Structure of $\sigma$–stratified Modules for Finite-dimensional Lie Algebras
II. BGG-resolution in the Simply–laced Case

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Structure of $\alpha$-stratified modules for finite-dimensional Lie algebras II. BGG-resolution in the simply-laced case

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Abstract

We construct the strong BGG-resolution for irreducible $\alpha$-stratified modules over finite-dimensional simple Lie algebras with simply-laced diagrams.

1 Introduction

This paper is a sequel of [1] where the submodule structure of $\alpha$-stratified (i.e. torsion free with respect to the subalgebra corresponding to a root $\alpha$) generalized Verma modules was studied. The results obtained there generalize the classical theorem of Bernstein-Gelfand-Gelfand on Verma modules inclusions. The crucial role in the study is played by the generalized Weyl group $W_\alpha$ that acts on the space of parameters of generalized Verma modules.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over the complex with a simply-laced Coxeter-Dynkin diagram (i.e. there is no multiple arrows). In the present paper for any such algebra we construct a strong BGG-resolution for $\alpha$-stratified irreducible modules in the spirit of [4, 7].

The structure of the paper is the following. In the section 2 we collect the notation and preliminary results. A weak generalized BGG-resolution is constructed in section 3. Here we follow closely [4]. Section 4 contains an extension lemma for $\alpha$-stratified modules which generalizes a well-known result of Rocha-Caridi for Verma modules ([7]). Our proof is analogous to the one of Humphreys ([8]) for Verma modules. In section 5 we study the maximal submodule of the generalized Verma module and construct a strong generalized BGG-resolution for $\alpha$-stratified irreducible modules in section 6. Finally, in section 7 we give a character formulae for certain $\alpha$-stratified irreducible modules.

2 Notation and preliminary results

Let $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ denotes all integers, $\mathbb{N}$ denotes all positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
Let $\pi$ be a basis of $\Delta$ containing $\alpha$, $\Delta_\pm = \Delta_\pm(\pi)$ be the set of positive (negative) roots with respect to $\pi$. For any $S \subset \pi$ let $\Delta_\pm(S)$ be a closed subset in $\Delta_\pm$ generated by $S$. Also let $\rho = \frac{1}{2} \sum_{\gamma \in \Delta_+} \gamma$. For $\lambda, \mu \in \mathfrak{h}^*$ we will say that $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\beta \in \pi} k_\beta \beta$, $k_\beta \in \mathbb{Z}_+$. Further $(\cdot, \cdot)$ will denote the standard form on $\mathfrak{h}^*$. If $\beta \in \Delta_+$ then $s_\beta \in W$ will denote a corresponding reflection in $\mathfrak{h}^*$: $s_\beta(\lambda) = \lambda - \frac{2(\lambda, \beta)}{(\beta, \beta)} \beta$.

Fix a basis $\{H_\beta, \beta \in \pi\}$ of $\mathfrak{h}$ normalized by the condition $\beta(H_\beta) = 2$ and a non-zero element $X_\gamma$ in each subspace $\mathfrak{g}_\gamma$, $\gamma \in \Delta$ such that $[X_\beta, X_{-\beta}] = H_\beta$, $\beta \in \pi$.

Denote $\mathfrak{n}_\pm = \sum_{\gamma \in \Delta_+} \mathfrak{g}_{\pm \gamma}$, $\mathfrak{n}_\pm^\circ = \sum_{\gamma \in \Delta_+ \setminus \{\alpha\}} \mathfrak{g}_{\pm \gamma}$, $\mathfrak{h}^\circ = \{h \in \mathfrak{h}|\alpha(h) = 0\}$. Then we have

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ = \mathfrak{g}_- + \mathfrak{n}_-^\circ + \mathfrak{h} + \mathfrak{n}_+^\circ$$

Where $\mathfrak{g}_-^\circ$ is generated by $\mathfrak{g}_{\pm \alpha}$. Also let $\mathfrak{n}_\alpha = \mathfrak{g}_-^\circ \cap \mathfrak{n}$ and thus $\mathfrak{g}^\circ = \mathfrak{g}_\alpha \oplus \mathfrak{n}_\alpha + \mathfrak{g}_{-\alpha}$. For $m \in \mathbb{Z}_+$ denote by $U(\mathfrak{g})^{(m)}$ the subspace in $U(\mathfrak{g})$ spanned by the elements of degree $m$ (with respect to the fixed PBW-basis above).

For a Lie algebra $\mathfrak{g}$ we will denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$.

Consider a linear space $\Omega = \mathfrak{n}^* \times \mathbb{C}$. For $(\lambda, p)$ and $(\mu, q)$ in $\Omega$ we say that $(\lambda, p) > (\mu, q)$ if $\lambda - \mu = \sum_{\beta \in \pi \setminus \{\alpha\}} n_\beta \beta$, $n_\beta \in \mathbb{Z}_+$ and $\lambda \neq \mu$.

