations of the graded algebra \(A(M)\).

Graded algebras furnished with a graded bracket satisfying 2.6 are called \textit{Gerstenhaber algebras} (see [KS1], [KS2]). From 2.7, it follows

\[
[X_1 \wedge \ldots \wedge X_m, Y_1 \wedge \ldots \wedge Y_n]^{s-n} = \sum_{k,l} (-1)^{k+l} [X_k, Y_l] \wedge \ldots \wedge \hat{X}_k \wedge \ldots \wedge X_m \wedge Y_1 \wedge \ldots \wedge \hat{Y}_l \wedge \ldots \wedge Y_n, \tag{2.8}
\]

where \(X_k, Y_l \in \mathcal{X}(M)\) and the hats stand for omissions. Let us note that the skew-symmetric Schouten-Nijenhuis bracket has its symmetric counterpart — the symmetric Schouten bracket \([\ , \ ]^s\) — defined on the space \(S(M)\) of symmetric multivector fields (see [B-V2], [DV-M]). It is an ordinary (non-graded) Lie bracket extending the Lie bracket of vector fields, satisfying the analog of 2.6 and

\[
[X_1 \vee \ldots \vee X_m, Y_1 \vee \ldots \vee Y_n]^{s} = \sum_{k,l} [X_k, Y_l] \vee \ldots \vee \hat{X}_k \vee \ldots \vee X_m \vee Y_1 \vee \ldots \vee \hat{Y}_l \vee \ldots \vee Y_n. \tag{2.9}
\]

It is well known that the map

\[
i : (S(M), [\ , \ ]^s) \longrightarrow (C^\infty(T^*M), \{\ , \ }_{p_M}),
i(X_1 \ldots \vee X_m) = i(X_1) \ldots i(X_m), \tag{2.10}
\]

where \(X_k \in \mathcal{X}(M)\) and \(i(X_k)(\theta_q) \overset{\text{def}}{=} < X_k(q), \theta_q >, i(f) \overset{\text{def}}{=} \pi_M^*(f)\) for \(f \in C^\infty(M)\), is an injective homomorphism of the symmetric Schouten bracket on
\( M \) into the canonical Poisson bracket \( \{ \cdot, \cdot \}_{PM} \) on the cotangent bundle \( \pi_M : T^* M \to M \).

Let us denote by \( \Omega^n (M) \) the space of \( n \)-forms on \( M \), by \( \Omega (M) \) – the exterior algebra of differential forms \( (\Omega (M) = \bigoplus_{n \in \mathbb{Z}} \Omega^n (M) \), with \( \Omega^n (M) = \{ 0 \} \) for \( n < 0 \), and by \( \Omega (M; TM) = \bigoplus_{n \in \mathbb{Z}} \Omega (M; TM) \) – the \( \Omega (M) \)-module of vector valued forms. Clearly, we have \( \Omega^n (M; TM) = \mathcal{X} (M) \) and the left and right actions of \( \Omega (M) \) on \( \Omega (M; TM) \) are given by

\[
\nu \wedge (\mu \circ X) = (\nu \wedge \mu) \circ X = (-1)^{\mu \nu} (\mu \circ X) \wedge \nu,
\]

for \( \mu, \nu \in \Omega (M), X \in \mathcal{X} (M) \).

We can extend usual insertion operators \( i (X) : \Omega^n (M) \to \Omega^{n-1} (M) \) and Lie derivatives \( \mathcal{L} (X) : \Omega^n (M) \to \Omega^n (M) \), defined for \( X \in \mathcal{X} (M) \), to insertions \( i (K) : \Omega^n (M) \to \Omega^{n+k-1} (M) \) and Lie differentials \( \mathcal{L} (K) : \Omega^n (M) \to \Omega^{n+k} (M) \), defined for \( K \in \Omega^k (M; TM) \), putting

\[
i (\mu \circ X) \nu \overset{\text{def}}{=} \mu \wedge i (X) \nu
\]

and

\[
\mathcal{L} (K) \overset{\text{def}}{=} i (K) \circ d + (-1)^k d \circ i (K),
\]

so that

\[
\mathcal{L} (\mu \circ X) \nu = \mu \wedge \mathcal{L} (X) \nu + (-1)^\mu d \mu \wedge i (X) \nu.
\]

We can also extend the insertion \( i (K) \) itself to the operator

\[
i (K) : \Omega^n (M; TM) \to \Omega^{n+k-1} (M; TM),
i (K) (\mu \circ X) \overset{\text{def}}{=} i (K) \mu \circ X.
\]

The Nijenhuis-Richardson bracket \( [\cdot, \cdot]^{N-R} \) is a graded Lie bracket on the graded space \( \Omega (M; TM) \), with elements of \( \Omega^n (M; TM) \) being of degree \( (n-1) \), defined by

\[
[K, L]^{N-R} \overset{\text{def}}{=} i (K) L - (-1)^{(k-1)(l-1)} i (L) K
\]

for \( K \in \Omega^k (M; TM), L \in \Omega^l (M; TM) \). We have the identity

\[
[i (K), i (L)] \overset{\text{def}}{=} i (K) \circ i (L) - (-1)^{(k-1)(l-1)} i (L) \circ i (K) = i ([K, L]^{N-R}).
\]

The Frölicher-Nijenhuis bracket \( [\cdot, \cdot]^{F-N} \) is a graded Lie bracket on the same space \( \Omega (M; TM) \), but, this time, with the grading which agrees with the form degree, defined on simple tensors \( \mu \circ X \in \Omega^\mu (M; TM) \) and \( \nu \circ Y \in \Omega^\nu (M; TM) \) by

\[
[\mu \circ X, \nu \circ Y]^{F-N} = \mu \wedge \nu \circ [X, Y] + \mu \wedge \mathcal{L} (X) \nu \circ Y - \mathcal{L} (Y) \mu \wedge \nu \circ X +
\]
The Frölicher-Nijenhuis bracket extends the usual bracket of vector fields (recall that $\Omega^3(M; TM) = X(M)$) and satisfies the following (cf. [K-M-S])

\[
\begin{align*}
[\mathcal{L}(K), \mathcal{L}(L)] & \overset{\text{def}}{=} \mathcal{L}(K) \circ \mathcal{L}(L) - (-1)^k \mathcal{L}(L) \circ \mathcal{L}(K) \\
& = \mathcal{L}([K, L]^F - N), \\
[\mathcal{L}(K), \iota(L)] & \overset{\text{def}}{=} \mathcal{L}(K) \circ \iota(L) - (-1)^k \iota(L) \circ \mathcal{L}(K) \\
& = \iota([K, L]^F - N) - (-1)^k \iota(L) \mathcal{L}(\iota(L) K).
\end{align*}
\]

Moreover,

\[
\begin{align*}
[K, \mu \wedge L]^F - N & = \mathcal{L}(K) \mu \wedge L + (-1)^k \mu \wedge [K, L]^F - N - (-1)^{(k+1)(k+1)} \mu \wedge \iota(L) K.
\end{align*}
\]

(see [Mi1], [DV-M], [K-M-S]).

### 3 Brackets on Poisson manifolds

Let us suppose that we are given a Poisson tensor on a manifold $M$, i.e., a bivector field $P \in \Lambda^2(M)$ such that

\[
[P, P]^S - N = 0.
\]

The corresponding Poisson bracket of functions

\[
\{f, g\}_P = < P, df \wedge dg >
\]

satisfies the Jacobi identity

\[
\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0
\]

(which is equivalent to 3.1) and the Leibniz rule

\[
\{f, gh\}_P = \{f, g\}_P h + g \{f, h\}_P.
\]

The standard examples of Poisson brackets in mechanics are associated with a phase space $T^* M$ or, more generally, with a symplectic manifold $(M, \omega)$, where $P = \omega^{-1}$ in the sense that the mapping

\[
P^\#: T^* M \rightarrow TM, \ P^\#(\mu) \overset{\text{def}}{=} i(\mu)P,
\]

is the inverse of

\[
\omega^1 : TM \rightarrow T^* M, \ \omega^1(X) \overset{\text{def}}{=} i(X)\omega.
\]
In canonical coordinates, with \( \omega = \sum_k \text{d}q_k \wedge \text{d}p_k \) we associate \( \mathcal{P} = \sum_k \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k} \) and the Poisson bracket \( \{ \cdot, \cdot \} \).

More general (degenerate) Poisson structures appear, for instance, in the process of Poisson reduction (e.g., The Kostant-Kirillov-Souriau bracket on the dual space of a Lie algebra), or, in the process of passing to semi-classical limits of quantum groups, when one encounters Poisson Lie structures (see [Dr]).

It is well known that assigning to a function \( f \) its Hamiltonian vector field \( \mathcal{H}^P(f) \overset{\text{def}}{=} \mathcal{P}^\#(df) \) gives a Lie bracket homomorphism:

\[
\mathcal{H}^P(\{f,g\}_P) = [\mathcal{H}^P(f),\mathcal{H}^P(g)]
\]  

and that

\[
\{f,g\}_P = \mathcal{L}(\mathcal{H}^P(f))g.
\]

Let us note that we will usually write \( \mathcal{P} \) instead of \( \mathcal{P}^\#(\mu) \) and \( \mathcal{H}\mu \) instead of \( \mathcal{H}^P(\mu) \), when it will be clear, what a Poisson tensor we have in mind.

