Geometry of the Dirac Quantization of Constrained Systems

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Geometry of the Dirac Quantization of Constrained Systems

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Abstract

Geometric properties of operators of quantum Dirac constraints and physical observables are studied in semiclassical theory of generic constrained systems. The invariance transformations of the classical theory – contact canonical transformations and arbitrary changes of constraint basis – are promoted to the quantum domain as unitary equivalence transformations. The operators of physical observables are constructed satisfying one-loop quantum gauge invariance and Hermiticity with respect to a physical inner product. Abelianization procedure on Lagrangian constraint surfaces of phase space is discussed in the framework of the semiclassical expansion.

1. Introduction

In this paper we shall discuss geometric properties of the quantum dynamical systems subject to first class constraints. At the classical level such systems are described by the canonical action of the generic form

\[ S = \int dt \left\{ p_i \dot{q}^i - H_0(q, p) - \lambda^\mu T_\mu(q, p) \right\} \]  

(1.1)

in the configuration space of coordinates and momenta \((q, p) = (q^i, p_\mu)\) and Lagrange multipliers \(\lambda^\mu\). The variation of the latter leads to the set of nondynamical equations – constraints

\[ T_\mu(q, p) = 0. \]  

(1.2)

The constraint functions \(T_\mu(q, p)\) belonging to first class satisfy the Poisson-bracket algebra

\[ \{T_\mu, T_\nu\} = U^{\lambda\nu}_{\mu\lambda} T_\lambda \]  

(1.3)

with the structure functions \(U^{\lambda\nu}_{\mu\lambda}(q, p)\) which can generally depend on phase-space variables of the theory.
The first class constraints indicate that the theory with the action possesses a local gauge invariance generated in the sector of phase-space variables by constraints themselves $T_\mu(q, p)$ and by certain transformations of Lagrange multipliers $[1]$. The dimensionality of the gauge group coincides with the dimensionality of the space of constraints, the both being enumerated by the gauge index $\mu$. If we denote the range of index $i$ by $n$, $i = 1, \ldots, n$, and that of $\mu$ by $m$, $\mu = 1, \ldots, m$, then the number of the physical dynamically independent degrees of freedom equals $n - m$: $2(n - m)$ physical phase-space variables originate from the initial $2n$ variables $(q^i, p_i)$ by restricting to $(2n - m)$-dimensional constraint surface (1.2) and then factoring on this surface out the action of $m$ gauge transformations generated by $T_\mu$.

Dirac quantization of the theory (1.1) consists in promoting initial phase-space variables and constraint functions to the operator level $(q, p, T_\mu) \rightarrow (\hat{q}, \hat{p}, \hat{T}_\mu)$ and selecting the physical states $|\Psi\rangle$ in the representation space of $(\hat{q}, \hat{p}, \hat{T}_\mu)$ by the equation

$$\hat{T}_\mu|\Psi\rangle = 0.$$  

Operators $(\hat{q}, \hat{p})$ satisfy canonical commutation relations $[\hat{q}^k, \hat{p}_l] = i\hbar \delta_l^k$ and the quantum constraints $\hat{T}_\mu$ as operator functions of $(\hat{q}, \hat{p})$ should satisfy the correspondence principle with classical $c$-number constraints and be subject to the commutator algebra

$$[\hat{T}_\mu, \hat{T}_\nu] = i\hbar \hat{U}^\lambda_{\mu\nu} \hat{T}_\lambda,$$

with some operator structure functions $\hat{U}^\lambda_{\mu\nu}$ standing to the left of operator constraints. This algebra generalizes (1.3) to the quantum level and serves as integrability conditions for equations (1.4).

Classically the theory (1.1) and this reduction to its physical sector has two types of invariances: the invariance with respect to canonical transformations of the initial phase-space variables and the geometric invariance with respect to the transformations of the basis of constraints. The latter property means that one and the same constraint surface (1.2) is determined not just by one specific choice of the set of constraint functions $T_\mu(q, p)$, but by the equivalence class of those differing from one another by linear recombinations

$$T'_\mu = \Omega^\nu_\mu T_\nu, \quad \Omega^\nu_\mu = \Omega^\nu_\mu(q, p), \quad \det \Omega^\nu_\mu \neq 0$$

with arbitrary invertible matrix function of $(q, p)$ acting in the vector space of gauge indices. A natural question arises whether these invariances can be preserved also in the Dirac quantization procedure?

The program of finding such quantum constraints for a generic constrained system has been partly implemented in $[2]$ in the lowest nontrivial order of semiclassical expansion in $\hbar$. The symbols of operators $\hat{T}_\mu, \hat{U}^\lambda_{\mu\nu}$ with linear in $\hbar$ quantum corrections
have been found and a partial answer was given to the question of the above type: the obtained operators turned out to be covariant with respect to contact canonical transformations of the initial phase space, provided the Dirac wavefunctions \( \langle q | \Psi \rangle = \Psi(q) \) in the coordinate representation of commutation relations for \((\hat{q}, \hat{p})\) transform as 1/2-weight densities on the configuration-space manifold of \(q^i\).

It turned out, however, that the operator algorithms for \((\hat{T}_\mu, \hat{U}^\lambda_{\mu\nu})\) involve not only their classical counterparts featuring in (1.3) but also the higher-order structure functions of the canonical gauge algebra \([3, 4]\). Generally the gauge algebra involves the whole hierarchy of structure functions and relations which begin with \(T_\mu(q, p)\) and (1.3)

\[
G = \{T_\mu, U^\alpha_{\mu\nu}, U^\alpha_{\mu\nu\lambda}, \ldots\}
\]

and at any new stage iteratively build up as consistency conditions for those of the previous stages. For example, the cyclic Jacobi identity \([T_\mu, \{T_\sigma, T_\lambda\}] + \text{cycle}(\mu, \sigma, \lambda) = 0\) applied to (1.3) results in the equation

\[
\{T_\mu, U^\alpha_{\sigma\lambda}\} + U^\beta_{\sigma\lambda}U^\alpha_{\mu\beta} + \text{cycle}(\mu, \sigma, \lambda) = U^\alpha_{\mu\nu\lambda}T_\beta,
\]

multiplied by \(T_\alpha\), necessarily generating a new structure function \(U^\alpha_{\mu\nu\lambda}\) antisymmetric in upper (and lower) indices \([4]\). For constraints forming the closed Lie algebra all higher-order structure functions are vanishing, but this property depends on the choice of basis of constraints: the rotation of the constraint basis (1.6) can convert the Lie algebra (even Abelian one with \(U^\alpha_{\sigma\lambda} = 0\)) into an open algebra with the infinite set of structure functions. Thus the invariance of the the theory with respect to transformations of the form (1.6) with arbitrary \(\Omega^\mu_\nu\) necessitates considering higher-order structure functions (1.7) and their operator realization \(G \to \hat{G} = \{\hat{T}_\mu, \hat{U}^\alpha_{\mu\nu}, \hat{U}^\alpha_{\mu\nu\lambda}, \ldots\}\).