Let $r \in \mathbb{C}$. Consider a linear space $B_r = \sum_{\beta \in \pi \setminus \{\alpha\}} \mathbb{C}_\beta + r \alpha$ with a fixed point $r \alpha$, a $\mathbb{Z}$-module $B_r = B_r \oplus \mathbb{Z} \alpha$ and let $\Omega_r = B_r \times \mathbb{C}$, $\Omega = B_r \times \mathbb{C}$.

In [1] we introduced the generalized Weyl group $W_\alpha$ acting on the space $\Omega_r$ in the following way.

Consider a partition of $\pi$: $\pi = \pi_1 \cup \pi_2$ where $\pi_1 = \{\gamma \in \pi | \alpha + \gamma \in \Delta, \}$, $\pi_2 = \{\gamma \in \pi | \alpha + \gamma \not\in \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi_1$ denote

$$n_{\beta}^\pm (\lambda, p) = \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p)$$

and define $(\lambda_\beta, p_\beta) \in \Omega$, where $\lambda_\beta = \lambda - n_{\beta}^- (\lambda, p) \beta$, $p_\beta = n_{\beta}^+ (\lambda, p)$.

For each $\beta \in \pi$ consider $\ell_\beta \in GL(\Omega)$ such that

$$\ell_\beta(\lambda, p) = \begin{cases} 
(\lambda, -p), & \beta = \alpha \\
(s_\beta \lambda, p), & \beta \in \pi_2 \setminus \{\alpha\} \\
(\lambda_\beta, p_\beta), & \beta \in \pi_1.
\end{cases}$$

Then $W_\alpha = \langle \ell_\beta, \beta \in \pi \rangle$.

It is easy to see that $W_\alpha$ is isomorphic to the Weyl group $W$. Moreover, there exists a root system $\Delta_{\alpha, r}$ in $\Omega_r$ with respect to which $W_\alpha$ is the Weyl group ([1]). We denote
by \( \sigma_\beta \) the reflection in \( \Omega_r \) corresponding to a root \( \beta \in \Delta_{\alpha,r} \). Also let \((\cdot, \cdot)_r\) denotes a corresponding non-degenerated form on \( \Omega_r \) and \( \zeta = \zeta_{\alpha,r}: \Delta \to \Delta_{\alpha,r} \) be a natural bijection.

Let \( p_r, i = 1, 2 \) be a natural projection on the \( i \)-th component of \( \Omega_r \).

For a \( \mathfrak{g}\)-module \( V \) with a Jordan-Hölder series let \( \mathcal{J}H(V) \) denotes the set of all irreducible subquotients of \( V \). A \( \mathfrak{g}\)-module \( V \) is called weight if

\[
V = \bigoplus_{\lambda \in \mathcal{J}^*} V_{\lambda}
\]

where \( V_{\lambda} = \{ v \in V | hv = \lambda(h)v \text{ for all } h \in \mathcal{J} \} \). If \( V_{\lambda} \neq 0 \) then \( \lambda \) is called a weight of \( V \). Denote by \( \text{supp} V \) the set of all weights of \( V \). A weight \( \lambda \) is called highest weight if \( V_{\lambda + \beta} = 0 \) for all \( \beta \in \Delta^+ \). A weight \( \mathfrak{g}\)-module \( V \) is called \( \alpha \)-stratified if \( X_\alpha \) and \( X_{-\alpha} \) act injectively on \( V \).

Let \( V \) be a weight \( \mathfrak{g}\)-module. A non-zero element \( v \in V \) is called \( \alpha \)-primitive (with respect to \( \mathfrak{g} \)) if \( v \in V_\lambda \) for some \( \lambda \in \mathcal{J}^* \) and \( \mathfrak{m}^\lambda \alpha v = 0 \).

It is known that \( c = (H_\alpha + 1)^2 + 4X_\alpha X_\alpha \) generates \( Z(\mathfrak{g}^\alpha) \). Let \( a, b \in \mathbb{C} \). Any such pair defines a unique indecomposable weight \( \mathfrak{g}^\alpha \)-module \( N(a,b) \) on which \( X_\alpha \) acts injectively and where \( a \) is an eigenvalue of \( H_\alpha \) and \( b \) is an eigenvalue of \( c \). The module \( N(a,b) \) has a \( \mathbb{Z} \)-basis \( \{ v_i \} \) such that \( X_\alpha v_i = v_{i+1}, H_\alpha v_i = (a + 2i)v_i \) and \( X_\alpha v_i = \frac{1}{4}(b - (a + 2i + 1)^2)v_i \).

One can easily check (see [1, Lemma 2.2]) that the module \( N(a,b) \) is torsion free if and only if \( b \neq (a + 2\ell + 1)^2 \) for all \( \ell \in \mathbb{Z} \).

