Modular Functor and Representation Theory of $\widehat{sl}_2$ at a Rational Level

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Modular Functor and Representation Theory of $\widehat{\mathfrak{sl}_2}$ at a Rational Level

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Abstract

We define a new modular functor based on Kac-Wakimoto admissible representations and the corresponding $D$-module on the moduli space of rank 2 vector bundles with parabolic structure. A new fusion functor arises which is related to representation theory of the pair “$\mathfrak{osp}(1|2), \mathfrak{sl}_2$” in the same way as the fusion functor for the Virasoro algebra is related to representation theory of the pair “$\mathfrak{sl}_2, \mathfrak{sl}_2$.”

1 Introduction

In this paper we define a new modular functor based on Kac-Wakimoto admissible representations over $\widehat{\mathfrak{sl}_2}$. The modular functor introduced by Segal [42] assigns a finite-dimensional vector space to the data consisting of a punctured curve, a rank 2 vector bundle and a collection of integral dominant highest weights attached to the punctures. Our modular functor does the same for the Segal’s data (with integral dominant highest weights replaced with admissible highest weights) extended by the lines in the fibers over the punctures. As the data “surface, vector bundle, punctures, lines in fibers over punctures” evolve, so does the corresponding finite dimensional vector space. This leads to a new $D$-module on the moduli space of rank 2 vector bundles with parabolic structure (fixed lines in certain fibers). The main feature of this $D$-module, as opposed to the standard one (see Tsuchiya-Ueno-Yamada [44], or Beilinson-Feigin-Mazur [4]), or Moore-Seiberg [37]), is that it is singular over a certain set of exceptional vector bundles. The latter is closely related to the Hitchin’s global nilpotent cone.
We also prove that our $D$-module has (in a proper sense) regular singularities at infinity and that dimension of the generic fiber can be calculated by the usual combinatorial algorithm: by pinching the surface the problem is reduced to the case of a sphere with $\geq 3$ punctures and further to a collection of spheres with 3 punctures. Dimension of the space attached to the datum “3 modules sitting at 3 points on a sphere” is calculated explicitly. It is a pure linear algebra calculation of dimension of the space of coinvariants of a certain infinite dimensional algebra in a certain infinite dimensional representation. As the result is amusing we will record it here.

First of all, and it is important, in the genus zero case, one can work with modules at a generic level, as opposed to admissible representations which only exist when the level is rational. It is in complete analogy with the usual WZW model, where the famous theory of Knizhnik-Zamolodchikov equations arises from a collection of the so-called Weyl modules sitting on a sphere (terminology is borrowed from [30]). The family of Weyl modules is good for the purpose of studying integrable representations because each integrable representation is a quotient of some Weyl module. This is no longer the case as far as admissible representations are concerned. A family of modules suitable for our needs is that of what we call generalized Weyl modules; the latter is defined to be a Verma module quotiented out by a singular vector.

Generalized Weyl modules are naturally parametrized by the symbols $(V^r_s; V^s_r) \in \mathbb{Z}/2\mathbb{Z}$. Here $V^r_s$ is to be thought of as the $r+1$-dimensional irreducible $sl_2$-module; meaning of $V^r_s$ will be explained soon. It is appropriate to keep in mind that the conventional Weyl module is defined to be the module induced from $V^r_s$. Therefore usually Weyl modules are labelled by $sl_2$-modules. In our situation Weyl modules are those related to symbols $(V^0_0, V^r_s)$.

According to Verlinde, dimensions of the spaces associated to 3 modules on a sphere are structure constants of Verlinde algebra. Result of calculation of Verlinde algebra in our situation is as follows:

\[
(V^r_s, V^1_1) \circ (V^0_0, V^s_r) = (V^{r+s+1}_s \otimes V^0_0) + (V^{r+s+2}_s \otimes V^0_0) + (V^{r+s+3}_s \otimes V^0_0) + \cdots + (V^{r+s+s}_s \otimes V^0_0).
\]

Recall that the usual Verlinde algebra built on Weyl modules is as follows:
\[ V_{s_1} \circ V_{s_2} = V_{s_1} \otimes V_{s_2}, \]
i.e. it is the Grothendieck ring if the category of finite dimensional representations of \( \mathfrak{sl}_2 \). Observe that our formula agrees with the latter on Weyl modules.

The first component of the right hand side of our formula is equally easy to interpret. It is known that the symbols \( V_x \) naturally parametrize finite dimensional representations of the simplest rank 1 superalgebra \( \mathfrak{osp}(1|2) \). The category of finite dimensional \( \mathfrak{osp}(1|2) \)-modules is a tensor category and (1) reads as follows: Verlinde algebra is isomorphic to the product of Grothendieck rings of the categories of finite dimensional representations of \( \mathfrak{osp}(1|2) \) and \( \mathfrak{sl}_2 \).

It is known in principle what to do when passing from modules to their quotients, in our case from generalized Weyl modules at a generic level to admissible representations at a rational level: one has to replace Lie algebras with quantized universal enveloping algebras at roots of unity and consider Grothendieck rings of the corresponding semisimple “quotient categories”. Examples: Verlinde algebra built on integrable \( \hat{\mathfrak{sl}}_2 \)-modules has to do with \( \mathfrak{sl}_2 \) in this way, and Verlinde algebra built on minimal representations of Virasoro algebra in this way has to do with 2 copies of \( \mathfrak{sl}_2 \). Calculation of coinvariants shows that one more example can be added to the list: Verlinde algebra built on admissible representations is related to the pair \((\mathfrak{osp}(1|2), \mathfrak{sl}_2)\) in exactly the same way as \( \text{Vir} - \text{Verlinde} \) algebra is related to the pair of \( \mathfrak{sl}_2 \)'s.

Interest in admissible representation originates in the fact that the characters of admissible representations at a fixed level give a representation of the modular group. However realization of this fact immediately gave rise to two puzzles:

(i) Given a representation of the modular group, Verlinde formula produces structure constants of Verlinde algebra; in the case of admissible representations some of the structure constants are negative. This does not make much sense as they are supposed to count dimensions.

(ii) Quantum Drinfeld-Sokolov reduction provides a functor from the category of \( \hat{\mathfrak{sl}}_2 \)-modules to the category of \( \text{Vir} - \text{modules} \), which sends admissible representations to minimal representations. It should give an epimorphism (or some weakened version of it) of a suitably defined Verlinde algebra for \( \hat{\mathfrak{sl}}_2 \) on the well-known Verlinde algebra for \( \text{Vir} \).

We are able to give an answer to (ii), and a partial answer to (i).
As far as (ii) is concerned, let us for simplicity step aside and consider \( \text{Vir} \)-modules at a generic (not necessarily rational) level. Then there is an analogue of a generalized Weyl module – Verma module quotiented out by a singular vector – and these are naturally parametrized by the symbols \((V_r, V_s)\). The desired epimorphism is given by:

\[
(V_r^e, V_s^e) \mapsto (V_r, V_s) + (V_{r-1}, V_s).
\]

This map is naturally related to the Drinfeld-Sokolov reduction in the following way. As we have fixed the category of representations, we have triangular decomposition of \(\mathfrak{sl}_2\); in particular we have 2 opposite nilpotent subalgebras, \(\mathfrak{ce}, \mathfrak{cf}\). Therefore there are in fact 2 Drinfeld-Sokolov functors, \(\phi_e, \phi_f\). It happens that the map above is induced by the direct sum \(\phi_e \oplus \phi_f\). Further, this map is even easier to interpret from the point of view of finite dimensional representation theory. Indeed, \(\mathfrak{osp}(1|2)\) contains \(\mathfrak{sl}(2)\) as the even part. Therefore there arises the forgetful morphism of the categories of finite dimensional representations \(\text{Rep}(\mathfrak{osp}(1|2)) \to \mathfrak{sl}(2)\). It is easy to see that the forgetful morphism is exactly our map \(V^e_r \to V^e_r \oplus V^e_{r-1}\).

As to (i), the situation is as follows. The structure constants naturally arrange in a tensor \(\{c^e_{ij}\}\), the indices running through a set of representations in question. Let us compare the set \(\{c^e_{ij}\}\) of the structure coefficients of our algebra and the set \(\{b^e_{ij}\}\) of structure coefficients of the algebra calculated by Verlinde formula:

If our \(c^e_{ij} = 0\), then \(b^e_{ij} = 0\). If \(c^e_{ij} \neq 0\), then \(b^e_{ij}\) is "most certainly" zero, however in some exceptional cases it is non-zero. The latter cases in our situation are interpreted in the following way. Recall that we have not only 3 modules, \(i, j, r\), but also 3 Borel subalgebras, \(\mathfrak{b}_i, \mathfrak{b}_j, \mathfrak{b}_r\), which vary. Now as \(c^e_{ij} \neq 0\), the fiber of our \(D\)-module is \(\neq 0\) (in fact it is 1-dimensional), if the 3 Borel subalgebras are pairwise different. If however 2 of them meet, the fiber usually vanishes, but sometimes survives. It survives if and only if \(b^e_{ij} \neq 0\). If non-zero, \(b^e_{ij}\) can be \(\pm 1\). There is no doubt that \(b^e_{ij}\) is a result of some cohomological calculation related to the \(D\)-module. Unfortunately we cannot make it more precise at the moment.

Just as in the usual case Weyl modules on a sphere produce a trivial vector bundle with the flat (Knizhnik-Zamolodchikov)connection, in our case we get a bundle with a flat connection on a space of the 2 times greater dimension. The extra coordinates come from the flag manifold, recall that we are dealing with moduli of vector bundles with parabolic structure. Horizontal sections of this connection satisfy a system of differential equations;
we get twice as many equations as there are KZ equations: half of them are indeed KZ equations and the other half comes from singular vectors in Verma modules over \( \hat{\mathfrak{sl}}_2 \). The latter is but natural – it is exactly one of the lessons of the pioneering work [6]. This allows to put the integral formulas for solutions of Knizhnik-Zamolodchikov equations, which we wrote in [20], in a proper context: they give horizontal sections of this new connection. We conjecture that our methods, in fact, provide all horizontal sections. The relation of our formulas to those in [43] is that the latter are necessarily polynomials as functions on the flag manifold while ours are not.

An important representation theoretic fact behind all the mentioned results is a theorem on singular support of admissible representations. It claims that a representation of \( \hat{\mathfrak{sl}}_2 \) is admissible if and only if its singular support is contained in the space of 1-forms with values in the nilpotent cone of \( \mathfrak{sl}_2 \). The result was discovered at least conjecturally and at least for the vacuum representation a few years ago by E.Frenkel and B.F. The proof we propose here uses, in particular, some results of our work [19]. As an application we get a construction of an infinite collection of elements of the annihilating ideal of admissible representations at a given level.

We wish to acknowledge that there has been a number of works approaching WZW model for admissible representation from different points of view, see for example [1, 17, 24, 38, 40]. It would be interesting to relate our integral formulas with those in [38] and the new Hopf algebra of [40] to the above mentioned \( \mathfrak{osp}(1, 2) \times \mathfrak{sl}_2 \) at roots of unity. To the best of our knowledge, Verlinde algebras proposed in these works do not solve (ii) above – those algebras are rather trivial when compared to the \( \mathfrak{vir}^{-} \)-analogue. Our starting point, see [19], was the work [1], where Verlinde algebra for admissible representations was first calculated (in the form equivalent but much less illuminating than the one described above), using the language which left completely open the problem of existence of a \( D \)-module, such that dimension of the fiber is calculated through this algebra.

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2 Notations and known results

2.1

Some notations from commutative algebra are as follows:

\[ \mathbb{C}[t] \text{ is a polynomial ring, } \mathbb{C}[[t]] \text{ is its completion by positive powers of } t; \mathbb{C}[t, t^{-1}] \text{ is a ring of Laurent polynomials and } \mathbb{C}((t)) \text{ is its completion by positive powers of } t. \]

By functions on the formal (punctured) neighborhood of a non-singular point on a curve we will mean a ring isomorphic to \( \mathbb{C}[[t]] \) (\( \mathbb{C}((t)) \) resp.); to specify such an isomorphism means to pick a local coordinate \( t \). The analogous meaning will be given to the phrase “sections of a vector bundle on the formal (punctured) neighborhood of a non-singular point on a curve”.

2.2

Set \( \mathfrak{g} = \mathfrak{sl}_2, \mathfrak{g} = \mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \). Choose a basis \( e, h, f \) of \( \mathfrak{g} \) satisfying the standard relations \([h, e] = 2e, [h, f] = -2f, [e, f] = h\). We say that

\[ \mathfrak{g}_e = Ce + Ch \text{ and } \hat{\mathfrak{g}}_e = \mathfrak{g} \otimes z\mathbb{C}[[z]] \oplus \mathfrak{b} \oplus \mathbb{C}c \text{ are standard Borel subalgebras of } \mathfrak{g} \text{ and } \hat{\mathfrak{g}} \text{ resp.;} \]

\[ \mathfrak{g}_> = Ce \text{ and } \hat{\mathfrak{g}}_> = \mathfrak{g} \otimes z\mathbb{C}[[z]] \oplus \mathfrak{g}_e \text{ are standard “maximal nilpotent subalgebras” of } \mathfrak{g} \text{ and } \hat{\mathfrak{g}} \text{ resp.}; \]

\( Ch \) and \( Ch \oplus \mathbb{C}c \) are standard Cartan subalgebras of \( \mathfrak{g} \) and \( \hat{\mathfrak{g}} \) resp.

The Verma module \( M_{\lambda, k} \) is a module induced from the character of \( \mathfrak{g} \otimes z\mathbb{C}[[z]] \oplus \mathfrak{b} \oplus \mathbb{C}c \) annihilating \( \mathfrak{g} \otimes z\mathbb{C}[z] \oplus \mathbb{C}c \) and sending \( h \) and \( c \) to \( \lambda \) and \( k \) resp. \( k \) is often referred to as a level. Generator of \( M_{\lambda, k} \) is usually denoted by \( v_{\lambda, k} \). A quotient of a Verma module is called highest weight module.
The algebra $\hat{g}$ is $\mathbb{Z}_2$-graded by assigning $f \otimes z^n \mapsto (1, -n)$, $e \otimes z^n \mapsto (-1, -n)$ and so is a Verma module (as well as its quotients): $M_{\lambda,k} = \bigoplus_{i,j} M_{\lambda,k}^{ij}$.

There is a canonical antiinvolution $\omega : \hat{g} \to \hat{g}$ interchanging $\hat{g}_>$ and $\hat{g}_<$ and constant on the Cartan subalgebra. For any highest weight module $V$ denote by $V^e$ and call contragredient the module equal to the restricted dual $V^*$ as a vector space with the following action of $\hat{g}$:

$$<gx, y> = <x, \omega(g)y>, g \in \hat{g}, x \in V^*, y \in V.$$ 

If a highest weight module $V$ is irreducible then it is isomorphic to $V^e$. A morphism of highest weight modules $V_1 \to V_2$ naturally induces the morphism of the corresponding contragredient modules: $V_1^e \to V_2^e$.

A morphism of Verma modules $M_{\lambda,k} \to M_{\mu,k}$ is determined by the image of $v_{\lambda,k}$. The image can be written as $Sv_{\mu,k}$ for a uniquely determined element $S$ of the universal enveloping algebra of $\hat{g} \otimes z^{-1} \mathbb{C}[z^{-1}] \oplus \mathbb{C}f$. If non-zero, the vector $Sv_{\mu,k}$, or even $S$ for this matter, is called singular. The singular vector can be equivalently defined as an eigenvector of the Cartan subalgebra of $\hat{g}$ annihilated by $\hat{g}_>$. In this form definition applies to an arbitrary $\hat{g}$-module.

### 2.3 Singular vector formula

It follows from Kac-Kazhdan determinant formula that a singular vector generically appears in the homogeneous components of degree either $n(-1,m)$, $m > 0, n > 0$ or $n(1,m)$, $m \geq 0, n > 0$. Denote the corresponding singular vectors by $S_{n,m}^1$ and $S_{n,m}^0$ resp.

Singular vectors $S_{n,m}^0$ were found in [33] in an unconventional form containing non-integral powers of elements of $\hat{g}$ (see also [3] for another approach):

$$S_{n,m}^0 = (e \otimes z^{-1})^{n+m\ell} f^{m+(m-1)\ell} (e \otimes z^{-1})^{n+(m-2)\ell} \cdots (e \otimes z^{-1})^{n-m\ell}, \quad (2)$$

$$S_{n,m}^1 = f^{n+m\ell} (e \otimes z^{-1})^{n+(m-1)\ell} f^{m+(m-2)\ell} \cdots f^{m-m\ell}, \quad (3)$$

where $t = k + 2$.

This form is not always convenient to calculate a singular vector. It is, however, a useful tool to derive properties of a singular vector. For example, denoting by $\pi : \hat{g} \to g$, $\hat{g} \otimes z^n \mapsto g$ the evaluation map, one uses (2, 3) to derive that (see [23], also [34] for the proof in a more general quantum case):
\[ \pi S_{nm}^0 = \left( \prod_{i=1}^{m} \prod_{j=1}^{n} P(-it - j) \right) e^n \tag{4} \]

\[ \pi S_{nm}^1 = \left( \prod_{i=1}^{m} \prod_{j=0}^{n-1} P(it + j) \right) f^n, \tag{5} \]

where \( P(t) = ef - (t + 1)b - t(t + 1) \).

### 2.4 Generalized Weyl modules and admissible representations

The structure of Verma modules over \( \hat{\mathfrak{g}} \) is known in full detail ([35]). Outside the critical level \( k = -2 \) a Verma module is generically irreducible. \( M_{\lambda, k} \) happens to be reducible if and only if it contains a singular vector. If \( M_{\lambda, k} \) is reducible then the following 2 cases arise:

(i) \( k \) is generic (not rational) and \( M_{\lambda, k} \) contains only one singular vector;

(ii) \( k + 2 = \frac{p}{q} > 0 \) is a ratio of 2 positive integers and \( M_{\lambda, k} \) contains infinitely many singular vectors.