It is known ([Fu], [M-M], [G-D], [Ka], [KS-M]) that the Poisson structure \( \mathcal{P} \) defines not only the Poisson bracket \( \{ \cdot, \cdot \}_P \) of functions, but also a Lie bracket \( \{ \cdot, \cdot \}_P \) on 1-forms, given by

\[
\{\mu,\nu\}_P = \mathcal{L}(\mathcal{P}_\mu)\nu - \mathcal{L}(\mathcal{P}_\nu)\mu - \text{d} \mu \wedge \nu > + \mu \wedge \nu >
\]

where \( < \cdot, \cdot > \) is the pairing between forms and multivector fields. In particular, \([\text{d}f,\text{d}g]_P = \text{d}\{f,g\}_P \) and \( \mathcal{P}^\# \) is a Lie bracket homomorphism:

\[
[\mathcal{P}_\mu,\mathcal{P}_\nu] = \mathcal{P}_{[\mu,\nu]}.
\]

The bracket \( \{ \cdot, \cdot \}_P \) on 1-forms can be extended to a graded Lie bracket on \( \Omega(M) \), as it was observed by Koszul [Ko] (see also [KS-M] and [Ka]), with \( n \)-forms being of degree \( (n-1) \), by the formula

\[
[\mu,\nu]_P = (-1)^p (\partial_P(\mu \wedge \nu) - \partial_P \mu \wedge \nu - (-1)^p \mu \wedge \partial_P \nu),
\]

where \( \partial_P = [i(P),\text{d}] = i(P)\text{d} - \text{d} \circ i(P) \) is the Poisson homology operator of Koszul and Brylinski (cf. [Vaj]). The following is essentially due to Koszul [Ko].

**Theorem 1 (Ko)** On the exterior algebra \( \Omega(M) \) of differential forms, the equation \( 3.11 \) defines a graded Lie bracket, called the Koszul-Schouten bracket, with \( n \)-forms being of bracket degree \( (n-1) \).

This bracket satisfies the Leibniz rule

\[
[\mu,\theta \wedge \nu]_P = [\mu,\theta]_P \wedge \nu + (-1)^{p-1} \theta \wedge [\mu,\nu]_P,
\]

where \( \mu \in \Omega^p(M) \) etc., and the exterior derivative is a derivation of the bracket

\[
\text{d}[\mu,\nu]_P = [\text{d}\mu,\nu]_P + (-1)^{p-1}[\mu,\text{d}\nu]_P.
\]
Moreover,

\[ [df, dg]_P = d\{f, g\}_P, \quad [df, g]_P = \{f, g\}_P, \quad [f, g]_P = 0 \]  \tag{3.14}

for functions \( f, g \in C^\infty(M) = \Omega^0(M) \), and the mapping \( \Lambda^P : \Omega(M) \rightarrow A(M) \),

\[ \Lambda^P : \Omega(M) \rightarrow A(M) \]  \tag{3.15}

defined by

\[ \Lambda^P(f) \overset{\text{def}}{=} f, \quad \Lambda^P(\mu_1 \wedge \ldots \wedge \mu_m) \overset{\text{def}}{=} P_{\mu_1} \wedge \ldots \wedge P_{\mu_m} \]  \tag{3.16}

for \( f \in C^\infty(M) \) and \( \mu_k \in \Omega^1(M) \), is a homomorphism of the Koszul-Schouten into the Schouten-Nijenhuis bracket:

\[ \Lambda^P(\mu, \nu)_P = [\Lambda^P(\mu), \Lambda^P(\nu)]^{\otimes^N}. \]  \tag{3.17}

It is easy to see that the mapping \( \Lambda^P \) is invertible if and only if \( P \) is nondegenerate. In this case, the inverse \( (\Lambda^P)^{-1} \) is given by

\[ (\Lambda^P)^{-1}(X_1 \wedge \ldots \wedge X_m) = \omega(X_1) \wedge \ldots \wedge \omega(X_m), \]  \tag{3.18}

where \( X_k \in \mathfrak{X}(M) \), \( \omega = P^{-1} \) is the symplectic form associated with \( P \) and \( \omega(X) = -i(X)\omega \) (cf. 3.5 and 3.6).

In [Mi1], Michor defines a 'generalized Hamiltonian mapping' on a symplectic manifold, using the unique extension \( P^\# : \Omega(M) \rightarrow \Omega(M; TM) \) of 3.5 into a derivation of degree \(-1\) on \( \Omega(M) \) with values in the \( \Omega(M) \)-module \( \Omega(M; TM) \) and putting \( H^P \overset{\text{def}}{=} P^\# \circ d \). We will continue writing in most cases \( P_\mu \) instead of \( P^\#(\mu) \), \( H_\mu \) instead of \( H^P(\mu) \), etc.

The Michor's construction is actually valid on any Poisson manifold (see [C-Y], [KS2], [B-M], [G-U2]). The mapping \( P^\# : \Omega(M) \rightarrow \Omega(M; TM) \) is characterized by the following

\begin{align*}
(1) & \quad P_f = 0 \quad \text{for } f \in C^\infty(M), \\
(2) & \quad P_\mu = i(\mu)P \quad \text{for } \mu \in \Omega^1(M), \\
(3) & \quad P_{\mu_1 \wedge \ldots \wedge \mu_m} = \sum_k (-1)^{k+1} \mu_1 \wedge \ldots \wedge \widehat{\mu_k} \wedge \ldots \wedge \mu_m \circ P_{\mu_k} 
\end{align*}  \tag{3.19}

for \( \mu_k \in \Omega^1(M) \).

As we mentioned already, \( P^\# \) is a derivation of degree \(-1\):

\[ P_{\mu \wedge \nu} = P_\mu \wedge \nu + (-1)^{\mu} \mu \wedge P_\nu \]  \tag{3.20}

and we have the generalized Hamiltonian map

\[ H^P : \Omega(M) \rightarrow \Omega(M; TM), \quad H^P(\mu) \overset{\text{def}}{=} P^\#(d\mu). \]  \tag{3.21}

Let us denote

\[ \langle \mu, \nu \rangle_P \overset{\text{def}}{=} (-1)^{\mu+1} (i(P)(\mu \wedge \nu) - i(P)\mu \wedge \nu - \mu \wedge i(P)\nu). \]  \tag{3.22}
Lemma 1

(1) \( <\mu,\nu>_P \in \Omega^{\mu+\nu-2} \). \hspace{1cm} (3.23)
(2) \( <\mu,\nu>_P = -(-1)^{\mu+1}(\nu+1) <\mu,\nu>_P \). \hspace{1cm} (3.24)
(3) \( <\mu,\nu \wedge \theta>_P = \mu \wedge + (-1)^{\mu\nu} \nu \wedge <\mu,\theta>_P \). \hspace{1cm} (3.25)
(4) \( <\mu,\nu>_P = i(P_\mu)\nu \). \hspace{1cm} (3.26)

Proof. Parts (1) and (2) are trivial. To prove the rest, let us assume that

\[ P = \sum_{j,k} c_{jk} X_j \otimes X_k, \quad X_j, X_k \in \mathcal{X}(M), \quad c_{jk} = -c_{kj} \in \mathbb{R}. \] \hspace{1cm} (3.27)

Then,

\[ <\mu,\nu>_P = \sum_{j,k} c_{jk} i(X_j)\mu \wedge i(X_k)\nu = i(\sum_{j,k} c_{jk} i(X_j)\mu \otimes X_k)\nu \] \hspace{1cm} (3.28)
and it is easy to see that \( <\mu,\nu>_P \) is a derivation of degree \( \mu - 2 \) with respect to \( \nu \), which proves (3).