Set \( \Omega = \{(\lambda, p) \in \Omega | p \neq \pm (\lambda(H_\alpha) + 2\ell) \text{ for all } \ell \in \mathbb{Z} \}, \Omega_\alpha = \Omega_\alpha \cap \Omega, \Omega_\beta = \Omega_\beta \cap \Omega \alpha \). Hence, if \((\lambda, p) \in \Omega\) then \( N((\lambda - \rho)(H_\alpha), p^2) \) is irreducible and torsion free.

Since \( \mathcal{J} = \mathcal{J}_\alpha \oplus \mathcal{J}^\alpha \), any element \( \lambda \in \mathcal{J}^\alpha \) can be written uniquely as \( \lambda = \lambda_\alpha + \lambda^\alpha \) where \( \lambda_\alpha \in \mathcal{J}^\alpha_\alpha \) and \( \lambda^\alpha \in (\mathcal{J}^\alpha)_\alpha^* \). Let \( a, b \in \mathbb{C} \) and \( \lambda \in \mathcal{J}^\alpha \) such that \((\lambda - \rho)(H_\alpha) = (\lambda_\alpha - \rho)(H_\alpha) = a \). Define a \( \mathcal{J}^\alpha \)-module structure on \( N(a,b) \) by letting \( hv = \lambda^\alpha(h)v \) for any \( h \in \mathcal{J}^\alpha \) and any \( v \in N(a,b) \). Thus \( N(a,b) \) becomes a \( \mathcal{J}^\alpha + \mathcal{J}^\alpha \)-module. Moreover, we can consider \( N(a,b) \) as \( D = \mathcal{J} + \mathcal{J}^\alpha + \mathcal{J}^\alpha + \mathfrak{m}^\alpha \)-module with a trivial action of \( \mathfrak{m}^\alpha \).

The generalized Verma module associated with \( \alpha, \lambda, b \) is defined as follows:

\[
M_\alpha(\lambda, b) = U(\mathfrak{g}) \otimes_{U(D)} N(a,b).
\]

Set \( M(\lambda, b) = M_\alpha(\lambda, b) \).

It will be more convenient to use a slightly different parametrization of generalized Verma modules replacing \( M(\lambda, b) \) by \( M(\lambda, p) \) where \( p^2 = b \). Thus we always have \( M(\lambda, p) = M(\lambda, -p) \).

Note that module \( M(\lambda, p) \) has a unique maximal submodule and it is \( \alpha \)-stratified if and only if \((\lambda, p) \in \Omega^\alpha \).

It follows from [2, Corollary 1.11] that module \( M(\lambda, p) \) admits a central character \( \theta_{(\lambda,p)} \in \mathcal{Z}(\mathfrak{g}) \), i.e. \( zv = \theta_{(\lambda,p)}(z)v \) for any \( z \in \mathcal{Z}(\mathfrak{g}) \) and \( v \in M(\lambda, p) \).

Denote by \( L(\lambda, p) \) the unique irreducible quotient of \( M(\lambda, p) \).

**Lemma 1.** \( L(\lambda, p) \simeq L(\lambda + k\alpha, p) \) for all \( k \in \mathbb{Z} \).
The following order on $\Omega_r$ was introduced in [1]: Let $(\lambda, p), (\mu, q) \in \Omega_r$ and $\beta \in \Delta_{\alpha, r}$. We will write $(\lambda, p) \rightarrow (\mu, q)$ if $(\mu, q) = \sigma_\beta(\lambda, p)$ and $(\beta, (\lambda, p)), \in \mathbb{N}$ for $\beta \neq \zeta(\alpha)$. Then $(\mu, q) \ll (\lambda, p)$ will describe the fact that there exists a sequence $\beta_1, \beta_2, \ldots, \beta_k$ in $\Delta_{\alpha, r}$ such that $(\mu, q) \rightarrow \beta_1 \rightarrow \sigma_{\beta_1}(\mu, q) \rightarrow \beta_2 \rightarrow \sigma_{\beta_2}(\mu, q) \rightarrow \beta_3 \rightarrow \ldots \rightarrow \sigma_{\beta_k}(\mu, q) \rightarrow (\lambda, p)$.

The main result of [1, Theorem 7.6] is the following theorem which describes the structure of $\alpha$-stratified generalized Verma module with respect to the order on $\Omega_r$.

**Theorem 1.** Let $(\lambda, p)$ and $(\mu, q) \in \Omega_r^\times$. The following statements are equivalent:

1. $M(\mu, q) \subseteq M(\lambda, p)$;
2. $L(\mu, q) \in \mathcal{J}H(M(\lambda, p))$;
3. There exists $k \in \mathbb{Z}$ such that $(\mu + k\alpha, q) \ll (\lambda, p)$.

Let $P^{++} = \{(\lambda, p) \in \Omega_r^\times | w(\lambda, p) \ll (\lambda, p) \text{ for all } w \in W_\alpha\}$.