It can of course happen that \( k + 2 = \frac{p}{q} < 0 \). We will not be interested in this case and confine to mentioning that here the situation is in a sense dual to (ii).

#### 2.4.1 Case (i)

\( M_{\lambda, k} \) contains a unique proper submodule \( M \) generated by the singular vector. \( M \) is, in fact, a Verma module.

**Definition.** The irreducible quotient \( V_{\lambda, k} \) is called *generalized Weyl module*. \( \square \)

There arises the exact sequence

\[ 0 \rightarrow M \rightarrow M_{\lambda, k} \rightarrow V_{\lambda, k} \rightarrow 0. \tag{6} \]

A simple property of Kac-Kazhdan equations [26] is that, given (6), the module \( M \) is irreducible and does not project on any generalized Weyl module. Note that if the composition series of a \( \hat{\mathfrak{g}} \)-module only consist of generalized Weyl modules then this module breaks into a direct sum of its components. (This can be proved by methods of Deodhar-Gabber-Kac [10].)
It is an exercise on Kac-Kazhdan equations to derive that the highest weight \((\lambda, k)\) of a generalized Weyl module \(V_{\lambda, k}\) belongs to either the line
\[
\lambda = -it + j - 1, \; k = t - 2,
\]
for some \(i \geq 0, j \geq 1\), or to the line
\[
\lambda = it - j - 1, \; k = t - 2,
\]
for some \(i, j \geq 1\); in both cases \(t\) is regarded as a parameter. Formula (7) cooresponds to the case when \(V_{\lambda, k}\) is obtained from \(M_{\lambda, k}\) by quotienting out the singular vector \(S^{0}_{i,j}\); analogously, (8) cooresponds to the case when \(V_{\lambda, k}\) is obtained from \(M_{\lambda, k}\) by quotienting out the singular vector \(S^{1}_{i,j}\).

We see that for a fixed level \(k\) generalized Weyl modules are parametrized by the triples consisting of a pair of nonnegative numbers, \(i, j\) in the formulas above, and an element taking one of the 2 values needed to distinguish between (7) and (8). To be more precise, denote by \(V_i\) the \(i + 1\)-dimensional irreducible representation of \(\mathfrak{g}\).

**Notation.** Assign to \(V_{\lambda, k}\) either the symbol \((V^{0}_{i}, V_{j-1})\), \(i \geq 0, j \geq 1\) if \((\lambda, k)\) satisfies (7), or the symbol \((V^{1}_{i-1}, V_{j-1})\), \(i, j \geq 1\) if \((\lambda, k)\) satisfies (8). \(\square\)

This gives us a one-to-one correspondence between the set of generalized Weyl modules at a fixed generic level and the set of symbols \((V^{\epsilon}_{i}, V_{j})\), where \(\epsilon\) is understood as an element of \(\mathbb{Z}/2\mathbb{Z}\).

Observe that the conventional Weyl module of the level \(k\) is defined to be the induced representation
\[
\text{Ind}_{\mathfrak{g}[z]] \to \mathfrak{g}}^{\mathfrak{g}[[z]]} \mathcal{C}_{\epsilon} V_n,
\]
where \(\mathfrak{g}[z]\) operates on \(V_n\) via the evaluation map \(\mathfrak{g}[z] \to \mathfrak{g}\) and \(\epsilon \mapsto k\). From our point of view the Weyl module is a quotient of the Verma module \(M_{n,k}\) by the submodule generated by the singular vector \(f^{n+1}v_{\lambda, k}\). In other words, Weyl modules are associated to the symbols \((V^{\epsilon}_{0}, V_{n})\). This partially explains appearance of \(\mathfrak{g}\)-modules in our notations.

**2.4.2 Case (ii)**

A Verma module contains infinitely many singular vectors and is embedded in finitely many other Verma modules. Among all singular vectors in \(M_{\lambda, k}\) there are 2 independent ones and these generate the maximal proper submodule. Although formally all such Verma
modules look alike a special role is played by those which can only embed (non-trivially) in themselves. Highest weights of such modules were called by Kac and Wakimoto admissible ([29]) and are described as follows.

Let \( k + 2 = p/q \), where \( p, q \) are relatively prime positive integers. The set of admissible highest weights at the level \( k = p/q - 2 \) is given by

\[
\Lambda_k = \{ \lambda(m,n) = m\frac{p}{q} - n - 1 : 0 < m \leq q, 0 \leq n \leq p - 1 \}.
\]

What is said above about the structure of Verma modules implies that any Verma module appears in the exact sequence of the form

\[
0 \leftarrow L_{\lambda_0,k} \leftarrow M_{\lambda_0,k} \xrightarrow{d_0} M_{\lambda_1,k} \oplus M_{\mu_1,k} \xrightarrow{d_1} M_{\lambda_2,k} \oplus M_{\mu_2,k} \xrightarrow{d_2} \cdots,
\]

where \( \lambda_0 \) is an admissible weight at the level \( k \) and \( L_{\lambda_0,k} \) is the corresponding irreducible module. \( L_{\lambda_0,k} \) is also called admissible. The exact sequence (9) is called Bernstein-Gel’fand-Gel’fand (BGG) resolution.

Again cohomological arguments show (see e.g. [29]) that if the composition series of a \( \hat{g} \)-module only consists of admissible representations then the module is completely reducible.

The parametrization of the set of admissible representations we are going to use is as follows. Two different generalized Weyl modules project onto one and the same admissible representation: formula (9) implies that the two modules projecting onto \( L_{\lambda_0,k} \) are \( M_{\lambda_0,k}/M_{\lambda_1,k} \) and \( M_{\lambda_0,k}/M_{\mu_1,k} \). Therefore two different triples \( (V_m^\epsilon, V_n^\eta) \) are related to the same admissible representation. Introduce the equivalence relation \( \approx \) by

\[
(V_m^\epsilon, V_n^\eta) \approx (V_m^{\epsilon+1}, V_n^{\eta+1}), \quad 0 \leq m \leq q - 1, 0 \leq n \leq p - 2.
\]

Denote by \( (V_m^\epsilon, V_n^\eta)^\sim \) the equivalence class of \( (V_m^\epsilon, V_n^\eta) \).

It easy to check that admissible representations are parametrized by the equivalence classes of the triples:

\[
\{ \text{admissible representations} \} \iff \{(V_m^\epsilon, V_n^\eta)^\sim\}.
\]

### 2.5

Considerable part of the above carries over to the arbitrary Kac-Moody algebra case. Here, for example, is the definition of an admissible representation. Drop the condition that \( g = sl_2 \), let \( M_{\lambda,k} \) be a Verma module over \( \hat{g} \) and \( L_{\lambda,k} \) be its irreducible quotient. Call
(\lambda, k) \text{ admissible if } M_{\lambda, k} \text{ satisfies the following projectivity condition: if composition series of a } \hat{g} \text{--module } W \text{ contains } L_{\lambda, k} \text{ then } M_{\lambda, k} \text{ non-trivially maps in } W.\n
Unfortunately we do not have a reasonable definition of a generalized Weyl module in the higher rank case. (Actually we have but cannot prove it.) This is one of the reasons for which we have to confine mostly to the } \mathfrak{sl}_2 \text{--case.}

\section{2.6 Loop modules}

We will also be using } \hat{g} \text{--modules different from Verma modules or corresponding irreducible ones.}

Denote by } \mathcal{F}_{\alpha\beta} \text{ a } \mathfrak{g} \text{--module with the basis } F_i, \ i \in \mathbb{Z} \text{ and the action given by}

\begin{align*}
\varepsilon F_i &= -(\alpha + i - \beta)F_{i+1}, \\
h F_i &= (2 \alpha + 2i - \beta)F_i, \\
f F_i &= (-\alpha - i)F_{i-1}.
\end{align*}

The space } \mathcal{F}_{\alpha\beta}^C = \mathcal{F}_{\alpha\beta} \otimes \mathbb{C}[z, z^{-1}] \text{ is endowed with the natural } \hat{g} \text{--module structure. The elements } F_{ij} = F_i \otimes z^j, \ i, j \in \mathbb{Z} \text{ serve as a natural basis in it.}

Recall (see 2.3) that } S_{nm}^1, S_{nm}^0 \text{ stand for a singular vector of degree } n(-1, m) \text{ or } n(1, m) \text{ resp. in a Verma module. The following formulas are proved by using (4,5):}

\begin{align*}
S_{nm}^1 F_{n, nm} &= \{ \prod_{i=1}^m \prod_{j=1}^n (-it + j - \alpha + \beta)(-it - j - \alpha) \{ \prod_{s=1}^n (\alpha + s) \} F_{00} \} \quad (11) \\
S_{nm}^0 F_{-n, m} &= \{ \prod_{i=1}^m \prod_{j=1}^n (it + j - \alpha + \beta)(-it - j - \alpha) \{ \prod_{s=1}^n (\alpha - \beta - s) \} F_{00}, \quad (12)
\end{align*}

where } t = k + 2.\n
\section{3 Construction of the modular functor.}

Although most of our results have to do with } \mathfrak{sl}_2 \text{, up to some point it is no extra effort to work in greater generality. So until sect.4, } \mathfrak{g} \text{ will stand for } \mathfrak{sl}_n \text{ unless otherwise stated.}

\subsection{3.1 Algebra } \hat{\mathfrak{g}}^A \text{ and categories of } \hat{\mathfrak{g}}^A \text{-modules}

\subsubsection{3.1.1}

Let } \mathcal{C} \text{ be a smooth compact algebraic curve and } \rho : \mathcal{E} \to \mathcal{C} \text{ be a rank } n \text{ vector bundle with a flat connection. The connection relates to a section } s \text{ of any bundle } \mathcal{A} \text{ associated with } \mathcal{E} \text{ the section } dc \text{ of } \Omega \otimes \mathcal{A} \text{ where } \Omega \text{ is the sheaf of differential forms over } \mathcal{C}. \text{ A typical example,
of $\mathcal{A}$ is the bundle $\text{End}\mathcal{E}$ of fiberwise endomorphisms of $\mathcal{E}$. The sheaf of sections of $\text{End}\mathcal{E}$ is naturally a sheaf of Lie algebras over $\mathcal{C}$.

For a point $P \in \mathcal{C}$ let $\mathfrak{g}^P$ be the algebra of sections of $\text{End}\mathcal{E}$ over the formal neighborhood of $P$. For a finite subset $\widetilde{A} = \{P_1, P_2, \ldots, P_m\} \subset \mathcal{C}$ set $\mathfrak{g}^\mathcal{A} = \bigoplus_{i=1}^m \mathfrak{g}^{P_i}$. Define $\hat{\mathfrak{g}}^\mathcal{A}$ to be the central extension of $\mathfrak{g}^\mathcal{A}$ by the cocycle

$$< x, y > = \sum_{i=1}^m \text{Res}_P \text{Tr} dx \cdot y.$$ 

In particular, we obtain the splitting

$$\hat{\mathfrak{g}}^\mathcal{A} = \mathfrak{g}^\mathcal{A} \oplus \mathcal{C} \cdot c. \quad (13)$$

Consider a finite set $\mathcal{A} = \{(P_1, b_1), \ldots, (P_m, b_m)\}$ where $P_i \in \mathcal{C}$ are pairwise different and $b_i$ is a Borel subalgebra of the algebra of traceless linear transformations of the fiber $\rho^{-1} P_i$ ($1 \leq i \leq m$). Let $\tilde{\mathcal{A}}$ be the projection of $\mathcal{A}$ on $\mathcal{C}$. Set $\hat{\mathfrak{g}}^\mathcal{A} = \hat{\mathfrak{g}}^\tilde{\mathcal{A}}$.

### 3.1.2

Given $\mathcal{A}$ as above, set $\mathfrak{n}_i = [b_i, b_i]$. Denote by $\hat{\mathfrak{g}}^{\mathcal{A}}_{\mathfrak{s}}$ the subalgebra consisting of sections $x(.)$ such that $x(P_i) \in \mathfrak{n}_i$, $1 \leq i \leq m$, and by $\hat{\mathfrak{g}}^{\mathcal{A}}_{\mathfrak{s}}$ the subalgebra spanned by the space of sections $x(.)$ such that $x(P_i) \in \mathfrak{b}_i$, $1 \leq i \leq m$, and the central element $c$. These are analogues of the maximal “nilpotent” and maximal “solvable” subalgebras for $\hat{\mathfrak{g}}^\mathcal{A}$, c.f.2.2.

Denote by $\mathcal{O}^{\mathcal{A}}_k$, $k \in \mathcal{C}$, the category of finitely generated $\hat{\mathfrak{g}}^\mathcal{A}$-modules satisfying the conditions:

(i) $c$ acts as multiplication by $k$;

(ii) the action of the subalgebra $\hat{\mathfrak{g}}^{\mathcal{A}}_{\mathfrak{s}}$ is locally finite.

In much the same way as in 2.2 one defines Verma and generalized Weyl modules over $\hat{\mathfrak{g}}^\mathcal{A}$:

**Definition.**

(i) We will say that $(\lambda, k)$ is a highest weight of $\hat{\mathfrak{g}}^\mathcal{A}$ if $\lambda$ is a functional on $\bigoplus_i b_i / \mathfrak{n}_i$ and $k$ is a number.

(ii) A highest weight $(\lambda, k)$ naturally determines a character of $\hat{\mathfrak{g}}^\mathcal{A}$ sending $c$ to $k$ and annihilating $\hat{\mathfrak{g}}^{\mathcal{A}}_{\mathfrak{s}}$. Denote by $\mathcal{C}_{\lambda, k}$ the corresponding 1-dimensional representation.

(iii) Define the Verma module $M^{\mathcal{A}}_{\lambda, k}$ to be the induced representation

$$\text{Ind}_{\hat{\mathfrak{g}}^{\mathcal{A}}_{\mathfrak{s}}}^{\hat{\mathfrak{g}}^\mathcal{A}} \mathcal{C}_{\lambda, k}. \quad \square$$

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There is an isomorphism
\[ M_{i,k}^A \cong \bigotimes_{i=1}^m M_{i,k}^{P_i,b_i}. \]

Suppose now that each \( M_{i,k}^{P_i,b_i} \) has at least one singular vector. If \( k \in \mathbb{C} \setminus \mathbb{Q} \) then this singular vector is unique for each \( i \). Quotienting out all of them one obtains the \textit{generalized Weyl module} \( V_{i,k}^A \). As above there is an isomorphism
\[ V_{i,k}^A \cong \bigotimes_{i=1}^m V_{i,k}^{P_i,b_i}. \]

If \( k \) is not a rational number then any generalized Weyl module is irreducible. Denote by \( \tilde{O}_k \) the full subcategory of \( O_k \) consisting of all \( \hat{g}^A \)-modules whose composition series consist of generalized Weyl modules. Again if \( k \) is not a rational number then \( \tilde{O}_k \) is semisimple.

If \( k \) is rational then there arises the admissible representation \( L_{\lambda,k}^A \) if \((\lambda,k)\) is admissible. If the composition series of a module \( V^A \) consists only of admissible representations, then \( V^A \) is completely reducible.

### 3.1.3

Let \( A \) be as in 3.1.1. Let \( \mathfrak{g}(\mathbb{C},A) \) be the Lie algebra of meromorphic sections of \( \text{End} \mathcal{E} \) holomorphic outside \( \tilde{A} \). The maps of restriction to formal neighborhoods give rise to the Lie algebra morphism
\[ \mathfrak{g}(\mathbb{C},A) \rightarrow \hat{\mathfrak{g}}^A. \] (14)

The splitting (13) provides us with the section \( s_A : \mathfrak{g}^A \rightarrow \hat{\mathfrak{g}}^A \). Composition of (14) with \( s_A \) gives the linear morphism
\[ \mathfrak{g}(\mathbb{C},A) \rightarrow \hat{\mathfrak{g}}^A. \] (15)

The residue theorem implies that (15) is a Lie algebra morphism (even though \( s_A \) is not!).

By (15), the standard pullback makes each object of \( M^A \in \mathcal{O}_k^A \) into a \( \mathfrak{g}(\mathbb{C},A) \)-module. Hence there arises the space of coinvariants
\[ (M^A)_{\mathfrak{g}(\mathbb{C},A)} = M/\mathfrak{g}(\mathbb{C},A)M. \]

### 3.2 Localization of \( \hat{\mathfrak{g}}^A \)-modules

#### 3.2.1

Let \( V^A \) be a \( \hat{\mathfrak{g}}^A \)-module and suppose for simplicity that \( A \) consists of 1 element \((P,b)\). Consider a family of the data \( \{P,\mathcal{E} \rightarrow C\} \) – let us not care about Borel subalgebras for
the moment. One expects that the corresponding family of vector spaces arranges then in a locally trivial vector bundle. An obstacle to get this is that we have defined $V^P$ up to an isomorphism but have not specified any such isomorphism. For example, an attempt to choose a basis in $V^P$ requires to choose (in particular) a local coordinate $z$ at $P$, such that $z(P) = 0$. Different choices of $z$ are essentially different as the group $\text{Diff}(P)$ of diffeomorphisms of the formal neighborhood of $P$ does not in general act on $V^P$. However the subgroup $\text{Diff}(P)_1 \subset \text{Diff}(P)$ of diffeomorphisms preserving the 1-jet of parameter does act on $V^P$. We see that $V^P$, in fact, depends on the 1-jet of parameter at $P$.

To take care of Borel subalgebras, let us recall that with an $n$-dimensional vector space $W$ one associates the flag manifold $F(W) = GL(n, \mathbb{C})/B$ and the base affine space $\text{Base}(W) = GL(n, \mathbb{C})/N$, where $B$ is a Borel subgroup and $N$ unipotent subgroup of $B$. The natural map $\text{Base}(W) \to F(W)$ is a principal $(\mathbb{C}^*)^n$-bundle.

Similar arguments applied to $\mathfrak{b}$ show that the module $V^A = V^P \mathfrak{b}$ depends on the quadruple $(P, \mathfrak{b}, j, x)$ such that $j$ is a 1-jet of parameter at $P$ and $x \in \text{Base}(\mathbb{C}^n)$ belongs to the preimage of $\mathfrak{b}$.