Since both \( <\mu,\cdot>_P \) and \( i(P_\mu) \) are derivations of degree \( \mu - 2 \), and due to (2) and induction, it is now sufficient to prove (4) for \( \mu,\nu \in \Omega^1(M) \). We have, according to 3.28,

\[ i(P_\mu)\nu = \sum_{j,k} c_{jk} <X_j,\mu><X_k,\nu> = <\mu,\nu>_P, \] \hspace{1cm} (3.29)

where \( <,> \) is the obvious pairing, and the lemma follows. \( \square \)

Theorem 2 ([B-M]) The Koszul-Schouten bracket 3.11 may also be written in the form

\[ [\mu,\nu]_P \begin{array}{c} = \end{array} \begin{array}{c} i(H_\mu)\nu - (-1)^\mu L(P_\mu)\nu \\
= \end{array} \begin{array}{c} i(H_\mu)\nu - (-1)^\mu i(P_\mu)d\nu + \text{di}(P_\mu)\nu \\
= \end{array} \begin{array}{c} <d\mu,\nu>_P - (-1)^\mu <\mu,d\nu>_P + \text{di} <\mu,\nu>_P. \end{array} \] \hspace{1cm} (3.30)

Proof. Since both the Koszul-Schouten bracket on the left-hand side and the bracket defined by the right-hand side of 3.30 are graded skew-symmetric and are graded derivations of degree \( (\mu-1) \) with respect to \( \nu \), it is sufficient to check 3.30 for functions and 1-forms. Both sides are clearly 0 for functions, so let us assume that \( \mu \in \Omega^1(M), \nu = f \in C^\infty(M) \). Then, in view of 3.11,

\[ [\mu,f]_P \begin{array}{c} = \end{array} \begin{array}{c} f\partial_P \mu - \partial_P (f\mu) = f i(P)d\mu - i (P)(df \wedge \mu + f d\mu) \\
= \end{array} \begin{array}{c} i(P)(\mu \wedge df) = \text{di}(P_\mu)df, \end{array} \]

which agrees with 3.30, since \( i(H_\mu)f = i(P_\mu)f = 0 \).

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For 1-forms $\mu, \nu \in \Omega^1(M)$, according to 3.9 and Lemma 1,
\[
\begin{align*}
\{\mu, \nu\}_P &= i(P_\mu) d\nu - i(P_\nu) d\mu + d < P, \mu > \\
&= < \mu, d\nu >_P - < \nu, d\mu >_P + d < \mu, \nu >_P \\
&= < d\mu, \nu >_P + < \mu, d\nu >_P + d < \mu, \nu >_P
\end{align*}
\]
and the theorem follows. $\square$

4 Graded extensions of Poisson brackets

There were several attempts to extend the Poisson bracket of functions to a graded Lie bracket on $\Omega(M)$ such that the degree of a differential form coincides with the degree with respect to $\{ , \}_P$. The next assumption is that the exterior derivative is a homomorphism of the extended $\{ , \}_P$ into the Koszul-Schouten bracket. The last implies (and is equivalent in the case of a nondegenerate Poisson structure; see Corollary 1 and Theorem 6) that the generalized Hamiltonian map 3.21 should be a homomorphism of the extended $\{ , \}_P$ into the Frölicher-Nijenhuis bracket.

For symplectic manifolds (but with a straightforward generalization to arbitrary Poisson manifolds), Michor considered in [Mi1] the brackets
\[
\{\mu, \nu\}^1 \overset{\text{def}}{=} i(H_\mu) d\nu, \quad \{\mu, \nu\}^2 \overset{\text{def}}{=} L(H_\mu) \nu,
\]
(4.1)
The first one is graded skew-symmetric but it does not satisfy the graded Jacobi identity, while the second one satisfies the Jacobi identity but it is not skew-symmetric (it is a prototype of a Loday bracket [KS2]). The direct skew-symmetrization
\[
\{\mu, \nu\}^3 \overset{\text{def}}{=} \frac{1}{2} (L(H_\mu) \nu - (-1)^{\mu\nu} L(H_\nu) \mu)
\]
(4.2)
turns out again not to satisfy the Jacobi identity. All the brackets differ by something exact and
\[
H_{\{\mu, \nu\}} = H_{\{\mu, \nu\}}^1 = H_{\{\mu, \nu\}}^2 = [H_\mu, H_\nu]^F-N.
\]
(4.3)
Similar brackets for general Poisson structures, described in [C-V], are again either not skew-symmetric or do not satisfy the graded Jacobi identity. The graded Jacobi identity is satisfied only modulo exact forms and therefore all these brackets define (in fact, the same) graded Lie brackets on the space $\Omega(M)/B(M)$ of co-exact forms ($B(M)$ denotes, clearly, exact forms). In this section, we show the proper form of a graded extension of the Poisson bracket $\{ , \}_P$ which is simultaneously graded skew-symmetric and satisfies the graded Jacobi identity.

Before we define the graded extension of the Poisson bracket $\{ , \}_P$ of functions, let us start with some lemmata concerning the mapping $P^\#: \Omega(M) \ni \mu \mapsto P_\mu \in \Omega(M; TM)$. 

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Lemma 2 For $\mu, \nu \in \Omega(M)$, we have

$$P_i (P_{\nu}) = (-1)^{\mu} (i (P_{\mu}) P_{\nu} + (-1)^{\mu \nu} i (P_{\nu}) P_{\mu}).$$

(4.4)

Proof. Being a graded commutator of derivatives of degrees $-1$ and $(\mu - 2)$,

$$[P^\#_i, i (P_{\nu})] \equiv P^\#_i \circ i (P_{\mu}) - (-1)^{\mu} i (P_{\mu}) \circ P^\#_i$$

is a graded derivative of degree $(\mu - 3)$ on $\Omega(M)$ with values in the $\Omega(M)$-module $\Omega(M; TM)$, vanishing on functions. We will show that

$$[P^\#_i, i (P_{\nu})] (\nu) \equiv P_i (P_{\nu}) \nu - (-1)^{\mu} i (P_{\mu}) P_{\nu} = (-1)^{\mu (\nu + 1)} i (P_{\nu}) P_{\mu}.$$  

(4.6)

First, observe that

$$\nu \mapsto F_\mu (\nu) \equiv (-1)^{\mu (\nu + 1)} i (P_{\nu}) P_{\mu}$$

is also a graded derivation of degree $(\mu - 3)$ on $\Omega(M)$ with values in the $\Omega(M)$-module $\Omega(M; TM)$, vanishing on functions. Indeed,

$$F_\mu (\nu \wedge \theta) = (-1)^{\mu (\nu + 1) + 1} i (P_{\nu \wedge \theta}) P_{\mu}$$

$$= (-1)^{\mu (\nu + 1) + 1} i (P_{\nu} \wedge \theta + (-1)^{\nu} \nu \wedge P_{\theta}) P_{\mu}$$

$$= (-1)^{\mu (\nu + 1) + 1} i (P_{\nu}) P_{\mu} \wedge \theta + (-1)^{\mu (\nu + 1) + 1} \nu \wedge i (P_{\theta}) P_{\mu}$$

$$= F_\mu (\nu \wedge \theta + (-1)^{\mu (\nu + 1) + 1} \nu \wedge F_\mu (\theta).$$

Hence, inductively, it is sufficient to check 4.6 for $\nu$ being a 1-form. Thus, let us take $\nu \in \Omega^1(M)$ and $\mu = \mu_1 \wedge \ldots \wedge \mu_m$ with $\mu_j \in \Omega^1(M)$. We have

$$P_{\mu} = \sum_{j} (-1)^{j+1} \mu^{(j)} \otimes P_{\mu_j}$$

(4.8)

and

$$i (P_{\mu}) \nu = \sum_{j} (-1)^{j+1} \mu^{(j)} < P_{\mu_j} \wedge \nu >$$

(4.9)

where $\mu^{(j)} \equiv \mu_1 \wedge \ldots \wedge \mu_j \wedge \ldots \wedge \mu_m$. Hence,

$$P_i (P_{\mu}) \nu = \sum_{i < j} (-1)^{j+1} \sum_{l < j} \mu^{(l)} < P_{\mu_j} \wedge \nu > \mu^{(l)} \otimes P_{\mu_l}$$

$$+ \sum_{j < l} (-1)^{j+1} \sum_{l < j} \mu^{(l)} < P_{\mu_j} \wedge \nu > \mu^{(l)} \otimes P_{\mu_l},$$

(4.10)

where, for $1 \leq l < j \leq m$,

$$\mu^{(l,j)} \equiv \mu_1 \wedge \ldots \wedge \mu_l \wedge \ldots \wedge \mu_j \wedge \ldots \wedge \mu_m.$$  

(4.11)
On the other hand, \( i (P_{\mu}) \nu = 0 \), since \( P_{\mu} \in \Omega^0 (M; TM) \), and
\[
(-1)^{\frac{1}{2}(\nu + 1)} i (P_{\nu}) P_{\mu} = i (P_{\nu}) \sum_l (1)^{l+1} \rho^l \odot P_{\rho}.
\]
\[
= \sum_{j < l} (1)^{l+1} < P_{\nu} \wedge \rho^j \odot P_{\rho},
\]
\[
+ \sum_{l < j} (1)^{l+1} < P_{\nu} \wedge \rho^j \odot P_{\rho}.
\]

We see from 4.10 and 4.12 that \( P_{i(P_{\mu}) \nu} = i (P_{\nu}) P_{\mu} \), since \( \nu \wedge \rho^j = -\rho^j \wedge \nu. \) \( \square \)

The following generalization of 3.10 will be used in the sequel.