In this paper we shall show that the operators constructed in \([2]\) really possess the expected properties of invariance with respect to the transformation of the constraint basis (1.6). Since we restrict ourselves with the one-loop (linear in \(h\)) approximation, we shall focus at the covariance of only the Dirac constraints \(\hat{T}_\mu\) and Dirac equations on physical states (1.4): the higher-order structure functions will not be important for us because they are responsible for multi-loop orders of the semiclassical expansion. We show that the covariance of equations (1.4) induces the weight properties of Dirac wavefunctions also in the space of gauge indices. These properties will guarantee the unitary equivalence of quantum theories starting with different choices of constraint bases, equipped with a correct physical inner product. We also consider a number of issues omitted in the previous paper \([2]\): operator realization of gauge invariant physical observables, the gauge independence of their matrix elements and their Hermitian conjugation properties in the physical inner product and also discuss an abelianization procedure for semiclassical constrained systems on Lagrangian manifolds of the phase space.
2. Operator realization of quantum constraints and physical observables

The operator realization of quantum Dirac constraints and lowest order structure functions was found in [2] in the form of the normal \(qp\)-ordering of their \(qp\)-symbols expanded up to the linear order in \(\hbar\). This representation implies that for any operator \(\hat{G} = \{\hat{T}_\mu, \hat{U}_{\mu\nu}^\alpha, \hat{U}_{\mu\nu\lambda}^{\alpha\beta}, \ldots\}\) one can put into correspondence its normal symbol – a \(c\)-number function on phase space \(\hat{G}(q, p)\) – such that the operator \(\hat{G}\) can be obtained from \(\hat{G}(q, p)\) by replacing its arguments with noncommuting operators with all the momenta standing to the right of coordinates. For a symbol expandable in momentum series

\[
\hat{G}(q, p) = \sum_{n=0}^{\infty} \hat{G}^{i_1 \ldots i_n}(q)p_{i_1} \cdots p_{i_n}
\]

(2.1)

this means that

\[
\hat{G} = \mathcal{N}_{qp} \hat{G}(q, p) \equiv \sum_{n=0}^{\infty} \hat{G}^{i_1 \ldots i_n}(q)p_{i_1} \cdots p_{i_n}.
\]

(2.2)

The one-loop (linear in \(\hbar\)) algorithms of [2] for two lowest-order operators \(\hat{G}\) have the form

\[
\hat{T}_\mu = \mathcal{N}_{qp} \left\{ T_\mu - i\hbar \frac{\partial^2 T_\mu}{2 \partial q^i \partial p_k} + \frac{i\hbar}{2} U_{\mu\nu}^\nu \right\} + O(\hbar^2),
\]

(2.3)

\[
\hat{U}_{\mu\nu}^\lambda = \mathcal{N}_{qp} \left\{ U_{\mu\nu}^\lambda - i\hbar \frac{\partial^2 U_{\mu\nu}^\lambda}{2 \partial q^i \partial p_k} - i\hbar \frac{U_{\mu\nu\sigma}}{2} \right\} + O(\hbar^2)
\]

(2.4)

involving, as it was mentioned in Introduction, the higher-order classical structure functions \(U^{\lambda\sigma}_{\mu\nu}\). As shown in [2], these operators have two important properties. First, they are covariant under contact canonical transformations of \((q, p)\)

\[
q^i = q^i(q'), \quad p_k = p_k + \frac{\partial q^{k'}}{\partial q^i} q^i, \quad G(q, p) = G'(q', p'),
\]

(2.5)

under which the constraints and structure functions (1.7) (all quantities bearing only gauge indices) behave like scalars. In the coordinate representation of canonical commutation relations the covariance of operators (2.3)-(2.4) can be written down as

\[
\left| \frac{\partial q'}{\partial q} \right|^{-1/2} \hat{G} \left| \frac{\partial q'}{\partial q} \right|^{1/2} = \hat{G}',
\]

(2.6)
where the operators $\hat{G}$ are constructed by the above algorithms from their primed classical counterparts. This transformation law obviously implies that the Dirac wavefunction satisfying quantum constraints (1.4) should be regarded a scalar density of $1/2$-weight

$$\Psi(q) = \left| \frac{\partial q'^{1/2}}{\partial q} \right| \Psi'(q')$$

in complete correspondence with the diffeomorphism invariance of the auxiliary inner product of unphysical states\(^1\)

$$\langle \Psi_1 | \Psi_2 \rangle = \int dq \, \Psi^*_1(q) \Psi_2(q).$$

The second important property of these operators is their anti-Hermitian part with respect to this auxiliary inner product. It is given by the trace of the structure functions and for Dirac constraints has the form

$$\hat{T}_\mu - \hat{T}_\mu^\dagger = i\hbar (U^\lambda_\mu)^\dagger + O(\hbar^2).$$

The algorithms (2.3)-(2.4) were derived in [2] solely as a solution of the commutator algebra (1.5). This helps to extend these algorithms for obtaining another class of operators - the operators of physical observables. Classically the physical observables $\mathcal{O}_I$ (enumerated by some index $I$) are defined as a functions on phase space, invariant under the action of canonical gauge algebra. This invariance generally holds in a weak sense, that is only on the constraint surface

$$\{ \mathcal{O}_I, T_\mu \} = U^\lambda_\mu T_\lambda,$$

the gauge transformation of $\mathcal{O}_I$ being a linear combination of constraints with some coefficients $U^\lambda_\mu = U^\lambda_\mu(q, p)$. Note that again due to the rotation of the constraint basis (1.6) we have to consider nonvanishing coefficients $U^\lambda_\mu$ which can always be generated even for strongly invariant observables by a transition to another basis of constraints.

