Noncommutative Contact Algebras

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1. Introduction

In [5], Gerstenhaber proposed a definition of the deformation of an algebra which associates to an associative algebra $\mathcal{A}$ an associative structure on the space $\mathcal{A}[[h]]$ of formal power series in $h$ with coefficients in $\mathcal{A}$. Here $h$ is a central element in $\mathcal{A}[[h]]$.

Stimulated by this work, several authors have studied such deformations from a geometrical point of view. In particular, Bayen et al. [1] proposed a notion of deformation quantization, which is a deformation of Poisson algebras. In this context, the question naturally arises of how to generalize these deformations for various geometric structures.

In this paper, we will introduce the notion of a noncommutative contact algebra. On a contact manifold $M$, the Lagrange bracket $\{\ ,\ \}_L$ defines a Lie bracket on the space $C^\infty(M)$ of all smooth functions on $M$, under which
$C^\infty(M)$ forms a Lie algebra. For a deformation of contact structures, one might attempt first to deform the Lie algebra $(C^\infty(M), \{,\}_L)$. However, there exists no non-trivial deformation of the Lie algebra $(C^\infty(M), \{,\}_L)$ in general [4].

In our process of “noncommutatizing” contact algebras, the deformation parameter will no longer be a central element. A distinguishing feature of contact algebras, as opposed to Poisson algebras, is the existence of the characteristic vector field. We will noncommutatize the characteristic vector field together with the algebra $C^\infty(M)$. Thus, the algebra we present here is a slight generalization of deformation quantization.

Our main purpose in this paper is also to show the existence of the noncommutative contact algebra on an arbitrary contact manifold.

2. Contact algebra

Let $M$ be a contact manifold of dimension $(2n+1)$ and $\omega$ its contact 1-form; $(d\omega)^n \wedge \omega \neq 0$. We denote by $C^\infty(M)$ the algebra of all smooth functions on $M$. There exist local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ such that $\omega$ can be written locally as

$$\omega = dz + \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j). \quad (1)$$

The contact 1-form $\omega$ gives a Jacobi structure on $M$ (cf. [6]); a pair $(\Lambda, D)$ consisting of a vector field $D \in \Gamma(TM)$ and a skew-symmetric bivector field $\Lambda$ on $TM$ satisfying the following;

(i) $[D, \Lambda] = 0$,
(ii) $[\Lambda, \Lambda] = 2D \wedge \Lambda$,
(iii) $\text{rank} \Lambda = 2n$,

where $[,]$ stands for the Schouten bracket. In terms of the local coordinates where the 1-form $\omega$ has the form (1), we have

$$\Lambda = \sum_{j=1}^{n} \partial_{x_j} \wedge \partial_{y_j} - \partial_z \wedge \frac{1}{2} \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}) \text{ and } D = \partial_z.$$ 

For $f, g \in C^\infty(M)$, set

$$\{f, g\} = \Lambda(df, dg) : C^\infty(M) \times C^\infty(M) \to C^\infty(M). \quad (2)$$

The bracket $\{,\}$ is skew-symmetric and satisfies

$$\{f, gh\} = \{f, g\} h + g\{f, h\}, \quad D\{f, g\} = D\{f, g\} + \{f, Dh\}. \quad (3)$$

We call the triple $(C^\infty(M), \{,\}, D)$ the contact algebra on $M$. 
We remark that Lichnerowicz [6] introduced a Lie algebra structure on 
$C^\infty(M)$ by setting
\[
\{f, g\}_L = \{f, g\} + fD(g) - D(f)g.
\] (4)
The bracket $\{ , \}_L$ given by (4) is called the Lagrange bracket on $M$.

3. Noncommutative contact manifold

We first generalize the notion of deformation quantization (cf. [9]): let $\mathcal{A}$ be
an associative, complete topological algebra. We denote by $a*b$ the product
of $a, b \in \mathcal{A}$. We set $[a, b] = a*b - b*a$ and $\mu^k = \underbrace{\mu \cdots \mu}_k$.

**Definition 1** $\mathcal{A}$ is called a formal $\mu$-regulated algebra if there exists an
element $\mu \in \mathcal{A}$ with $\mu = \mu$ and a closed linear subspace $B$ such that
\begin{align}
(A.1) \quad [\mu, \mathcal{A}] &\subset \mu*\mathcal{A} + \mathcal{A} \\
(A.2) \quad [\mathcal{A}, \mathcal{A}] &\subset \mu*\mathcal{A} \\
(A.3) \quad \mathcal{A} &\supset \prod_{k=0}^{\infty} \mu^k * B \text{ (topological direct sum)} \\
(A.4) \quad \text{The mappings } \mu^* : \mathcal{A} \rightarrow \mu*\mathcal{A}, \quad *\mu : \mathcal{A} \rightarrow \mathcal{A} + \mu \text{ defined by} \\
&\quad a \rightarrow \mu^* a, a \rightarrow a*\mu \text{ respectively are linear isomorphisms.}
\end{align}

$\mu$ is called the regulator. By (A.2), the product $*$ defines a commutative
associative product on the factor space $\mathcal{A}/\mu k\mathcal{A}$ identified with $B$. We denote
the commutative product on $B$ by $a \cdot b$.