**Lemma 3** For \( \mu, \nu \in \Omega (M) \), we have
\[
[P_{\mu}, P_{\nu}]^{F-N} = P_{d_i (P_{\mu}) \nu} + (-1)^{(\mu + 1)(\nu + 1)} i (P_{\nu}) P_{d \rho} - i (P_{\mu}) P_{d \nu} = H_i (P_{\mu}) \nu + (-1)^{(\mu + 1)(\nu + 1)} i (P_{\nu}) H_{\rho} - i (P_{\rho}) H_{\nu}.
\]

**Proof.** Both sides of the equation 4.13 vanish, if \( \mu \) or \( \nu \) is a function. For \( \mu, \nu \in \Omega^1 (M) \), the equation 4.13 is equivalent to 3.10. Indeed, the left-hand side reduces to the usual bracket of vector fields and the right-hand side equals, according to 4.4,
\[
P_{d_i (P_{\nu}) \nu} = P_{d_i (P_{\nu}) \mu} + P_{i (P_{\nu}) d \rho} = P_{[\mu, \nu]}^{F-N}.
\]

We have, on one hand (cf. 2.21),
\[
[P_{\mu}, P_{\nu} \wedge \omega]^{F-N} = [P_{\mu}, \nu \wedge \omega \wedge P_{\nu}]^{F-N} = \mathcal{L} (P_{\mu}) \nu \wedge P_{\nu} + (-1)^{(\mu - 1)(\nu + 1)} \nu \wedge [P_{\mu}, P_{\omega}]^{F-N} - (-1)^{\nu + 1} \nu' \wedge i (P_{\omega}) P_{\rho} - (1)^{(\mu + 1)^{\nu} \wedge i (P_{\omega}) P_{\rho}} + (-1)^{\nu + 1} \nu' \wedge [P_{\mu}, P_{\rho}]^{F-N} + (-1)^{\nu + 1} \nu' \wedge i (P_{\rho}) P_{\nu}.
\]

On the other hand,
\[
P^# (d_i (P_{\mu}) (\nu' \wedge \nu)) + (-1)^{(\mu + 1)^{\nu} i (P_{\nu} \wedge \omega)} H_{\rho} - i (P_{\mu}) P_{d_i (\nu' \wedge \omega)})
\]
\[
= P^# ((d_i (P_{\mu}) (\nu') \wedge \nu + (-1)^{(\mu + 1)^{\nu}} (P_{\rho}) \nu' \wedge d \nu) + (-1)^{\nu + 1} \nu' \wedge i (P_{\rho}) d \nu) + (-1)^{(\mu + 1)^{\nu} \wedge i (P_{\rho}) d \nu} + (1)^{(\mu + 1)^{\nu} \wedge i (P_{\rho}) d \nu} + (-1)^{(\mu + 1)^{\nu} \wedge i (P_{\rho}) d \nu} + (1)^{(\mu + 1)^{\nu} \wedge i (P_{\rho}) d \nu} - i (P_{\mu}) (H_{\rho} \wedge \nu + d \nu \wedge P_{\nu} \wedge \nu' \wedge P_{\nu} + d \nu + \nu' \wedge H_{\rho})
\]
\[
= P^{\nu'} (i (P_{\mu}) d \nu + (-1)^{(\mu + 1)^{\nu} d \nu} + (1)^{(\mu + 1)^{\nu} d \nu} - (i (P_{\mu}) d \nu' + (-1)^{(\mu + 1)^{\nu} d \nu}) - P_{\nu} (d \nu + (i (P_{\mu}) d \nu') \wedge P_{\nu}).
\]

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where we used \( g_0 \) and the inductive assumption. Since the last coincides with \( g_1 \), the lemma follows.

**Corollary 1** The space \( \mathcal{Z}(M) \) of closed forms is a graded Lie subalgebra of the Koszul-Schouten algebra \( (\Omega(M), [\cdot, \cdot]_P) \) and the map

\[
P^\# : (\mathcal{Z}(M), [\cdot, \cdot]_P) \longrightarrow (\Omega(M; TM), [\cdot, \cdot]^{F-N})
\]

is a homomorphism of graded Lie algebras:

\[
P_{[\mu, \nu]} = [P_\mu, P_\nu]^{F-N} \quad \text{for } d\mu = 0, \, d\nu = 0.
\]

**Proof.** According to 3.30 and 4.13, \([\mu, \nu]_P = di(P_\mu)\nu\) for \( \mu, \nu \) being closed and

\[
P_{[\mu, \nu]} = H_i(P_\mu)\nu = [P_\mu, P_\nu]^{F-N}.
\]

\(\square\)

Let us define a bracket operation on differential forms on a given Poisson manifold \((M, P)\), putting

\[
\{\mu, \nu\}_P \overset{\text{def}}{=} \mathcal{L}(H_\mu)\nu + d\mathcal{L}(P_\mu)\nu
\]

\[
= i(P_\mu)d\nu + di(P_\mu)d\nu - (-1)^{\mu\nu} di(P_\nu)d\mu
\]

\[
= d\mu, \nu \cap P + d\mu, \nu \cap P + (-1)^{\mu\nu} d\mu, \nu \cap P.
\]

The notation is justified by the fact that on functions the bracket 4.19 coincides with the Poisson bracket 3.8. The bracket 4.19 is manifestly graded

\[
\{\mu, \nu\}_P \in \Omega^{\mu + \nu}(M),
\]

graded skew-symmetric

\[
\{\mu, \nu\}_P = -(-1)^{\mu\nu}\{\nu, \mu\}_P,
\]

and differs from the Michor’s brackets 4.1 by exact forms.
Theorem 3 We have, for $\mu, \nu \in \Omega(M),$

\begin{align}
(1) \quad H_{[\mu,\nu]} &= [H_\mu, H_\nu]^{F-N} \\
(2) \quad P_{[\mu,\nu]} &= [P_\mu, H_\nu]^{F-N} + (-1)^{\mu \nu} [H_\mu, P_\nu]^{F-N} \\
(3) \quad H_{(\mu,\nu)} &= [H_\mu, H_\nu]^{R-N} + [H_\mu, P_\nu]^{F-N} + (-1)^{\mu+1} [P_\mu, H_\nu]^{F-N} \\
(4) \quad P_{(\mu,\nu)} &= [P_\mu, P_\nu]^{F-N} + (-1)^{\mu} (H_\mu) P_\nu - (-1)^{\mu+1} i (H_\nu) P_\mu.
\end{align}

Proof. The proof depends on obvious calculations with the use of Lemmata 2 and 3. We present only the calculations for the identities (4.22) and (4.23) which will be used in the sequel.

Putting $\mu := d\mu$ and $\nu := d\nu$ in 4.13, we get

$$[H_\mu, H_\nu]^{F-N} = H_i (H_\mu) d\nu$$

and, by definition,

$$H_{(\mu,\nu)} = P_{d\{\mu,\nu\}} = H_i (H_\mu) d\nu.$$

To prove 4.23, we can write, using 4.13 and 4.4,

$$P_{[\mu,\nu]} = P_{d\{\mu,\nu\}} + H_i (H_\mu) d\nu - (-1)^{\mu \nu} H_i (H_\mu) d\mu + (-1)^{\mu+1} (i (H_\mu) H_\nu + (-1)^{(\mu+1)\nu+1}) i (H_\mu) H_\nu + (-1)^{\mu} (P_\mu, H_\nu) [P_\mu, H_\nu]^{F-N} + (-1)^{\mu+1} i (H_\mu) H_\nu + (-1)^{\mu} [P_\mu, H_\nu]^{F-N} + (-1)^{\mu+1} i (H_\mu) H_\nu.$$

Theorem 4 The graded skew-symmetric bracket 4.19 satisfies the graded Jacobi identity

$$\{\{\mu, \nu\}, \theta\} = \{\mu, \{\nu, \theta\}\} - (-1)^{\mu \nu} \{\nu, \{\mu, \theta\}\}.$$

Proof. According to 2.5, the Jacobi identity 4.28 is equivalent to

$$\mathcal{L}(H_{[\mu,\nu]} + d\mathcal{L}(P_{[\mu,\nu]} = [\mathcal{L}(H_\mu) + d\mathcal{L}(P_\mu), \mathcal{L}(H_\nu) + d\mathcal{L}(P_\nu)].$$

It is easy to see that the right-hand side of 4.29 equals

$$[\mathcal{L}(H_\mu), \mathcal{L}(H_\nu)] + d[\mathcal{L}(P_\mu), \mathcal{L}(H_\nu)] + (-1)^{\mu} d[\mathcal{L}(H_\mu), \mathcal{L}(P_\nu)],$$

where the brackets are the graded commutators of derivations, and further, in view of 2.20, to

$$\mathcal{L}(H_\mu) + d\mathcal{L}(P_\mu) + \mathcal{L}(H_\nu) + d\mathcal{L}(P_\nu).$$

(4.30)
According to 4.22 and 4.23, the last equals
\[ L(H_{[\mu, \nu]} \chi) + dL(P_{[\mu, \nu]} \chi) \]  
and the theorem follows. \( \square \)

**Remark.** One can prove the identities 4.24 and 4.25 using the methods similar to those of the above proof and the fact that we already know that the Koszul-Schouten bracket satisfies the graded Jacobi identity. Then, we can obtain 4.22 and 4.23 as consequences. However, this clever method fails for generalizations, when the Lie derivatives \( L(K) \) do not determine \( K \). Thus we used a direct method in proving Lemma 2, which has the advantage that it is valid in general.