In addition to their weak invariance (2.10) we shall assume that the classical observables commute with one another or form a closed Lie algebra in a weak sense

$$\{ \mathcal{O}_I, \mathcal{O}_J \} = U^L_{ij} \mathcal{O}_L + U^\lambda_\mu T_\lambda, \quad U^L_{ij} = \text{const.}$$

In this case, from the viewpoint of commutator algebra the physical observables do not differ from constraints. The only difference is that unlike constraints they are not

\(^1\)This inner product diverges for physical states satisfying quantum Dirac constraints and, therefore, plays only an auxiliary role. It also appears as a truncated inner product in the bosonic sector of the extended configuration space of the BFV (BRST) quantization [4].
constrained to vanish. Therefore, to promote the classical observables to the quantum level, \( (\mathcal{O}_I, U^\lambda_{1\mu}) \rightarrow (\hat{\mathcal{O}}_I, \hat{U}^\lambda_{1\mu}) \), and enforce the quantum gauge invariance of their operators

\[
[\hat{\mathcal{O}}_I, \hat{T}_\mu] = i\hbar \hat{U}^\lambda_{1\mu} \hat{T}_\lambda,
\]

one can use the algorithm analogous to (2.3) solving this commutator algebra

\[
\hat{\mathcal{O}}_I = \mathcal{N}_V \left\{ \mathcal{O}_I - \frac{i\hbar}{2} \frac{\partial^2 \mathcal{O}_I}{\partial q^i \partial p_i} + \frac{i\hbar}{2} U^\lambda_{1\lambda} + \frac{i\hbar}{2} U^J_{1J} + O(\hbar^2) \right\}, 
\]

\[
\hat{U}^\lambda_{1\mu} = \mathcal{N}_V \left\{ U^\lambda_{1\mu} - \frac{i\hbar}{2} \frac{\partial^2 U^\lambda_{1\mu}}{\partial q^i \partial p_i} - \frac{i\hbar}{2} U^{\lambda\sigma}_{1\mu\sigma} + O(\hbar^2) \right\}
\]

with higher-order structure functions \( U^{\lambda\sigma}_{1\mu\sigma} \) of the classical algebra (2.10) and (2.11) (derivable by the method mentioned in Introduction).

The quantum observables (2.13) solve the closed commutator algebra (2.12). The proof of this statement goes by collecting the observables together with constraints into one set and repeating the derivation of [2]. The only thing to check is if the resulting commutator algebra does not contain nonvanishing components \( \hat{U}^K_{1\mu} \) of the operator structure functions (violating the weak quantum gauge invariance of observables (2.12)). This component can get a nonvanishing contribution only due to a higher-order structure functions \( U^{K\lambda}_{1\mu\lambda} + U^{K\lambda}_{1\mu\lambda} \) of the classical algebra (2.10) and (2.11). But it is easy to show that for a closed Lie algebra (2.11) the nonvanishing components of the second-order structure functions cannot have nongauge upper indices \( (U^{J\lambda}_{1\lambda} = 0, U^{JL}_{1J} = 0) \) and, therefore the quantum observables remain weakly invariant. For the same reason eq. (2.14) does not involve the contraction \( U^{\lambda J}_{1\mu J} \).

The quantum observables like constraints have anti-Hermitian part with respect to the auxiliary inner product (2.8). It is given by two contractions of structure functions \( U^\lambda_{1\lambda} \) and \( U^J_{1J} \). For all reasonable compact groups generating algebras of observables the latter is vanishing \( U^J_{1J} = 0 \), but \( U^\lambda_{1\lambda} \) is generally nontrivial, depends on the choice of constraint basis and violates Hermeticity of observables in the auxiliary inner product. It is however inessential, because only the physical inner product must generate real expectation values of observables, and this will be shown to be true below.

\section{Quantum transformation of the constraint basis}

Under the linear transformation of the classical constraint basis (1.6) the structure functions transform as

\[
U^{\kappa\sigma}_{\mu\nu} = \Omega^\alpha_{\mu} \Omega^\beta_{\nu} \Omega^\lambda_{\alpha\beta} \Omega^{-1}_{\lambda} + 2 \{ \Omega^\alpha_{[\mu; T_\beta]} \Omega^\beta_{\nu] \alpha} \} + \{ \Omega^\kappa_{\mu}, \Omega^\sigma_{\nu} \} T_\alpha \Omega^{-1}_{\beta},
\]

where
so that
\[
U_{\mu \lambda}^{\nu} = \Omega_{\nu \rho}^{\nu} U_{\nu \rho}^{\nu} + \{\Omega_{\nu}^{\nu}, T_{\alpha}\} - \{\ln \Omega, \Omega_{\nu}^{\nu} T_{\alpha}\}, \quad (3.2)
\]
\[
\Omega \equiv \det \Omega_{\rho}^{\rho}. \quad (3.3)
\]

The quantum constraints \(\hat{T}_\mu^\nu\) based on the transformed basis of classical constraints (1.6) and structure functions (3.1) take on the use of the algorithm (2.3) (with primed quantities) the form
\[
\hat{T}_\mu^\nu = \mathcal{N}_{ip} \left[ \hat{\Omega}_{\nu}^{\nu} \hat{T}_\nu - i \hbar \frac{\partial \hat{\Omega}_{\nu}^{\nu}}{\partial p_k} \frac{\partial \hat{\Omega}_{\mu}^{\nu}}{\partial q_k} - \frac{i \hbar}{2} \{\ln \hat{\Omega}, \hat{\Omega}_{\nu}^{\nu} \hat{T}_\nu\} + O(\hbar^2) \right], \quad (3.4)
\]
where \(\hat{T}_\nu\) is a normal \(qp\)-symbol of constraints in the original basis and
\[
\hat{\Omega}_{\mu}^{\nu} \equiv \hat{\Omega}_{\mu}^{\nu} - \frac{i \hbar}{2} \frac{\partial^2 \hat{\Omega}_{\mu}^{\nu}}{\partial q_k \partial p_k} + O(\hbar^2), \quad \hat{\Omega} \equiv \det \hat{\Omega}_{\rho}^{\rho}. \quad (3.5)
\]

From the two simple identities for operators and their normal \(qp\)-symbols
\[
\hat{F}_1 \hat{F}_2 = \mathcal{N}_{ip} \left[ \hat{F}_1 \hat{F}_2 - i \hbar \frac{\partial \hat{F}_1}{\partial p_k} \frac{\partial \hat{F}_2}{\partial q_k} + O(\hbar^2) \right],
\]
\[
\hat{\Omega}^{-1/2} \hat{F} \hat{\Omega}^{1/2} = \mathcal{N}_{ip} \left[ \hat{F} - i \hbar \frac{\partial \hat{\Omega}}{\partial q_k} + O(\hbar^2) \right]
\]
it is then easy to find the final form of the transformation law for quantum constraints under the transformation of their classical basis (1.6)
\[
\hat{T}_\mu^\nu = \hat{\Omega}^{-1/2} \hat{\Omega}_{\nu}^{\nu} \hat{T}_\nu \hat{\Omega}^{1/2}, \quad (3.6)
\]
\[
\hat{\Omega}_{\mu}^{\nu} = \mathcal{N}_{ip} \left\{ \Omega_{\mu}^{\nu} - \frac{i \hbar}{2} \frac{\partial^2 \Omega_{\mu}^{\nu}}{\partial q_k \partial p_k} + O(\hbar^2) \right\}, \quad \hat{\Omega} \equiv \det \hat{\Omega}_{\rho}^{\rho}. \quad (3.7)
\]

