Mixing for Finite Systems of Coupled Tent Maps

Gerhard Keller

Vienna, Preprint ESI 388 (1996)  

Supported by Federal Ministry of Science and Research, Austria  
Available via http://www.esi.ac.at

September 30, 1996
Mixing for finite systems of coupled tent maps

Gerhard Keller
Mathematisches Institut, Universität Erlangen-Nürnberg

October 1, 1996

Abstract
It is shown that a finite system of coupled mixing tent maps has a unique absolutely continuous invariant measure and is exact with respect to this measure provided the coupling strength does not exceed a certain value $\epsilon_{uni}$ which is independent of the size of the system.

1 Introduction and main result

Systems of coupled maps were widely studied during the last years, mostly by physicists, see [10, 5] for reviews. On the mathematical side two approaches to their understanding emerged: Initiated by Bunimovich and Sinai [3] several authors constructed models of statistical mechanics equivalent to infinite systems of coupled maps and studied invariant measures in this setting. Recent expositions of this type of approach are given in [2, 1]. On the other hand the dynamics of coupled maps were analyzed in terms of the transfer operator [8, 9], see also [7] for a brief review. In this note we continue this line of research. More specifically, we study finite systems of coupled tent maps $\tau : [0, 1] \to [0, 1], \tau(x) = \alpha(\frac{1}{2} - |x - \frac{1}{2}|)$ with $1 < \alpha \leq 2$. Let $L$ be a finite set and $X = [0, 1]^L$. Denote by $S_0 : X \to X$ the $L$-fold direct product of $\tau$ with itself, $(S_0 x)_i = \tau x_i$ ($i \in L$). Couplings among the individual positions $x_i$ are described by stochastic $L \times L$-matrices $A_\epsilon = E + \epsilon B$ that give rise to maps $\Phi_\epsilon : X \to X$ by $\Phi_\epsilon(x) = A_\epsilon \cdot x$. In general the matrix $B$ itself may depend on $x$, but in this note we restrict ourselves to the case where $B$ is state independent. So $\Phi_\epsilon$ extends to a linear map on $\mathbb{R}^L$ which is close to the identity. Now the coupled system is described by the map $S_\epsilon = \Phi_\epsilon \circ S_0$. In order to be definite we assume that $\|B\|_{\infty} = 2$, where $\|\cdot\|_{\infty}$ denotes the maximal row sum norm, i.e. the matrix norm associated to the max-norm $|\cdot|_{\infty}$ on $\mathbb{R}^L$. See Remark 3f for a comment on this particular choice.

The following theorem is a special case of the main result of [9] which generalizes previous work from [8]:

*Part of this work was done during the workshop on hyperbolic dynamical systems with singularities in September 1996 at the Erwin Schrödinger Institute at Vienna. I thank the ESI for its generous support.
1 Theorem Let $\tau$ be a tent map with slope $\alpha \in (1, 2]$. Assume that there is some $M \in \mathbb{Z}$ such that $\alpha^M > 2$ and $\tau^m(\frac{1}{2}) \neq \frac{1}{2}$ for $m = 1, \ldots, M - 1$. Then there exists $\epsilon_{ex} > 0$ depending only on $\tau$ such that for each $\epsilon \in [0, \epsilon_{ex}]$, each finite set $L$ and each linear coupling $\Phi_\epsilon$ defined as above holds: $S_\epsilon$ has an invariant probability density $h_\epsilon$ (w.r.t. Lebesgue measure), and there is some $r \in \mathbb{Z}$ such that $S_\epsilon^r$ is exact on each of its ergodic components.