Properties of the extended Poisson bracket are collected in the following.

**Theorem 5** For the bracket \( \mathbf{4.19} \), we have

1. Given \( \mu, \nu \in \Omega(M) \) and \( f \in C^\infty(M) \),
\[ \{ f, \mu \} = L(H_{[f, \mu]} \mu) \quad \text{and} \quad d\{ \mu, \nu \} = (-1)^{\mu} \{ \mu, d\nu \} \]  
2. In particular, the graded subspace \( Z(M) \) of closed forms is a commutative Lie ideal of \( (\Omega(M), \{ , \}) \).
3. Given \( f_0, \ldots, f_n, g_0, \ldots, g_m \in C^\infty(M) \),
\[ \{ f_0, \ldots, f_n, g_0 \wedge g_1 \wedge \ldots \wedge d g_m \} = \{ f_0, g_0 \} \wedge f_1 \wedge \ldots \wedge d f_n \wedge g_0 \wedge \ldots \wedge d g_m \]  
4. Instead of the classical Leibniz rule, we have
\[ \{ \mu, \theta \wedge \nu \} = \{ \mu, \theta \} \wedge \nu + (-1)^{\mu} \theta \wedge \{ \mu, \nu \} \]  
and the generalized Leibniz rule
\[ \{ \mu, \theta \wedge \nu \wedge \beta \} = \{ \mu, \theta \wedge \nu \} \wedge \beta + (-1)^{\mu} \theta \wedge \{ \mu, \nu \wedge \beta \} \]  
\[ +(-1)^{\mu+\nu} \{ \mu, \nu \wedge \beta \} \wedge \nu - \{ \mu, \beta \} \wedge \nu \wedge \beta \]  
\[ -(-1)^{\mu} \theta \wedge \{ \mu, \nu \} \wedge \beta - (-1)^{\mu+\nu} \theta \wedge \nu \wedge \{ \mu, \beta \}. \]
Proof. The part (1) follows directly by definition 4.19. To prove (2), let us denote \( \phi = df_0 \wedge \ldots \wedge df_n, \gamma = dg_0 \wedge \ldots \wedge dg_m, \phi^{(k)} = df_0 \wedge \ldots \wedge df_k \wedge \ldots \wedge df_n, \phi^{(l,k)} = df_0 \wedge \ldots \wedge df_k \wedge \ldots \wedge df_n, \) etc. We have

\[
\{f_0, f_1 \wedge \ldots \wedge df_n, g_0 \wedge \ldots \wedge dg_m\}_P = \{f_0, \phi^{(l)}(g), \gamma^{(l)}\}_P
\]

\[
i (P_{g})_{\gamma} + d \left( P_{f_{0}\phi^{(e)}} \right)_{\gamma} - (-1)^{nm}d \left( P_{g_{0}\gamma^{(e)}} \right)_{\phi}
\]

\[
i \left( \sum_{k} (-1)^{k} \phi^{(k)} \wedge H_{f_{0}} \right)_{\gamma} + d \left( f_0 \left( \sum_{k > 0} (-1)^{k+1} \phi^{(0,k)} \wedge H_{f_{0}} \right)_{\gamma} \right)
\]

\[
= \sum_{k,l} (-1)^{k+l} \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(l)} + d \left( f_0 \sum_{k > 0, l} (-1)^{k+l+1} \{ f_k, g_l \}_P \phi^{(0,k)} \wedge \gamma^{(l)} \right)
\]

\[
= \sum_{k,l} (-1)^{k+l} \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(l)} - \sum_{k > 0, l} (-1)^{k+l} \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(l)}
\]

\[
- \sum_{l > 0, k} (-1)^{k+l} \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(l)} - f_0 \sum_{k > 0, l} (-1)^{k+l} d \{ f_k, g_l \}_P \phi^{(0,k)} \wedge \gamma^{(l)}
\]

\[
= \{ f_0, g_l \}_P \phi^{(0,l)} \wedge \gamma \left( P \right) - \sum_{k,l > 0} (-1)^{k+l} \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(l)}
\]

\[
- \sum_{k > 0, l} (-1)^{k+l} d \{ f_k, g_l \}_P \phi^{(0,k)} \wedge \gamma^{(l)}
\]

\[
- g_0 \sum_{l > 0, k} (-1)^{n+k+l} d \{ f_k, g_l \}_P \phi^{(k)} \wedge \gamma^{(0,l,k)}
\]

The identity 4.35 we get by obvious calculations. Then, using 4.35, we get

\[
\{ \mu, \theta \wedge \nu \wedge \beta \}_P - (-1)^{\mu} \theta \wedge \{ \mu, \nu \wedge \beta \}_P
\]

\[
- \{ \mu, \theta \wedge \nu \}_P \wedge \beta + (-1)^{\mu} \theta \wedge \{ \mu, \nu \}_P \wedge \beta
\]

\[
= \{ (-1)^{\mu} \theta (-1)^{\mu+1} \mathcal{L}(P_{\theta}) \theta \wedge d \beta + (-1)^{\mu+1} d \theta \wedge \mathcal{L}(P_{\theta}) \beta \wedge \nu \}
\]

which is equivalent to 4.36. \( \square \)

The next theorem shows certain relations between the extended Poisson bracket and other graded Lie brackets.
Theorem 6  (1) The exterior derivative
\[ d : (\Omega(M), \{ , \}_P) \rightarrow (\Omega(M), [ , ]_P) \]  \hspace{1cm} (4.37)
is a homomorphism of \( \{ , \}_P \) into the Koszul-Schouten bracket:
\[ d\{\mu, \nu\}_P = [d\mu, d\nu]_P. \]  \hspace{1cm} (4.38)
Moreover, we have the exact sequence of graded Lie algebra homomorphisms
\[ 0 \rightarrow \mathcal{Z}(M) \rightarrow (\Omega(M), \{ , \}_P) \xrightarrow{\delta} (\mathcal{Z}(M), [ , ]_P) \rightarrow \mathcal{H}(M) \rightarrow 0, \]  \hspace{1cm} (4.39)
where on the closed forms \( \mathcal{Z}(M) \) (on the left) and on the De Rham cohomology space \( \mathcal{H}(M) \) we put the trivial brackets.

(2) The generalized Hamiltonian map
\[ H^P : (\Omega(M), \{ , \}_P) \rightarrow (\Omega(M; TM), [ , ]^{P-N}) \]  \hspace{1cm} (4.40)
is a homomorphism of \( \{ , \}_P \) into the Frölicher-Nijenhuis bracket:
\[ H_{\{\mu, \nu\}} - [H_\mu, H_\nu]_P^{P-N}. \]  \hspace{1cm} (4.41)

(3) The total Hamiltonian map
\[ G^P : (\Omega(M), \{ , \}_P) \rightarrow (A(M), [ , ]^{S-N}), \]  \hspace{1cm} (4.42)
given by \( G^P \overset{\text{def}}{=} \Lambda_P \text{cd}, \) (see 3.16) is a homomorphism of \( \{ , \}_P \) into the Schouten-Nijenhuis bracket:
\[ G_{\{\mu, \nu\}} = [G_\mu, G_\nu]^{S-N}. \]  \hspace{1cm} (4.43)

Proof. The part (1) follows easily by 3.30 and 4.19. Note that \( \mathcal{Z}(M) \) is a commutative Lie ideal with respect to \( \{ , \}_P \) by Theorem 5 (1).

The part (2) just repeats 4.22, and (3) follows immediately, when we combine 4.38 with 3.17. \( \square \)

Let us remark, that the exact sequence 4.39 generalizes 1.2. Indeed, we have \( \mathcal{Z}^0(M) = \mathcal{H}^0(M) \) and locally Hamiltonian vector fields \( \text{LHam}(\omega) \) form the Lie algebra isomorphic to \( (\mathcal{Z}^1(M), [ , ]_P) \) in the case of a symplectic form \( \omega = P^{-1} \).

Also the exact sequences considered by Michor [Mi1] and using the generalized Hamiltonian map may be obtained from 4.39 in the symplectic case, since we can pass to the same bracket on co-exact forms, and the mapping
\[ \Omega^m (M) \ni \mu \mapsto \frac{(-1)^m}{m} [i[H_\mu]\omega] \in H^{m+1}(M) \]  \hspace{1cm} (4.44)
equals, in the symplectic case,
\[ \Omega^m (M) \ni \mu \mapsto [d\mu] \in H^{m+1}(M). \]  \hspace{1cm} (4.45)
The equation 4.34 shows that our bracket is given by a bilinear differential operator of order 2, and the equation 4.36 that we have only a generalized Leibniz rule. In fact, we cannot expect the classical Leibniz rule
\[ \{\mu, \theta \wedge \nu\}_P = \{\mu, \theta\}_P \wedge \nu + (-1)^{\theta} \theta \wedge \{\mu, \nu\}_P, \]
(4.46)
as shows the following.