For $\beta \in \pi$ denote by $\Delta_\beta$ a root subsystem of rank 2 generated by $\alpha$ and $\beta$.

In this paper we discuss the construction of analogues of the weak and the strong BGG-resolutions for irreducible modules $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

### 3 Cohomological part of the weak BGG-resolution

Let $P = \Delta_+(\pi \setminus \{\alpha\})$ and let $\mathfrak{P}$ be a subalgebra of $\mathfrak{S}$ generated by all root subspaces

$$\mathfrak{S}_{-\beta}, \beta \in P.$$

An element $(\lambda, p)$ will be called minimal if

$$pr_1((\lambda, p) - \sigma_\beta(\lambda, p)) = \beta$$

holds for every $\beta \in \pi \setminus \alpha$. In this section we fix a minimal element $(\lambda, p)$.

Consider the subalgebra $\mathfrak{P}$ as a module over a subalgebra $\mathfrak{a} = \mathfrak{N}_+^\times + \mathfrak{H}$ under the following action:

$$h \cdot a = [h, a] + \lambda(h)(a)$$

for any $h \in \mathfrak{H}$ and $a \in \mathfrak{P}$, and

$$b \cdot a = \begin{cases} [b, a], & [b, a] \in \mathfrak{P}; \\ 0, & [b, a] \notin \mathfrak{P} \end{cases}$$

for all $b \in \mathfrak{N}^\circ$ and $a \in \mathfrak{P}$. Clearly, this action can be naturally extended to the action on the external powers $\bigwedge_k \mathfrak{P}$ for all $k \in \mathbb{N}$.
Let \( \varepsilon \) be a unique eigenvalue on \( M(\lambda, p) \) of a quadratic Casimir operator

\[
C = H + \sum_{\alpha \in \Delta_k} X_{-\alpha} X_{\alpha},
\]

where \( H \) is a certain fixed element in \( \mathfrak{g} \). Note that this eigenvalue is determined uniquely by \((\lambda, p)\) via generalized Harish-Chandra homomorphism \([3]\).

Define \( U_z = U(\mathfrak{g})/(C - \varepsilon) \) and consider the following \( \mathfrak{g} \)-modules:

\[
D_k = U_z \otimes_{U(\mathfrak{g})} \bigwedge^k \mathfrak{g},
\]

where \( k \in \mathbb{Z}_+ \).

Following \([4]\), for \( k \in \mathbb{N} \) define the homomorphisms \( d_k : D_k \to D_{k-1} \) as follows

\[
d_k(X \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_k) = \sum_{i=1}^k (-1)^{i+1} X X_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k + \sum_{1 \leq i < j \leq k} (-1)^{i+j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k.
\]

Since \( d_k \circ d_{k+1} = 0 \) we immediately obtain that the sequence

\[
0 \leftarrow D_0 / \text{Im} \eta \leftarrow D_0 \xleftarrow{\eta} D_1 \xleftarrow{d_1} D_2 \xleftarrow{d_2} \cdots
\]

is a complex. Here \( \eta \) is a natural projection. We will denote this complex by \( V_\alpha(\lambda, \varepsilon) \).

**Theorem 2.** The complex \( V_\alpha(\lambda, \varepsilon) \) is exact.

**Proof.** The algebra \( U_z \) inherits the natural gradation on \( U(\mathfrak{g}) \) by the degree of the monomials. Using that we can define a gradation on \( D_k \). For \( l \geq k \) let \( D^{(l)}_k \) be a subspace spanned by the elements \( x \otimes y \) where \( x \) is an element in \( U_z \) of degree less or equal \( l - k \) and \( y \in \bigwedge^k \mathfrak{g} \). It is clear that \( d_k(D^{(l)}_k) \subset D^{(l-1)}_{k-1} \) and thus \( d_k \) induces a homomorphism

\[
d^{(l)}_k : D^{(l)}_k / D^{(l-1)}_{k-1} \to D^{(l)}_{k-1} / D^{(l-1)}_{k-1}.
\]

Also set \( M^{(l)} = D^{(l)}_0 / \text{Im} d^{(l)}_1 \) and let \( \eta^{(l)} \) be a corresponding induced homomorphism.

It is sufficient to show for every \( l \) the exactness of the complex

\[
0 \leftarrow M^{(l)} \xleftarrow{\eta^{(l)}} D^{(l)}_0 \xleftarrow{d^{(l)}_1} D^{(l)}_1 \xleftarrow{d^{(l)}_2} D^{(l)}_2 \xleftarrow{d^{(l)}_3} \cdots
\]

with \( \dot{D}^{(l)}_k = D^{(l)}_k / D^{(l-1)}_k \).
By the PBW theorem for every $k \in \mathbb{Z}_+$ one can write:

$$D_k = \left( U(\mathfrak{m}_-) \otimes \wedge^k \mathfrak{p} \right) \oplus \left( \sum_{m \geq 1} X^m \otimes U(\mathfrak{m}_-) \otimes \wedge^k \mathfrak{p} \right)$$

and hence

$$\hat{D}_k^{(1)} \simeq \left( U(\mathfrak{m}_-)^{(l-k)} \otimes \wedge^k \mathfrak{p} \right) \oplus \left( \sum_{m=1}^{l-k} X^m \otimes U(\mathfrak{m}_-)^{(l-k-m)} \otimes \wedge^k \mathfrak{p} \right).$$