One concludes that we do get a locally trivial vector bundle after pull-back to the space of pairs “1-jet of parameter at $P$, element of the maximal torus of the Borel group related to $\mathfrak{b}$”. Let us be more precise now.

3.2.2

Let $\tilde{\pi} : \mathcal{C}_S \to S$ be a family of smooth projective curves and $\rho_S : \mathcal{E}_S \to \mathcal{C}_S$ be a rank $n$ vector bundle. There arise 2 more bundles:

(i) the bundle $\text{Base}(\rho_S) : \text{Base}(\mathcal{E}_S) \to \mathcal{C}_S$ with the fiber over any $x \in \mathcal{C}_S$ equal to the base affine space of the vector space $\rho_S^{-1} x$;

(ii) the $\mathbb{C}^*$-bundle $J^{(1)}(\mathcal{C}_S) \to \mathcal{C}_S$ of 1-jets of coordinates along fibers of $\tilde{\pi}$.

Consider the fibered product $\text{Base}(\mathcal{E}_S) \times_{\mathcal{C}_S} J^{(1)}(\mathcal{C}_S)$ and the natural map

$$\pi : \text{Base}(\mathcal{E}_S) \times_{\mathcal{C}_S} J^{(1)}(\mathcal{C}_S) \to S.$$  

Pick a non empty finite set $A_S$ of sections of $\pi$ satisfying the condition:

for any $s \in S$ the natural projection of the set $A_S(s) = \{a(s), a \in A_S\}$ on $\tilde{\pi}^{-1}(s)$ is an injection.

Pick an arbitrary curve, say $\mathcal{C}_{s_0}$, from our family. Consider a highest weight module $M^A$ over $\mathfrak{g}^A$, where we write $A$ instead of the lengthy $A_S(s_0)$; what follows is obviously
independent of the choice of \( s_0 \).

By 3.1.2 and 3.2.1, we get a \( \mathfrak{g}^{A_{s}(s)} \)-module \( M^{A_{s}(s)} \) for any \( s \in S \) and the collection \( \{ M^{A_{s}(s)}, s \in S \} \) arranges in a locally trivial vector bundle. With each \( s \in S \) we can further associate a vector space, that is the space of coinvariants

\[
(M^{A_{s}(s)})_{g(\pi^{-1}S, A_{s}(s))},
\]

see 3.1.3.

**Theorem 3.2.1** Suppose the collection \( \psi = (M^{A}, \pi, A_{S}) \), satisfying the conditions imposed above, is given. Then there is a twisted \( \mathcal{D} \)-module (that is a sheaf of modules over a certain algebra of twisted differential operators) on \( S \) such that its fiber over \( s \in S \) is

\[
(M^{A_{s}(s)})_{g(\pi^{-1}S, A_{s}(s))}.
\]

This theorem is an immediate consequence of [4] and [5, 8]. Briefly the construction is as follows. Take a vector field \( \xi \) on \( U \subset S \). It lifts to a meromorphic vector field on \( C_{S} - A_{S}(S) \) over \( U \), and further to a meromorphic vector field on \( \pi^{-1}(U) \subset \text{Base}(C_{S}) \times C_{S}, J^{(1)}(C_{S}) \); denote this vector field by \( \xi^{*} \). Trivializing the infinitesimal neighborhood of \( A_{S}(U) \subset C_{S} \) by choosing, locally with respect to \( U \subset S \), coordinates in the fibers, one gets vertical components \( \{ \xi^{*}_{\text{vert; } i} \} \), so that \( \xi^{*}_{\text{vert; } i} \) is the vertical component in the formal neighborhood of the \( i \)-section. Projecting \( \xi^{*}_{\text{vert; } i} \) on \( \text{Base}(C_{S}) \) one gets some element of \( U(\mathfrak{g}) \), say \( u_{i} \); projecting \( \xi^{*}_{\text{vert; } i} \) on \( J^{(1)}(C_{S}) \) one gets some vector field, say \( v_{i} \). Both \( u_{i}, v_{i} \) act on our \( \mathfrak{g}^{A} \)-module \( M^{A} \): \( u_{i} \) naturally, \( v_{i} \) by means of the Sugawara construction. Going over definitions one gets that this well defines a twisted \( \mathcal{D} \)-module with the fiber as in the theorem. \( \square \)

Denote the constructed \( \mathcal{D} \)-module by \( \Delta_{\psi}(M^{A}) \).

In the case when \( M^{A} \) is an admissible representation the following result is valid.

**Theorem 3.2.2** If \( n = 2 \) and \( M^{A} \) is an admissible \( \mathfrak{g}^{A} \)-module then \( \Delta_{\psi}(M^{A}) \) is holonomic for almost any vector bundle \( E_{S} \) (i.e. as a sheaf \( \Delta_{\psi}(M^{A}) \) is isomorphic to a sheaf of sections of a certain finite rank vector bundle over some open set in \( S \)).

**Proof.**

To prove this theorem essentially means to show that the spaces \( (M^{A_{s}(s)})_{g(\pi^{-1}S, A_{s}(s))}, s \in S \), are all finite dimensional. That will be done in 4.3.3, Proposition 4.3.2 in the higher
genus case and in 4.4.2, Proposition 4.4.2 for \( \mathbb{CP}^1 \). We will also give there a precise meaning to the phrase “almost any vector bundle” in Theorem 3.2.2. □

Results of 4.7 will show that the standard combinatorial algorithm can be used to calculate the dimension of the fiber of our \( \mathcal{D} \)-module using the dimensions of the spaces of coinvariants on a sphere with 3 punctures. The latter dimensions will be calculated in 4.5.4.

4 The spaces of coinvariants

In this section we will be concerned with the space of coinvariants \((M^A)_{g(\mathcal{C},\mathcal{A})}\) (or spaces closely related to it) in the case when \( M^A \) is either a generalized Weyl module or an admissible representation. The standard tool to get finiteness results about coinvariants is the notion of singular support.

4.1 Singular support and coinvariants

Let \( \mathfrak{a} \) be a Lie algebra. Universal enveloping algebra \( U\mathfrak{a} \) is filtered in the standard way so that the associated graded algebra is \( S\mathfrak{a} \). Let \( V \) be an \( \mathfrak{a} \)-module carrying a filtration compatible with the filtration of \( U\mathfrak{a} \). Then the graded space, \( GrV \), becomes naturally an \( S\mathfrak{a} \)-module.

**Definition** Singular support, \( SSV \), of \( V \) is the zero set of the vanishing ideal of the \( S\mathfrak{a} \)-module \( GrV \). □

Obviously, \( SSV \) is a conical subset of \( \mathfrak{a}^* \).

For a subalgebra \( \mathfrak{n} \subset \mathfrak{a} \), call \( V \) an \( (\mathfrak{a},\mathfrak{n}) \)-module if it is an \( \mathfrak{a} \)-module and \( \mathfrak{n} \) acts on \( V \) locally nilpotently. Typical example: any module from the \( \mathcal{O} \)-category is a \( (\mathfrak{g},\mathfrak{g}^\vee) \)-module.

**Lemma 4.1.1 (see [4])** Let \( \mathfrak{a} \) be a Lie algebra and \( \mathfrak{p} \subset \mathfrak{a} \) be its subalgebra. Denote by \( \mathfrak{p}^- \) the annihilator of \( \mathfrak{p} \) in \( \mathfrak{a}^* \). Let \( V \) be a finitely generated \( (\mathfrak{a},\mathfrak{n}) \)-module. If \( SSM \cap \mathfrak{p}^- = \{0\} \) and \( \dim \mathfrak{a}/\mathfrak{n} \oplus \mathfrak{p} < \infty \) then \( \dim M_{\mathfrak{p}} < \infty \).

Recall that from now on \( \mathfrak{g} = \mathfrak{sl}_2 \) unless otherwise stated.
### 4.2 Singular support of $\hat{g}^A$–modules

Observe that there is an involution $\sigma$ of $\hat{g}$ sending $f$ to $e \otimes z^{-1}$ and $e \otimes z^{-1}$ to $f$, see 2.2 for notations. There arises the involution, also denoted by $\sigma$, acting on the algebras $\hat{g}^A$ and their duals. This involution is not canonical but we do not have to care as our considerations here are purely local.

Denote by $\Omega^A$ the space of $g$–valued differential forms on the formal neighborhoods of the points from $A$. There is a natural embedding $\Omega^A \hookrightarrow (\hat{g}^A)^*$ (“take the traces and then sum up all the residues!”)

We will make use of 2 subspaces of $\Omega^A$: $\Omega^A_{\text{reg}}$ is all regular forms and $\Omega^A_{\text{nilp}}$ is all forms with values in the nilpotent cone.

**Theorem 4.2.1** (i) If $M^A$ is a generalized Weyl module then $SSM^A = \Omega^A_{\text{reg}} \cup \sigma \Omega^A_{\text{reg}}$.

(ii) If $M^A$ is an admissible representation then $SSM^A = \Omega^A_{\text{nilp}} \cup \sigma \Omega^A_{\text{nilp}}$.

**Remark 4.2.2** (i) It is easy to see that although $\sigma$ is not determined uniquely the spaces $\sigma \Omega^A_{\text{reg}}$, $\sigma \Omega^A_{\text{nilp}}$ are canonical. For example $\sigma \Omega^A_{\text{reg}}$ is the space of forms such that:

- they have at most order 1 pole at $\tilde{A}$;
- their residue at each $P_i \in \tilde{A}$ belongs to $v_i$;
- at each $P_i \in \tilde{A}$ their constant term belongs to $b_i$.

(ii) Statement (ii) of Theorem 4.2.1 is easy to invert using cohomological characterization of admissible representations obtained in [19]. We will not need this result in what follows.

**Proof** of Theorem 4.2.1.

First of all, as the statements are purely local, we will assume that $\# \mathcal{A} = 1$ and forget about the curve.

(i) It follows from the definition of singular support that if $V$ is an $\sigma$-module on 1 generator $v$, then $SSV$ is the zero set of all symbols of elements of $U(\mathfrak{a})$ annihilating $v$. For example, if $V$ is a Verma module over $\hat{g}$, then $SSV$ consists of all $x \in \hat{g}^*$ satisfying $x(g) = 0$ for all $g \in \hat{g}_\geq$. This system is easy to solve and get the following: $SSV$ consists of all differential $g$–valued forms on the formal neighborhood of 0 having at most pole at 0, the residue lying in the $[\mathfrak{b}, \mathfrak{b}]$, where $\mathfrak{b} \subset \mathfrak{g}$ is the Borel our module is induced from.

If however $V$ is a generalized Weyl module, then one more equation is to be added, that is the one coming from the unique singular vector $S$ in the corresponding Verma module.
Symbol of $S$ is equal to $e_{-1}^m f^n$ (in natural notations). So we obtain one more equation and solving it get the desired result.

(ii) Here we have to consider 2 singular vectors and because of non-commutativity of the universal enveloping algebra argument becomes more involved. Consider first what is known as vacuum module. Such a module arises when the level $k + 2 = p/q$, and in notations of sect.2.4.2, (10) it is $(V_{q-1}^1, V_{p-2})$, or, in more standard notations from representation theory (see the same section, few lines before), $L_{0,k}$. An essential property of this module is that its generator is annihilated by $g$ and so, by (i), $SSL_{0,k}$ consists of forms regular at 0. One more equation arises, the one coming from another singular vector in the corresponding Weyl module $V_{0,k}$. This vector can be calculated using (5), its weight is $(p - 1)(-1, q)$.

Start with the case $p = 2$. The mentioned property of the vacuum module implies that under the action of $g$, this singular vector generates a $g$—submodule isomorphic to $g$. On the other hand, this singular vector is naturally identified with an element of $U(\bigoplus_{i>0} g \otimes z^{-i})$. (As we are dealing with a Weyl module, we can disregard elements of $g$.

Lemma 4.2.3 Symbol of the singular vector belongs to $S^q(g \otimes z^{-1})$.

Postpone proof of this lemma for a moment and use it to derive the statement of the theorem in this case. It follows that the symbols of the elements of the $g$—submodule generated by the singular vector determine an embedding of $g$ in $S^q(g \otimes z^{-1})$. By the well-known theorem of B.Kostant there is only one such embedding. It is given by

$$g \ni g \mapsto C_{(v-1)/2} g,$$

where $C$ is the quadratic Casimir element. (It is easy to see that for the vacuum representation under cconsideration $q$ is necessarily odd.) Therefore we get $dim g = 3$ more equations and immediately see that the set of common zeros is given by 1: $C = 0$, where $C$ is regarded as a quadratic function on $g$. As is well-known, the last equation determines the nilpotent cone of $g$. This proves the desired result at one point, 0; to complete proof one notices that the submodule generated by the singular vector is closed under the differentiation $d/dz$ (by Sugawara construction).

If $p > 2$, then the arguments remain essentially unchanged. By looking at the singular vector formula (5) one realizes that the symbol of the singular vector simply gets raised
to the $p-1$-st power, and instead of the embedding $\mathfrak{g} \to S^q(\mathfrak{g} \otimes z^{-1})$, the singular vector determines an embedding $V_{2(p-1)} \to S^q(\mathfrak{g} \otimes z^{-1})$. So, instead of zeros of a collection of functions we get zeros of the $p-1$-st powers of the same collection of functions. Theorem for vacuum representations follows.

**Proof** of Lemma4.2.3. This seemingly benign statement is not quite trivial. The equivalent reformulation is as follows: degree (we set $\deg g = 1$, $g \in \hat{\mathfrak{g}}$) of the symbol of the singular vector equals $q$. The easiest way to see that this is true, is to apply this singular vector to an element of a certain module $\mathcal{F}_{\alpha,\beta}^{C^*}$, and calculate the result using (12), see 2.6. Observe that in that formula the singular vector, $S^0_{1q}$ is understood as an element of $U(\hat{\mathfrak{g}}_<)$. To kill elements of $\mathfrak{g}$, we impose 1 linear condition on parameters $\alpha, \beta$, see again (12). One parameter survives, say $\alpha$, and the result happens to be a polynomial of degree $q$. □

This completes proof of Theorem4.2.1 for vacuum representations. To include other admissible representations we need 2 more ingredients: calculation of coinvariants for 3 admissible modules on the sphere, and a construction of annihilating ideals in admissible representations. End of proof is to be found in sect.5.2.3.

### 4.3 Finiteness of coinvariants – the higher genus case

#### 4.3.1 Hitchin’s theorem.

First recall a well-known result of Hitchin, [25]. With a vector bundle $\mathcal{E} \to \mathcal{C}$ associate the map

$$H(\mathcal{E}): H^0(\mathcal{C}, \Omega \otimes \text{End}\mathcal{E}) \to \bigoplus_{i=1}^n H^0(\mathcal{C}, \Omega^{\otimes i}), \quad (16)$$

$$X \mapsto \text{Tr} X^i$$

Call a bundle $\mathcal{E}$ *exceptional* if $\ker H(\mathcal{E}) \neq 0$. Obviously $\ker H(\mathcal{E})$ is exactly the space of global differential forms with values in nilpotent endomorphisms of the vector bundle $\mathcal{E}$.

**Theorem 4.3.1** (Hitchin [25]) Zero set of the map (16) is a maximal Lagrangian submanifold in the cotangent bundle of the moduli space of vector bundles over $\mathcal{C}$. In particular, exceptional vector bundles form a positive codimension algebraic subset of the moduli space of vector bundles.
For us, importance of Theorem 4.3.1 is in that generically a vector bundle does not allow a non-trivial global differential form with coefficients in nilpotent endomorphisms of the bundle.

### 4.3.2 Subtracting lines from rank 2 vector bundles.

An analogue of subtracting a point from a line bundle (or, better to say, from its divisor) is an operation of subtracting a line from a rank 2 vector bundle.

To a rank 2 vector bundle $\mathcal{E} \to \mathcal{C}$ one can associate a module over the sheaf of regular functions – the sheaf of sections of $\mathcal{E}$; denote this sheaf by $\text{Sect}(\mathcal{E})$. This establishes a one-to-one correspondence between rank 2 vector bundles and rank 2 locally free modules over the sheaf of regular functions. Now fix a line, $l$, in a fiber of $\mathcal{E}$ over some point $P \in \mathcal{C}$. Denote by $S(l)$ a sheaf such that:

1. $S(l)|_U = \text{Sect}(\mathcal{E})|_U$ if $P$ does not belong to $U$;
2. $S(l)|_U, P \in U$, is the space of meromorphic sections of $\mathcal{E}$ over $U$ regular outside $P$, having at most order 1 pole at $P$ and such that their residue at $P$ belongs to the fixed line $l$.

It is obvious that $S(l)$ is a rank 2 locally free module. Therefore it defines a rank 2 vector bundle. Denote this vector bundle by $\mathcal{E}(l)$. If a collection of lines $l_1, l_2, \ldots, l_m$ is subtracted, then denote the corresponding vector bundle by $\mathcal{E}(l_1 + \cdots + l_m)$.

Suppose we have a moduli space of rank 2 vector bundles with parabolic structure and fixed determinant. Elements of such a space are isomorphism classes of the data (vector bundle $\mathcal{E}$, fixed lines $l_1, \ldots, l_m$ in some fibers.) It is rather clear that the map $(\mathcal{E}, l_1, \ldots, l_m) \mapsto (\mathcal{E}(l_1 + \cdots + l_m), l_1, \ldots, l_m)$ is a homeomorphism of 2 moduli spaces with different determinants.

**Definition.** Call the data $(\mathcal{E}, l_1, \ldots, l_m)$ generic if $\mathcal{E}(l_1 + \cdots + l_s)$ is not exceptional for any subset $\{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, m\}$. □

It follows from Theorem 4.3.1 that the set of generic vector bundles is open and everywhere dense.