**Theorem 7** For a non-trivial Poisson tensor \( P \) there is no graded Lie bracket \( \{ , \}_P \) on \( \Omega(M) \) extending the Poisson bracket of functions and satisfying simultaneously 4.38 and the Leibniz rule 4.46.

**Proof.** Combining 4.38 and 4.46, we get easily
\[
(-1)^{\theta} \theta \wedge \{\mu, \nu\}_P = \{\mu, \theta\}_P \wedge \nu + (-1)^{\theta} \theta \wedge \{\mu, \nu\}_P, \]
(4.47)
Putting now \( \theta := d\theta \) in 4.47, we get
\[ \{\mu, d\theta\}_P \wedge \nu = (-1)^{\theta} d\{\mu, \theta\}_P \wedge \nu \]
(4.48)
and hence
\[ \{\mu, d\theta\}_P = (-1)^{\theta} d\{\mu, \theta\}_P = (-1)^{\theta} \{\mu, \theta\}_P. \]
(4.49)
(cf. 4.33), since \( \dim(M) > 1 \) for non-trivial \( P \). Now, putting \( \theta := \theta \wedge \nu \) in 4.49 and using the Leibniz rule 4.46, we get
\[ (-1)^{\theta} \theta \wedge \{\mu, \nu\}_P + (-1)^{\theta+\nu} \{\mu, \theta\}_P \wedge \nu = 0. \]
(4.50)
Hence, \( d\theta \wedge \{\mu, d\nu\}_P = 0 \) and, consequently, \( \{\mu, d\nu\}_P = d\{\mu, \nu\}_P = 0 \) for all \( \mu \) and \( \nu \) being functions. This implies \( P = 0 \). \( \square \)

It is interesting that we have not only the homomorphism 4.40 but also an embedding of the Frölicher-Nijenhuis bracket into the extended canonical Poisson bracket on the corresponding cotangent bundle. Namely, let \( \pi_M : T^* M \longrightarrow M \) be the cotangent bundle over \( M \) and let \( P_M \) be the canonical Poisson structure on \( T^* M \) associated with the canonical symplectic form \( \omega_M \). Recall that we have the homomorphism of Lie algebras 2.10.

**Theorem 8** The mapping
\[
J^* : (\Omega(M;TM), \{ , \}_P \wedge \nu) \longrightarrow (\Omega(T^* M), \{ , \}_{P_M}), \]
(4.51)
given on simple tensors by \( J^*(\mu \otimes X) \overset{\text{def}}{=} i(X)\pi_M^* (\mu) \), is an injective homomorphism of graded Lie algebras.

Composing \( J^* \) with the mappings \( H^{P_M} \) and \( G^{P_M} \) (cf. 4.40 and 4.42), we get the injective homomorphisms of graded Lie algebras:
\[
H : (\Omega(M;TM), \{ , \}_P \wedge \nu) \longrightarrow (\Omega(T^* M;TT^* M), \{ , \}_P \wedge \nu), \]
\[ H \overset{\text{def}}{=} H^{P_M} \circ J^*, \]
(4.52)
\[ G : (\Omega(M; TM), [ , ]^N) \rightarrow (A(T^*M), [ , ]^S) \]
\[ G \overset{\text{def}}{=} G^P \circ \mathcal{J}. \]  

(4.53)

For the proof, depending on obvious direct calculations with the use of 4.34, we refer to [G-U2]. Let us only remark that, due to the embeddings 4.51 and 2.10, we can regard \((\Omega(T^*M), [ , ]^P)\) as a common generalization of the Frölicher-Nijenhuis and the symmetric Schouten brackets.

Also \((\Omega(T^*M; TT^*M), [ , ]^P)\), in the presence of 4.52, may be viewed as such a common generalization, which was first observed by Dubois-Violette and Michor [DV-M]. On the other hand, the embedding 4.53 implies that one can regard the Frölicher-Nijenhuis algebra (over \(M\)) as a subalgebra of the Schouten algebra (over \(T^*M\)), so that 4.52 may serve as a definition of \([ , ]^P\).

Let us finish this section with results showing that our extension of the Poisson bracket behaves well also with respect to the tangent lifts. The tangent lifts (vertical \(v_T\) and complete \(d_T\)) of tensors over a manifold \(M\) to tensors over its tangent bundle \(\tau_M : TM \rightarrow M\) have been considered, for example, in [Y-I], [G-U1] and [G-U2] (for definitions and basic facts we refer to [G-U1]). In particular,

\[ v_T \overset{\text{def}}{=} \tau_M^* : \Omega(M) \rightarrow \Omega(TM) \]

(4.54)
is a homomorphism of exterior algebras and the complete lift \(d_T : \Omega(M) \rightarrow \Omega(TM)\) is a \(v_T\)-derivation of degree 0:

\[ d_T(\mu \wedge \nu) = d_T(\mu) \wedge v_T(\nu) + v_T(\mu) \wedge d_T(\nu), \]

(4.55)

commuting with the exterior derivative

\[ d_T(d\mu) = dd_T(\mu). \]

(4.56)
The identity 4.55 is also true for multivector fields and, for \(X \in \mathfrak{X}(M)\) and a differential form \(\mu\),

\[ d_T(i(X)\mu) = i(d_T(X))(d_T(\mu)) \quad v_T(i(X)\mu) = i(v_T(X))(d_T(\mu)) \quad \text{(4.57)} \]

(see [G-U1]).

The complete tangent lift preserves the Schouten-Nijenhuis bracket:

\[ [d_T(X), d_T(Y)]^S = d_T([X, Y]^S), \]

(4.58)

so the complete lift \(d_T(P)\) of a Poisson tensor \(P\) on \(M\) is a Poisson tensor on \(TM\) (see [G-U1], [Co1] and [Co2]).

**Theorem 9 (1)** The complete lift of differential forms on a Poisson manifold \((M, P)\) preserves the graded extensions of Poisson brackets:

\[ d_T(\{\mu, \nu\}_P) = \{d_T(\mu), d_T(\nu)\}_{d_T(P)}, \]

(4.59)
where $\delta_T(P)$ is the Poisson tensor on $TM$ being the complete lift of $P$.

(2) The following cotangent Poisson lift

$$\mathcal{J}^P \overset{\text{def}}{=} \mathcal{J}^* \circ \mathcal{H}^P : (\Omega(M), \{, \}) \to (\Omega(T^*M), \{, \}_{P_M}),$$

(4.60)

where $P_M$ is the canonical Poisson tensor on the cotangent bundle $T^*M$, is a homomorphism of graded Lie algebras:

$$\mathcal{J}^P (\{\mu, \nu\}_P) = \{\mathcal{J}^P (\mu), \mathcal{J}^P (\nu)\}_{P_M}.$$  

(4.61)

**Proof.** According to 4.19 and the fact that the complete lift of forms commutes with the exterior derivative, it suffices to show that

$$\delta_T(\langle \mu, \nu \rangle_P) = \langle \delta_T(\mu), \delta_T(\nu) \rangle_{\delta_T(P)}.$$  

(4.62)

For $P = \sum_{j,k} c_{jk} X_j \otimes X_k$, we have (cf. 4.57)

$$\delta_T(P) = \sum_{j,k} c_{jk} (\delta_T(X_j) \otimes \nu_T(X_k) + \nu_T(X_j) \otimes \delta_T(X_k)).$$  

(4.63)

and

$$\delta_T(\langle \mu, \nu \rangle_P) = \delta_T(\sum_{j,k} c_{jk} \langle X_j, \nu \rangle \wedge \mu)$$

$$= \sum_{j,k} c_{jk} \langle \delta_T(X_j), \nu \rangle \wedge \delta_T(\mu) + \nu_T(X_j) \wedge \delta_T(\mu)$$

$$= \sum_{j,k} c_{jk} \langle \delta_T(X_j), \nu_T \rangle \wedge \delta_T(\mu) + \nu_T(X_j) \wedge \delta_T(\mu)$$

$$+ \mu \wedge \delta_T(X_j) \wedge \nu = \langle \delta_T(\mu), \delta_T(\nu) \rangle_{\delta_T(P)},$$

where we used 4.55 and 4.57.