Suppose now that $\tau$ is mixing, which is the case if and only if $\alpha > \sqrt{2}$. Then $S_\epsilon$, a direct product of mixing systems, is mixing, and in particular $S_\epsilon$ has a unique invariant probability density. By a perturbation argument (see[8]) one can conclude from this that for each $d \in \mathbb{Z}$ there is some $\epsilon_{uni}(d) > 0$ such that for $\epsilon \in [0, \epsilon_{uni}(d)]$ and for each $L$ with $\#L = d$ also the coupled system $S_\epsilon$ is mixing and has a unique invariant density. However, the technique used in [8] does not permit better estimates than $\epsilon_{uni}(d) \sim \frac{1}{d}$, and it seems that in order to obtain a dimension independent lower estimate of $\epsilon_{uni}$ one has to proceed differently. Using a geometric approach we prove in this note:

2 Theorem Let $\tau$ be a tent map with slope $\alpha > \sqrt{2}$. There exists $\epsilon_{uni} > 0$ depending only on $\tau$ such that for each $\epsilon \in [0, \epsilon_{uni}]$, each finite set $L$ and each linear coupling $\Phi_\epsilon$ defined as above holds: The invariant probability density $h_\epsilon$ of $S_\epsilon$ is unique, and $(S_\epsilon, h_\epsilon, dx)$ is an exact dynamical system.

3 Remarks

a) For $\alpha < \sqrt{2}$ the single map $\tau$ is not mixing and this property persists under small perturbations. Therefore $S_\epsilon$ cannot be ergodic for small $\epsilon$ as soon as $\#L > 1$.

b) For $\alpha > \sqrt{2}$ numerical estimates based on [9] show that $\epsilon_{ex} > 0.08 \cdot (\alpha - \sqrt{2})$.

c) It is an immediate consequence of the spectral decomposition of the transfer operator $S_\epsilon^r$ for $S_\epsilon$ that under the assumptions of Theorem 2 the dynamical system $(S_\epsilon, h_\epsilon, dx)$ has exponentially decaying correlations for smooth observables (in fact, for observables which are functions of bounded variation), see [8].

d) In [8, 9] Theorem 1 was proved for more general $\tau$, namely for piecewise $C^2$ maps $\tau$ for which there is an integer $k$ such that $\inf_y |(\tau^k)'(y)| > 1$. I expect that also Theorem 2 continues to hold for such $\tau$ provided they are mixing.

e) Reference [8] treats only the case $\inf_y |\tau'(y)| > 2$ (and thus excludes tent maps from consideration), and it was only in [9] that Theorem 1 was proved under the assumption $\inf_y |(\tau^k)'(y)| > 1$ for some $k > 1$.

f) Some further remarks on $B$ are in order: $B$ itself may depend on $\epsilon$, but for definiteness we assume that $\|B\|_\infty = 2$. This is motivated by the following: Denote for the moment by $v$ the vector with all entries equal to $\frac{1}{2}$. As $A_\epsilon$ is a stochastic matrix, this leads to the estimate

$$|\Phi_\epsilon(x) - x|_\infty = |\Phi_\epsilon(x - v) - (x - v)|_\infty \leq \epsilon \cdot |B(x - v)|_\infty \leq \epsilon \cdot \frac{1}{2} \cdot \|B\|_\infty = \epsilon$$

for all $x \in X$, and it follows from a Neumann series expansion that $\|\Phi_\epsilon^{-1} - \text{Id}\|_\infty \leq \frac{\epsilon}{1 - \epsilon}$.
g) As no assumptions on the decay of the coefficients $b_{ij}$ with $|i - j|$ “large” are made, also globally coupled maps as studied in [6, 4] are covered by Theorems 1 and 2. Hence any finite lattice of globally coupled mixing tent maps with coupling strength $\epsilon < \epsilon_{uni}$ has a unique exact (and hence asymptotically stable) absolutely continuous invariant probability measure, whereas the invariant measure for the limit model ($\#L \to \infty$) is unstable, see [4].

Outline of the proof of Theorem 2: The strategy of the proof is as follows: Using a telescoping argument we show in Proposition 5 that any measurable set $G$ which is invariant under some power $S_0^r$ contains a hypercube modulo Lebesgue null sets. (From this it is not hard to conclude that all sets of the asymptotic algebra of $S$ are open modulo 0, but we shall not make explicit use of this observation.) The second step essentially consists in proving that $S_h$ is topologically mixing: We show that continuous forward images of hypercubes contain again hypercubes of larger size. As the phase space $X$ is bounded this argument must stop after finitely many steps, and we shall see in Proposition 7 that some $n^r$-th forward image of the initial hypercube must contain a hypercube $(c, u)^d$ for some $u > c$ which does not depend on $n$, $r$, and $G$. As the initial hypercube was contained modulo 0 in the $S_0^r$-invariant set $G$, this shows that there can be only one $S_0^r$-invariant set, so that all powers $S_0^r$ are ergodic. But in view of Theorem 1 this proves Theorem 2.