### 4.3.3 Finiteness of coinvariants.

Suppose we are in the situation of 3.1.2: we have an admissible $\hat{\mathfrak{g}}^A$-module $M^A$ on the curve $\mathcal{C}$ with a vector bundle $\mathcal{E} \to \mathcal{C}$. As $A$ is a collection of borel subalgebras $\mathfrak{b}_1, \ldots, \mathfrak{b}_m$
operating in fixed fibers, we have parabolic structure – lines \( l_1, \ldots, l_m \) in the corresponding fibers preserved by the \( b_i \)'s. Call the data \((\mathcal{E}, A)\) generic if the data \((\mathcal{E}, l_1, \ldots, l_m)\) is generic in the sense of 4.3.2.

Recall that we are interested in the space of coinvariants \( M^A_{\mathfrak{g}(C, A)} \), where \( \mathfrak{g}(C, A) \) is an algebra of endomorphisms of the bundle \( \mathcal{E} \) regular outside points from the corresponding \( \tilde{A} \), see 3.1.3 and 3.2.2, Theorem 3.2.1.

**Proposition 4.3.2** Let \((\mathcal{E}, A)\) be generic. Then

\[
\dim M^A_{\mathfrak{g}(C, A)} < \infty.
\]

**Proof.** One extracts from definitions that the annihilator \( \mathfrak{g}(C, A)^{-} \) of the algebra \( \mathfrak{g}(C, A) \) is the space \( \Omega_{C,A}(\mathcal{E}) \) of global meromorphic \( \text{End}(\mathcal{E}) \)-valued differential forms regular outside \( \tilde{A} \subset C \).

By Theorem 4.2.1(ii) we get that \( SSM^A \cap \mathfrak{g}(C, A)^{-} = \Omega_{nilp}(\mathcal{E}) \cup \sigma \Omega_{nilp}(\mathcal{E}) \), where \( \Omega_{nilp}(\mathcal{E}) \) is the space global nilpotent transformations of \( \mathcal{E} \), and \( \sigma \) is the twist introduced in 4.2.

Genericity condition means that \( \Omega_{nilp}(\mathcal{E}) = 0 \), see 4.3.1 and 4.3.2.

On the other hand it is easy to see that the operation of subtracting a line generates the twist \( \sigma \) on endomorphisms. (In fact one has to compose subtracting of a line with a reflection in the fiber, but this does not change the isomorphism class of the bundle.) Therefore genericity condition also implies that \( \sigma \Omega_{nilp}(\mathcal{E}) = 0 \).

Hence we get that \( SSM^A \cap \mathfrak{g}(C, A)^{-} = 0 \). And as the space \( \mathfrak{g}_{\geq}^A + \mathfrak{g}(C, A) \) is of finite codimension in \( \mathfrak{g}^A \), application of Lemma 4.1.1, see 4.1, completes the proof. \( \square \)

In order to study quadratic degenerations we will need the following stronger finiteness result. Along with the set \( A = \{(P_1, b_1), \ldots, (P_m, b_m)\} \), consider the set \( A_2 = \{(P_{m+1}, b_{m+1}), (P_{m+2}, b_{m+2})\} \) such that the points \( P_1, \ldots, P_{m+2} \in C \) are different. Denote by \( \mathfrak{g}(C, A, A_2) \) the subalgebra of \( \mathfrak{g}(C, A) \) consisting of functions taking values in \( n_i = [b_i, b_i] \) at point \( P_i \), \( i = m + 1, m + 2 \).

**Proposition 4.3.3** If \((\mathcal{E}, A \cup A_2)\) is generic and \( M^A \) is admissible, then

\[
\dim (M^A)_{\mathfrak{g}(C,A,A_2)} < \infty.
\]
Proof. We are again going to apply Lemma 4.1.1. Observe that \( g(\mathcal{C}, A, A_2)^- \) consists of meromorphic forms on \( \mathcal{C} \) with values in \( \text{End}(\mathcal{E}) \), regular outside \( \{P_1, \ldots, P_{m+2}\} \subset \mathcal{C} \), having at most order 1 poles at \( P_{m+1}, P_{m+2} \), their residues at the latter points lying in \( \mathfrak{b}_1 \) (\( \mathfrak{b}_2 \) resp.).

By Theorem 4.2.1(ii), \( g(\mathcal{C}, A, A_2)^- \cap \text{SSM}^A \) consists of forms with values in nilpotent endomorphisms, satisfying the above listed global conditions. This implies, in particular, that actually residues of our forms belong to \( \mathfrak{n}_{m+1}, \mathfrak{n}_{m+2} \) at \( P_{m+1}, P_{m+2} \) resp..

Given an element \( \omega \in g(\mathcal{C}, A, A_2)^- \cap \text{SSM}^A \), subtract some lines from \( \mathcal{E} \) so as to make \( \omega \) be everywhere regular. Genericity condition implies then that \( \omega = 0 \), and application of Lemma 4.1.1 completes the proof. □

4.4 Finiteness of coinvariants – the case of \( \text{CP}^1 \)

4.4.1 Generic vector bundles on \( \text{CP}^1 \)

Let \( O(n) \) be the degree \( n \) line bundle over \( \text{CP}^1 \). It is known, e.g. [39], that any rank 2 vector bundle over \( \text{CP}^1 \) is a direct sum \( O(r) \oplus O(s) \) for some \( r, s \).

As there are no moduli, it is hard to speak about generic vector bundles. Nevertheless we will call \( O(r) \oplus O(s) \) exceptional if \( |r - s| > 1 \). Here is a justification.

Lemma 4.4.1 Let \( \mathcal{E} = O(r) \oplus O(s) \) and \( (\mathcal{E}, l_1, \ldots, l_m) \), \( m \geq |r - s| \), a vector bundle with parabolic structure. Then generically with respect to \( l_1, \ldots, l_m \) the bundle \( \mathcal{E}(l_1 + \cdots + l_m) \) is not exceptional:

\[
\mathcal{E}(l_1 + \cdots + l_m) = \begin{cases} 
O(p + 1) \oplus O(p) & \text{if } r + s - m = 2p + 1 \\
O(p) \oplus O(p) & \text{if } r + s - m = 2p.
\end{cases}
\]

Lemma 4.4.1 seems to be common knowledge, although we failed to find a reference with its proof.

Proceed just like we did in 4.3.2: call \((\mathcal{E}, l_1, \ldots, l_m)\) generic if \( \mathcal{E}(l_{i_1} + \cdots + l_{i_s}) \) is not exceptional for any subset \( \{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, m\} \).

4.4.2 Finiteness of coinvariants

A specific feature of the genus zero case is that we do not necessarily have to consider admissible representations – generalized Weyl modules, see 2.4.1, will also do.
Let us again consider a vector bundle $\mathcal{E}$ over $\mathbb{CP}^1$ and a $\mathfrak{g}^A$–module $M^A$. As in 4.3.3, $A$ determines a parabolic structure on $\mathcal{E}$, say $(\mathcal{E}, l_1, \ldots, l_m)$. Call the data $(\mathcal{E}, A)$ generic if $(\mathcal{E}, l_1, \ldots, l_m)$ is also.

**Proposition 4.4.2** If $(\mathcal{E}, A)$ is generic and $M^A$ is either admissible or generalized Weyl module, then

$$\dim (M^A)_{\mathfrak{g}(\mathbb{CP}^1, A)} < \infty.$$ 

**Proof** is a simplified version of the proof of Proposition 4.3.2 in 4.3.3. The new features are as follows: to include generalized Weyl modules one uses Theorem 4.2.1(i) in addition to Theorem 4.2.1(ii); instead of the Hitchin’s theorem one uses the “observation” that $O(n)$ has no non-zero global sections if $n < 0$. □

**Corollary 4.4.3** If $M^A$ is a generalized Weyl module then there is a holonomic twisted $D$-module living in the space $(J^{(1)}(\mathbb{CP}^1) \times J^{(1)}(\mathbb{CP}^1))^\times m$ with the fiber $M^A_{\mathfrak{g}(\mathbb{CP}^1, A)}$.

**Proof.** Repeating word for word proof of Theorem 3.2.2 one derives from Proposition 4.4.2 existence of a twisted $D$–module on the space $(\text{Base}(\mathbb{C}^2) \times J^{(1)}(\mathbb{CP}^1))^\times m$. But for $s_2$, the flag manifold is $\mathbb{CP}^1$ and the base affine space (Base($\mathbb{C}^2$) is also the space of 1-jets of parameter $J^{(1)}(\mathbb{CP}^1))^\times m$. □

As in 4.3.3, we want to prove a generalization of Proposition 4.4.2 in order to prepare grounds for studying quadratic degeneration.

Along with $A = \{(P_1, b_1), \ldots, (P_m, b_m)\}$ consider 2 sets $A_1 = \{(P_{m+1}, b_{m+1})\}$ and $A_2 = \{(P_{m+1}, b_{m+1}), (P_{m+2}, b_{m+2})\}$ such that $P_1, \ldots, P_{m+2}$ are different points in $\mathcal{C}$.

With $A_1$ and $A_2$ associate the following 2 subalgebras of $\mathfrak{g}(\mathbb{CP}^1, A)$: $\mathfrak{g}(\mathbb{CP}^1, A, A_1)$ consists of all functions taking values in $n_{m+1} = [b_{m+1}, b_{m+1}]$ at the point $P_{m+1}$; $\mathfrak{g}(\mathbb{CP}^1, A, A_2)$ consists of all functions taking values in $n_i = [b_i, b_i]$ at the point $P_i$, $i = m+1, m+2$.

**Proposition 4.4.4** Let $(\mathcal{E}, A_2)$ be generic. Then

(i) If $M^A$ is a generalized Weyl module over $\mathfrak{g}^A$, then $\dim (M^A)_{\mathfrak{g}(\mathbb{CP}^1, A_1)} < \infty$;

(ii) If $M^A$ is an admissible representation of $\mathfrak{g}^A$, then $\dim (M^A)_{\mathfrak{g}(\mathbb{CP}^1, A_2)} < \infty$.

**Proof.** of (ii) repeats almost word for word that of Proposition 4.3.3 in 4.3.3 with simplifications analogous to those indicated in the proof of Proposition 4.4.2.
As to (i), its proof is again application of the same technique in a slightly different form: one has to take a form \( \omega \in \mathfrak{g}(\mathbb{CP}^1, A, A_2)^- \cap SSM^A \) and to subtract lines from \( \mathcal{E} \) so as to make \( \omega \) into a form with either one pole (at \( P_{m+1} \)) or 2 poles (one of them is again at \( P_{m+1} \)) in such a way that the bundle obtained is \( O(n) \oplus O(n) \). The 2 cases are of course distinguished by the parity of the difference between the degrees of the determinant of \( \mathcal{E} \) and \( \omega \). In both cases it is easy to prove that \( \omega = 0 \) using the fact that any differential form with trivial coefficients has at least 2 poles. □

### 4.4.3 Holonomic D-module on \( (\mathbb{C} \times \mathbb{C})^m \)

We will now get rid of twisted differential operators in Corollary 4.4.3 under the assumption that the vector bundle \( \mathcal{E} \to \mathbb{CP}^1 \) is trivial. Consider the set \( A' = A \cup \{P_{\infty}, b_{\infty}\} \).

Attach to the point \( (P_{\infty}, b_{\infty}) \) the module \( (V_0, V_0) \) known as the vacuum representation, see 2.4.1 for notations. \( (P_{\infty}, b_{\infty}) \) can be redefined as the module induced from the trivial representation (see also 2.4.1) and therefore there is an isomorphism \( M_A^A(\mathbb{CP}^1, A) \approx M_A^A(\mathbb{CP}^1, A) \). Now consider the twisted \( D \)-module with fiber \( M_A^A(\mathbb{CP}^1, A) \) on the space \( (J^{[1]}(\mathbb{CP}^1) \times J^{[1]}(\mathbb{CP}^1))^\times m + 1 \). Restrict it to the space \( (J^{[1]}(\mathbb{CP}^1) \times J^{[1]}(\mathbb{CP}^1))^\times m \) by having the point \( (P_{\infty}, b_{\infty}) \) fixed. The result of this operation is that the bundles in question trivialize: \( \mathbb{CP}^1 - b_{\infty} = \mathbb{CP}^1 - P_{\infty} = \mathbb{C} \) and \( J^{[1]}(\mathbb{C}) = \mathbb{C}^n \times \mathbb{C} \). Further pushing forward by “integrating along \( \mathbb{C}^n \)” one gets a \( D \)-module over the space \( (\mathbb{C} \times \mathbb{C})^m \). Observing that it is appearance of the bundle \( J^{[1]}(\mathbb{CP}^1) \to \mathbb{CP}^1 \) which was responsible for the twisting of the \( D \)-module, one argues that we get a usual holonomic \( D \)-module on \( (\mathbb{C} \times \mathbb{C})^m \) with fiber \( M_A^A(\mathbb{CP}^1, A) \). In particular, we get a bundle with flat connection over an open subset of \( (\mathbb{C} \times \mathbb{C})^m \).

**Notation.** Denote the constructed in this way bundle with flat connection by \( \Delta(M_A^A) \).

□

We are unable to describe this open subset explicitly at present. However it follows from the requirement that \( (\mathcal{E}, A) \) be generic in all our finiteness results that the diagonals should be thrown away, meaning that \( P_i \neq P_j \) and \( b_i \neq b_j \) for all \( i \neq j \).

One may want to write down differential equations satisfied by horizontal sections of this bundle. We will show in 5.3 that horizontal sections satisfy a system of \( 2m \) differential equations of which \( m \) equations are Knizhnik-Zamolodchikov equations and the other \( m \) are obtained from singular vectors of the Verma module projecting onto \( M_A^A \).
Everything said here holds true for an admissible representation. It is easy to see that the bundle associated with an admissible representation is a quotient of the just constructed bundle for the corresponding generalized Weyl module.

4.5 Calculation of the dimensions of coinvariants. Fusion algebra

Let $\mathcal{E} \to \mathbb{CP}^1$ be the rank 2 trivial vector bundle and $M^A$ be a $\mathfrak{g}^A$–module. Here we will calculate the dimension of the space $(M^A)_{\mathfrak{g}(\mathbb{CP}^1,A)}$, $iA = 3$, in the following 2 cases: (i) the level $k$ is not rational and $M^A$ is a generalized Weyl module; (ii) $k + 2 = p/q$, $p$ and $q$ being positive integers, and $M^A$ is an admissible representation. Without loss of generality we can:

fix a coordinate $z$ on $\mathbb{CP}^1$; assume that $A = \{(0, b_0), (1, b_1), (\infty, b_\infty)\}$, where $b_0 = C e \oplus C h$, $b_\infty = C f \oplus C h$ and $b_1 = C (e - h - f) \oplus C (h + 2 f)$.

(In fact, for any $b_0 \neq b_\infty$ we can always choose a basis of $\mathfrak{g}$ so that $b_0, b_\infty$ are as above. As to $b_1$, there really is some freedom but it is easy to see that all the calculations below are independent of the choice. We have set $b_1 = (\exp f) b_0 (\exp - f)$.)

4.5.1 The generic level case

So by 3.1.2 we are given three irreducible generalized Weyl modules $V^0_{\lambda_0, k}, V^1_{\lambda_1, k}, V^\infty_{\lambda_\infty, k}$. Recall, see 2.4.1, that generalized Weyl modules are parametrized by symbols $(V^\varepsilon, V^\nu)$, where $m, n$ are nonnegative integers, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $V_m$ is an $m + 1$-dimensional $\mathfrak{g}$–module. Therefore we can and will assume that we have

$$(V^\varepsilon_{m_i}, V^\nu_{n_i}), \ i = 0, 1, \infty.$$

It is convenient to interpret the result of calculation of $\dim \big( \bigotimes_{i=0,1,\infty} (V^\varepsilon_{m_i}, V^\nu_{n_i}) \big)_{\mathfrak{g}(\mathbb{CP}^1,A)}$ in terms of the fusion algebra. The latter is defined as follows. Suppose that for any pair of generalized Weyl modules, say $(V^\alpha_{r_i}, V^\nu_{s_i}), \ i = 0, 1$, there is only finite number of $(V^\alpha_{r_\infty}, V^\nu_{s_\infty})$ such that

$$\dim \big( \bigotimes_{i=0,1,\infty} (V^\alpha_{r_i}, V^\nu_{s_i}) \big)_{\mathfrak{g}(\mathbb{CP}^1,A)} \neq 0.$$

Now view the symbols $(V^\varepsilon_{m_i}, V^\nu_{n_i})$ as generators of a free abelian group. Then there naturally arises an algebra (over $\mathbb{Z}$) with the operation of multiplication $\circ$ defined by

$$(V^\varepsilon_{r_0}, V^\nu_{s_0}) \circ (V^\alpha_{r_1}, V^\nu_{s_1}) = \sum_{(r_0, s_0, r_\infty, s_\infty)} \dim \big( \bigotimes_{i=0,1,\infty} (V^\varepsilon_{m_i}, V^\nu_{n_i}) \big)_{\mathfrak{g}(\mathbb{CP}^1,A)} (V^\alpha_{r_\infty}, V^\nu_{s_\infty}).$$

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The algebra defined in this way is called fusion algebra. Of course structure constants of the fusion algebra determine the dimensions of the spaces of coinvariants.

One last piece of notation: in the following theorem we formally set \((X \oplus Y, Z) = (X, Z) + (Y, Z)\) and \((X, Y \oplus Z) = (X, Y) + (X, Z)\). Recall also that in the category of \(\mathfrak{g}\)-modules one has

\[ V_f \otimes V_s \approx V_{r+s} \oplus V_{r+s-2} \oplus \cdots \oplus V_{|r-s|}. \]

**Theorem 4.5.1** (i) For any triple of generalized Weyl modules the space \((V_{c_i}^\alpha, V_{n_i})\) is finite dimensional.