Formally setting
\[
[\mu^{-1}, a] = -\mu^{-1} * [\mu, a] * \mu^{-1},
\] (5)
we see by (A.1) and (A.4) that $[\mu^{-1}, a]$ is well-defined. Since $[\mu^{-1}, \mathcal{A}] \subset \mathcal{A}$,
ad($\mu^{-1}$) is a derivation of $(\mathcal{A}, *)$, which also induces a derivation of $(B, \cdot)$.

According to the decomposition (A.3), we can write
\begin{align}
a \ast \ b &\ast \ = \ a \cdot b + \mu^* \pi_1(a, b) + \cdots + \mu^k \pi_k(a, b) + \cdots, \\
ad(\mu^{-1})a &\ast \ = \ \xi_0(a) + \mu \ast \xi_1(a) + \cdots + \mu^k \ast \xi_k(a) + \cdots,
\end{align}
(6)
uniquely for any $a, b \in B$.

**Remark 1** Since $\mathrm{ad}(\mu^{-1})$ is a derivation of $(\mathcal{A}, *)$, $\xi_0$ is a derivation of
$(B, \cdot)$. Note that the commutator bracket $[a, b]$ is a biderivation of $\mathcal{A} \times \mathcal{A}$ to
$\mathcal{A}$. Therefore, the skew part $\pi_1^{-}$ of $\pi_1$ is a skew-biderivation of $B \times B$ to $B$.

**Definition 2** Let $M$ be a contact manifold and $(C^\infty(M), \{ , \}, D)$ the
contact algebra on $M$. A formal $\mu$-regulated algebra $\mathcal{A} = B[[\mu]]$ of formal power
series in $\mu$ with coefficients in $B$ is called the noncommutative contact algebra
on $M$ if $B = C^\infty(M)$, $\pi_1 = \{ , \}$, and $\xi_0 = D$ in (6).
Example 1 A typical example of a noncommutative contact algebra is the noncommutative 3-sphere given in [9], which is obtained by quantizing the Hopf fibration $\pi : S^3 \to S^2$.

Example 2 Another example of a noncommutative contact algebra is an algebra $\Sigma^0(T^*N)$ of symbols of order 0 on the cotangent bundle $T^*N$ over a compact manifold $N$ modulo the algebra $\Sigma^{-\infty}(T^*N)$ of symbols of order $-\infty$ (cf. [10]). This gives a noncommutative contact algebra defined on the unit cosphere bundle $S^*N$.

Theorem 1 For the contact algebra $(C^\infty(M), \{ , \}, D)$ on a contact manifold $M$, there exists a noncommutative contact algebra $C^\infty(M)[[\mu]]$.

Theorem 1 shows that every contact algebra extends to a formal $\mu$-regulated algebra, i.e. every contact algebra is deformation quantizable.

4. Proof of Theorem 1

Let $M$ be a contact manifold. Consider the direct product manifold $R_+ \times M$, with $R_+$ the set of positive real numbers. Define a closed 2-form $\Omega = d(\rho \omega)$ on $R_+ \times M$, where $\rho$ is the coordinate function on $R_+$. It is easy to see that $\Omega$ is a symplectic 2-form on $R_+ \times M$. Thus, it defines a Poisson bracket $\{ , \}$ on $C^\infty(R_+ \times M)$. We see that

$$\{f, g\} = \rho^{-1}\{f, g\}$$

for $f, g \in C^\infty(M)$, where $\{ , \}$ is given by (2). Let $B = C^\infty(M)[[\rho^{-1}]]$ be the set of all formal power series $f$ in $\rho^{-1}$ with coefficients in $C^\infty(M)$:

$$f(p) = f_0(p) + \rho^{-1}f_1(p) + \cdots + \rho^{-k}f_k(p) + \cdots.$$  

It is obvious that the bracket $\{ , \}$ in (7) gives a Poisson bracket on $B$ such that

$$\{B, B\} \subset \rho^{-1}B.$$  

We now consider a deformation quantization of $B$: an associative algebra $B[[h]]$ with a star product $*$ such that $h$ is in the center and for $f, g \in B$

$$\tilde{f} * \tilde{g} = \tilde{\pi}_0(\tilde{f}, \tilde{g}) + h\tilde{\pi}_1(\tilde{f}, \tilde{g}) + \cdots + h^k\tilde{\pi}_k(\tilde{f}, \tilde{g}) + \cdots,$$

where

$$\tilde{\pi}_0(\tilde{f}, \tilde{g}) = \tilde{f} \cdot \tilde{g}, \quad \tilde{\pi}_1(\tilde{f}, \tilde{g}) = \frac{1}{2}\{\tilde{f}, \tilde{g}\}_*.$$  