2 Proofs of Propositions 5 and 7

Throughout the rest of the paper we assume w.l.o.g. that $I = \{1, \ldots, d\}$. Let $Z^m$ be the partition of $X$ into maximal pieces on which $S^m$ is affine. As $S = \Phi \circ S_0$, we have $Z_0^1 = Z_0^1$, and $Z_0^1$ is the collection of all $d$-dimensional rectangles $I_1 \times \ldots \times I_d$ with $I_1, \ldots, I_d \in [0, 1] \setminus \{0, \frac{1}{2}, \{\frac{1}{2}, 1\}. (Note that for our purposes it is unessential whether the point $\frac{1}{2}$ belongs to the left or to the right half of $[0, 1]$.) As $Z^m = Z^1 \vee S^{-1}Z^{m-1}$, all elements of $Z^m$ are convex polytopes. (Although this is not essential, it is a nice feature of our special piecewise affine system which helps the geometric intuition.)

Denote by $Z^m_1[x]$ that element of $Z^1$ which contains $x$. Then $S_1Z^m_1[x] \subseteq Z^{m-1}_1[S_1x]$ with equality if and only if $Z^{m-1}_1[S_1x] \subseteq Z^m_1[x]$, and this is equivalent to $(Z^{m-1}_1[S_1x])^C \cap \partial(S_1Z^m_1[x]) = \emptyset$.

Let $d\mu(x) = h_1(x) \, dx$ denote an absolutely continuous invariant probability measure for $S_1$ whose existence is guaranteed by Theorem 1. In fact, a bit more is known about its density $h_1$ (see [8, 9]): As a function of its $d$ variables, $h_1$ is of bounded variation, and it follows from a version of the Sobolev embedding theorem (e.g. [11]) that $h_1 \in L^{1+1/jd}_{Lebesgue}$. Furthermore, measures $\mu_1$ of this type are the only absolutely continuous invariant measures for $S_1$.

For $N, n \in \mathbb{Z}$ we define $Y_{r,N,n}$ as the set of all points $x \in X$ that, under the action of $S_h$, approach the singularity hyperplanes of $S_1$ too fast in the following sense:

$$Y_{r,N,n} := \left\{ x \in X : \exists 0 \leq k \leq n - 1 \text{ s.th. } (S^N_1[S_1^{k+1}x])^C \cap \partial(S_1Z^1_1[S_1^kx]) \neq \emptyset \right\}.$$ 

Note that $S_h^N[Z^N_1[S_1^kx]] = Z^N_1[S_1^kx]$ for $x \in X \setminus Y_{r,N,n}$.

The following lemma is a variation of a folklore estimate.
4 Lemma Let \( \tau \) be a tent map with slope \( \alpha > 1 \). There is \( \varepsilon_1 > 0 \) such that for all \( \delta > 0 \) and all \( d \geq 1 \) there is an integer \( N = N(\delta, d) \) such that

\[
\forall n \geq 0 \forall \varepsilon \in [0, \varepsilon_1] : \mu_\varepsilon (Y_{\varepsilon, N, n}) < \delta.
\]

Proof: Let \( g \) denote an inverse branch of \( S_{\varepsilon} \). Then \( Dg = \alpha^{-1} \cdot A_{\varepsilon} \), whence

\[
\|Dg\|_\infty = \|A_{\varepsilon}\|_\infty \leq \alpha^{-1} \cdot (1 + \varepsilon \cdot \|B\|_\infty) = \alpha^{-1} \cdot (1 + 2\varepsilon).
\]