Similarly to (1.6) this transformation involves a linear recombination of constraint operators with operator-valued matrix \(\hat{\Omega}_{\mu}^{\nu}\) (standing to the left of constraints). This matrix is obtained from its classical counterpart \(\Omega_{\mu}^{\nu}(q, p)\) by the algorithm (3.7) similar to (2.3); its symbol involves analogous quantum corrections except the anti-Hermitian part. In addition to linear combinations the transformation (3.6) includes the canonical transformation generated by the square root of its determinant \(\hat{\Omega}^{1/2}\). This canonical transformation implies that the physical states satisfying Dirac constraints (1.4) transform contragrediently to (3.6)
\[
|\Psi\rangle' = \hat{\Omega}^{-1/2} |\Psi\rangle
\]
and turn out to be scalar densities of weight \(-1/2\) in the space of gauge indices. This property has been observed for systems subject to constraints linear in momenta in [5]
and, as we see, turns out to be true for a generic case at least in the one-loop order of the semiclassical expansion.

It is easy to repeat now similar calculations for the operators of physical observables \( (2.13) \). In view of the transformation law for the trace of the observable structure function \( U^{\alpha}_{i\lambda} = U^{\lambda}_{i\alpha} - \{\ln \det \Omega, \mathcal{O}_f\} \) these calculations immediately show that the operators of observables also transform canonically

\[
\hat{\mathcal{O}}_f = \hat{\Omega}^{-1/2} \hat{\mathcal{O}}_f \hat{\Omega}^{1/2}. \tag{3.9}
\]

Obviously, the theories differing by the choice of constraint basis should be unitarily equivalent. This means that the physical inner product of states \( |\Psi\rangle \) should contain a measure depending on this choice and transforming contragrediently to (3.8). In the next section we show that semiclassical states and their inner product really satisfy the properties compatible with the transformation law of the above type.

### 4. Semiclassical physical states

Semiclassical expansion of the operator symbols of the above type makes sense when the corresponding quantum states also have a semiclassical form. In the coordinate representation semiclassical wavefunctions

\[
\Psi(q) = P(q) \exp \left[ \frac{i}{\hbar} S(q) \right] \tag{4.1}
\]

are characterized by the Hamilton-Jacobi function \( S(q) \) and preexponential factor \( P(q) \) expandable in \( \hbar \)-series beginning with the one-loop order \( O(\hbar^0) \). The action of the operator \( \hat{F} \) on such functions reads as

\[
\hat{F} \Psi(q) = \left[ \hat{F} \left( q, \frac{\partial S}{\partial q} \right) + O(\hbar) \right] P \exp \left[ \frac{i}{\hbar} S(q) \right] \tag{4.2}
\]

where, as in (2.2), \( \hat{F} \) is a normal \( qp \)-symbol of \( \hat{F} \).

The general semiclassical solution of quantum constraints (1.4) with operators (2.3) was found in [6, 2, 7] in the form of the two-point kernel \( K(q, q') \) "propagating" the

---

\( ^2 \)This corresponds to the fact that the tree-level part is entirely contained in the exponential and is \( O(\hbar^{-1}) \). When the Hamiltonian \( H_0 \) in eq.\((1.1)\) is nonvanishing the wavefunction \((4.1)\), its Hamilton-Jacobi function, two-point kernel, etc. are time-dependent. In what follows we shall, however, omit the time label, because we will be mainly interested in constraint properties rather than the dynamical ones. Another way to view this is to parametrize time and the conjugated Hamiltonian \( H_0(q, p) \) among the phase-space variables and regard the Hamilton-Jacobi and Schrodinger equations as one extra classical and quantum constraint correspondingly [2, 7].
initial data from the Cauchy surface throughout the whole superspace of $q$

$$K(q, q') = P(q, q') \exp \left[ \frac{i}{\hbar} S(q, q') \right].$$  \hfill (4.3)

In both expressions (4.1) and (4.3) the phase in the exponential satisfies the Hamilton-Jacobi equation

$$T_\mu \left( q, \frac{\partial S}{\partial q} \right) = 0,$$  \hfill (4.4)

while the one-loop preexponential factor is subject to continuity type equation originating from the full quantum constraint in the approximation linear in $\hbar$

$$\frac{\partial}{\partial q^i} \left( \nabla^i \mu P^2 \right) = U^\lambda_\mu P^2, \quad \nabla^i \mu \equiv \frac{\partial T_\mu}{\partial p_i} \bigg|_{p} = \partial S / \partial q.$$  \hfill (4.5)

For a two-point kernel the Hamilton-Jacobi function coincides with the principal Hamilton function $S(q, q')$ (action on the extremal joining points $q$ and $q'$) and the solution of the continuity equation can be found as a generalization of the Pauli-Van Vleck-Morette ansatz for the one-loop preexponential factor [8, 7] of the Schrodinger propagator. This generalization is nothing but a Faddeev-Popov gauge-fixing [9] procedure for a matrix of mixed second-order derivatives of the principal Hamilton function

$$S_{ik'} = \frac{\partial^2 S(q, q')}{\partial q^i \partial q^{k'}}$$  \hfill (4.7)

which is degenerate in virtue of the Hamilton-Jacobi equations (4.4) giving rise to the left zero-value eigenvectors (4.6) and analogous right zero-vectors [6, 2]

$$\nabla^i \mu S_{ik'} = 0, \quad S_{ik'} \nabla^k \mu = 0, \quad \nabla^k \mu \equiv \frac{\partial T_\mu(q', p')}{\partial p'_k} \bigg|_{p' = -\partial S / \partial q}.$$  \hfill (4.8)

The preexponential factor reads

$$P = \left[ \frac{\det F_{ik'}}{J(q)J(q') \det c_{\mu\nu}} \right]^{1/2},$$  \hfill (4.9)

where $F_{ik'}$ is a nondegenerate matrix of the initial action Hessian (4.7) supplied with a gauge-breaking term

$$F_{ik'} = S_{ik'} + \chi^\mu_\nu e_{\mu\nu} \chi^\nu_{k'},$$  \hfill (4.10)

and $J(q)$ and $J(q')$ are the Feynman-DeWitt-Faddeev-Popov "ghost" determinants [10, 9] compensating for the inclusion of this term. The gauge-breaking term and
ghost determinants are constructed with the aid of two sets of arbitrary covectors $(\chi^\mu_i, \chi^\nu_i)$ ("gauge" conditions) satisfying the only requirement of the nondegeneracy of their ghost operators [6, 7, 2]

$$
J_\mu(q) = \chi^\mu_i \nabla^i, \quad J(q) \equiv \text{det} J_\mu(q) \neq 0,
$$

$$
J_\mu(q') = \chi^{\mu'}_i \nabla'^i, \quad J(q') \equiv \text{det} J_\mu(q') \neq 0.
$$

(4.11)

The invertible gauge-fixing matrix $c_{\mu\nu}$ and its determinant (contribution of Nielsen-Kallosh ghosts) are the last ingredients of the generalized Pauli-Van Vleck-Morette ansatz (4.9).