(ii) The fusion algebra is well-defined, multiplication being given by the following formula

\[
(V_{r_1}^\alpha, V_{s_1}) \circ (V_{r_2}^\beta, V_{s_2}) = (V_{r_1+r_2}^{\alpha+\beta}, V_{s_1} \otimes V_{s_2}) + (V_{r_1+r_2-1}^{\alpha+\beta+1}, V_{s_1} \otimes V_{s_2}) + \cdots + (V_{|r_1-r_2|}^{\alpha+\beta}, V_{s_1} \otimes V_{s_2}).
\]

**4.5.2**

**Proof of Theorem 4.5.1.**

Throughout the proof \(A\) will stand for \(\{(0, b_0), (1, b_1)\}\), \(A_i\) for \(\{(\infty, b_\infty)\}\). Along with the algebras \(\mathfrak{g}(\mathbb{CP}^1, A), \mathfrak{g}(\mathbb{CP}^1, A, A_1)\) (see 4.4) introduce the algebra \(\tilde{\mathfrak{g}}(\mathbb{CP}^1, A, A_1) \subset \mathfrak{g}(\mathbb{CP}^1, A)\) consisting of all functions taking values in \(\mathfrak{b}_\infty\) at the point \(\infty\).

Of course \(\mathfrak{g}(\mathbb{CP}^1, A, A_1) \subset \tilde{\mathfrak{g}}(\mathbb{CP}^1, A, A_1)\) is an ideal and \(\dim \tilde{\mathfrak{g}}(\mathbb{CP}^1, A, A_1)/\mathfrak{g}(\mathbb{CP}^1, A, A_1) = 1\). Define \(\tilde{h}_\infty\) to be a basis element of \(\tilde{\mathfrak{g}}(\mathbb{CP}^1, A, A_1)/\mathfrak{g}(\mathbb{CP}^1, A, A_1)\). It is a standard (and simple) fact of Lie algebra cohomology theory that \(\tilde{h}_\infty\) acts on \((M^A)_{\mathfrak{g}(\mathbb{CP}^1, A, A_1)}\).

**Lemma 4.5.2** Let \(M^A\) be a generalized Weyl module. The element \(\tilde{h}_\infty\) has a simple spectrum as an operator acting on \((M^A)_{\mathfrak{g}(\mathbb{CP}^1, A, A_1)}\). Further, if \(M^A = (V_{r_1}^\alpha, V_{s_1}) \otimes (V_{r_2}^\beta, V_{s_2})\) then the set of eigenvalues of \(\tilde{h}_\infty\) is the set of the highest weights of the modules appearing in the right-hand side of Theorem 4.5.1(ii).

Proof of this lemma is essentially the same as that of Theorem 4.4 in [19] and mostly consists of solving a system of 2 equations related to 2 singular vectors – one in \((V_{r_1}^\alpha, V_{s_1})\), another in \((V_{r_2}^\beta, V_{s_2})\). We will discuss it in 4.5.3. Derivation of Theorem 4.5.1 from Lemma 4.5.2 is again very similar to that of Theorem 3.2 from Theorem 4.4 in loc. cit and uses Verma modules as follows.
Lemma 4.5.3  

(i) Let $(M^A)^\mu \in (M^\mu)_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)}$ be the eigenspace related to the eigenvalue $\mu$ of $\mathfrak{h}_\infty$. Then $(M^A)^\mu \approx (M^A \otimes M^\infty_{\mu,k})_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)}$.

(ii) Projection of a Verma module $M^\infty_{\mu,k}$ onto a generalized Weyl module $W$ induces an isomorphism of the coinvariants

$$(M^A \otimes M^\infty_{\mu,k})_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)} \approx (M^A \otimes W)_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)}.$$ 

Proof of Lemma 4.5.3

(i) A Verma module sitting at a point is induced from the 1-dimensional representation of the algebra of functions on the formal disk whose value at the point belong to the corresponding Borel subalgebra. Therefore (i) follows from Frobenius duality.

(ii) Consider the resolution of $W$ by Verma modules (see 2.4.1, formula (6)):

$$0 \rightarrow M \rightarrow M^\infty_{\mu,k} \rightarrow W \rightarrow 0$$

and tensor it with $M^A$. There arises the long exact sequence of homology groups of which we consider the following part:

$$(M^A \otimes M)_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)} \rightarrow (M^A \otimes M^\infty_{\mu,k})_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)} \rightarrow (M^A \otimes W)_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)} \rightarrow 0.$$

Since $M^\infty_{\mu,k}$ projects onto a Weyl module, the Verma module $M$ does not, see 2.4.1. Lemma 4.5.2 now gives that $(M^A \otimes M)_{\mathfrak{g}(\mathbb{C}P^1, A, A_1)} = \{0\}$. $\square$

To complete the proof of Theorem 4.5.1 observe that Lemma 4.5.2 and Lemma 4.5.3 together is a reformulation of Theorem 4.5.1. $\square$

Corollary 4.5.4  Let $\sharp A = 1$ and let $A_1$ and $A_2$ be as in 4.4. The following conditions are equivalent

(i) $M^A$ is a direct sum of generalized Weyl module;

(ii) $SS M^A = \Omega^A_{\text{reg}} \cup \sigma \Omega^A_{\text{reg}}$;

(iii) For any Verma module $W^A_1$ $\dim(M^A \otimes W^A_1)_{\mathfrak{g}(\mathbb{C}P^1, A_1, A_2)} < \infty$.

4.5.3

Here we sketch the proof of Lemma 4.5.2. First of all replace $M^A$ with the corresponding Verma module $\tilde{M}^A$. Then pass from the space $(\tilde{M}^A)_{\mathfrak{g}(\mathbb{C}P^1, A_1, A_2)}$ to its dual, that is to
the space of \( g(\mathbb{CP}^1, A, A) \)–invariant functionals on \( \tilde{M}^A \). Choose \( h \otimes (1 - z^{-1}) \) to be a representative of \( \tilde{h}_{\infty} \). Let \( \Psi \) be the eigenvector of \( h \otimes (1 - z^{-1}) \). By definition \( \Psi \) is a linear functional on \( M^{0,b}_0 \otimes M^{1,b}_1 \). It is an exercise on Frobenius duality to show that such a functional exists and unique.

Define \( F \) to be the following linear functional on \( M^{0,b}_0 \): \( F(w) = \Psi(w \otimes v_{\lambda_1}) \), where, as usual, \( v_{\lambda_1} \) is the vacuum vector of \( M^{1,b}_1 \). As \( M^{0,b}_0 \) is \( \mathbb{Z}_+ \times \mathbb{Z}_+ \)–graded (see 2.2), we denote by \( F_{ij} \) the restriction of \( F \) to the \((i, j)\)–component. Direct calculations show that with respect to the natural action of \( g \) on \( M^{0,b}_0 \):

\[
\bigoplus_{i,j \in \mathbb{Z}} CF_{ij} \cong \mathcal{F}^{\mathbb{C}^*}_{\alpha \beta}
\]

where \( \alpha = \lambda_{\infty} - \lambda_1 - \lambda_0 - 2 \), \( \beta = \lambda_1 \)

The functional \( F \) factors through the projection \( M^{0,b}_0 \to V^{0,b}_0 \) if and only if it vanishes on the singular vector of \( M^{0,b}_0 \). In other words, if this singular vector, say \( S \), has degree \((i, j)\) then the following equation holds

\[
SF_{ij} = 0.
\]

The latter equation can be written down and solved explicitly using formulas (11 or 12). Similar arguments go through for the module \( M^{1,b}_1 \) giving another equation, say

\[
S'F_{i'j'} = 0.
\]

Simultaneous solutions to these 2 equations give the desired result. By the way, as (11, 12) show, each of the expressions \( SF_{ij} \), \( S'F_{i'j'} \) splits in a product of linear factors; therefore geometrically the solution is a collection of intersection points of 2 families of lines in the plane. \( \Box \)

### 4.5.4 The rational level case

Suppose \( k + 2 = p/q \), \( p \) and \( q \) being positive integers. Now instead of 3 generalized Weyl modules sitting at 3 points in \( \mathbb{CP}^1 \) we are given 3 admissible representations sitting at 3 points on \( \mathbb{CP}^1 \). Recall, see 2.4.2, that admissible representations are parametrized by symbols \((V_m^r, V_n)\), \( 0 \leq m \leq q - 1, 0 \leq n \leq p - 2 \) modulo the relation \((V_m^r, V_n) = (V_{q-1-m}^r, V_{p-2-n})\). Denote by \((V_m^r, V_n)\) an equivalence class of \((V_m^r, V_n)\). We assume that \((V_m^r, V_n)\) satisfies the same bilinear condition \((V_m^r, V_n)\) in Theorem 4.5.1 does.

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The definition of the fusion (Verlinde) algebra in this case repeats word for word that in 4.5.1.

Recall finally that Kazhdan-Lusztig fusion functor [30] gives

\[ V_r \otimes_k V_s = V_{m-n} \oplus V_{m-n+2} \oplus \cdots \oplus V_{\min\{2k-r-s, r+s\}}. \]

The following theorem was proved in [19] in an equivalent but much less illuminating form.

**Theorem 4.5.5** (i) For any triple of admissible representations the space \((V^r_m, V^n_s)\) is finite dimensional.

(ii) The fusion algebra is well-defined, multiplication being given by the following formula

\[
(V_{r_1}, V_{s_1}) \circ (V_{r_2}, V_{s_2}) = \\
(V_{r_1-r_2}, V_{s_1} \otimes_{p-2} V_{s_2}) \circ (V_{r_2+r_2}, V_{s_1} \otimes_{p-2} V_{s_2}) + \cdots + \\
(V_{r_1}, V_{s_1} \otimes_{p-2} V_{s_2}) \circ (V_{r_2}, V_{s_2} \otimes_{p-2} V_{s_2})
\]

where \(N = \min\{2q-2-r-s, r+s\}\).

It is an easy exercise to derive this theorem from Theorem 4.5.1. For future purposes, however, we now sketch its original proof. Set \(A = \{(\infty, b_\infty), (0, b_0), (1, b_1)\}\). In addition to the algebras \(g(CP^1, A), g(CP^1, A, A_2)\) as in 4.4, we introduce an algebra \(\tilde{g}(CP^1, A, A_2) \subset g(CP^1, A)\). The latter consists of all functions whose values at the points 0 (1 resp.) belong to \(b_0\) (\(b_1\) resp.). Obviously \(g(CP^1, A, A_2) \subset \tilde{g}(CP^1, A, A_2)\) is an ideal and the quotient algebra \(\tilde{g}(CP^1, A, A_2)/g(CP^1, A, A_2)\) is commutative and 2-dimensional. This algebra naturally operates on the space \((M^A)_{\tilde{g}(CP^1, A, A_2)}\). Let \(\tilde{h}_0, \tilde{h}_1\) be a basis of \(\tilde{g}(CP^1, A, A_2)/g(CP^1, A, A_2)\).

**Lemma 4.5.6** ([19])

(i) \(\dim(M^A)_{\tilde{g}(CP^1, A, A_2)} < \infty\) if and only if \(M^A\) is an admissible representation.

(ii) Let \(M^A\) be an admissible representation. The elements \(\tilde{h}_0, \tilde{h}_1\) have simple spectra as operators acting on \((M^A)_{\tilde{g}(CP^1, A, A_2)}\). Their eigenvalues recover the structure constants of the fusion algebra.

"Inserting" Verma modules and using BGG resolution one derives Theorem 4.5.5 from Lemma 4.5.6 in a way similar to that we used in 4.5.1.

Another important corollary of Lemma 4.5.6 is as follows.
Corollary 4.5.7 Let $\tilde{\lambda} \Lambda = 1$ The following conditions are equivalent (i) $M^A$ is a sum of admissible representations;

(ii) $SSM^A = \Omega^A_{\text{nilp}} \cup \sigma \Omega^A_{\text{nilp}}$;

(iii) $\dim(M^A)_{(\mathbb{C}P^1, \Lambda_1, \Lambda_2)} < \infty$.

4.5.5 Classical and quantum $\mathfrak{osp}(1|2)$. Fusion algebra as a Grothendieck ring

**A.** $\mathfrak{osp}(1|2)$ is a rank 1 superalgebra – one of the superanalogues of $\mathfrak{sl}_2$. It can be defined as an algebra on 2 odd generators, $x_+, x_-$, one even generator, $h$, and relations

$$[x_+, x_-] = h, \ [h, x_\pm] = \pm x_\pm.$$

Even part of this algebra is $\mathfrak{sl}_2$ and is generated by $x_\pm^2$; odd part is $V_1$ as an $\mathfrak{sl}_2$–module, its basis is $\{x_+, x_-, 1\}$.

From this it is easy to obtain the following classification of all simple finite dimensional $\mathfrak{osp}(1|2)$–modules. (It is even simpler to do this in the way modelling the $\mathfrak{sl}_2$–case – by starting with Verma modules and then quotienting out a singular vector; for details see [32]). Each $\mathfrak{osp}(1|2)$–module $W$ is a sum of an even and odd part $W = \text{even} \ W \oplus \text{odd} \ W$; each $W$ is an $\mathfrak{sl}_2$–module, i.e. direct sum of $V_n$'s. These are generalities. But in reality each irreducible $\mathfrak{osp}(1|2)$–module is of one of the 2 following types:

$$V_n^0 \text{ such that } \text{even} \ V_n^0 = V_n, \ \text{odd} \ V_n^0 = V_{n-1};$$

$$V_n^1 \text{ such that } \text{even} \ V_n^0 = V_{n-1}, \ \text{odd} \ V_n^0 = V_n.$$  

The fact that the dimensions of the even and odd parts are different by 1 is a consequence of the fact that odd part of the algebra is $V_1$.

We see that each irreducible $\mathfrak{osp}(1|2)$–module is odd-dimesional; further $V_n^0$ and $V_n^1$ are isomorphic as modules and obtained from each other by the change of parity. This is the category of finite dimensional representations of $\mathfrak{osp}(1|2)$; denote it $\text{Rep}(\mathfrak{osp}(1|2))$. As in the $\mathfrak{sl}_2$–case, one proves that $\text{Rep}(\mathfrak{osp}(1|2))$ is semisimple.

The universal enveloping algebra $U\mathfrak{osp}(1|2)$ is in fact a Hopf algebra, for example the comultiplication is given by the standard formula $g \mapsto g \otimes 1 + 1 \otimes g, \ g \in \mathfrak{osp}(1|2)$. This makes $\text{Rep}(\mathfrak{osp}(1|2))$ a tensor category: $\text{Rep}(\mathfrak{osp}(1|2)) \times \text{Rep}(\mathfrak{osp}(1|2)) \to \text{Rep}(\mathfrak{osp}(1|2))$, $A, B \mapsto A \otimes B$, where the $\mathfrak{osp}(1|2)$–module structure on $A \otimes B$ is determined through the
comultiplication (and the rule of sign). Decomposing the tensor product of 2 irreducible modules one gets the Grothendieck ring of $\text{Rep}(\mathfrak{osp}(1|2))$.

**Lemma 4.5.8**

$$V^\alpha_{r_1} \otimes V^\beta_{r_2} = V^{\alpha+\beta}_{r_1+r_2} + V^{\alpha+\beta+1}_{r_1+r_2-1} + V^{\alpha+\beta-2}_{r_1+r_2-2} + \cdots + V^{\alpha+\beta}_{|r_1-r_2|}.$$ 

**Proof.** Direct calculations show that $V^\alpha_{r_1} \otimes V^\beta_{r_2}$ contains one and only one singular (annihilated by $x_+)$ vector of each weight from $|r_1 - r_2|$ to $r_1 + r_2$ and that the submodules generated by these vectors are irreducible. Proof is completed by counting dimensions. □

Theorems 4.5.1 and 4.5.5 provide us with 2 commutative algebras. Here we interpret these algebras as Grothendieck rings of certain categories. Start with the algebra of Theorem 4.5.1 and denote it $A^\text{gen}$. Obviously $A^\text{gen} = A_0 \otimes A$, where $A$ is the Grothendieck ring of the category of finite-dimensional representations of $\mathfrak{g}$ (its multiplication law is defined by the formula preceding Theorem 4.5.1) and $A_0$ is the algebra with basis $V_i^\alpha$, $i \geq 0, \alpha \in \mathbb{Z}/2\mathbb{Z}$, multiplication being given by

$$V^\alpha_{r_1} \circ V^\beta_{r_2} = V^{\alpha+\beta}_{r_1+r_2} + V^{\alpha+\beta+1}_{r_1+r_2-1} + V^{\alpha+\beta-2}_{r_1+r_2-2} + \cdots + V^{\alpha+\beta}_{|r_1-r_2|}. \quad (19)$$

Comparing (19) with Lemma 4.5.8 we get the following.

**Proposition 4.5.9** $A_0$ is the Grothendieck ring of the category of finite-dimensional representations of the superalgebra $\mathfrak{osp}(1|2)$.

Appearance of $\mathfrak{osp}(1|2)$ here, although artificial as it may seem to be, has deep reasons behind it. To see this we will analyze the rational level case using quantized enveloping algebras.

**Remark 4.5.10** It follows from Lemma 4.5.8 that the forgetful functor $\text{Rep}(\mathfrak{osp}(1|2)) \to \text{Rep}(\mathfrak{sl}_2), V^\alpha_m \mapsto V_m \oplus V_{m-1}$ induces an epimorphism of the Grothendieck rings.

B. Both $U\mathfrak{sl}_2$ and $U\mathfrak{osp}(1|2)$ admit quantization, $U_t\mathfrak{sl}_2$ and $U_t\mathfrak{osp}(1|2)$ resp. Let us remind the relevant formulas. The Drinfeld-Jimbo (see [9, 28]) algebra $U_t\mathfrak{sl}_2$, $t \in \mathbb{C}$ is defined to be an associative algebra on generators $E, F, K^{\pm 1}$ and relations

$$EF - FE = \frac{K - K^{-1}}{t - t^{-1}}, \quad KEK^{-1} = t^2 E, \quad KFK^{-1} = t^{-2} F.$$
$U_{t\mathfrak{osp}(1|2)}$ is similarly defined [32] as an associative algebra on generators $X_+, X_-, K^{\pm 1}$ and relations

$$X_+ X_- + X_- X_+ = \frac{K - K^{-1}}{t - t^{-1}}, \quad K X_\pm K^{-1} = t^{\pm 1} X_\pm.$$

The representation theory of $\mathfrak{sl}_2$ and $\mathfrak{osp}(1|2)$ “deforms to” the representation theory of $U_{t\mathfrak{sl}_2}$ and $U_{t\mathfrak{osp}(1|2)}$ resp. We will continue denoting by $V_m$ the $m+1$-dimensional module over $U_{t\mathfrak{sl}_2}$, and by $V_m^\alpha$, $V_m^\beta$ the 2 $(2m+1)-$ dimensional modules over $U_{t\mathfrak{osp}(1|2)}$. For generic $t$ these modules are irreducible, the categories of finite dimensional representations, $Rep(U_{t\mathfrak{sl}_2})$ and $Rep(U_{t\mathfrak{osp}(1|2)})$, generated by these modules are semisimple.