Fix \( \varepsilon_1 > 0 \) such that \( \lambda := \alpha^{-1} \cdot (1 + 2\varepsilon_1) < 1 \). Then \( \|Dg\|_\infty \leq \lambda \) for all \( \varepsilon \in [0, \varepsilon_1] \) (independently of \( d \)) and \( \text{diam}(Z^n_{\varepsilon}[x]) \leq \lambda^n \) for all \( x \in X \) and \( n \geq 0 \). Making use of the \( S_{\varepsilon}\)-invariance of \( \mu_\varepsilon \) and of the fact that \( h_\varepsilon \in L_{\text{Lebesgue}}^{1+1/d} \) we can estimate

\[
\mu_\varepsilon (Y_{\varepsilon, N, n}) \leq \sum_{k=0}^{n-1} \mu_\varepsilon \{ x \in X : (Z^n_{\varepsilon+k}[S_{\varepsilon}^k x]) \cap \partial(S, Z^n_{\varepsilon}[S_{\varepsilon}^k x]) \neq \emptyset \}
\]

\[
= \sum_{k=0}^{n-1} \mu_\varepsilon \{ x \in X : (Z^n_{\varepsilon+k} S_{\varepsilon} x) \cap \partial(S, Z^n_{\varepsilon} x) \neq \emptyset \}
\]

\[
\leq \sum_{j=0}^{n-1} \sum_{z \in \mathbb{Z}^j} ||h_\varepsilon||_{1+1/d} \cdot \{ x \in Z^j_{\varepsilon} : (Z^n_{\varepsilon+k} S_{\varepsilon} x) \cap \partial(S, Z^n_{\varepsilon} x) \neq \emptyset \}
\]

\[
\leq \text{const}_d \cdot \lambda^{\frac{n}{2+1}}
\]

where \( |.| \) denotes Lebesgue measure. For each fixed \( d \) an integer \( N = N(\delta, d) \) can be chosen such that \( \text{const}_d \cdot \lambda^{\frac{n}{2+1}} < \delta \).

5 Proposition Let \( \tau \) be a tent map with slope \( \alpha > 1 \) and let \( \varepsilon \in [0, \varepsilon_1] \) as in Lemma 4. If \( S_{\varepsilon}^r G \subseteq G \) for some measurable set \( G \) with \( \mu_\varepsilon (G) > 0 \) and some \( r \in \mathbb{Z} \), then there is an open set \( Q \subset X \) such that \( |Q \setminus G| = 0 \). (Again \( |.| \) denotes Lebesgue measure.)

Proof: Let \( \delta := \frac{1}{4} \mu_\varepsilon (G) \) and fix \( N = N(\delta, d) \) as in Lemma 4 so that \( \mu_\varepsilon (Y_{\varepsilon, N, n}) < \delta \) for all \( n \geq 0 \). Let \( H_n := G \setminus Y_{\varepsilon, N, r} \) and \( \overline{H} := \cap_{n \geq 0} \cup_{n \geq k} H_n \). Then \( \overline{H} \subseteq G \) and \( \mu_\varepsilon (\overline{H}) \geq \inf_{n} \mu_\varepsilon (H_n) \geq \delta > 0 \), in particular \( |\overline{H}| > 0 \).

Let

\[
f_n(x) := E_{L^d} [\chi_{\overline{H}} Z^n_{\varepsilon} | x] = \frac{|\overline{H} \cap Z^n_{\varepsilon}[x]|}{|Z^n_{\varepsilon}[x]|}.
\]

Then \( f_n(x) \to \chi_{\overline{H}}(x) \) for Lebesgue-a.e. \( x \) by Lévy’s theorem, and there exists a point \( x_0 \in \overline{H} \) such that

\[
\lim_{n \to \infty} \frac{|\overline{H} \cap Z^n_{\varepsilon}[x_0]|}{|Z^n_{\varepsilon}[x_0]|} = \chi_{\overline{H}}(x_0) = 1.
\]