Notice now that under the transformation of the basis (1.6) the vectors (4.6) defined on the Lagrangian manifold of phase-space $p = \partial S / \partial q$ transform covariantly with respect to their gauge indices $(\nabla^i)_\mu = \Omega^i \nabla^i$. Therefore the ghost determinants transform as densities

$$
J' = (\text{det} \Omega^i_\mu) J,
$$

(4.12)

whence it follows that the two-point kernel with respect to each of its arguments transforms in accordance with the law (3.8) in which the action of the operator $\hat{\Omega}^{-1/2}$ semiclassically boils down to the multiplication with $[\text{det} \Omega^i_\mu(q, \partial S / \partial q)]^{-1/2}$.

5. **The physical inner product: Hermiticity and gauge independence**

The auxiliary inner product (2.8) cannot serve as an inner product for physical states because it is not well defined. In view of quantum constraints the physical states have a distributional nature $|\Psi\rangle = "\delta(\hat{T})" |\Psi_{\text{aux}}\rangle$ with somehow determined $m$-dimensional delta-function of non-abelian operators $\hat{T}_\mu$, and their naive bilinear combinations are divergent because $"\delta(\hat{T})"^2 \sim \delta(0)"\delta(\hat{T})"$. At most the auxiliary vectors $|\Psi_{\text{aux}}\rangle$ participating in the construction of the physical states can be required to be square-integrable in $L^2$ sense and thus induce a finite inner product for $|\Psi\rangle$ (which is the idea of the so-called refined algebraic quantization of constrained systems [11, 12]). Another approach may consist in the unitary map from the Dirac quantum states to wavefunctions of the reduced phase space quantization, which have a trivial inner product inducing a correct physical product in the Dirac quantization scheme. The latter turns out to be the integral over $(n-m)$-dimensional physical subspace of the coordinate space of $q$ (superspace) with certain measure. For constrained systems of the general form it was constructed in the one-loop approximation and for semiclassical
states (4.1) looks like [13, 6, 2]

\[
(\Psi |\Psi) = \int dq \Psi^\dagger(q) \delta(\chi(q)) J(q, \partial S/\partial q) \Psi(q) + O(h).
\]

Here \( \chi(q) = \chi^\mu(q) \) is a set of gauge conditions delta-function of which

\[
\delta(\chi) = \prod_\mu \delta(\chi^\mu(q)),
\]

determines the \((n - m)\)-dimensional physical subspace \( \Sigma \) embedded in superspace and

\[
J(q, p) = \det J^\mu_\nu(q, p), \quad J^\mu_\nu(q, p) = \{ \chi^\mu, T_\nu \}.
\]

The geometry of the physical space embedding, considered in much detail in [7], can be better described in special coordinates on superspace \( \vec{q}^i = (\xi^A, \theta^\mu) \), in which \( \xi^A, A = 1, \ldots n - m \), serve as intrinsic coordinates on \( \Sigma \) (physical configuration coordinates), and \( \theta^\mu \) is determined by gauge conditions:

\[
\begin{align*}
q^i \to \vec{q}^i &= (\xi^A, \theta^\mu), \quad q^i = e^i(\xi^A, \theta^\mu), \quad \theta^\mu = \chi^\mu(q),
\end{align*}
\]

The equation of the surface \( \Sigma \) in the new coordinates is \( \theta^\mu = 0 \), so that its embedding equations coincide with the above reparametrization equations at \( \theta^\mu = 0 \), \( e^i(\xi) = e^i(\xi, 0) \)

\[
\Sigma : q^i = e^i(\xi), \quad \chi^\mu(e^i(\xi)) \equiv 0.
\]

The relation between the integration measures on superspace \( dq = d^\mu q \) and on \( \Sigma \), \( d\xi = d^{\mu-m} \xi \)

\[
d\xi = dq \delta(\chi) M, \quad M = (\det[e^i_A, e^i_\mu])^{-1},
\]

involves the Jacobian of this reparametrization, built of the basis of vectors tangential and normal to \( \Sigma \):

\[
e^i_A = \partial e^i/\partial \xi^A, \quad e^i_\mu = \partial e^i/\partial \theta^\mu.
\]

Note that \( m \) covectors normal to the surface can be chosen as gradients of gauge conditions

\[
\begin{align*}
\chi^\mu_i &= \frac{\partial \chi^\mu}{\partial q^i}, \quad \chi^\mu_i e^i_\nu = \delta^\mu_\nu,
\end{align*}
\]

that can be identified with auxiliary covectors participating in the algorithm for the pre-exponential factor (4.9). With this identification the Faddeev-Popov operator \( J(q, \partial S/\partial q) \)
coincides with the operator \( J^\mu (q) \) of this algorithm (which explains the use of the same notation).

On the same footing with \( (e^i_A, \epsilon^i_\mu) \) as a full local basis one can also choose the set \( (e^i_A, \nabla^i_\mu) \) with vectors \( \nabla^i_\mu \) transversal to \( \Sigma \) given by eq.(4.6). The normal vectors of the first basis when expanded in the new basis

\[
e^i_\mu = J^{-1}_\mu e^i + \Omega^A_\mu e^i_A
\]

have one coefficient of expansion always determined by the inverse of the Faddeev-Popov matrix \( J^{-1}_\mu \) and, thus, independent of the particular parametrization of \( \Sigma \) by internal coordinates. The second coefficient is less universal and depends on a particular choice of this parametrization. Missing information about \( \Omega^A_\mu \) does not prevent, however, from finding the relation between the determinants of matrices of the old and new bases

\[
det [e^i_A, \nabla^i_\mu] = \frac{J}{M}. \tag{5.10}
\]

The quantity standing on the right hand side of this relation plays an important role in the unitary map between the Dirac and reduced phase space quantizations. As shown in [6, 2, 7] it maps the two-point kernel (4.3) with prefactor (4.9) on the one-loop unitary evolution kernel \( K(t, \xi | t', \xi') \) of the Schrödinger equation in reduced phase space quantization

\[
K(t, \xi | t', \xi') = \text{const} \ K(q, q') (J'/M')^{1/2} \bigg|_{q = \epsilon(t), q' = \epsilon(t')}. \tag{5.11}
\]