We will denote by $s_\alpha \mathfrak{m}_-$ a subalgebra generated by $\mathfrak{m}_-$ and $X_\alpha$. Let $\mathfrak{m}_+^\alpha$ ($s_\alpha \mathfrak{m}_+^\alpha$ resp.) be a subalgebra generated by $X_\beta, \beta \in \Delta_+, \beta \notin \Delta_+ (\pi \setminus \{\alpha\})(\beta \in s_\alpha \Delta_+, \beta \notin s_\alpha \Delta_+ (\pi \setminus \{\alpha\})$ resp.) and let $S_j(\mathfrak{p})$ be a set of all homogeneous elements of degree $j$ in the symmetric algebra of $\mathfrak{p}$. Then

$$\hat{D}_k^{(1)} \simeq \left( \sum_{j=0}^{l-k} U(\mathfrak{m}_+^\alpha)^{(l-j-k)} S_j(\mathfrak{p}) \otimes \wedge^k \mathfrak{p} \right) \oplus \left( \sum_{j=0}^{l-k} U(s_\alpha \mathfrak{m}_+^\alpha)^{(l-j-k)} S_j(\mathfrak{p}) \otimes \wedge^k \mathfrak{p} \right).$$

For any homogeneous element $u \in U(\mathfrak{m}_+^\alpha)$ ($u \in U(s_\alpha \mathfrak{m}_+^\alpha)$ resp.) of degree $l - j - k$ we have that $d_k^{(1)}(uS_j(\mathfrak{p}) \otimes \wedge^k \mathfrak{p}) \subset uS_{j+1}(\mathfrak{p}) \otimes \wedge^{k-1} \mathfrak{p}$. Therefore the element $u$ generates a complex which is in fact the Koszul complex ([6]) and hence is exact. Using the PBW theorem we conclude that the complex (1) decomposes into a direct sum of exact complexes and therefore is exact. The theorem is proved. \hfill \Box

For a weight $\mathfrak{g}$-module $V$ consider a formal character

$$\text{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu.$$

Corollary 1.

$$\text{ch} D_0/\text{Im} d_1 = \sum_{i \geq 1} (-1)^{i+1} \text{ch} D_i.$$

4 Extension lemma

In this section we prove an analogue of the Extension lemma ([7], [8]) for $\alpha$-stratified generalized Verma modules.

Recall that $\alpha$-stratified generalized Verma modules are the important objects in the category $\mathcal{O}^\alpha$ which was studied in [2], [5]. This category has properties similar to those of the classical category $\mathcal{O}$. It was shown, in particular, that $\mathcal{O}^\alpha$ has enough projective objects. Let $P(\lambda, p)$ be the projective cover of $L(\lambda, p)$. 
Theorem 3. Let \((\lambda, p), (\mu, q) \in \Omega_c^+\). If
\[
\text{Ext}_{\mathcal{O}^\circ}(M(\mu, q), M(\lambda, p)) \neq 0
\]
then \((\mu, q) \ll (\lambda, p)\).

Proof. The proof is based on the properties of the category \(\mathcal{O}^\circ\) ([5]) and is analogous to the proof of the extension lemma in [8]. □

Consider a subgroup \(W_\alpha^+ \subset W_\alpha\) generated by all \(l_\beta, \beta \in \pi \setminus \{\alpha\}\). Since \(W_\alpha^+\) is a reflection group we have a well-defined notion of the length \(l(w)\) for any \(w \in W_\alpha^+\).

Corollary 2. For \((\lambda, p) \in \mathcal{P}++\) and \(w_1, w_2 \in W_\alpha^+\) with \(l(w_1) = l(w_2)\) holds
\[
\text{Ext}_{\mathcal{O}^\circ}(M(w_1(\lambda, p)), M(w_2(\lambda, p))) = 0.
\]

5 The structure of the maximal submodule of \(M(\lambda, p)\)

The main result of this section is the following

Theorem 4. The module \(D_0/\text{Im} \, d_1\) is irreducible.

Corollary 3. If \((\lambda, p) \in \mathcal{P}++\) and \(\mathcal{N}\) is the maximal submodule of \(M(\lambda, p)\) then
\[
\mathcal{N} = \sum_{\gamma \in \pi \setminus \{\alpha\}} M(\sigma_\gamma(\lambda, p)).
\]

Proof. Follows immediately from theorem 2 and theorem 4. □

To prove the theorem 4 we will need several lemmas.

