Constrained Hamiltonian Systems as Implicit Differential Equations

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1 Introduction.
Differential equations of analytical mechanics are presented as implicit differential equations. An algorithm for extracting the integrable part of an implicit differential equation is formulated and adapted to generalized Dirac systems generated by Morse families. This work follows an earlier paper on integrability of Dirac systems [3]. There is a minor change in the formulation of the general integrability algorithm in Section 6 and the integrability of Dirac systems is discussed in terms of Morse families.

2 Preliminary definitions.
The reader is assumed to be familiar with basic concepts of differential geometry. We review below the definitions of concepts fundamental for the discussion of implicit differential equations and their integrability.

The tangent bundle of a differential manifold \( M \) of dimension \( m \) is a differential manifold \( TM \) of dimension \( 2m \). The underlying set of \( TM \) is the set of equivalence classes of differentiable curves in \( M \) called vectors. Two curves \( \gamma: I \to M \) and \( \gamma': I' \to M \) are equivalent if

\[
\gamma'(0) = \gamma(0) \tag{1}
\]

and

\[
D(f \circ \gamma')(0) = D(f \circ \gamma)(0), \tag{2}
\]

for each differentiable function \( f: M \to \mathbb{R} \). The sets \( I \) and \( I' \) are open neighbourhoods of \( 0 \in \mathbb{R} \). The symbol \( D \) denotes the the first derivative of a function. The equivalence class of a curve \( \gamma: I \to M \) is denoted by \( t \gamma(0) \). The mapping

\[
t \gamma: \mathbb{R} \to TM
\]

\[
: s \mapsto t \gamma(s + \cdot)(0) \tag{3}
\]

is called the tangent prolongation of the curve \( \gamma \). The mapping

\[
\tau_M: TM \to M
\]

\[
:t \gamma(0) \mapsto \gamma(0) \tag{4}
\]

is called the tangent bundle projection.
Let $\eta: M \to N$ be a differentiable mapping. The mapping

$$T_\eta: TM \to TN$$

defined by

$$T_\eta(t\gamma(0)) = t(\eta \circ \gamma)(0)$$

is called the \textit{tangent mapping} of $\eta$.

A vector $v \in TM$ is said to be \textit{tangent} to a set $C \subseteq M$ if there is a representative $\gamma: I \to M$ with $\text{im} (\gamma) \subseteq C$. The set of all vectors tangent to $C$ is denoted by $TC$.

3 \textbf{Submanifolds of symplectic manifolds.}

Let $(P, \omega)$ be a symplectic manifold and let $\beta: TP \to T^*P$ be the natural isomorphism characterized by

$$\langle w, \beta(v) \rangle = \langle v \wedge w, \omega \rangle$$

for each $v \in TP$ and each $w \in TP$ such that $\tau_P(w) = \tau_P(v)$. Let $V \subseteq T_P P$ be a vector subspace. The symbol $T_P P$ denotes the set $\tau_P^{-1}(p)$. In general, the fibre $\eta^{-1}(p)$ of a fibration $\eta: Y \to P$ over a point $p$ will be denoted by $Y_p$. The \textit{polar} $V^*$ is the subspace

$$\{ q \in T_P^*P; \forall v \in V, \langle v, q \rangle = 0 \}.$$

We denote by $V^*$ the \textit{symplectic polar}

$$\beta^{-1}(V^*) = \{ w \in T_P P; \forall v \in V, \langle v \wedge w, \omega \rangle = 0 \}.$$

If $C \subseteq P$ is a submanifold, then $T^*C$ will denote the set

$$\bigcup_{p \in C} (T_P C)^*.$$

We recall that a submanifold $C \subseteq P$ is said to be \textit{isotropic} if $T^*C \supseteq TC$. A submanifold $C \subseteq P$ is said to be \textit{coisotropic} if $T^*C \subseteq TC$. The set $T^*C$ is called the \textit{characteristic distribution} of a coisotropic submanifold $C \subseteq P$. The characteristic distribution is Frobenius integrable. Its integral foliation is called the \textit{characteristic foliation} of $C$. In Dirac’s terminology a coisotropic submanifold of the phase space is a \textit{first class constraint}. A submanifold $C \subseteq P$ is called a \textit{second class constraint} if the symplectic form $\omega$ restricted to $C$ is nondegenerate. A submanifold $C \subseteq P$ is said to be \textit{Lagrangian} if $T^*C = TC$. For an intrinsic definition of the class of a submanifold of a symplectic manifold see [4].