As \( x_0 \in \overline{H} \), there is a sequence of indices \( n_1 < n_2 < n_3 < \ldots \) (depending on \( x_0 \)) such that \( x_0 \in H_{n_i} \) for all \( i \). In particular, \( x_0 \not\in Y_{\varepsilon, N, r_{n_i}} \), which implies that \( S_{\varepsilon}^{r_{n_i}} Z^n_{\varepsilon}[S_{\varepsilon} x_0] = Z^n_{\varepsilon}[S_{\varepsilon} x_0] \) for all \( i \). Therefore, and because \( S_{\varepsilon}^{r_{n_i}} | Z^n_{\varepsilon}[S_{\varepsilon} x_0] \) is affine,

\[
\lim_{i \to \infty} \frac{|S_{\varepsilon}^{r_{n_i}} \overline{H} \cap Z^n_{\varepsilon}[S_{\varepsilon} x_0]|}{|Z^n_{\varepsilon}[S_{\varepsilon} x_0]|} = \lim_{i \to \infty} \frac{|\overline{H} \cap Z^n_{\varepsilon}[x_0]|}{|Z^n_{\varepsilon}[x_0]|} = 1.
\]
As $Z^N_\epsilon$ is finite, there exists some $Z^* \in Z^N_\epsilon$ such that $Z^N_\epsilon[S^n[x_0] = Z^*$ for infinitely many $n_i$, and it follows that

$$\frac{|G \cap Z^*|}{|Z^*|} = \lim_{i \to \infty} \frac{|S^n[G \cap Z^*]|}{|Z^*|} \geq \lim_{i \to \infty} \frac{|S^n[H \cap Z^*]|}{|Z^*|} = 1.$$

Hence $|Z^* \setminus G| = 0$, and as $(Z^*)^c \neq \emptyset$, this finishes the proof of the lemma. \(\square\)

The following corollary is a rather direct consequence of the last Proposition. As we do not make explicit use of it, we skip its proof.

6 Corollary In the situation of Proposition 5 the set $G$ is open modulo a Lebesgue null set.

With the next Proposition we make the step from local exactness to global exactness:

7 Proposition Let $\tau$ be a tent map with slope $\alpha > \sqrt{2}$. There is $\epsilon_2 \in (0, \epsilon_1]$, $\epsilon_1$ from Lemma 4, such that for all $\epsilon \in [0, \epsilon_2]$ and for all $d \geq 1$ holds: If $S^d[G \subseteq \epsilon$ for some measurable set $G$ with $\mu(G) > 0$ and some $r \in Z$ and if $G$ contains an open set $Q$ modulo Lebesgue measure zero, then $G$ contains the hypercube $(c, u)^d$ modulo Lebesgue measure zero where $u > c$ is a point in $[0, 1]$ that depends only on $\tau$ (and influences the choice of $\epsilon_2$).

In order to formalize the subsequent geometric arguments we introduce the following notation: For $a, b \in \mathbb{R}^d$ let $a \leq b$ if $a_i \leq b_i$ for all $i = 1, \ldots, d$. For $a, b \in \mathbb{R}^d$, $a \leq b$, let $R(a, b) := \{x \in \mathbb{R}^d : a \leq x \leq b\}$. Denote $1 = (1, \ldots, 1) \in \mathbb{R}^d$.

8 Lemma Let $a, b \in \mathbb{R}^d$, $a \leq b$, and denote $\tilde{\epsilon} := \frac{\epsilon}{1-\epsilon}$. Then

a) $\Phi, R(a, b) \supseteq R(\Phi, a + \tilde{\epsilon}b - a|_1, \Phi, a + (b - a) - \tilde{\epsilon}|b - a|_1)$.

b) $\Phi, R(a, b) \supseteq R(a + \tilde{\epsilon}1, b - \tilde{\epsilon}1)$.