Let \(K = \Delta_-(\pi) \setminus \mathcal{P}\) and \(K(\mathfrak{g})\) be a subalgebra generated by \(X_\beta, \beta \in K\).

Lemma 2. Let \((\mu, q) \in \Omega_c^+\). If \(\beta \in K\) and \((\beta, \alpha) \neq 0\) then \(X_\beta\) acts injectively on \(L(\mu, q)\).

Proof. Suppose that there exists a non-zero \(v \in L(\mu, q)\) such that \(X_\beta v = 0\). Since \((\alpha, \beta) \neq 0\) then either \(\alpha + \beta \not\in \Delta\) or \(\alpha - \beta \not\in \Delta\). Thus, either \(X_\beta X_\alpha v = 0\) or \(X_\beta X_{-\alpha} v = 0\). Viewing \(\alpha\)-stratified module \(L(\mu, q)\) as a module over Lie algebra \(\mathfrak{g} = X_\beta, X_{-\beta} \simeq \mathfrak{sl}(2, \mathbb{C})\) and using the fact that \(L(\mu, q)\) is \(X_{-\beta}\)-finite we obtain that \(L(\mu, q)\) contains irreducible finite-dimensional \(\mathfrak{sl}(2, \mathbb{C})\)-submodules of the same dimension and with different highest weights which is impossible. Lemma is proved. □

Lemma 3. Let \((\mu, q) \in \Omega_c^+\) and \(0 \neq v \in M(\mu, q)_{\mu-%}\). Then for \(\beta \in K\) and \(k \geq 1\) an element \(X_\beta^k v\) is not \(\alpha\)-primitive.
Proof. If \((\alpha, \beta) \neq 0\) then the statement follows from lemma 2.

Suppose now that \((\alpha, \beta) = 0\) and consider the maximal (with respect to the height of roots) \(\gamma \in \Delta_+\) such that \(\gamma \neq \alpha, \gamma + \beta \in K\) and \((\beta + \gamma, \alpha) \neq 0\). The existence of such \(\gamma\) is obvious.

Let \(k\) be the minimal positive integer for which \(X_\beta^k v\) is \(\alpha\)-primitive. Then

\[
0 = X_\gamma X_\beta^k v = a X_{\beta + \gamma} X_{\beta}^{k-1} v + \ldots
\]

with \(a \neq 0\). It follows from PBW theorem that \(X_\gamma X_\beta^k v = 0\) which contradicts lemma 2. \(\square\)

Let \(M\) be a \(\mathfrak{g}\)-module. A non-zero weight element \(v \in M\) will be called quasi-primitive if there exists a submodule \(N \subset M\) such that \(v\) becomes \(\alpha\)-primitive in the quotient \(M/N\).

**Lemma 4.** Let \((\mu, q) \in \Omega^*\), \(N \subset M(\mu, q), 0 \neq v \in M(\mu, q)_{\mu - \rho}\) and \(K(\mathfrak{g})v \cap N \neq 0\). Then \(K(\mathfrak{g})v\) contains a quasi-primitive element.

**Proof.** Since module \(N\) is \(\alpha\)-stratified and finitely generated one can choose a set of generators \(w_1, \ldots, w_i\) (which are not necessary \(\alpha\)-primitive) of \(N\) such that \(w_i \in U(\mathfrak{g}_-) v\) for all \(i\). Let \(0 \neq v' \in K(\mathfrak{g})v \cap N\). There exists \(k > 0\) for which

\[
X^k_\alpha v' \in \sum_i U(\mathfrak{g}_-) w_i.
\]

We obtain a contradiction now from the PBW theorem since \(v' \in K(\mathfrak{g})v\). This completes the proof of lemma. \(\square\)

**Lemma 5.** Let \((\mu, q) \in \Omega^*\) and \(0 \neq v \in M(\mu, q)_{\mu - \rho}\). Then \(K(\mathfrak{g})v\) has no quasi-primitive elements except \(\bigcap X^k_\alpha v, k \geq 0\).

**Proof.** It follows from theorem 1 that if \(0 \neq v' \in M(\mu, q)_\nu, \nu \leq \mu - \rho\) is \(\alpha\)-primitive for all \(q\) then \(v' \notin K(\mathfrak{g})v\). On the other hand, a direct calculation shows that for any \(\tau \in \mathfrak{h}^*\) the existence of a non-zero \(\alpha\)-primitive element in \(K(\mathfrak{g})v\) of weight \(\mu - \tau\) is equivalent to the system of linear equations on \(\mu\). This implies that the only \(\alpha\)-primitive elements in \(K(\mathfrak{g})v\) are \(\bigcap X^k_\alpha v, k \geq 0\).