The deformations $U_{t\mathfrak{sl}_2}$ and $U_{t\mathfrak{osp}(1|2)}$ are especially remarkable in that they afford simultaneous deformation of the Hopf algebra structure. We get 2 tensor categories $Rep(U_{t\mathfrak{sl}_2})$ and $Rep(U_{t\mathfrak{osp}(1|2)})$. What has been said implies that the Grothendieck rings of $Rep(U_{t\mathfrak{sl}_2})$ and $Rep(U_{t\mathfrak{osp}(1|2)})$ are isomorphic to the Grothendieck rings of the corresponding classical objects if $t$ is generic.

If however $t$ is a root of unity, things change dramatically. Suppose for simplicity that $t$ is a primitive $l$-th root of unity, $l$ being odd. Then

(i) $V_m$ is irreducible if and only if $m < l; \quad (20)$

(ii) $V_m^\prime$ is irreducible if and only if $m < l; \quad (21)$

(Both statements are proved by direct computations.)

What is even more important is that the categories $Rep(U_{t\mathfrak{sl}_2})$ and $Rep(U_{t\mathfrak{osp}(1|2)})$ are no longer semisimple. For example, tensor product of 2 irreducible representations is not semisimple. Things, however, are still very much under control.

Lemma 4.5.11 Let $t$ be a primitive $l$-th root of unity, $l$ being odd, $m, n < l$. Then

(i) $V_m \otimes V_n = V_{|m-n|}^{m-n} \oplus V_{|m-n|+2}^{m-n+2} \oplus \cdots \oplus V_{\min{2,(l-1)-(m-n,m+n)}} \oplus W,$

where $W$ is not semisimple.

(ii) $V_m^\alpha \otimes V_n^\beta = V_{|m-n|}^{\alpha+\beta} \oplus V_{|m-n|+1}^{\alpha+\beta+1} \oplus \cdots \oplus V_{\min{2,(l-1)-(m-n,m+n)}}^{\alpha+\beta} \oplus W,$

where $W$ is not semisimple.
Sketch of Proof. (i) is well-known, see [41]. We will however review both cases as at our level of brevity there will no difference between them. First, direct calculations as in the proof of Lemma 4.5.8 show that regardless of $t$ at each weight space there can always be only one singular vector. Now decomposition of Lemma 4.5.8, statements (20, 21) and this uniqueness result show that the submodules $V_{i-1+i}$ and $V_{i-1-i}$ (or $V_{i-1+i}$ and $V_{i-1-i}$), $i \leq m + n - l + 1$ are non-trivially tangled. Other $V_j$ coming from generic $t$ are still irreducible and appear as direct summands. □

Definition.

(i) Define $\text{Rep}(U_t \mathfrak{sl}_2)^{(l)}$ and $\text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)}$ to be subcategories of $\text{Rep}(U_t \mathfrak{sl}_2)$ and $\text{Rep}(U_t \mathfrak{osp}(1|2))$ resp. consisting of direct sums of irreducible modules $V_m$ (or $V_m^a$ resp.), $m < l$.

(ii) Define functors

$$\text{Rep}(U_t \mathfrak{sl}_2)^{(l)} \times \text{Rep}(U_t \mathfrak{sl}_2)^{(l)} \to \text{Rep}(U_t \mathfrak{sl}_2)^{(l)}, \ A, B \mapsto A \boxtimes B,$$

$$\text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)} \times \text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)} \to \text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)}, \ A, B \mapsto A \boxtimes B,$$

by taking the usual tensor product and then throwing away $W$ in the right hand side of formulas in Lemma 4.5.11. □

We get tensor categories $\text{Rep}(U_t \mathfrak{sl}_2)^{(l)}$ and $\text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)}$.

C. It is easy now to interpret the fusion algebra at the rational level in terms of the Grothendieck rings of $\text{Rep}(U_t \mathfrak{sl}_2)^{(l)}$ and $\text{Rep}(U_t \mathfrak{osp}(1|2))^{(l)}$. In view of Lemma 4.5.11 and Definition above, Theorem 4.5.5 reads as follows.

**Proposition 4.5.12** Fusion algebra at the level $k + 2 = p/q$ is a quotient of the tensor product of the Grothendieck rings of the categories $\text{Rep}(U_t \mathfrak{sl}_2)^{(p-1)}$ and $\text{Rep}(U_t \mathfrak{osp}(1|2))^{(q)}$. Further, the fusion algebra always contains the Grothendieck ring of $\text{Rep}(U_t \mathfrak{osp}(1|2))^{(q)}$ via the classes of symbols $V_m^a, V_0$.

**4.5.6 Kac-Moody vs. Virasoro**

Virasoro algebra, $\text{Vir}$, is defined to be a vector space with basis $\{L_i, z, \ i \in \mathbb{Z}\}$ and bracket

$$[L_i, L_j] = (i - j)L_{i+j} + \delta_{i-j} \frac{i^3 - i}{12} z.$$
Representation theory of Virasoro algebra is to a great extent parallel to that of $\hat{g}$. We will confine to essentials, making reference to [1,5].

One defines the Verma module $M_{h,c}$, where $(h,c)$ is a highest weight, i.e. eigenvalues of $L_0$, determined by the vacuum vector; $c$ is sometimes referred to as level. A Verma module is reducible if and only if it contains a singular vector. $M_{h,c}$ generically has no singular vectors. By the Kac determinant formula, there is a family of hyperbolas labelled by pairs of positive integers $m,n$ in the plane with coordinates $(h,c)$ such that if $M_{h,c}$ contains a singular vector, then $(h,c)$ belongs to one of these hyperbolas; generically along the hyperbolas the singular vector is unique. Denote the singular vector arising in $M_{h,c}$ as $(h,c)$ gets on the hyperbola with the label $m,n$ by $S_{mn}$. There arises the $Vir$-analogue of the generalized Weyl module $M_{h,c}/<S_{mn}=0>$. Attach to $M_{h,c}/<S_{mn}=0>$ the symbol $(V_{n-1},V_{m-1})$. Further, for $c$ fixed there arises a one-to-one correspondence between the $Vir$-analogues of generalized Weyl modules and symbols $(V_{n-1},V_{m-1})$. This has all been in precise analogy with 2.4.1.

It has hardly been written anywhere, but is nevertheless known that the $Vir-$ analogue of the fusion algebra from 4.5.1, i.e. at a generic level, is as follows:

$$(V_{n_1},V_{m_1}) \circ (V_{n_2},V_{m_2}) = (V_{n_1} \otimes V_{n_2},V_{m_1} \otimes V_{m_2}).$$  \hfill (22)\hspace{1cm}

(The interested reader can prove this result using methods of [16]; our treatment of the $\hat{g}$-fusion algebra in 4.5.1 is also a direct analogue of these.)

There is a functor sending $\hat{g}$-modules to $Vir$-modules - quantum Drinfeld-Sokolov reduction. One of the prerequisites for it is a choice of a nilpotent subalgebra of $\mathfrak{sl}_2$. The two obvious possibilities are $C_e$ and $C_f$. Denote the corresponding functors $\phi_e$ and $\phi_f$. It can be extracted from [13] that both functors send generalized Weyl modules to generalized Weyl modules. In our terminology one gets

$$\phi_e: (V^0_m,V_n) \mapsto (V_m,V_n),$$

$$\phi_f: (V^0_m,V_n) \mapsto (V_{m-1},V_n),$$

where the symbol $V_{-1}$, if arises, is understood as zero.

The $Vir-$analogue of admissible representations is the celebrated minimal representations. The latter can be defined as quotients of generalized Weyl modules by repeating
It is known that minimal representations arise only when
\[ c = c_{pq} = 1 - \frac{6(p - q)^2}{pq}, \]
where \( p, q \) are relatively prime positive integers. There are again 2 generalized Weyl modules projecting on a given minimal representation. Therefore minimal representations are labelled by equivalence classes of symbols \((V_m, V_n)\). It can be shown that the equivalence relation is as follows: \((V_m, V_n) \approx (V_{m-2}, V_{n-2})\) for \( c = c_{pq} \). From this and (22) one can easily calculate the fusion algebra. We will not write down the relevant formulas here and confine to mentioning that the algebra is related to the product of Grothendieck rings of 2 quantum \( U_t(st_2) \) at appropriate roots of unity in much the same way as the fusion algebra for \( \hat{g} \) is related to the product of Grothendieck rings of \( U_t(osp(1|2)) \) and \( U_t(st_2) \). Recall also that the \( Vir \)-fusion algebra was calculated in [6]; mathematically acceptable exposition can be found in [16].

Another property of the Drinfeld-Sokolov reduction is that both \( \phi_e \) and \( \phi_f \) send admissible representations at the level \( k = 2 - p/q \) of \( \hat{g} \) to minimal representations of \( Vir \) at the level \( c_{pq} \), see [18].

**Proposition 4.5.13** The functor \( \phi_e \oplus \phi_f \) determines an epimorphism of the \( \hat{g} \)-fusion algebra onto the \( Vir \)-fusion algebra at both generic and rational levels.

**Proof.** The generic level case follows from Remark 4.5.10 and formula (22) above. In the rational level case, the statement follows from the fact that both, \( \hat{g} \)- and \( Vir \)-, fusion algebras are obtained from their generic level counterparts by imposing the equivalence relations and the 2 equivalence relations agree with each other. \( \square \)

### 4.6 Fusion functor.

This part is an announcement, proofs will appear elsewhere.

Suppose we have a trivial vector bundle \( E \rightarrow \mathbb{C}P^1 \), \( A = \{(P_1, b_1), (P_2, b_2)\}, \( B = \{(P_3, b_3)\} \), so that \((E, A \cup B)\) is generic. There is a construction which to a \( \hat{g}^A \)-module associates a \( \hat{g}^B \)-module. This construction is a natural adjustment of the Kazhdan-Lusztig tensoring [30] to our needs.
Denote by \( g(\mathbb{CP}^1, A, B) \) the subalgebra of \( g(\mathbb{CP}^1, A) \) consisting of functions taking values in \( \mathfrak{n}_3 = [\mathfrak{b}_3, \mathfrak{b}_3] \) just like we did in 4.4.2. For a \( \mathfrak{g}^A \)-module \( M^A \), denote by \( M^A_N \) the subspace of \( (M^A)^* \) annihilated by \( g(\mathbb{CP}^1, A, B)^N \). Obviously \( M^A_1 \subset M^A_{N+1} \), \( N \geq 1 \). Set

\[
F^{A-B}(M^A) = \bigcup_{N \geq 1} M^A_N.
\]

One can show that the vector space \( F^{A-B}(M^A) \) affords in a natural way a structure of an \( \mathfrak{g}^B \)-module at the same level; this is easy to show in the spirit of [30, 4]. Using our methods one can show that

(i) if \( M^A \) is from the \( \mathcal{O} \)-category, or further a generalized Weyl module, or further an admissible representation, then \( F^{A-B}(M^A) \) is also as a \( \mathfrak{g}^B \)-module;

(ii) the arising in this way Grothendieck rings coincide with those in Theorem 4.5.1 or Theorem 4.5.5 if the level is generic or rational resp..

This generalizes the statement for the integrable representations, see [11].

**Problem.** Describe the arising tensoring in the spirit of Kazhdan-Lusztig.

### 4.7 Quadratic degeneration

#### 4.7.1

The setup here will the following version of 3.2.2:

(i) \( \bar{\pi} : C_S \to S \) be a family of curves over a formal disk \( S \), such that the fiber over the generic point of \( S \) (“outside origin”) is a smooth projective curve, and over the origin, \( O \), the fiber is a curve \( C_O \), with exactly one quadratic singularity;

(ii) \( \rho_S : \mathcal{E}_S \to C_S \) is a rank 2 vector bundle.

As in 3.2.2, we complete these data to the localization data with logarithmic singularities, say \( \tilde{\psi} \). In the standard way, Theorem 3.2.1 rewrites to give a \( D \)-module over \( S \) with logarithmic singularities at \( O \); call it \( \Delta_{\tilde{\psi}}(M^A) \). This is because \( Spec(S) = \mathbb{C}[[t]] \) and vector fields vanishing at \( q = 0 \) are exactly those which can be lifted to \( C_S \).

Along with the family \( \bar{\pi} : C_S \to S \) consider the family \( \bar{\pi}^\vee : C^\vee_S \to S \), obtained from \( \bar{\pi} : C_S \to S \) by replacing the singular fiber \( C_O \) with its normalization \( C^\vee_O \) (i.e. be tearing \( C_O \) apart at the self-intersection point). There is a projection \( C^\vee_O \to C_O \) and the preimage of the self-intersection point \( a \in C_O \) consists of 2 points \( a_0, a_\infty \in C^\vee_O \).

It is obvious that the datum \( \mathcal{E} \to C_S \) is equivalent to the data “\( \rho^\vee_S : \mathcal{E}^\vee_S \to C^\vee_S \), equivalence \( (\rho_S^\vee)^{-1}(a_0) \approx (\rho_S^\vee)^{-1}(a_\infty) \)”. The localization data with logarithmic singularities \( \tilde{\psi} \) rewrites to give a “normalized” localization data \( \psi^\vee \).
In addition fix 2 different lines $l_0, l_\infty$ in the fiber of $\mathcal{E}_s$ over the point $a \in \mathcal{C}_O$. This determines 2 Borel subalgebras, $\mathfrak{b}_0, \mathfrak{b}_\infty$ operating in the fiber over $a$.

After normalization these additional data determine the line $l_0$ and the Borel subalgebra $\mathfrak{b}_0$ operating in the fiber of $\mathcal{E}_s^\vee$ over $a_0$, as well as the line $l_\infty$ and the Borel subalgebra $\mathfrak{b}_\infty$ operating in the fiber over $a_\infty$. We also get a distinguished Cartan subalgebra $\mathfrak{h} = \mathfrak{b}_0 \cap \mathfrak{b}_\infty$. Set $A^\vee = A \setminus \{(a_0, \mathfrak{b}_0), (a_\infty, \mathfrak{b}_\infty)\}$.

Now with a $\mathfrak{g}^A$—module $M^A$ at the level $k$ and an admissible weight $\lambda \in \mathfrak{h}^*$ we associate the $\mathfrak{g}^A$—module $M^A \otimes L_{P_k}^\mathfrak{b}_0 \mathfrak{b}_0 \otimes L_{P_k}^\mathfrak{b}_\infty \mathfrak{b}_\infty$. We get a $D$—module for the “normalized” localization data:

$$\bigoplus \lambda \Delta_{\mathfrak{g}^A}(M^A \otimes L_{P_k}^\mathfrak{b}_0 \mathfrak{b}_0 \otimes L_{P_k}^\mathfrak{b}_\infty \mathfrak{b}_\infty).$$

**Proposition 4.7.1** Generically with respect to $l_0, l_\infty$, if $\Delta_{\mathfrak{g}^A}(M^A)$ is smooth then $\bigoplus \lambda \Delta_{\mathfrak{g}^A}(M^A \otimes L_{P_k}^\mathfrak{b}_0 \mathfrak{b}_0 \otimes L_{P_k}^\mathfrak{b}_\infty \mathfrak{b}_\infty)$ is also and there is an isomorphism of $D$—modules

$$\Delta_{\mathfrak{g}^A}(M^A) \cong \bigoplus \lambda \Delta_{\mathfrak{g}^A}(M^A \otimes L_{P_k}^\mathfrak{b}_0 \mathfrak{b}_0 \otimes L_{P_k}^\mathfrak{b}_\infty \mathfrak{b}_\infty).$$

**4.7.2 Proof**

(i) Begin with the genus zero case. Observe that the algebra of regular functions on the neighborhood of the point $a$ is $\mathbb{C}[t_0, t_\infty][[t]] < t_0 t_\infty = t >$ where $t$ is a coordinate on $S$; $\mathcal{C}_O^\vee$ in this case is just a union of 2 spheres. Therefore the set $A$ splits in two: $A'$ and $A''$, each of which has to do with one of the spheres.

Hence the algebra $\mathfrak{g}(\pi^{-1}(s), A)$ can be degenerated into the following one as $s$ “approaches” $O$:

$$(\mathfrak{g}(\mathbb{C}^1, A', (P_0, \mathfrak{b}_0)) + \mathfrak{h}) \oplus \mathfrak{h} (\mathfrak{g}(\mathbb{C}^1, A'', (P_\infty, \mathfrak{b}_\infty))).$$

Meaning of the last expression is as follows: recall, see 4.4, that $\mathfrak{g}(\mathbb{C}^1, A', (P_0, \mathfrak{b}_0))$ consists of functions regular outside $\tilde{A}$ and sending $P_0$ to $\mathfrak{n}_0$; $\mathfrak{g}(\mathbb{C}^1, A'', (P_\infty, \mathfrak{b}_\infty))$ is defined similarly with $P_0, \mathfrak{n}_0$ replaced with $P_\infty, \mathfrak{n}_\infty$; further the algebra $\mathfrak{g}(\mathbb{C}^1, A', (P_0, \mathfrak{b}_0)) + \mathfrak{h}$ is the algebra of functions sending $P_0$ to $\mathfrak{b}_0$, the same is true for $\mathfrak{h} + \mathfrak{g}(\mathbb{C}^1, A'', (P_\infty, \mathfrak{b}_\infty))$; finally “$\oplus \mathfrak{h}$” means direct product over $\mathfrak{h}$.