Proof: Observe that $\Phi, 1 = 1$.

a) $\Phi^{-1}R(\Phi, a + \tilde{\epsilon}b - a|_1, \Phi, a + (b - a) - \tilde{\epsilon}|b - a|_1)$

$= \Phi^{-1}\{\Phi, a + \tilde{\epsilon}b - a|_1 + x : 0 \leq x \leq b - a - 2\tilde{\epsilon}|b - a|_1\}$

$= \{a + \tilde{\epsilon}b - a|_1 + (\Phi^{-1}x - x) + x : 0 \leq x \leq b - a - 2\tilde{\epsilon}|b - a|_1\}$

$\subseteq \{a + x : 0 \leq x \leq b - a\} = R(a, b)$

where we used for the last step that $\|\Phi^{-1} - \text{Id}\|_\infty \leq \frac{\epsilon}{1-\epsilon} = \tilde{\epsilon}$.

b) $\Phi^{-1}R(a + \tilde{\epsilon}1, b - \tilde{\epsilon}1)$

$= \{\Phi^{-1}a + \tilde{\epsilon}1 + \Phi^{-1}x : 0 \leq x \leq b - a - 2\tilde{\epsilon}1\}$

$= \{a + \tilde{\epsilon}1 + \Phi^{-1}(a + x) - (a + x) + x : 0 \leq x \leq b - a - 2\tilde{\epsilon}1\}$

$\subseteq R(a, b)$
Proof of Proposition 7: Let $c = \frac{1}{2}$ be the critical point of $\tau$ and denote by $y$ the orientation reversing fix point of $\tau$. Then $\tau^2 c < c < z_\tau < \tau c$, and there is some point $u \in (z_\tau, \tau c)$ such that $\tau^2 c < c < \tau u < z_\tau < u < \tau^2 u < \tau c$. Observe that $\alpha \cdot |u - c| = |\tau u - \tau c| < |c - \tau c|$, in particular $|u - c| < \frac{1}{2} \left( \frac{1}{2} - \frac{a}{2} \right) = \frac{2}{2} \left( \frac{1}{2} - 1 \right) \leq \frac{1}{2}$. Denote by $\check{u}$ the symmetric point of $u$, i.e. $\check{u} = 1 - u$, $\tau \check{u} = \tau u$. Fix $\epsilon_3 < \frac{1}{2}$ such that for all $\epsilon \in [0, \epsilon_3]$

$$
\begin{align*}
\tau (\tau u + \check{\epsilon}) - \check{\epsilon} & \geq u \\
\tau (\tau c - \check{\epsilon}) + \check{\epsilon} & \leq c \\
\alpha \cdot |u - c| & \leq |\tau c - c| - \check{\epsilon}
\end{align*}
$$

where $\check{\epsilon} = \frac{u - c}{1 - \epsilon}$ as in the preceding lemma. Let $K := |u - c|^{-1} \geq 4$ and choose $\epsilon_4 \in [0, \epsilon_3]$ such that $\frac{\alpha^2}{2} (1 - 6K \epsilon_4) > 1$.

Let $\epsilon_2 = \min\{\epsilon_{xy}, \epsilon_1, \epsilon_4\}$. From now on we fix some $\epsilon \in [0, \epsilon_2]$ and some $L = \{1, \ldots, d\}$. We construct a sequence of hyper-rectangles $R_n = R(a^{(n)}; b^{(n)}) \subseteq S^n Q$ with the following three properties:

$(A_n)$ For $n > 0$ and each $i = 1, \ldots, d$ holds:

$$
|b_i^{(2n)} - a_i^{(2n)}| \geq \frac{\alpha^2}{2} (1 - 6K \epsilon) \cdot |b_i^{(2n-2)} - a_i^{(2n-2)}| \quad \text{or}
$$

$$(a_i^{(2n)}, b_i^{(2n)}) \text{ contains at least one of } (c, u) \text{ and } (\check{u}, c).$$

$(B_n)$ If $n > 0$ and if $(a_i^{(2n-2)}, b_i^{(2n-2)})$ contains $(c, u)$ or $(\check{u}, c)$, then $(a_i^{(2n)}, b_i^{(2n)}) \supseteq (c, u)$.

$(C_n)$ Either a) there is $\ell^{(2n)} \in (0, |u - c|)$ such that $b_i^{(2n)} - a_i^{(2n)} = \ell^{(2n)}$ for all $i \in L$ or b) $|u - c| \leq |b_i^{(2n)} - a_i^{(2n)}| \leq \frac{\epsilon}{2}$ for all $i \in L$.