The second part is trivial, since both $\mathcal{H}^P$ and $\mathcal{J}^*$ are homomorphisms of graded Lie algebras (see 4.41 and Theorem 8). □

5 Extensions of Poisson brackets to multivector fields

We know already from Theorem 1 that the mapping $\Lambda^P : \Omega(M) \to A(M)$ is a homomorphism of the Koszul-Schouten into the Schouten-Nijenhuis bracket. Moreover, if the Poisson tensor $P$ is nondegenerate and $\omega = P^{-1}$ is the associated symplectic form, then $\Lambda^P(\omega) = P$ and

$$\Lambda^P(i(\mu)) = i(\omega)\Lambda^P(\mu).$$  

(5.1)
Using the Koszul-Schouten bracket on sections of the cotangent bundle $T^*M$, we can develop a contravariant version of differential calculus exchanging the role of forms and multivector fields (see [M-X], [G-U2], [KS-M], [Va]). For example, the ‘exterior derivative’ $d_P : A^n (M) \rightarrow A^{n+1} (M)$ is given by the Cartan formula

$$d_P X(\mu_1, \ldots, \mu_{n+1}) = \sum_k (-1)^{k+1} P_{\mu_k} (X(\mu_1, \ldots, \hat{\mu_k}, \ldots, \mu_{n+1}))$$

$$+ \sum_{k<l} (-1)^{k+l} X([\mu_k, \mu_l] P_{\mu_k}, \mu_1, \ldots, \hat{\mu_k}, \ldots, \hat{\mu_l}, \ldots, \mu_{n+1}).$$

It is well known that

$$d_P (X) = [P, X]^{S-N} \text{ and } \Lambda^P (d\mu) = -d_P (\Lambda^P (\mu)). \quad (5.2)$$

The $\Omega(M)$-module $\Omega(M; TM)$ of vector valued forms is replaced by the $A(M)$-module $A(M; T^*M)$ of 1-form valued multivector fields (or multivector valued 1-forms) and we can define, instead of insertions $i(K)\mu$ with $K \in \Omega(M; TM)$, $\mu \in \Omega(M)$, insertions $i(\kappa)X$ for $\kappa \in A(M; T^*M)$ and $X \in A(M)$ in the natural way. We have even an analog of the Frölicher-Nijenhuis bracket on $A(M; T^*M)$ (see [G-U2]).

Using the Poisson tensor $P$, we can define a mapping

$$\lambda^P : \Omega(M) \rightarrow A(M; T^*M), \quad (5.3)$$

putting $\lambda^P (f) \overset{\text{def}}{=} 0$ for $f \in C^\infty (M)$ and

$$\lambda^P (\mu_1 \wedge \ldots \wedge \mu_m) \overset{\text{def}}{=} \sum_k (-1)^{k+1} P_{\mu_k} \wedge \ldots \wedge \hat{P}_{\mu_k} \wedge \ldots \wedge P_{\mu_m} \otimes \mu_k \quad (5.4)$$

for $\mu_k \in \Omega^1 (M)$.

It is easy to see that $\lambda^P$ is a $\Lambda^P$-derivation of degree $-1$:

$$\Lambda^P (\mu \wedge \nu) = \lambda^P (\mu) \wedge \lambda^P (\nu) + (-1)^{\mu} \Lambda^P (\mu) \wedge \lambda^P (\nu) \quad (5.5)$$

(cf. 3.20). Fixing $P$, we will write usually $\lambda_\mu$ instead of $\lambda^P (\mu)$ and $\Lambda_\mu$ instead of $\Lambda^P (\mu)$.

**Theorem 10**

$$\Lambda^P ([\mu, \nu]_P) = i (\lambda_\mu) d_P (\Lambda_\nu) + (-1)^\mu d_P (i (\lambda_\mu)\Lambda_\nu)$$

$$-d_P (i (\lambda_\mu) d_P (\Lambda_\nu)). \quad (5.6)$$

*In particular, Ker($\Lambda^P$) is a Lie ideal of $(\Omega (M), [\ , \ ]_P)$.*

**Proof.** First, let us see that

$$\Lambda i (P_\mu) \nu = -i (\lambda_\mu) \Lambda_\nu. \quad (5.7)$$
Indeed, both sides of 5.7 vanish for $\mu$ or $\nu$ being a function, so let us assume that $\mu = \mu_1 \wedge \ldots \wedge \mu_m$, $\nu = \nu_1 \wedge \ldots \wedge \nu_n$, with $\mu_k, \nu_l \in \Omega^1(M)$. Then,

$$\Lambda_i (p_{\mu})^\nu =$$

$$= \sum_{k, l} (-1)^{k+l} \tilde{P}_{\mu_k} \wedge \nu_l > \tilde{P}_{\mu_1} \wedge \ldots \wedge \tilde{P}_{\mu_k} \wedge \tilde{P}_{\nu_1} \wedge \ldots \wedge \tilde{P}_{\nu_l} \wedge \ldots \wedge \tilde{P}_{\nu_n}$$

$$- i \left( \sum_k (-1)^{k+1} \tilde{P}_{\mu_k} \wedge \ldots \wedge \tilde{P}_{\mu_m} \otimes \mu_k \right) \tilde{P}_{\nu_1} \wedge \ldots \wedge \tilde{P}_{\nu_n} = - i (\lambda_{\mu}) \Lambda_{\nu}.$$

Now, the theorem is a direct consequence of 4.19 and 5.2. □

On the image $\text{A}_P(M) \overset{\text{def}}{=} \Lambda^P(\Omega(M))$ of $\Lambda^P$, which is the exterior algebra generated over $C^\infty(M)$ by Hamiltonian vector fields and a graded Lie subalgebra of $(\text{A}(M), [\cdot, \cdot]^s-N)$, let us define the Poisson bracket $\{ \cdot, \cdot \}_P$, putting

$$\{ \Lambda_{\mu}, \Lambda_{\nu} \}_P \overset{\text{def}}{=} \Lambda_{\{\mu,\nu\}_P}. \tag{5.8}$$

On functions -- we get the original Poisson bracket, on hamiltonian vector fields -- we get 0, but it is not clear in general that the bracket 5.8 is well defined.

**Theorem 11** The bracket on $\text{A}_P(M)$, given by 5.8, is well defined and we have the formula

$$\{ f, \text{H}_{f_1} \wedge \ldots \wedge \text{H}_{f_n}, g, \text{H}_{g_1} \wedge \ldots \wedge \text{H}_{g_m} \}_P$$

$$= \{ f, g \}_P \text{H}_{f_1} \wedge \ldots \wedge \text{H}_{f_n} \wedge \text{H}_{g_1} \wedge \ldots \wedge \text{H}_{g_m} \tag{5.9}$$

$$- \sum_{k, l > 0} (-1)^{k+l} \{ f_k, g_l \}_P \text{H}_{f_1} \wedge \ldots \wedge \text{H}_{f_n} \wedge \text{H}_{g_1} \wedge \ldots \wedge \text{H}_{g_m}$$

$$- f_l \sum_{k > 0, l} (-1)^{k+l} [\text{H}_{f_k}, \text{H}_{g_l}] \wedge \text{H}_{f_1} \wedge \ldots \wedge \text{H}_{f_n} \wedge \text{H}_{g_1} \wedge \ldots \wedge \text{H}_{g_m}$$

$$- g_k \sum_{l > 0, h} (-1)^{n+k+l} [\text{H}_{f_k}, \text{H}_{g_l}] \wedge \text{H}_{f_1} \wedge \ldots \wedge \text{H}_{f_n} \wedge \text{H}_{g_1} \wedge \ldots \wedge \text{H}_{g_m},$$

where $f_k, g_l \in C^\infty(M)$ and $\text{H}_{f_k}, \text{H}_{g_l}$, etc., are the corresponding Hamiltonian vector fields.
Moreover, the map

\[-d_P : (A_P(M), \{ , \}^P) \longrightarrow (A(M), \{ , \}^{S-N}),\]

\[-d_P(X) = -[P, X]^{S-N}\]  \hspace{1cm} (5.10)

is a homomorphism of graded Lie algebras.

If \( P \) is nondegenerate, then \( A_P(M) = A(M) \) and the bracket 5.8, defined on all multivector fields, is an 'integral' of the Schouten-Nijenhuis bracket and can be written in terms of the associated symplectic form \( \omega = P^{-1} \) as follows:

\[
\{X,Y\}_P = -d_P(X), d_P(Y) >_\omega + (-1)^r d_P < d_P(X), Y >_\omega + d_P < X, d_P(Y) >_\omega,
\]  \hspace{1cm} (5.11)

where

\[
<X, Y >_\omega \overset{\text{def}}{=} (-1)^{r+1}(i(\omega)(X \wedge Y) - i(\omega)X \wedge Y) = i(\omega)X \wedge Y - X \wedge i(\omega)Y.
\]  \hspace{1cm} (5.12)

**Proof.** It follows from 5.6 that \( \Lambda_{\mu} = 0 \) implies \( \Lambda_{\{\mu,\nu\}} = 0 \), so the bracket 5.8 is well defined. We get now 5.9 from 4.34, since

\[
\Lambda_{j_0j_1\ldots j_r} = f_{j_0}H_{j_1} \wedge \ldots \wedge H_{j_r},
\]  \hspace{1cm} (5.13)

and \( H_{\{f,g\}} = [H_f, H_g] \).

The last part is a direct consequence of 4.19, 5.1 and 5.2. \( \square \)

6. **Generalizations**

All above can be done in a more general setting. We just replace the tangent bundle \( \tau_M : TM \longrightarrow M \), furnished with the Lie bracket \( \{ , \} \) on its sections (vector fields), with an arbitrary Lie algebroid over \( M \).