Therefore the coinvariants degenerate into the space

$$((M^{A'})_{\mathfrak{g}(\mathbb{C}^1, A', (P_0, \mathfrak{b}_0))} \otimes (M^{A''})_{\mathfrak{g}(\mathbb{C}^1, A'', (P_\infty, \mathfrak{b}_\infty))})^\mathfrak{h},$$

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where \( \mathfrak{h} \) acts by means of the diagonal embedding; this makes sense as the fibers are identified.

By Proposition 4.4.1, the space

\[
(M^A)^{\mathfrak{g}}_{\mathbb{C}P^1, A^e, (P_0, \mathfrak{b}_0)} \otimes (M^{A^e})^{\mathfrak{g}}_{\mathbb{C}P^1, A^e, (P_\infty, \mathfrak{b}_\infty)}
\]

is finite dimensional. It is easy to extract from Lemma 4.5.2 that as an \( \mathfrak{h} \)-module this space is semisimple and therefore is isomorphic to

\[
\bigoplus_{\lambda} ((M^A \otimes M^\mathfrak{f}_{\lambda, k} \otimes M^{P_\infty, \mathfrak{b}_\infty})_{\mathfrak{g}}(C^\mathfrak{v}_{\lambda, \lambda})).
\]

By Lemma 4.5.6, in the last formula \( \lambda \) can be chosen to be admissible and the Verma modules can be replaced with the corresponding admissible representations.

This proves that \( \bigoplus_{\lambda} \Delta_{\psi} \nu(M^A \otimes L^\mathfrak{f}_{\lambda, k} \otimes L^{P_\infty, \mathfrak{b}_\infty}) \) is smooth and gives a morphism

\[
\Delta_{\psi} (M^A) \to \bigoplus_{\lambda} \Delta_{\psi} \nu(M^A \otimes L^\mathfrak{f}_{\lambda, k} \otimes L^{P_\infty, \mathfrak{b}_\infty}).
\]

That this is an isomorphism can be shown in the standard way constructing the inverse map using the formal character of \( L_{\lambda, k} \), see [4].

(ii) The higher genus case is not much different. For example, pinching makes a torus into a sphere. Therefore in this case proof is literally the same. It also proves an analogue of Lemma 4.5.6 for a torus. This provides a basis for induction.

In genus \( \geq 2 \) at an appropriate place instead of Proposition 4.4.4 one has to make reference to Proposition 4.3.3 and then use induction. □

### 4.7.3 Remarks

(i) Meaning of Proposition 4.7.1 is that the dimension of the generic fiber of the \( D \)-module \( \Delta_{\psi} (M^A) \) can be calculated by the usual combinatorial algorithm: by pinching the surface and further inserting inserting all possible representations the problem is reduced to the case of a sphere with three punctures and in the latter case the complete results are available.

(ii) In the genus 0 case the analogue of Proposition 4.7.1 for generalized Weyl modules is valid. To see this it is enough to examine part (i) of the proof and convince oneself that the only requirement on \( M^A \) used there was that \( M^A \) be generalized Weyl module; in fact at an appropriate place instead of Lemma 4.5.6 one has to use Lemma 4.5.3.
(iii) Quadratic degeneration for generalized Weyl modules on the sphere allows to write horizontal sections of the corresponding bundle as a product of vertex operators. This will be explained in sect.5.

5 Screening operators and correlation functions

In this section we will study in detail the situation described in 4.4.3: we have the trivial rank 2 bundle $E \to \mathbb{CP}^1$, a generalized Weyl module $M^A$, and a holonomic $D$-module $\Delta(M^A)$ on the space $\mathbb{C}^m \times \mathbb{C}^m$ with fiber $(M^A)_{\mathfrak{g}(\mathbb{CP}^1,A)}$. For the reasons which will become clear later we replace this bundle with the dual one, its fiber being $((M^A)^*)_{\mathfrak{g}(\mathbb{CP}^1,A)}$. Denote the corresponding $D$-module by $\Delta(M^A)^*$. Using our results on quadratic degeneration we rewrite horizontal sections of the corresponding bundle with flat connection as matrix elements of vertex operators, which serves the two-fold purpose: we find that the differential equations satisfied by horizontal sections are provided by the singular vectors of the corresponding Verma module and write down integral representations for solutions to these differential equations.

5.1 Vertex operators and correlation functions

An alternative to the language of coinvariants in the genus zero case is the language of vertex operators.

**Definition.** A vertex operator is a $\mathfrak{g}$-morphism

$$Y : \mathcal{F}_{a\beta}^C \otimes V_1 \to V_2,$$

(23)

where $\mathcal{F}_{a\beta}^C$ is a loop module (see 2.6) and $V_1, V_2 \in \mathcal{O}_k$ are highest weight modules. □

In other words, a vertex operator is an embedding

$$\mathcal{F}_{a\beta}^C \hookrightarrow \text{Hom}_C(V_1 \to V_2).$$

The space $\mathcal{F}_{a\beta}^C$ has the basis \{\(F_{ij} = F_i \otimes z^j, i, j \in \mathbb{Z}\}\}, where \{\(F_i, i \in \mathbb{Z}\}\} is a basis in $\mathcal{F}_{a\beta}$, see 2.6. Given a vertex operator $Y$, consider the generating function

$$Y(x, z) = x^{\Delta_1}z^{\Delta_2} \sum_{i,j=-\infty}^{\infty} F_{ij} x^{-i} z^{-j},$$

the “monodromy coefficients” $\Delta_1, \Delta_2$ are defined by:

$$\Delta_1 = \frac{-\lambda_2 + \lambda_1 + \beta}{2}, \quad \Delta_2 = \frac{-C(\lambda_2) + C(\lambda_1) + C(\beta)}{2},$$

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where \( \lambda_i \) is the highest weight of \( V_i \) and \( C(\lambda) = \lambda(\lambda + 2)/2 \). (\( \Delta_1, \Delta_2 \) will later appear as genuine monodromy coefficients of a certain flat connection.)

The formal series \( Y(x, z) \) is, of course, an element of \( \text{Hom}_C(V_1, V_2 \otimes x^{\Delta_1} z^{\Delta_2} C[[x^{\pm 1}, z^{\pm 1}]] \). Further, for any \( g \in \mathfrak{g} \) the commutator \([g \otimes z^n, Y(x, z)]\) is also a well-defined element of \( \text{Hom}_C(V_1, V_2 \otimes x^{\Delta_1} z^{\Delta_2} C[[x^{\pm 1}, z^{\pm 1}]] \). For the standard basis of \( \mathfrak{g} \), see 2.2, one derives from the definition of a vertex operator that

\[
[e \otimes z^n, Y(x, z)] = z^n (-x^2 \frac{\partial}{\partial x} + \beta x) Y(x, z),
\]

\[
[f \otimes z^n, Y(x, z)] = z^n \frac{\partial}{\partial x} Y(x, z),
\]

\[
[h \otimes z^n, Y(x, z)] = z^n (2x^2 \frac{\partial}{\partial x} - \beta x) Y(x, z).
\]

We conclude that for any \( g \in \mathfrak{g} \) there is a differential operator \( D_g(x) \) in \( x \) such that

\[
[g \otimes z^n, Y(x, z)] = z^n D_g(x) Y(x, z),
\]

Suppose now we are given a collection of vertex operators

\[
Y_i : \mathcal{F}_{\lambda_i, \mu_i} \otimes V_{i-1/2} \to V_{i+1/2}, 1 \leq i \leq m.
\]

The product of the corresponding generating functions \( Y_m(x_m, z_m) \cdots Y_2(x_2, z_2) Y_1(x_1, z_1) \) is a well-defined element of \( \text{Hom}_C(V_{1/2}, V_{m+1/2} \otimes \prod_i x_i^{\Delta_i} z_i^{\Delta_i} C[[x_i^{\pm 1}, z_i^{\pm 1}, \ldots, z_m^{\pm 1}]] \). The matrix element

\[
\langle v^*, Y_m(x_m, z_m) \cdots Y_2(x_2, z_2) Y_1(x_1, z_1) v \rangle, \quad v \in V_{1/2}, v^* \in V^*_{m+1/2}
\]

is, therefore, a formal Laurent series in \( x_i, z_i \), \( 1 \leq i \leq m \).

**Definition** Suppose \( Y_i(x_i, z_i), 1 \leq i \leq m \) are as above. Then the matrix element

\[
\Psi(x_1, \ldots, x_m, z_1, \ldots, z_m) = \langle v^*, Y_m(x_m, z_m) \cdots Y_2(x_2, z_2) Y_1(x_1, z_1) v \rangle
\]

is called **correlation function** if \( V_{1/2}, \ldots, V_{m+1/2} \) are irreducible generalized Weyl modules, \( V_{1/2} \) is the vacuum module, \( v \) is the highest weight vector of \( V_{1/2} \) and \( v^* \) is the dual to the highest weight vector of \( V_{m+1/2} \). (The latter condition is meaningful in view of the weight space decomposition of a highest weight module.) \( \Box \)

A correlation function has been understood as a formal power series. We will show that, in fact, it is a holomorphic function satisfying a certain holonomic system of partial differential equations. In order to do that we will interpret vertex operators as horizontal sections of a line bundle with a flat connection provided by three modules on \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).
5.2 From coinvariants to vertex operator algebra

5.2.1

We return to the setup of 4.5.1. In the cartesian product \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) fix coordinate system \((x, z)\). Attach to the point \(x\) in the first factor the Borel subalgebra \(\mathfrak{b}_x\) spanned by the vectors \(e_x = e - xh - x^2f, \; h_x = h + 2xf\). This means, in particular, that \(\mathfrak{b}_0\) is the standard Borel subalgebra \(\mathfrak{ce} \oplus \mathfrak{ch}\) (see 2.2) and \(\mathfrak{b}_\infty\) is the opposite one. Set \(A = \{(0, 0), (x, z), (\infty, \infty)\}\). Let \(V^A = V_0^{\mathfrak{b}_0, 0} \otimes V_1^{\mathfrak{b}_z, z} \otimes V_\infty^{\mathfrak{b}_\infty, \infty}\) be a generalized Weyl module over \(\hat{\mathfrak{g}}^A\). Consider the space of invariants \((V^A)^* \otimes \mathfrak{g}(\mathbb{CP}^1, A)\). By Theorem 4.5.1 this space is either 0- or 1-dimensional. Suppose the latter possibility is the case. Then by Theorem 3.2.2 we get a line bundle with flat connection over \(\mathbb{C}^* \times \mathbb{C}^*\) whose fiber over the point \((x, z) \in \mathbb{C}^* \times \mathbb{C}^*\) is \(((V^A)^*) \otimes \mathfrak{g}(\mathbb{CP}^1, A)\). There arises an embedding

\[ V_1^{\mathfrak{b}_z, z} \hookrightarrow \text{Hom}_\mathbb{C}(V_0^{\mathfrak{b}_0, 0} \otimes V_\infty^{\mathfrak{b}_\infty, \infty}, \mathbb{C}). \]

The dual space \((V_\infty^{\mathfrak{b}_\infty, \infty})^*\) as a \(\hat{\mathfrak{g}}\)–module is isomorphic to the contragredient module \((V_\infty^{\mathfrak{b}_\infty, \infty})^c\), see 2.2. As the level is generic, the latter module is irreducible and is, therefore, isomorphic to a certain generalized Weyl module \(V_\infty^{\mathfrak{b}_0, 0}\). Hence we get an embedding

\[ V_1^{\mathfrak{b}_z, z} \hookrightarrow \text{Hom}_\mathbb{C}(V_0^{\mathfrak{b}_0, 0}, V_\infty^{\mathfrak{b}_0, 0} \otimes x^{\Delta_1} z^{\Delta_2} \mathbb{C}[x^{\pm 1}, z^{\pm 1}]), \]

where \(\Delta_1, \Delta_2\) are monodromy coefficients of the flat connection. We conclude that any \(w \in V_1^{\mathfrak{b}_z, z}\) can be looked upon as a certain generating function

\[ w(x, z) = x^{\Delta_1-n} z^{\Delta_2-l} \sum_{i,j \in \mathbb{Z}} w_{ij} x^{-i} z^{-j} \]

of a family of operators \(\{w_{ij} \subset \text{Hom}_\mathbb{C}(V_0^{\mathfrak{b}_0, 0}, V_\infty^{\mathfrak{b}_0, 0})\}\), where \((n, l)\) is a bidegree of \(w\) as an element of \(V_1^{\mathfrak{b}_z, z}\).

Lemma 5.2.1 Suppose \(v_1 \in V_1^{\mathfrak{b}_z, z}\) is the highest weight vector. Then

(i) \(v_1(x, z)\) is a generating function of a certain vertex operator as in 5.1;
(ii) any vertex operator is obtained in this way.

Proof is a direct and simple calculation using definitions, see also 4.5.3 formula (17).
Let us now relate correlation functions to horizontal sections of the bundle built on the generalized Weyl module $M^A$, $\Delta(M^A)^*$, see beginning of sect.5 for notations. Suppose that $M^A$ is the tensor product of “individual” generalized Weyl modules

$$\otimes_{i=1}^m V_{i}^{z_i, b_{x_i}}.$$ 

Consider all possible correlation functions

$$<v^*, v_m(x_m, z_m)\cdots v_1(x_1, z_1)v >,$$

where $v_i(x_i, z_i)$ is a generating function of a vertex operator related to the highest weight vector $v_i \in V_{i}^{z_i, b_{x_i}}$.

**Corollary 5.2.2** Let $M^A$ be as above. Over a suitable open contractible subset $U$ of $\mathbb{C}^m \times \mathbb{C}^m$, there is an isomorphism between the space of horizontal sections of the bundle $\Delta(M^A)^*$ and the space of correlation functions

$$<v^*, v_m(x_m, z_m)\cdots v_1(x_1, z_1)v >.$$ 

**Proof.** Intertwining properties of vertex operators imply a correlation function is a horizontal section of $\Delta(M^A)^*$ in a formal sense. This give a map in one direction. A map in the opposite direction in provided by quadratic degeneration, see Proposition 4.7.1. □

5.2.2

By Lemma 5.2.1 coinvariants recover vertex operators. In fact they give us much more: the collection of generating functions $w(x, z)$, $w \in V_{1}^{b_{x}, z}$ affords a kind of vertex operator algebra structure. We will not discuss the latter in detail (see [22]) and only explain how one can get explicit formulas for $w(x, z)$, $w \in V_{1}^{b_{x}, z}$ in terms of the vertex operator $v_1(x, z)$ related to the highest weight vector $v_1$.

For any $g \in \mathfrak{g}$ set $g(i) = g \otimes z^i \in \mathfrak{g}$. Define the current $g(z)$ to be $g(z) = \sum_{i \in \mathbb{Z}} g(i) z^{-1-i} \in \mathfrak{g} \otimes \mathbb{C}[[z^{\pm 1}]]$. Define $g(z)^{(l)}$ to be the $l$-th (formal) derivative of $g(z)$ with respect to $z$. For any $g(z)^{(l)}$ set

$$(g(z)^{(l)})_+ = \left(\frac{d}{d z}\right)^l \sum_{i=0}^{\infty} g_{-i+1} z^i, \ (g(z)^{(l)})_- = g(z)^{(l)} - (g(z)^{(l)})_+.$$ 

Observe that for any $w(x, z) \in \text{Hom}_{\mathbb{C}}(V_{0}^{b_{0}, 0}, V_{\infty}^{b_{0}, 0} \otimes x^{\Delta_1} z^{\Delta_2} \mathbb{C}[[x^{\pm 1}, z^{\pm 1}]])$ and any $g \in \mathfrak{g}$, the products $(g(z)^{(l)})_- w(x, z), w(x, z)(g(z)^{(l)})_+$ are also well-defined elements of $\text{Hom}_{\mathbb{C}}(V_{0}^{b_{0}, 0}, V_{\infty}^{b_{0}, 0} \otimes x^{\Delta_1} z^{\Delta_2} \mathbb{C}[[x^{\pm 1}, z^{\pm 1}]])$.

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Define for any $g \in \mathfrak{g}$, $w(x, z) \in \text{Hom}_{\mathbb{C}}(V_{b, 0}^{b_0, 0}, V_{\infty}^{b_0, 0} \otimes z^{\Delta_1} z^{\Delta_2} \mathbb{C}[x, z])$

\[ g(z)^{(k)}w(x, z) := (g(z)^{(l)})^{-w}w(x, z) + w(x, z)(g(z)^{(l)})_{1} \]  

(29)

**Lemma 5.2.3** Let $g \in \mathfrak{g}$, $w \in V_{1}^{b, n}, w(x, z)$ the corresponding element of $\text{Hom}_{\mathbb{C}}(V_{0}^{b_0, 0}, V_{\infty}^{b_0, 0} \otimes x^{\Delta_1} z^{\Delta_2} \mathbb{C}[x, z])$. Then

(i) $(g \cdot w)(x, z) = [g, w(x, z)];$

(ii) $(g(-l) \cdot w)(x, z) = (1/(l - 1)!): g(z)^{(l-1)} w(x, z); l > 0;$

**Proof** is a direct calculation of matrix elements of the operator $(g(-l) \cdot w)(x, z)$ based on the definition of the space of coinvariants. 

**5.2.3 Application: annihilating ideals and singular support of admissible representations.**

There is a well-known theorem of M. Duffo (see e.g. [7]), which establishes essentially a one-to-one correspondence between 2-sided primitive ideals of $\mathbf{U}(\mathfrak{g})$ (one actually has to quotient out a maximal ideal of the center) and the annihilating ideals of irreducible highest weight $\mathfrak{g}$-modules. There are various reasons for which this correspondence does not survive for $\mathfrak{g}$-modules. Ideas from conformal field theory provided a substitute – a remarkable construction of elements in a certain completion of $\hat{\mathbf{U}}(\hat{\mathfrak{g}})$ of $\mathbf{U}(\hat{\mathfrak{g}})$ which annihilate some classes of irreducible highest weight modules. This construction is well-known for integrable highest weight modules. Our results allow us to extend it to admissible representations over $\hat{\mathfrak{g}}_2$. As an application, we will finish proof of the theorem on singular support in (almost) one line.