4 \textbf{Affine symplectic spaces.}

An \textit{affine space} is a triple $(M, V, \alpha)$, where $M$ is a set, $V$ is a real vector space of finite dimension and $\mu$ is a mapping $\mu: M \times M \to V$ such that

1. $\mu(x_3, x_2) + \mu(x_2, x_1) + \mu(x_1, x_3) = 0$;
2. the mapping $\mu(\cdot, x): M \to V$ is bijective for each $x \in M$.

The set $M$ is said to be an \textit{affine space modelled} on the vector space $V$.

The tangent bundle $TM$ of an affine space $M$ is identified with $M \times V$. The cotangent bundle $T^*M$ is identified with $M \times V^*$ and the tangent bundle $TT^*M$ is identified with $M \times V^* \times V \times V^*$.

The cotangent bundle $T^*M$ is a symplectic affine space. The canonical symplectic form $\omega$ is defined by

$$\langle (x, p, \dot{x}, \dot{p}) \wedge (x, p, \delta x, \delta p), \omega \rangle = \langle \delta x, \dot{p} \rangle - \langle \dot{x}, \delta p \rangle$$

for vectors $(x, p, \dot{x}, \dot{p})$ and $(x, p, \delta x, \delta p)$ in $TT^*M$. 

2
5 Implicit differential equations of analytical mechanics.

An implicit first order differential equation in a differential manifold \( P \) is a submanifold \( D \) of the tangent bundle \( TP \). A curve \( \gamma: I \to P \) is called a solution of a differential equation \( D \subset TP \) if \( \gamma'(s) \in D \) for each \( s \in I \subset \mathbb{R} \).

The phase space of a mechanical system is a symplectic manifold \((P, \omega)\). Implicit differential equations of analytical mechanics are submanifolds of the tangent bundle \( TP \) of the phase space. Three types of differential equations are encountered in modern analytical mechanics.

Let \( H: P \to \mathbb{R} \) be a differentiable function on the phase space \( P \). The set

\[
D = \left\{ w \in TP; \; p = \tau_P(w), \forall u \in \mathbb{T}_wP \langle u \wedge w, \omega \rangle = \langle u, dH \rangle \right\}
\]

(12)

is the image of the Hamiltonian vector field

\[
X: P \to TP
\]

characterized by

\[
i_X \omega = -dH.
\]

(14)

This is the simplest type of a differential equation of analytical mechanics called a Hamiltonian system. The image of a vector field is called an explicit differential equation. Explicit differential equations are integrable in the sense defined in the next section.

Let \( C \subset P \) be a submanifold of the phase space and let \( H: C \to \mathbb{R} \) be a differentiable function. The set

\[
D = \left\{ w \in TP; \; p = \tau_P(w) \in C, \forall u \in \mathbb{T}_wC \langle u \wedge w, \omega \rangle = \langle u, dH \rangle \right\}
\]

(15)

is a generalized Hamiltonian system [6] known as a Dirac system. Truly implicit differential equations of this type appear in gauge independent formulations of the dynamics of charged particles [4].

Let \( \eta: Y \to P \) be a differential fibration and let \( G: Y \to \mathbb{R} \) be a differentiable function. Under certain regularity conditions [7] the set

\[
D = \left\{ w \in TP; \; \exists y \in Y_{\eta(w)} \forall u \in \mathbb{T}_yY \langle T\eta(u) \wedge w, \omega \rangle = \langle u, dG \rangle \right\}
\]

(16)

is a submanifold of \( TP \) of dimension equal to the dimension of \( P \). In this case the function \( G \) is called a Morse family of functions on fibres of \( \eta \) and is represented by the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{G} & \mathbb{R} \\
\downarrow \downarrow & \downarrow & \downarrow \\
\eta & \downarrow & \eta \\
P & \downarrow & \mathbb{R}
\end{array}
\]

(17)

The implicit differential equation \( D \) will be called a generalized Dirac system generated by the Morse family \( G \). The regularity conditions are satisfied in examples discussed in Section 9.