This kernel is given by the well-known Pauli-Van Vleck-Morette ansatz [8] involving the principal Hamilton function of physical variables \( S(t, \xi | t', \xi') \) and solving the Schrödinger equation in the linear in \( \hbar \) approximation\(^3\)

\[
K(t, \xi | t', \xi') \equiv \left[ \det \frac{i}{2\pi \hbar} \frac{\partial^2 S}{\partial \xi^A \partial \xi^{B'}} \right]^{1/2} e^{\frac{i}{\hbar} S(t, \xi | t', \xi')}. \tag{5.12}
\]

The relation (5.11) obviously implies the map between the wavefunctions of the Dirac \( \Psi(q) \) and reduced phase space \( \Psi(t, \xi) \) quantizations

\[
\Psi(t, \xi) = \left( \frac{J}{M} \right)^{1/2} \left. \Psi(q) \right|_{q = \epsilon(t)} \tag{5.13}
\]

\(^3\)To avoid notational confusion we reintroduce the time labels in the left-hand side of eq.(5.11). Moments of time \( t \) and \( t' \) are explicitly contained on the right-hand side of this equation for systems with nonvanishing Hamiltonian \( H_0(q, p) \). For the so-called parametrized systems with \( H_0(q, p) = 0 \) time enters only through the embedding functions \( q = \epsilon(\xi, t), q' = \epsilon(\xi', t') \) because for such systems canonical gauge conditions should explicitly depend on time to generate evolution in reduced phase space theory [7, 2].
which is unitary provided the physical inner products of these states coincide in both schemes

\[(\Psi_1 | \Psi_2)_{\text{red}} = (\Psi_1 | \Psi_2). \quad (5.14)\]

But for a simple inner product of physical states in reduced phase space scheme

\[(\Psi_1 | \Psi_2)_{\text{red}} \equiv \int d\xi \Psi_1^*(\xi)\Psi_2(\xi) \quad (5.15)\]

and the physical inner product (5.1) of the Dirac wavefunctions this equality holds in virtue of the relation (5.6) between the integration measures on superspace and the physical space \(\Sigma\).

This explains the nature of the physical inner product in the Dirac quantization. Its measure contains the Faddeev-Popov determinant which depends on the choice of the constraint basis and transforms under the transition (1.6) to another basis as a density of weight 1 in the space of gauge indices (4.12) as compared to -1/2 weight of the Dirac wavefunctions (3.8). This proves the invariance of the physical inner product under this transformation and shows that at the quantum level it is not only canonical but also unitary. Our purpose now, till the end of this section, will be to discuss the Hermiticity properties of physical observables relative to this inner product and the gauge independence of their matrix elements.

The semiclassical physical inner product (5.1) can be rewritten as an auxiliary inner product of physical states with a nontrivial operatorial measure

\[(\Psi_1 | \Psi_2) = \langle \Psi_1 | J\delta(\dot{\chi}) | \Psi_2 \rangle + O(\hbar). \quad (5.16)\]

Here the operator ordering in operators of the ghost determinant and gauge conditions is unimportant because it effects the multiloop orders \(O(\hbar)\) that go beyond the scope of this paper. The matrix element of the physical observable \(\hat{O}_I\) is therefore

\[(\Psi_1 | \hat{O}_I | \Psi_2) \equiv (\Psi_1 | \hat{O}_I \Psi_2) = \langle \Psi_1 | J\delta(\dot{\chi})\hat{O}_I | \Psi_2 \rangle. \quad (5.17)\]

To check the Hermiticity of \(\hat{O}_I\) we have to show that the expression

\[
(\hat{O}_I | \Psi_1 | \Psi_2) - (\Psi_1 | \hat{O}_I | \Psi_2) = \langle \Psi_1 | \hat{O}_I J\delta(\dot{\chi}) | \Psi_2 \rangle - \langle \Psi_1 | J\delta(\dot{\chi})\hat{O}_I | \Psi_2 \rangle, \quad (5.18)
\]

where a dagger denotes Hermitian conjugation with respect to the auxiliary inner product, is vanishing. From (2.13) it follows that \(\hat{O}_I^\dagger = \hat{O}_I - i\hbar U^\dagger_{I\lambda} + O(\hbar^2)\) (we consider the algebras of observables with \(U^\dagger_{I\lambda} = 0\)), and the expression above takes the form

\[
\langle \Psi_1 | [\hat{O}_I, J\delta(\dot{\chi})] - i\hbar U^\dagger_{I\lambda} | \Psi_2 \rangle = O(\hbar^2), \quad (5.19)
\]
which, as shown in Appendix A, is vanishing in the one-loop approximation. Thus, the physical observables are semiclassically Hermitian with respect to the physical inner product (5.1).

Another important property of this product and matrix elements of observables is their independence of the choice of gauge conditions $\chi^\mu(q)$ participating in their construction. The gauge independence of the inner product itself is based, as shown in [2, 14], on the fact that it can be rewritten as an integral over $(n - m)$-dimensional surface $\Sigma$ of certain $(n - m)$-form which is closed in virtue of the Dirac constraints on physical states

$$ (\Psi_2|\Psi_1) = \int_\Sigma \omega^{(n-m)}, \quad d\omega^{(n-m)} = 0. \quad (5.20) $$

It follows then from the Stokes theorem that this integral is independent of the choice of $\Sigma$ or equivalently of the choice of gauge conditions specifying the physical subspace. The form $\omega^{(n-m)}$ in the one-loop approximation equals

$$ \omega^{(n-m)} = \frac{d\Psi_2^* \Psi_1}{(n-m)!} \Psi_2^* \nabla_{i_1}^{n-m+1} \cdots \nabla_{i_m}^{n-m} \Psi_1, \quad (5.21) $$

and its closure is a corollary of the continuity equation (4.5) for $\Psi_2^* \Psi_1$

$$ \frac{\partial}{\partial q^i} (\nabla_{\mu}^i \Psi_2^* \Psi_1) = U_{\mu\lambda}^\lambda \Psi_2^* \Psi_1. \quad (5.22) $$