Before constructing the $R_n$ we show how Proposition 7 follows: As $\frac{\alpha^2}{2} (1 - 6K \epsilon) > 1$, properties $(A_n)$ and $(B_n)$ imply that there is some $n_0$ such that $S^n Q \supseteq R_n \supseteq (c, u)^d$ for all $n \geq n_0$. As $Q \subseteq G$ modulo Lebesgue measure zero and as $S^r Q \subseteq G$ for some $r \in \mathbb{Z}$, it follows that $(c, u)^d \subseteq G$ modulo Lebesgue measure zero.

Now let $R_0 = R(a^{(0)}, a^{(0)} + \ell^{(0)} \mathbf{1})$ be some hypercube contained in the open set $Q$. Properties $(A_0)$-$\text{(C}_0\text{)}$ are trivially satisfied. Suppose that $R_0, \ldots, R_{2n}$ are constructed with properties $(A_k)$-$\text{(C}_k\text{)}$ ($k = 0, \ldots, n$). Denote $L_k := \{i \in L : a_i^{(k)} < c < b_i^{(k)}\}$, and for each $i \in L$ let $H_i := \{x \in \mathbb{R}^d : x_i = c\}$. Then $R_{2n}$ is divided by the family $(H_i : i \in L_{2n})$ into $2^d L_{2n}$ hyper-rectangles. Among them there is one, call it $\check{R}_{2n}$, such that

$$
(a_i^{(2n)}; \check{b}_i^{(2n)}) = (a_i^{(2n)}, \check{b}_i^{(2n)}) \quad \text{for } i \in L \setminus L_{2n} \text{ and}
$$

$$
|\check{b}_i^{(2n)} - a_i^{(2n)}| \geq \frac{1}{2} |b_i^{(2n)} - a_i^{(2n)}| \quad \text{and } c \in \{a_i^{(2n)}, \check{b}_i^{(2n)}\} \quad \text{for } i \in L_{2n}.
$$

As $S_0 |R_{2n}$ is the direct product of monotone branches of $\tau$, also $S_0 \check{R}_{2n}$ is a hyper-rectangle, and in view of Lemma 8 we can choose a hyper-rectangle $R_{2n+1}$ contained in $S_0 \check{R}_{2n} = \Phi_{\epsilon} (S_0 \check{R}_{2n})$ as follows:


1st case: Property \((C_n a)\) is satisfied. Then the lengths of different edges of \(R_{2n}\) differ at most by a factor 2, and because of Lemma 8a) \(R_{2n+1} \subseteq S, R_{2n}\) can be chosen such that

\[
|b_i^{(2n+1)} - a_i^{(2n+1)}| = (1 - 4\varepsilon) \cdot \frac{\alpha}{2} |b_i^{(2n)} - a_i^{(2n)}| \quad \text{for } i \in L_{2n}, \text{ and}
\]

\[
|b_i^{(2n+1)} - a_i^{(2n+1)}| = (1 - 4\varepsilon) \cdot \alpha |b_i^{(2n)} - a_i^{(2n)}| \quad \text{for } i \in L \setminus L_{2n}.
\]

We claim that \(L_{2n} \cap L_{2n+1} = \emptyset\): In fact, suppose that there is some \(i \in L_{2n} \cap L_{2n+1}\). Then \(b_i^{(2n+1)} \geq \tau c - \varepsilon\) by Lemma 8b as \(i \in L_{2n}\), and \(a_i^{(2n+1)} < \varepsilon\) as \(i \in L_{2n+1}\). Hence

\[|\tau c - \varepsilon - \varepsilon| \leq |b_i^{(2n+1)} - a_i^{(2n+1)}| \leq \alpha \cdot \ell(2n) < \alpha \cdot |u - c|,\]

which contradicts the choice of \(u\) and \(\varepsilon_3\).