We define the following 1-parameter conformal symplectomorphism on $(R_+ \times M, \Omega)$:

$$\phi_t(\rho, x) = (e^t \rho, x), \quad t \in R.$$  

Setting $\phi_t^*(h) = e^t h$, we extend $\phi_t^*$ on $B$ to $B[[h]]$:
Lemma 1 Let \( B[[\hbar]] \) be a deformation quantization of \((B,\cdot,\{,\},s)\). Then, the map \( \phi_s^\hbar \) is an automorphism of \((B[[\hbar]],\ast)\) if and only if \( \tilde{\pi}_k \) satisfies
\[
\tilde{\pi}_k (\rho^{-l} f, \rho^{-m} g) \in \rho^{-(k+l+m)} C^\infty(M)
\]
for all \( f, g \in C^\infty(M) \) and for all \( k, l, m \geq 0 \).

We note the following:

**Proposition 1** Let \( B[[\hbar]] \) be a deformation quantization of \((B,\cdot,\{,\},s)\). If \( \phi_s^\hbar \) is an automorphism of \((B[[\hbar]],\ast)\), then the subalgebra of all \( \phi_s^\hbar \)-invariant elements in \( B[[\hbar]] \) is a noncommutative contact algebra on \( M \).

**Proof.** Set \( \mu = \hbar \rho^{-1} \). Denote by \( \mathcal{B}[[\hbar]] \) the space of \( \phi_s^\hbar \)-invariant elements of \( B[[\hbar]] \). We have \( \mathcal{B}[[\hbar]] = C^\infty(M)[[\mu]] \) For \( f, g \in C^\infty(M) \), we have
\[
f \ast g = f \cdot g + \frac{\mu}{2} \{f, g\} \pmod{\mu^2} ,
\]
\[
[\mu^{-1}, f] = \xi_\mu(f) \pmod{\mu},
\]
which gives Proposition 1.

To obtain Theorem 1, it suffices to show the following:

**Lemma 2** There exists a deformation quantization \( B[[\hbar]] \) of \((B,\cdot,\{,\},s)\) with the property \((13)\).

**Proof.** Since \( \Omega \) is a symplectic form on \( R_+ \times M \), the Poisson algebra \((C^\infty(R_+ \times M),\cdot,\{,\},s)\) is deformation quantizable \((2,3,8)\). We recall the stepwise construction of the bilinear map \( \tilde{\pi}_k \) in \((9)\). As in \((8)\), given \( \{\tilde{\pi}_j\}_{j=0}^{k-1} \), \( \tilde{\pi}_k \) is determined by the following equations:
\[
\tilde{\pi}_k^-(f, gh) = g \tilde{\pi}_k^-(f, h) - \tilde{\pi}_k^-(f, g) h = \langle \langle f, g \rangle^-, h \rangle_k^+ + \langle \langle f, h \rangle^-, g \rangle_k^+ + \langle \langle g, h \rangle^+, f \rangle_k^-
\]
\[
\tilde{\pi}_k^+(f, gh) = \tilde{\pi}_k^+(h, gf) - \tilde{\pi}_k^+(f, g) h + \tilde{\pi}_k^+(h, g) f = \langle \langle f, g \rangle^+, h \rangle_k^+ - \langle \langle h, g \rangle^+, f \rangle_k^+ + \langle \langle h, f \rangle^-, g \rangle_k^-,
\]
where \( \tilde{\pi}_i(f, g) = \frac{1}{2} \{\tilde{\pi}_i(f, g) \pm \tilde{\pi}_i(g, f)\} \), and
\[
\langle \langle f, g \rangle^\pm, h \rangle_k^\pm = \sum_{\sum_{i,j = \pm}} \tilde{\pi}_i^\pm(f, g, h) \quad (m \geq 2).
\]

On each coordinate neighborhood, we can construct \( \tilde{\pi}_k \) inductively as a bidifferential operator satisfying \((14)\) and \((16)\) \((cf.\ [8])\). A partition of unity argument yields Lemma 2.

Since there always exists a contact one-form on every orientable, compact 3-manifold \((cf.\ [7])\), we have
Corollary 1 Let $M$ be a orientable compact 3-manifold. Then there exists a noncommutative contact algebra on $M$.

References