The evaluation of the matrix element of the observable $\hat{O}$ semiclassically involves the evaluation of the quantity

$$ (\Psi_2|\hat{O}\Psi_1) = \int_\Sigma \omega^{(n-m)} \hat{O}_1(q, \partial S_1/\partial q) + O(\hbar), \quad (5.23) $$

which will be also gauge independent provided the continuity equation holds for the quantity $\Psi_2^*(q)\Psi_1(q)\partial S_1/\partial q$. But this equation will also be a corollary of (5.22) because, as shown in Appendix A, on the Lagrangian manifold of phase space

$$ \nabla_{\mu}^i \frac{\partial}{\partial q^i} \left[ \hat{O} \left( q, \frac{\partial S}{\partial q} \right) \right] \equiv \{ O, T_{\mu} \} \bigg|_{p = \partial S/\partial q} = 0 \quad (5.24) $$

in view of gauge invariance (2.10) of observables. Thus, as it should have been expected from the theory of gauge fields [10, 9] the gauge independence of the physical matrix

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4In eqs. (5.21)-(5.23) the product $\Psi_2^* \Psi_1 = P_2 P_1 \exp[i(S_1 - S_2)/\hbar]$ involves two different Hamilton-Jacobi functions, so that it seems ambiguous on which Lagrangian manifold $(p = \partial S_1/\partial q$ or $p = \partial S_2/\partial q$) all the relevant quantities should be constructed. One should remember, however, that in semiclassical expansion the integral (5.20) is calculated by the stationary phase method in which a dominant contribution comes from the stationary point satisfying $\partial S_1/\partial q = \partial S_2/\partial q$. This makes these Lagrangian surfaces to coincide in the leading order, their difference being treated perturbatively as expansion in $\hbar$. 

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elements or expectation values of observables follows from the gauge invariance of the latter. This is true not only at the formal path-integral quantization level, but also in the operatorial Dirac quantization scheme.

6. Conclusions

Thus we see that, despite a complicated non-abelian nature of the formalism, Dirac quantization of generic constrained systems is remarkably consistent and reveals rich geometrical structures beyond the lowest order semiclassical approximation. Geometrical covariance of operators and physical states takes place not only in the coordinate representation space of the theory, but also in the space of gauge transformations. The quantum formalism turns out to be covariant in the sense of unitary equivalence with respect generic symmetries of a classical theory including arbitrary change of the constraint basis. The operators of physical observables turn out to be Hermitian with respect to a physical inner product and their matrix elements are gauge independent in full correspondence with similar properties in the classical domain. All these properties were obtained perturbatively in the one-loop approximation of semiclassical expansion but, no doubt, they can be extended to multi-loop orders, though, apparently by the price of growing technical complexity.

It should be emphasized that this remarkably general picture of quantum invariances was obtained in the Dirac quantization of constrained systems, when all quantum constraints are imposed on physical states being a quantum truncation of the BFV(BRST) quantization with $C \bar{P}$-ordered form of the nilpotent BRS operator acting in the extended relativistic phase space of original $(q, p)$ and ghost canonical variables $(\bar{C}, \bar{P})$ [7, 2]. Another unitary inequivalent quantization based on the normal Wick ordering in the ghost sector is more widely applied now in field theoretical models, including strings and low-dimensional CFT [15]. This quantization after truncation to $(q, p)$-sector results in only a half of the initial constraints imposed on physical states [4], and for them the geometric properties considered here are much less known if at all available at such a general level not resorting to a harmonic oscillator decomposition of fields. Thus it seems interesting to try extending the above geometric methods to this quantization scheme.

As far as it concerns the present results, they have important implications in cosmological problems as an operatorial justification and proof of gauge independence and unitarity of the covariant effective action algorithms for distribution functions in the quantum ensemble of the tunnelling state (or no-boundary state) universes [7, 16]. The latter serve as a ground of the quantum origin of the early inflationary Universe and give important predictions at the overlap of quantum cosmology, inflation theory and
A. Hermiticity of observables

To prove eq.(5.19) note that

$$\langle \Psi_1 | [\hat{O}_1, J \delta(\chi)] | \Psi_2 \rangle = i\hbar \int dq \{ \mathcal{O}_1, J \delta(\chi) \} \Psi_1^\dagger \Psi_2 + O(\hbar^2). \quad (A.1)$$

The Poisson bracket commutator here can be transformed by using the cyclic Jacobi identity and the weak gauge invariance of constraints (2.10)

$$\int dq \{ \mathcal{O}_1, J \delta(\chi) \} \Psi_1^\dagger \Psi_2 = \int dq \delta(\chi) J \left[ U^\mu_{i \mu} + J^{-1 \mu}_\nu \{ T_\nu, \{ \chi^\mu, \mathcal{O}_1 \} \} \right] \Psi_1^\dagger \Psi_2$$

$$+ \int dq \frac{\partial \delta(\chi)}{\partial \chi^\mu} J \{ \mathcal{O}_1, \chi^\mu \} \Psi_1^\dagger \Psi_2. \quad (A.2)$$

The last term here can be integrated in a special coordinate system on superspace defined by eqs.(5.4) and (5.6)

$$\int dq \frac{\partial \delta(\chi)}{\partial \chi^\mu} J \{ \mathcal{O}_1, \chi^\mu \} \Psi_1^\dagger \Psi_2 = - \int dq \delta(\chi) M \frac{\partial}{\partial \theta^\mu} \left[ \frac{J}{M} \{ \mathcal{O}_1, \chi^\mu \} \Psi_1^\dagger \Psi_2 \right]. \quad (A.3)$$

To transform this expression further let us derive several useful identities. First of all, note that any function of phase space variables $f(q, p)$ when restricted to the Lagrangian manifold defined by the Hamilton-Jacobi function $S$ becomes a function on superspace $f(q, \partial S/\partial q)$. Its derivative with respect to coordinate $\theta^\mu$ (of the new coordinate system)

$$\frac{\partial}{\partial \theta^\mu} \left[ f \left( q, \frac{\partial S}{\partial q} \right) \right] = \varepsilon^i_\mu \left( \frac{\partial f}{\partial q^i} + \frac{\partial f}{\partial p_k} \frac{\partial S}{\partial q^k} \frac{\partial^2 S}{\partial q^i} \right), \quad (A.4)$$

can be simplified to

$$\frac{\partial}{\partial \theta^\mu} \left[ f \left( q, \frac{\partial S}{\partial q} \right) \right] = J^{-1 \mu}_\nu \{ f, T_\nu \} + \Omega_{\mu A} \frac{\partial}{\partial \xi^A} \left[ f \left( q, \frac{\partial S}{\partial q} \right) \right]. \quad (A.5)$$

in view of eq.(5.9) and the differentiated version of the Hamilton-Jacobi form of constraints