Therefore we can apply the same reasoning as above to \(R_{2n+1}\). The ratios of the lengths of the different edges of \(R_{2n+1}\) differ again by a factor of at most 2 because \(L_{2n} \cap L_{2n+1} = \emptyset\), and we obtain \(R_{2n+2}\) with

\[
|b_i^{(2n+2)} - a_i^{(2n+2)}| = (1 - 4\varepsilon) \cdot \frac{\gamma}{2} |b_i^{(2n+1)} - a_i^{(2n+1)}| \quad \text{for } i \in L_{2n+1}, \text{ and}
\]

\[
|b_i^{(2n+2)} - a_i^{(2n+2)}| = (1 - 4\varepsilon) \cdot \alpha |b_i^{(2n)} - a_i^{(2n+1)}| \quad \text{for } i \in L \setminus L_{2n+1}.
\]

Reducing, if necessary, the size of \(R_{2n+2}\) in directions \(i \in L \setminus (L_{2n} \cup L_{2n+1})\) we finally obtain a hyper-rectangle \(R_{2n+2}\) such that

\[
|b_i^{(2n+2)} - a_i^{(2n+2)}| = (1 - 4\varepsilon)^2 \cdot \frac{\gamma^2}{2} |b_i^{(2n)} - a_i^{(2n)}| = (1 - 4\varepsilon)^2 \cdot \frac{\gamma^2}{2} \ell(2n) =: \ell(2n+2)
\]

for all \(i \in L\). If \(\ell(2n+2) > \frac{1}{2}\), we reduce \(R_{2n+2}\) further to a hypercube with \(\ell(2n+2) = \frac{1}{2}\).

Then \((A_n + 1)\) holds as \(K \geq 4\), and \((C_{n+1})\) is satisfied because \(R_{2n+2}\) is a hypercube. \((B_{n+1})\) is checked later.

2nd case: Property \((C_n b)\) is satisfied.

Compared to the 1st case two modifications of the induction argument are necessary: As \(R_{2n}\) is no longer a hypercube, the lengths of two different sides of \(R_{2n}\) may have a ratio greater than 2. However, because of \((C_n b)\), these ratios are bounded by \(K\), so that the factors \((1 - 4\varepsilon)\) in (2) must be replaced by \((1 - 2K\varepsilon)\). Similarly, the corresponding ratios of \(R_{2n+1}\) are bounded by \(2K\) so that we have \((1 - 4K\varepsilon)\) instead of \((1 - 4\varepsilon)\) in (3). For \(i \in L \setminus (L_{2n} \cap L_{2n+1})\) we thus have

\[
|b_i^{(2n+2)} - a_i^{(2n+2)}| \geq (1 - 6K\varepsilon) \cdot \frac{\gamma^2}{2} |b_i^{(2n)} - a_i^{(2n)}| > |u - c|.
\]

On the other hand, for \(i \in L_{2n} \cap L_{2n+1}\) holds as in the 1st case: \(a_i^{(2n+1)} < \varepsilon < \tau c - \varepsilon \leq b_i^{(2n+1)}\) and hence

\[
(a_i^{(2n+2)}, b_i^{(2n+2)}) \supseteq (\tau(\tau c - \varepsilon) + \varepsilon, \tau c - \varepsilon) \supseteq (c, u)
\]

by Lemma 8b. If, for some \(i\), this construction leads to \(|b_i^{(2n+2)} - a_i^{(2n+2)}| > \frac{1}{2}\), then the interval \((a_i^{(2n+2)}, b_i^{(2n+2)})\) contains \((c, u)\) or \(\hat{c}, c)\), and we reduce it to that one. Thus \((A_{n+1})\) and \((C_{n+1})\) are satisfied.
In both cases also \((B_{n+1})\) is satisfied: Suppose that \((a_i^{(2n)}, b_i^{(2n)})\) contains \((e, u)\) or \((\hat{u}, c)\). Then \(a_i^{(2n+1)} \leq \tau u + \hat{c}\) and \(b_i^{(2n+1)} \geq \tau e - \hat{c}\) so that
\[
(a_i^{(2n+2)}, b_i^{(2n+2)}) \supseteq (\tau (\tau e - \hat{c}) + \hat{c}, \tau (\tau u + \hat{c}) - \hat{c}) \supseteq (e, u).
\]

\[\square\]

References