It is known that the tangent bundle \( TP \) together with the derived two-form \( d_T\omega \) form a symplectic manifold \((TP, d_T\omega)\) [6]. All types of implicit differential equations of analytical mechanics are Lagrangian submanifolds of this symplectic manifold.
Let $W$ be a vector space and let $K: P \to W$ be a differentiable mapping such that $C = K^{-1}(0)$ is a submanifold of $P$. Let $\hat{H}: P \to \mathbb{R}$ be a differentiable function. The Dirac system $(15)$ generated by the function $H = \hat{H}|C$ can be treated as a generalized Dirac system generated by the Morse family

\[
P \times W^* \overset{G}{\longrightarrow} \mathbb{R}
\]

where $G$ is the function

\[
G: P \times W^* \to \mathbb{R}
: (p, \lambda) \mapsto \hat{H}(p) + \langle K(p), \lambda \rangle
\]

and

\[
\eta: P \times W^* \to P
: (p, \lambda) \mapsto p
\]

is the canonical projection. The representation of Dirac systems as generalized Dirac systems provides an unification of the analysis of integrability.

6 Integrability of implicit differential equations.

An implicit differential equation $D \subset TP$ is said to be integrable if for each $v \in D$ there is a solution $\gamma: I \to P$ such that $t\gamma(0) = v$.

**Proposition 1.** If $D \subset TP$ is integrable, then

\[
D \subset T(\tau_P(D)).
\]

**Proof:** Let $v \in D$ and let $\gamma: I \to P$ be a solution of $D$ such that $t\gamma(0) = v$. For each $s \in I$ we have $t\gamma(s) \in D$. It follows that $\gamma(s) \in \tau_P(D)$ for each $s \in I$. Consequently $t\gamma(s) \in T(\tau_P(D))$ for each $s \in I$ and $v = t\gamma(0) \in T(\tau_P(D))$.

**Proposition 2.** If $C = \tau_P(D)$ is a submanifold of $P$ and if the mapping

\[
\tau: D \to C
: v \mapsto \tau_P(v)
\]

is a surjective submersion, then the condition $D \subset TC$ is sufficient for integrability of the implicit differential equation $D \subset TP$.

**Proof:** Let $v$ be an element of $D$ and let $p = \tau_P(v)$. Let $\sigma: C \to D$ be a (local) section of $\tau: D \to C$ such that $\sigma(p) = v$. If $\varepsilon: D \to TC$ is the canonical injection, then $X = \varepsilon \circ \sigma: C \to TC$ is a section of $\tau_C: TC \to C$, hence a vector field on $C$. Let $\gamma: I \to C$ be an integral curve of $X$ such that $\gamma(0) = p$. Then $\mathrm{im}(t\gamma) \subset D$ and $t\gamma(0) = X(p) = v$. 

Proposition 2 suggests a simple algorithm for extracting the integrable part of a sufficiently regular implicit differential equation.

Let $D \subset TP$ be an implicit differential equation. We consider the sequence of sets

$$C^0 = \pi_P(D), C^1 = \pi_P(D \cap TC^0), \ldots, C^k = \pi_P(D \cap TC^{k-1}), \ldots$$

(23)

It may happen that after a finite number of steps the sets are all equal to a non-empty set $\overline{C}$. This set satisfies the equality

$$\overline{C} = \pi_P(D \cap TC).$$

(24)

If $C$ is a submanifold of $P$ and if the mapping

$$\pi: D \cap TC \to \overline{C}$$

$$: v \mapsto \pi_P(v)$$

is a surjective submersion, then $\overline{D} = D \cap T\overline{C}$ is the integrable part of $D$.

This integrability algorithm is a modification of the algorithm described in [3].