Now suppose that \(v' \in (K(\mathfrak{g})v)_{\xi}\) is quasi-primitive and \((K(\mathfrak{g})v)_{\xi}\) has no quasi-primitive elements if \(\xi > \tau\). Consider the minimal generating system \(G\) in \(\Delta_+ \setminus \{\alpha\}\) containing \(\gamma \in \pi \setminus \{\alpha\}\). Then obviously \(X_\gamma v' = 0\) for all \(\gamma \in \pi \setminus \{\alpha\}\). If \(\gamma \in G \setminus \pi\) then \((\gamma, \alpha) \neq 0\). Let \(b \simeq sl(2, \mathbb{C})\) be a subalgebra generated by \(X_{\pm \gamma}\) and \(N\) be a \(b\)-module generated by \(v'\).

Suppose that \(X_\gamma v' \neq 0\). Since \(v'\) is quasi-primitive it implies that \(v' \notin X_{-\gamma} N\) and thus \(N\) has a finite-dimensional subfactor. Using the fact that our module is \(\alpha\)-stratified and the fact that \((\gamma, \alpha) \neq 0\) we easily obtain a contradiction from \(sl(2)\)-theory. Hence \(v'\) is \(\alpha\)-primitive and thus belongs to \(\bigcap X^k_\alpha v\) for some \(k \geq 0\). \(\square\)

**Lemma 6.** Let \(V\) be a quotient of \(M(\mu, q), 0 \neq v \in M(\mu, q)_{\mu - \rho}\) and \(v \in \mathfrak{h}^*\) be a weight of \(V\). Then \(\dim V_v \geq \dim (K(\mathfrak{g})v)_v\) where \((K(\mathfrak{g})v)_v = K(\mathfrak{g})v \cap M(\mu, q)_v\). Moreover, if \(\dim V_v = \dim (K(\mathfrak{g})v)_v\) for infinitely many weights \(v_i\) of \(V\), where \(v_i - v_j \notin \mathbb{Z}_0\) for all \(i \neq j\), then module \(V\) is irreducible.
Proof. Follows immediately from lemmas above. \qed

6 Strong BGG-resolution

In this section we follow [4, 7] to construct the strong BGG-resolution for irreducible $\alpha$-stratified module $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

Let $(\lambda, p) \in P^{++}$. For $k \geq 0$ denote

$$(W_\alpha^+)^k = \{ w \in W_\alpha^+ | l(w) = k \}$$

and set

$$C_k = \sum_{w \in (W_\alpha^+)^k} M(w(\lambda, p)).$$

Define a map $D_i : C_i \to C_{i-1}$ using the matrix $(d_{w_1w_2}^i)$, $w_1 \in (W_\alpha^+)^i$, $w_2 \in (W_\alpha^+)^{i-1}$ where $d_{w_1w_2}^i = s(w_1, w_2)$ if $w_1 > w_2$ (with respect to Bruhat order) and zero otherwise. Here the numbers $s(w_1, w_2)$ are defined as in [4, Lemma 10.4]. Set $m = |\Delta_+(\pi \setminus \{\alpha\})|.$

Theorem 5. Let $\eta : M(\lambda, p) \to L(\lambda, p)$ be a natural projection. Then the sequence

$$0 \leftarrow L(\lambda, p) \xleftarrow{\eta} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} \ldots \xrightarrow{D_m} C_m \leftarrow 0$$

is exact.

Proof. It follows from the construction that this sequence is a complex.

To show the exactness in each term we will follow the proof of [7, Corollary 10.6].

Let $\mathcal{K}$ be a category of all weight $\mathfrak{G}$-modules having central character. Clearly every module $V \in \mathcal{K}$ has a decomposition

$$V = \sum_{\chi \in Z^*(\mathfrak{G})} V(\chi),$$

where $V(\chi)$ is a component with central character $\chi$. Let $\theta \in Z^*(\mathfrak{G})$ be a central character of $M(\lambda, p)$ and let $F_\theta : \mathcal{K} \to \mathcal{K}$ be a functor such that $F_\theta(V) = V(\theta)$ for all $V \in \mathcal{K}$.

Obviously, there exists a minimal element $(\mu, q) \in P^{++}$ and a finite-dimensional $\mathfrak{G}$-module $U$ such that $Y = F_\theta(L(\mu, q) \otimes U)$ contains an $\alpha$-primitive element with parameters $(\lambda, p)$. Moreover, the dimension of $Y_{\lambda-\rho}$ equals 1.

We will show that in fact $Y \simeq L(\lambda, p)$. Suppose that $Y$ is not irreducible and $N$ is some non-trivial submodule of $Y$. Then it follows from lemma 6 that the dimension growth of
$Y/N$ is strictly less then the dimension growth of any irreducible module $L(\lambda', p')$ in $\mathcal{K}$.

The obtained contradiction implies that $Y \simeq L(\lambda, p)$.