$$\nabla^\mu_{\nu} \frac{\partial^2 S}{\partial q^i \partial q^k} = \frac{\partial T_\mu}{\partial q^k} \quad (A.6)$$
Similar derivation shows that the gauge transformation of this function on the Lagrangian manifold, generated by the vector flow $\nabla_i^\mu$, coincides with the Poisson bracket of $f(q,p)$ with the constraint
\[
\nabla_i^\mu \frac{\partial}{\partial q^i} \left[ f \left( q^\nu, \frac{\partial S}{\partial q^\nu} \right) \right] = \{ f, T_\mu \}, 
\]
evaluated certainly at $p = \partial S / \partial q$. With these identities and using the derivatives of the measure $M$ (5.6)
\[
\frac{\partial M}{\partial \theta^\mu} = -M \frac{\partial e_i^\mu}{\partial q^i}, \quad \frac{\partial M}{\partial \xi A} = -M \frac{\partial e^i_A}{\partial q^i} 
\]
($e_i^\mu$ and $e^i_A$ are defined by (5.7)), the relation (5.9) and the Jacobi identity for Poisson brackets one can obtain the following gauge derivative
\[
\frac{\partial}{\partial \theta^\mu} \left( \frac{J}{M} \right) = \frac{J}{M} J^{-1 \nu} \mu \left( \frac{\partial \nabla_i^\nu}{\partial q^i} - U_{i \nu}^\lambda \right) + \frac{\partial}{\partial \xi A} \left( \frac{J}{M} \Omega_{\mu}^A \right). 
\]
Using this equation for $J/M$ and applying (A.6) and (A.7) to other quantities in (A.3) we finally obtain
\[
\int dq \delta(\chi) M \frac{\partial}{\partial \theta^\mu} \left[ \frac{J}{M} \left\{ \mathcal{O}_1, \chi^\mu \right\} \Psi_1^* \Psi_2 \right] = \int dq \delta(\chi) \frac{J}{M} J^{-1 \mu} \nu \left\{ T_\mu, \left\{ \chi^\nu, \mathcal{O}_1 \right\} \right\} \Psi_1^* \Psi_2 
\]  
\[+ \int dq \delta(\chi) \frac{J}{M} J^{-1 \mu} \nu \left\{ \mathcal{O}_1, \chi^\nu \right\} \left[ \frac{\partial}{\partial q^i} (\nabla_i^\mu \Psi_1^* \Psi_2) - U_{i \nu}^\lambda \Psi_1^* \Psi_2 \right] \]
\[+ \int d\xi \frac{\partial}{\partial \xi A} \left[ \frac{J}{M} \Omega_{\mu}^A \left\{ \mathcal{O}_1, \chi^\mu \right\} \Psi_1^* \Psi_2 \right]. 
\]
The second term here is vanishing in view of the continuity equation (or more precisely $O(t \hbar)$ for different $\Psi_1$ and $\Psi_2$). The third total derivative term is vanishing in view of zero boundary conditions for physical wavefunctions at the infinity of the physical space $\Sigma$. Collecting equations (A.2), (A.3) and (A.10) together we see that the matrix element of the commutator reduces to the one term containing the trace of the structure function $U_{i \mu}^\nu$ which exactly cancels out in the equation (5.19). This proves the Hermiticity of observables in the physical inner product.

**B. Abelianization procedure on Lagrangian manifolds**

It is well known that when the structure functions of the Poisson-bracket algebra of constraints (1.3) are not constants this algebra is open [3]: the commutator of two consequtive transformations $\delta_\mu f \equiv \{ f, T_\mu \}$ of any function on phase space
\[
\{ \{ f, T_\mu \}, T_\nu \} - \{ \{ f, T_\nu \}, T_\mu \} = U_{\mu \nu}^\lambda \left\{ f, T_\lambda \right\} + \{ f, U_{\mu \nu}^\lambda \} T_\lambda 
\]  
(B.1)
is a linear combination of these transformations only on the constraint surface $T_\lambda(q,p) = 0$. Semiclassically, the restriction to this surface takes place on the Lagrangian manifold of phase space

$$ p_i = \frac{\partial S}{\partial q^i} $$

defined by the Hamilton-Jacobi function of a semiclassical state, satisfying the Hamilton-Jacobi equation (4.4). As was shown in Appendix A, the action of the constraint generators on $f(q,p)$ (in the sense of Poisson brackets) on this surface (A.7) can be generated by directional derivatives along a special set of vectors $\nabla^i_\mu$ (4.6). Therefore these vectors can be regarded as gauge generators on the Lagrangian manifold of phase space. They have a closed Lie-bracket algebra [2]

$$ \nabla^i_\mu \frac{\partial \nabla^k_\nu}{\partial q^i} - \nabla^i_\nu \frac{\partial \nabla^k_\mu}{\partial q^i} = U^\lambda_{\mu\nu} \nabla^k_\lambda $$

and, therefore, can be abelianized [17] by recombining these vectors with the aid of the matrix of the inverse Faddeev-Popov operator constructed out of some admissible gauge conditions $J^\mu_i \equiv \nabla^i_\nu \partial \chi^\nu / \partial q^i$

$$ \mathcal{R}^i_\mu = J^{-1\mu}_i \nabla^i_\nu \left[ \mathcal{R}^i_\mu \frac{\partial}{\partial q^i}, \mathcal{R}^i_\nu \frac{\partial}{\partial q^i} \right] = 0. $$

Abelian generators allow one to construct preferred parametrization of the coordinate manifold (5.4) with a special internal coordinates on a physical space $\Sigma$. Note that the new coordinates $\xi^A$ in (5.4) as functions of the original coordinates are not necessarily gauge invariant. Abelianization procedure can render them invariant as follows. Demand that the reparametrization functions in (5.4) satisfy the equations

$$ \frac{\partial e^i(\xi,\theta^\mu)}{\partial \theta^\mu} = \mathcal{R}^i_\mu, \quad (B.5) $$

which are integrable in view of the abelian nature of $\mathcal{R}^i_\mu$. Then the identity $\delta^i_\mu \partial \xi^A / \partial q^i = 0$ implies the gauge invariance of $\xi^A(q)$

$$ \nabla^i_\nu \frac{\partial \xi^A}{\partial q^i} = 0. \quad (B.6) $$

Many equations above simplify with this preferred parametrization of physical space $\Sigma$. Indeed, eq. (B.5) implies that $\Omega^A_\mu = 0$ in the equation (5.9) and the equation (A.9) for the factor $J/M$ performing the map from the Dirac quantization to the reduced phase space quantization takes the form of the continuity equation for a one-loop prefactor

$$ \frac{\partial}{\partial q^i} \left( \nabla^i_\mu J/M \right) = U^\lambda_{\mu\nu} J/M, \quad (B.7) $$

which means that this factor with the measure $M$ constructed in this parametrization represents a kinematical solution of the continuity equation.
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