Let us recall that a *Lie algebroid* (see [Pr] and [Ma]) over a manifold \( M \) is a triple \( (\tau, \{ , \}, a_\tau) \), where \( \tau : E \longrightarrow M \) is a vector bundle over \( M \), \( \{ , \}_\tau \) is a Lie bracket on sections \( \Gamma(E) \) of \( \tau \), and \( a_\tau : E \longrightarrow TM \) is a vector bundle morphism (called the anchor map), such that

1. The anchor map induces on sections a Lie algebra homomorphism:

\[
a_\tau([X,Y]_\tau) = [a_\tau(X), a_\tau(Y)].
\]  \hspace{1cm} (6.1)

2. For any \( f \in C^\infty(M) \) and \( X, Y \in \Gamma(E) \), we have the following Leibniz rule

\[
[X, fY]_\tau = f[X,Y]_\tau + a_\tau(X)(f)Y.
\]  \hspace{1cm} (6.2)
The tangent bundle itself, with $\pi = \pi_M$, $a = \text{id}_{TM}$ and the usual Lie bracket on $\Gamma(TM) = \mathcal{X}(M)$ is a canonical Lie algebroid.

Another significant example is a Lie algebroid structure on the cotangent bundle $\pi_M : T^*M \to M$ over a Poisson manifold, with the Lie bracket on sections of $T^*M$ (1-forms) given by the bracket $[\ , \ ]_P$ (cf. 3.9) and the anchor map $\rho^\#$ (cf. 3.5). Given a Lie algebroid, we can generalize the standard calculus of differential forms and vector fields. We replace the exterior algebras of multivector fields and differential forms by the exterior algebras

$$\bigwedge^k(M) \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R} \quad \text{and} \quad \bigwedge^k(E) \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R}.$$ 

and

$$\bigwedge^k\bigwedge^k\mathbb{R} \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R}$$

and the anchor map $\rho^\#$. Given a Lie algebroid, we can generalize the standard calculus of differential forms and vector fields. We replace the exterior algebras of multivector fields and differential forms by the exterior algebras

$$\bigwedge^k(M) \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R} \quad \text{and} \quad \bigwedge^k(E) \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R}.$$

and

$$\bigwedge^k\bigwedge^k\mathbb{R} \overset{\text{def}}{=} \bigwedge^k \bigwedge \mathbb{R}$$

We have $d_e^2 = 0$ and, for $X \in \Gamma(E)$, one defines the insertion operator $i_X : \Phi^\alpha(\pi) \to \Phi^{\alpha-1}(\pi)$ and the Lie derivative $\mathcal{L}(X) \overset{\text{def}}{=} i_X \circ d_e + d_e \circ i_X$ in the obvious way. All the standard formulae, like

$$d_e(\mu \wedge \nu) = d_e(\mu) \wedge \nu + (-1)\mu \wedge d_e(\nu),$$

or

$$\mathcal{L}(X) \circ \mathcal{L}(Y) - \mathcal{L}(Y) \circ \mathcal{L}(X) = \mathcal{L}([X,Y]_\tau),$$

hold and, quite parallel to the standard definitions, we can define the (generalized) Schouten-Nijenhuis, Nijenhuis-Richardson, and Frölicher-Nijenhuis brackets (see [G-U2]).

The Schouten-Nijenhuis bracket $[\ , \ ]_\tau^{S-N}$ is defined on $\Phi(\pi)$ and satisfies the formulae analogous to 2.6 and 2.7. The Nijenhuis-Richardson $[\ , \ ]_\tau^{N-R}$ and the Frölicher-Nijenhuis $[\ , \ ]_\tau^{F-N}$ brackets are defined on the space

$$\Phi_1(\pi) \overset{\text{def}}{=} \bigoplus_{k \in \mathbb{Z}} \Phi^k(\pi), \quad \Phi^{k}(\pi) \overset{\text{def}}{=} \Gamma(\bigwedge^k E^* \otimes E),$$

with formally the same definitions as 2.16 and 2.18 in the classical case. One important difference in the general case is the fact that the space $\Phi^k(\pi)$ of $k$-forms is usually not generated, as a $C^\infty(M)$-module, by exact ‘forms’, as in the
case $\Phi^k(\tau) = \Omega(M)$. This is possible, since the exterior derivative $d_\tau$ depends on the anchor $a_\tau$ and the Lie bracket $[\cdot, \cdot]$, which may be quite degenerated. It makes inductive proofs harder and it is also the reason, why the definition of the generalized Frölicher-Nijenhuis bracket via the identity 2.20 is not possible: $\mathcal{L}(K) : \Phi(\tau) \rightarrow \Phi(\tau)$ may be trivial for a non-trivial $K \in \Phi(\tau)$.

For the Lie algebroid structure on the cotangent bundle $\pi_M : T^*M \rightarrow M$ over a Poisson manifold $(M, P)$, the exterior derivative, we get, is exactly $d_P = [P, \cdot]^{s-N}$, as was proved, independently in various contexts (see [B-V1], [KS-M] and [Hu]), and the generalized Schouten-Nijenhuis bracket is, in this case, the Koszul-Schouten bracket 3.11 ([Ko], [KS-M], [KS1], [Kr1], [Kr2]).

In general, a Poisson tensor for a Lie algebroid $(\tau, [\cdot, \cdot], a_\tau)$ is a tensor $P \in \Phi^2(\tau)$ satisfying $[P, P]^{s-N} = 0$. The formula 3.8 defines then a Lie algebroid structure on the dual bundle $\pi : E^* \rightarrow M$, with the anchor $a_\tau = a_\tau \circ P^\#$ (cf. [KS-M] and [M-X]). Hence, we have Lie algebroid structures on both: the original bundle $\pi : E \rightarrow M$ and the dual bundle $\pi : E^* \rightarrow M$ which form a nice structure of a triangular Lie bialgebroid in the sense of Mackenzie and Xu [M-X].

As in the classical case, a (generalized) Poisson tensor defines analogously to 3.2 a Poisson bracket $\{\cdot, \cdot\}_P$ on functions on $M$. We can also define the mapping $P^\# : \Phi(\pi) \rightarrow \Phi_1(\pi)$ and the generalized Hamiltonian map $H^P : \Phi(\pi) \rightarrow \Phi_1(\pi)$ similarly to 3.19 and 3.21 and to define an extension of $\{\cdot, \cdot\}_P$ to a graded Lie bracket on $\Phi(\pi)$. Most of results of Sections 4 and 5 can be obtained mutatis mutandis, since the proofs can be almost immediately adapted to the general case. Let us summarize some of these results in the following.

**Theorem 12** Given a Poisson tensor $P$ for a Lie algebroid $(\tau, [\cdot, \cdot], a_\tau)$, the formulae

$$[\mu, \nu]_P \overset{\text{def}}{=} i(H_\mu)\nu - (-1)^{\nu} \mathcal{L}(P_\mu)\nu$$

and

$$\{\mu, \nu\}_P \overset{\text{def}}{=} \mathcal{L}(H_\mu)\nu + d_\tau \mathcal{L}(P_\mu)\nu$$

define graded Lie algebra structures on $\Phi(\pi)$, with elements of $\Phi^k(\pi)$ being of degree $(k-1)$ with respect to the bracket $\{\cdot, \cdot\}_P$, and of degree $k$ with respect to $\{\cdot, \cdot\}_P$.

The maps:

$$\Lambda^P : (\Phi(\pi), [\cdot, \cdot]_P) \rightarrow (\Phi(\pi), [\cdot, \cdot]_P^{s-N}),$$

$$d_\tau : (\Phi(\pi), \{\cdot, \cdot\}_P) \rightarrow (\Phi(\pi), [\cdot, \cdot]_P),$$

$$H^P : (\Phi(\pi), \{\cdot, \cdot\}_P) \rightarrow (\Phi_1(\pi), [\cdot, \cdot]_P^{s-N})$$

are homomorphisms of graded Lie algebras and we have the following exact sequence of homomorphisms of graded Lie algebras

$$0 \rightarrow Z(\pi) \rightarrow (\Phi(\pi), \{\cdot, \cdot\}_P) \xrightarrow{d_\tau} (Z(\pi), [\cdot, \cdot]_P) \rightarrow H(\pi) \rightarrow 0,$$

26
where the $d_{-}$-closed elements $Z(\pi)$ of $\Phi(\pi)$ and the space of $d_{-}$-cohomology $H(\pi)$ are taken with the trivial brackets.

Moreover, the equation

$$\{\Lambda^{P}(\mu), \Lambda^{P}(\nu)\}_{P} \equiv \Lambda^{P}(\{\mu, \nu\}_{P})$$

(6.15)

defines properly a graded Lie bracket on the graded Lie subalgebra

$$\Phi_{P}(\pi) \equiv \Lambda^{P}(\Phi(\pi))$$

of $(\Phi(\pi), [,]_{S-N}^{S-N})$. For this bracket, the mapping $X \mapsto -[P, X]_{S-N}^{S-N}$ is a homomorphism into the generalized Schouten-Nijenhuis bracket $[,]_{S-N}^{S-N}$.

References