Let $\varepsilon$ be an eigenvalue of $C$ on $L(\mu, q)$. Consider an exact complex $V_\alpha(\mu, \varepsilon)$. Applying the functor $F_\alpha(\cdot \otimes U)$ to $V_\alpha(\mu, \varepsilon)$ we obtain the following exact complex:

$$0 \to L(\lambda, p) \xrightarrow{\eta} B_0 \xleftarrow{d_1} B_1 \xleftarrow{d_2} B_2 \xleftarrow{d_3} \cdots$$

where $B_i = F_\alpha(D_i \otimes U), i \geq 0$.

Using [4, Proposition 9.6] and theorem 3 we conclude that

$$B_i \simeq C_i, i \geq 0.$$

Following [7, Lemmas 10.2,10.5] there exists a sequence of isomorphisms $\nu^i : B_i \to C_i$ which makes the following diagram commutative:

$$\cdots \longrightarrow B_2(\lambda, p) \xrightarrow{d_2} B_1(\lambda, p) \xrightarrow{d_1} B_0(\lambda, p) \xrightarrow{\eta} L(\lambda, p) \longrightarrow 0$$

$$\nu_2^i \downarrow \quad \nu_1^i \downarrow \quad \nu_0^i \downarrow \quad 1 \downarrow$$

$$\cdots \longrightarrow C_2(\lambda, p) \xrightarrow{d_2^i} C_1(\lambda, p) \xrightarrow{d_1^i} C_0(\lambda, p) \xrightarrow{\eta} L(\lambda, p) \longrightarrow 0.$$  

This completes the proof of the theorem. $\square$

7 Character formulae

In this section we use the strong BGG-resolution to obtain a character formulae for a $\mathfrak{g}$-module $L(\lambda, p)$ with $(\lambda, p) \in P^{++}$.

For $\nu \in \mathfrak{h}^*$ let

$$\mathfrak{h}_\nu = \nu + \sum_{\beta \in \varpi \setminus \{\alpha\}} \mathbb{Z}\beta.$$  

Set for any $\nu \in \text{supp } V$

$$\text{ch}^{a, \nu}(V) = \sum_{\mu \in \mathfrak{h}_\nu} (\dim V_\mu') e^{\mu}.$$

**Lemma 7.** Let $V$ be an $\alpha$-stratified $\mathfrak{g}$-module and $\nu \in \text{supp } V$ then

$$\text{ch}(V) = \left( \sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \text{ch}^{a, \nu}(V).$$

*Proof.* Follows from the fact that $X_{\pm \alpha}$ act injectively on $V$. $\square$

Let $\varphi : \mathfrak{h}^* \to \mathfrak{h}_0$ be a natural projection along the root $\alpha$. Set $\Delta' = \{ \varphi(\beta) | \beta \in \Delta_+ \}$. It is easy to see (see for example [9]) that for any $(\mu, q) \in \Omega$

$$\text{ch}^{a, \mu-\beta}(M(\mu, q)) = e^{\mu-\beta} \prod_{\beta \in \Delta'} (1 - e^{-\beta})$$

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and thus
\[ \text{ch}(M(\mu, q)) = e^{\mu - \rho} \prod_{\beta \in \Delta^+ \setminus \{\alpha\}} (1 - e^{-\beta}) \left( \sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \]
by lemma 7.

Set \( \rho' = \frac{1}{2} \sum_{\beta \in P} \beta \).

**Theorem 6.** Let \((\lambda, p) \in P^{++}\). Then there exists an element \( a(\lambda, p) \in \mathfrak{g}^* \) such that
\[
\text{ch}(L(\lambda, p)) = \left( \sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \left( \prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1} \right) \times \left( \sum_{w \in W_\alpha^+} (-1)^{i(w)} e^{w(\lambda + \pi(\lambda, p) + \rho') - \pi(\lambda, p)} \right) \left( \sum_{w \in W_\alpha^+} (-1)^{i(w)} e^{w(\rho')} \right)^{-1}.
\]

**Proof.** It follows from theorem 5, that the character \( \text{ch}(L, p) \) satisfies the following alternating formulae:
\[
\text{ch} L(\lambda, p) = \sum_{i \geq 0} (-1)^i \sum_{w \in (W_\alpha^+)^{(i)}} \text{ch} M(w(\lambda, p)).
\]

Thus using the character formulae for \( M(\mu, q) \) above we obtain
\[
\text{ch} L(\lambda, p) = \left( \sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \left( \prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1} \right) \times \sum_{i \geq 0} (-1)^i \sum_{w \in (W_\alpha^+)^{(i)}} e^{\pi_{\alpha}(w(\lambda, p)) - \pi(\lambda, p)} \prod_{\beta \in P} (1 - e^{-\beta})^{-1}.
\]

Since the group \( W_\alpha^+ \) is an affine reflection group in every \( \Omega \), the result follows from the classical Weyl character formulae for finite-dimensional modules ([10, Theorem 7.5.9]). \( \square \)

Note that the element \( a(\lambda, p) \) in theorem 6 is determined uniquely by the element in \( \Omega \), with respect to which the group \( W_\alpha^+ \) is linear.

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References


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