Let $\text{Vac}$ be the vacuum representation at level $k$: $\text{Vac} = \text{Ind}_{\mathfrak{g}_2 \oplus \mathfrak{e} \oplus \mathfrak{c}^{\infty}} \mathbb{C}$. Consider an embedding

\[ \text{Vac} \to \text{Hom}(V; V \otimes \mathbb{C}[[z, z^{-1}]]) \],  

(30)

where $V$ is a generalized Weyl module. Observe that since $\text{Vac}$ is induced from the trivial $\mathfrak{g}$-module, there is no dependence on Borel subalgebra, and no $x$ in notations.

Suppose now $k$ becomes rational. Then $\text{Vac}$, $V$ become reducible. The map (30) comes from the invariant functional, and this functional pushes forward on the corresponding irreducible modules if and only if it vanishes on the singular vectors. When this is possible is the question addressed in sect.4.5.4, Theorem 4.5.5. The coinvariant survives if and only
if $V$ projects onto an admissible representation ($V_{ac}$ always does so). The tiny part of Theorem 4.5.5 we have used simply means that $V_{ac}$ is the identity of the fusion algebra.

On the other hand, if $S$ is the unique singular vector appearing in $V_{ac}$, when $k$ becomes rational, then the "field" $S(z)$ related to $S$ under (30) is zero. Therefore all its Fourier components act as 0 on the quotient of $V$. Of course for the same reason all elements of the submodule of $V_{ac}$ generated by $S$ will analogously give rise to a huge collection of operators annihilating admissible representations. Explicit formulas for these elements can in principle be written down using Lemma5.2.3.

**Example** If $k$ is a positive integer, then $S = (e \otimes z^{-1})^{k+1}$. Formulas of the previous section show that in this case $S(z) = e(z)^{k+1}$. The same is easy to do for an arbitrary affine Lie algebra to get $e_{\ell}(z)^{k+1}$, where $e_{\ell}$ is the highest root vector. This is the formula mentioned in the beginning of this section.

**Back to Proof of Theorem4.2.1** Consider elements $[g, S], g \in \mathfrak{g}$. Their symbols were calculated in the beginning of proof of Theorem4.2.1, and the set of their common zeros was shown to lie in the nilpotent cone.

Consider now $\lim_{z \to 0}[g, S](z)v, g \in \mathfrak{g}$ and identify it with an element of $U(\hat{\mathfrak{g}} \subset )$ (divide out $\nu!$), where $v$ is the highest weight vector of $V$. From formulas provided by Lemma5.2.3 it follows that the symbol of this element is essentially the same as that of $[g, S]$. (There will actually be one more term which is easy to kill using another singular vector.) Therefore common zeros of elements $\lim_{z \to 0}[g, S](z)v, g \in \mathfrak{g}$ also lie in the nilpotent cone. \(\square\)

### 5.3 Differential equations satisfied by correlation functions

We return to the setup of 5.1 and consider a correlation function

$$\Psi(x_1, \ldots, x_m, z_1, \ldots, z_m) = \langle v^* , Y_m(x_m, z_m) \cdots Y_2(x_2, z_2) Y_1(x_1, z_1) v \rangle,$$

coming from the product of vertex operators

$$Y_i : \mathcal{F}^C_{\lambda_i, \mu_i} \otimes V_{i-1/2} \to V_{i+1/2}, 1 \leq i \leq m.$$  

Using Lemma 5.2.1 we assume that there are generalized Weyl modules $V_i, 1 \leq i \leq m$ with highest weight vectors $v_i, 1 \leq i \leq m$ such that $Y_i(x, z) = v_i(x, z)$. An advantage of
this point of view is that for any collection of elements \( w_i \in V_i \), \( 1 \leq i \leq m \) we can consider the matrix element
\[
< v^*, w_m(x_m, z_m) \cdots w_2(x_2, z_2)w_1(x_1, z_1)v >.
\]

**Lemma 5.3.1** For any \( w_i \in V_i \), \( 1 \leq i \leq m \)
\[
< v^*, w_m(x_m, z_m) \cdots w_2(x_2, z_2)w_1(x_1, z_1)v > = D \cdot \Psi(x_1, \ldots, x_m, z_1, \ldots, z_m),
\]
where \( D \) is a differential operator in \( x \)'s with coefficients in rational functions in \( z \)'s.

**Proof.** Start with the function
\[
< v^*, v_m(x_m, z_m) \cdots v_{i+1}(x_{i+1}, z_{i+1})(g(-l)v_i)(x_i, z_i)v_{i-1}(x_{i-1}, z_{i-1}) \cdots v_1(x_1, z_1)v >, \ l > 0.
\]
By Lemma 5.2.3 (ii) it rewrites as
\[
< v^*, v_m(x_m, z_m) \cdots v_{i+1}(x_{i+1}, z_{i+1})(g(z)^{l-1})v_i(x_i, z_i)-v_i(x_i, z_i)g(z)^{l-1})v_{i-1}(x_{i-1}, z_{i-1}) \cdots v_1(x_1, z_1)v >, \ l > 0.
\]
Then commute all \( g_i \), \( i < 0 \) through to the right and all \( g_i \), \( i \geq 0 \) to the left in a standard way, c.f.[21] and use commutation relations (24,25,26). The case \( l = 0 \) is treated in a similar and simpler way using Lemma 5.2.3 (i). Further argue by induction using again Lemma 5.2.3. □

By definition each \( V_i \) is a quotient of a Verma module and therefore there are elements, singular vectors in the corresponding Verma module (see 2.2) \( S_i \in \hat{U}(\mathfrak{g}) \) such that \( S_iv_i = 0 \), \( 1 \leq i \leq m \). On the other hand, by Lemma 5.3.1 there are differential operators \( D_i \), \( 1 \leq i \leq m \) such that
\[
D_i < v^*, v_m(x_m, z_m) \cdots v_{i+1}(x_{i+1}, z_{i+1})(S_iv_i)(x_i, z_i)v_{i-1}(x_{i-1}, z_{i-1}) \cdots v_1(x_1, z_1)v >, \ 1 \leq i \leq m.
\]
We arrive to the following result.

**Lemma 5.3.2** The correlation function
\[
\Psi(x_1, \ldots, x_m, z_1, \ldots, z_m) = < v^*, v_m(x_m, z_m) \cdots v_{2}(x_2, z_2)v_1(x_1, z_1)v >
\]
satisfies the system of equations
\[
D_i\Psi(x_1, \ldots, x_m, z_1, \ldots, z_m) = 0, \ 1 \leq i \leq m.
\]
(31)
Observe that, although there are in general no explicit formulas for $D_i$, the fact that $[D_i, D_j] = 0$ is an obvious consequence of the definition.

We have obtained $m$ equations our function of $2m$ variables satisfies. The rest is, of course, the Knizhnik-Zamolodchikov equations. Let us write them down explicitly. Recall that we can look upon $\Psi(x_1, \ldots, x_m, z_1, \ldots, z_m)$ as a function of $z_1, \ldots, z_m$ with coefficients in a completed tensor product of $m\ g$-modules. (The variables $x_1 \ldots x_m$ are responsible for that, see (24,25,26).) For any $A = \sum_s a_s \otimes b_s \in g \otimes g$ denote by $A_{ij}, 1 \leq i, j \leq m$ an operator acting on the $m$-fold tensor product of $g$-modules by the formula

$$A_{ij} \cdot w_1 \otimes \cdots w_m = \sum_s w_1 \otimes a_s w_i \otimes \cdots b_j w_j \otimes \cdots w_m.$$

The formula (27) implies that $A_{ij}$ is a differential operator in $x_i, x_j$. Set $\Omega = ef + f e + h^2/2$.

**Lemma 5.3.3 ([31])**

*The correlation function $\Psi = \Psi(x_1, \ldots, x_m, z_1, \ldots, z_m)$ satisfies the system of Knizhnik-Zamolodchikov equations

$$\frac{(k + 2)}{2} \frac{\partial}{\partial z_i} \Psi = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Psi, \ 1 \leq i \leq m. \quad (32)$$

There is no need to prove this lemma here as one can repeat word for word the known proofs. However we point out that if one considers a highest weight module $V$ as a module over the semi-direct product of $\hat{g}$ and the Virasoro algebra $Vir$ then $V$ is annihilated by the element $d/dz - L_{-1}$, where $L_{-1}$ is one of the Sugawara elements. One then shows that the singular vectors $(d/dz - L_{-1})v$, where $v$ is a highest weight vector of $V$, give rise to the equations (32) in exactly the same way the singular vectors $S_i$ gave rise to the equations (31). An immediate consequence of this proof is that the system of equations (32,31) is consistent.

### 5.4 Screening operators and integral representations of correlation functions

#### 5.4.1

Suppose a function $\Psi_{v,i} = \Psi(x_1, \ldots, x_m, z_1, \ldots, z_m)$ is the matrix element of the product of vertex operators

$$\Psi_{v,i} = \langle v^* \pi \circ Y(x_m, z_m) \cdots Y_1(x_1, z_1)v_c \rangle,$$
satisfying the same conditions as the expression in (28), see 5.1, except that instead of assuming that \( V_{n+1} \) is a generalized Weyl module we assume that \( V_{n+1} \) is a contragredient Verma module, see 2.2. (Why “old” will become clear in a moment.) It is easy to see that

\[ \Psi_{\text{old}} = \Psi(x_1, \ldots, x_n, z_1, \ldots, z_n) \]

satisfies the same system of equations (31,32). Suppose in addition that there is a projection \( \pi : V_{n+1} \to W \) onto another contragredient Verma module \( W \). Denoting by \( w^* \) an element dual to the highest weight vector \( w \in W \) one can consider the matrix element

\[ \Psi_{\text{new}} = \langle w^*, \pi \circ Y(x_m, z_m) \cdots Y_1(x_1, z_1)v_0 \rangle. \]

We again observe that \( \Psi_{\text{old}} \) is a solution to the same system (31,32). This new solution can be calculated as follows.

There arises the dual map \( \pi^* : W^* \to V_{n+1}^* \) and by definition there is an element \( S \) of \( U(\hat{\mathfrak{g}}_{\geq}) \) such that \( \pi^*(w^*) = Sv^* \). We now take the definition of \( \Psi_{\text{new}} \), replace in it \( \pi^*(w^*) \) with \( Sv^* \) and get

\[ \Psi_{\text{new}} = \langle S \cdot v^*, \pi \circ Y(x_m, z_m) \cdots Y_1(x_1, z_1)v_0 \rangle. \]

Then we commute \( S \) through to the right. The intertwining properties of vertex operators tell us that

\[ \Psi_{\text{new}} = S^t \cdot \Psi_{\text{old}}, \]

where \( ^t \) signifies the canonical antiinvolution on a Lie algebra \( (g_1 g_2 \cdots g_n \to g_n g_{n-1} \cdots g_1) \) and the action is determined by the following condition: if \( g \in \mathfrak{g} \) then

\[ (g \otimes z^n) \cdot \Psi_{\text{old}} = \sum_{i=1}^{m} D_g(x_i) z_i^n \Psi_{\text{old}}, \]

see (27).

We intend to use (34) in the case when \( \pi \) and therefore \( S \) do not exist!

### 5.4.2 Screening operators

Let \( V_{\lambda, k} \) be a highest weight module and \( v \in V_{\lambda, k} \) a highest weight vector. If the obvious integrality conditions are satisfied then the vectors \( f^{\lambda + 1} v, (e \otimes z^{-1})^{k-\lambda + 1} v \) are singular
and give rise to embeddings of the type $W \hookrightarrow V_{\lambda_{i,k}}$. Now take 3 highest weight modules $V_{\lambda_{i,k}}$, $i = 0, 1, \infty$ attach them to 3 point in $\mathbb{CP}^1$ and consider the space of coinvariants

$$(\otimes_{i=0,1,\infty} V_{\lambda_{i,k}})^g(\mathbb{CP}^1_{\{0,1,\infty\}}).$$

Of course an embedding $W \hookrightarrow V_{\lambda_{i,k}}$ gives rise to a map

$$(W^{b_{i \infty}} \otimes_{i=0,1,\infty} V_{\lambda_{i,k}})^g(\mathbb{CP}^1_{\{0,1,\infty\}}) \hookrightarrow (\otimes_{i=0,1,\infty} V_{\lambda_{i,k}})^g(\mathbb{CP}^1_{\{0,1,\infty\}}).$$

It is remarkable that even if the embedding $W \hookrightarrow V_{\lambda_{i,k}}$ does not exist the last map still does. In the language of vertex operators this phenomenon was explained in great detail in [20].

Therefore with each of the formal singular vectors $f^\lambda v$ or $(e \otimes z^{-1})^k - \lambda v$ we have associated an operator acting on coinvariants. Call these operators screenings and denote them $R_1$ and $R_0$ respectively.

Let us calculate the action of the screenings explicitly. By definition

$R_j(\otimes_{i=0,1,\infty} V_{\lambda_{i,k}})^g(\mathbb{CP}^1_{\{0,1,\infty\}})$ only depends on $V_{\lambda_{i,k}}$ so we will be simply writing $R_j(V_{\lambda_{i,k}})$.

Now formulas for the related singular vectors $(f^\lambda v$, $(e \otimes z^{-1})^k - \lambda v$) and a very simple calculation using the formulas (7,8) give the following result:

$$R_1((V_m^0, V_n)) = (V_{m-1}^1, V_n)$$

$$R_1((V_m^1, V_n)) = (V_{m+1}^0, V_n)$$

$$R_0((V_m^0, V_n)) = (V_{m+1}^1, V_n)$$

$$R_0((V_m^1, V_n)) = (V_{m-1}^0, V_n)$$

Suppose we are given 2 generalized Weyl modules and a vertex operator acting between them. Suppose in addition that this vertex operator is related to a highest weight in the third generalized Weyl module, say $(V_m^c, V_n)$. Theorem 4.5.1 tells us that given such a vertex operator our screenings give us all the others of the type $(V_m^c, V_n)$ – we cannot only change the value of $n$. But then there is the standard screening operator $- S$ – which takes care of $n$, see e.g. [14]. So these three $- R_1, R_2, S$ – provide us with all vertex operators. This has an important application to the calculation of correlation functions.

Start with a simple correlation function given by the product of vertex operators, each of which is characterized by the condition $m = 0$. Then applying $S$ an appropriate number of times one gets all vertex operators and, hence, all correlation functions in spirit of Varchenko-Schechtman, see [2].

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Now take a Varchenko-Schechtman correlation function $\Psi_{\text{old}}$. It comes from a product of vertex operators:

$$\Psi_{\text{old}} = \langle v^*, Y(x_m, z_m) \cdots Y_1(x_1, z_1)v_0 \rangle,$$

$$Y_i : \mathcal{F}_{\lambda_{\mu_i}}^\ast \otimes V_{i-1/2} \to V_{i+1/2}, \quad 1 \leq i \leq m.$$  

Let $W_i,$ $0 \leq i \leq m$, be words on 2 letters $R_1$ and $R_2$. Replacing $V_{i+1/2}$ with $W_i(V_{i+1/2})$ we get a new correlation function $\Psi_{\text{new}}$. Doing this with all $\Psi_{\text{old}}$ and sufficiently many $W_i,$ $0 \leq i \leq m$ we get all solutions to (31,32). In principle all these solutions can be written down explicitly. It is especially simple to do so in the case when we keep $V_{i+1/2},$ $0 \leq i \leq m-1$, and only change $V_{m+1/2}$.

So assume that $\Psi_{\text{old}}$ is as above and replace $V_{m+1/2}$ with $R_j(V_{m+1/2}),$ $j = 0,1$. Then by (33) one is to expect that

$$\Psi_{\text{new}} = \langle X^\alpha \cdot v^*, \circ Y(x_m, z_m) \cdots Y_1(x_1, z_1)v_0 \rangle,$$

where $X$ is either $e$ or $f \otimes z$ if $j = 1$ or 0 resp., and $\alpha$ is either $\lambda + 2$ or $k - \lambda + 2$ resp., where $\lambda$ is the highest weight of $V_{m+1/2}$.

Of course if $\alpha$ is not a nonnegative integer then the last formula does not make much sense. Nevertheless using it and (34) as a motivation we arrive to

$$\Psi_{\text{new}} = X^\alpha \Psi_{\text{old}}.$$  

Now the right-hand side of the last equality does make sense: $X$ is a first order differential operator, see 5.4.1, therefore we can set in a rather straightforward manner

$$X^\alpha \Psi_{\text{old}} = \int t^{-\alpha-1} \{ \exp(-Xt)\Psi_{\text{old}} \} dt$$

and get a nice integral operator, for details see [20].

This procedure can be easily iterated to provide the functions

$$\int \prod_{i=1}^n t_i^{-\alpha_i-1} \{ \exp(-X_1t_1) \exp(-X_2t_2) \cdots \exp(-X_nt_n)\Psi_{\text{old}} \} \prod_{i=1}^n dt_i, \quad (39)$$

where $X_1, X_2, \ldots$ is either $e, f \otimes z, e, \ldots$ or $f \otimes z, e, f \otimes z, \ldots$.

**Lemma 5.4.1** Functions (39) are solutions to (31,32).
Proof is same as the proof of the analogous statement in [19]. In fact it is an easy exercise to make the heuristic arguments which have led us to the formula (39) into a precise proof. □

Integrating functions (39) with respect to $x'$s (or doing something similar but more esoteric) one is supposed to get the Dotsenko-Fateev correlation functions for the Virasoro algebra. It would be interesting to do this explicitly and compare the result with the calculations in [24].

**Conjecture 5.4.2**  
(i) Formulas (39) provide all solutions to the system (31,32).  
(ii) If the level $k$ is rational, then there arises a subbundle of the bundle in question, the one with fiber $((L^A)^*)^0 \mathbb{C}^{1,A}$, where $L^A$ is the corresponding admissible representation. We conjecture that in this case formulas (39) actually give horizontal sections of the latter bundle.

**References**


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