7 Integrability of Dirac systems. Let $(P, \omega)$ be a symplectic phase space of a mechanical system and let

$$D = \left\{ w \in TP; p = \pi_P(w) \in C, \forall u \in T_pC \langle u \wedge w, \omega \rangle = \langle u, dH \rangle \right\}$$

(26)

be a Dirac system generated by a Hamiltonian $H: C \to \mathbb{R}$ defined on a submanifold $C \subset P$. The set

$$D_p = \left\{ w \in T_pP; \forall u \in T_pC \langle u \wedge w, \omega \rangle = \langle u, dH \rangle \right\}$$

(27)

is an affine subspace of $T_pP$ modelled on the vector subspace

$$T_p^*C = \left\{ w \in T_pP; \forall u \in T_pC \langle u \wedge w, \omega \rangle = 0 \right\}.$$  

(28)

The image $\pi_P(D)$ is the submanifold $C$. The mapping $\tau$ defined in formula (22) is a submersion since local sections of this mapping are easily constructed. It follows that the condition $D \subset TC$ is sufficient for integrability of the Dirac system $D$.

For the space $T_pC$ we use the representation

$$T_pC = (T_p^*C)^*$$

$$= \left\{ w \in T_pP; \forall u \in T_p^*C \langle u \wedge w, \omega \rangle = 0 \right\}.$$  

(29)

We denote by $\ker(dH(p))$ the space

$$\left\{ u \in T_pP; u \in T_pC, \langle u, dH \rangle = 0 \right\}.$$  

(30)

Lemma 1. If $T_p^*C \subset \ker(dH(p))$, then $D_p \subset T_pC$.

Proof: Let $w \in D_p$. From the definition (27) we have $\langle u \wedge w, \omega \rangle = \langle u, dH \rangle$ for each $u \in T_pC$. If $T_p^*C \subset \ker(dH(p))$, then $\langle u \wedge w, \omega \rangle = 0$ for each $u \in T_p^*C$. Hence, $w \in (T_p^*C)^* = T_pC$. 

\[ \blacksquare \]
Lemma 2. If $D_p \subset T_p C$, then $T_p^* C \subset T_p C$.

Proof: Let $v \in T_p^* C$. If $w \in D_p$, then $w + v \in D_p$ since $D_p$ is an affine space modelled on $T_p C$. If $D_p \subset T_p C$, then $w \in T_p C$ and $w + v \in T_p C$. Hence, $v \in T_p C$. □

Lemma 3. If $D_p \subset T_p C$, then $T_p^* C \subset \ker (dH(p))$.

Proof: If $D_p \subset T_p C$, then $D_p \cap T_p C$ is not empty. If $w \in D_p \cap T_p C$ and $u \in T_p C \cap T_p^* C$, then $\langle u, dH \rangle = \langle u \wedge w, \omega \rangle = 0$ since $u \in T_p C$ and $w \in T_p^* C$. It follows that $u \in \ker (dH(p))$. □

Theorem 1. A Dirac system

$$ D = \left\{ w \in TP; \ p = \tau_p(w) \in C, \forall u \in T_p C, \langle u \wedge w, \omega \rangle = \langle u, dH \rangle \right\} $$

is integrable if and only if the submanifold $C \subset P$ is coisotropic and the Hamiltonian function $H: C \rightarrow \mathbb{R}$ is constant on leaves of the characteristic foliation of $C$.

Proof:

1. If $C$ is coisotropic and $H$ is constant on leaves of the characteristic foliation of $C$, then $T_p^* C \subset \ker (dH(p))$ for each $p \in C$. By Lemma 1 this implies $D_p \subset T_p C$ for each $p \in C$. Hence, $D$ is integrable.

2. If $D$ is integrable, then $D_p \subset T_p C$ for each $p \in C$. By Lemma 2 and Lemma 3 this implies that $C$ is coisotropic and $H$ is constant on leaves of the characteristic foliation of $C$. □

If the Dirac system $D$ is not integrable, then the algorithm for extracting its integrable part can be tried. An adaptation of this algorithm to Dirac systems is described in [3].

8 Integrability of generalized Dirac systems. Let

$$ Y \xrightarrow{G} \mathbb{R} $$

be a Morse family generating the implicit differential equation

$$ D = \left\{ w \in TP; \ \exists y \in Y_r(w) \forall u \in T_y Y \langle T\eta(u) \wedge w, \omega \rangle = \langle u, dG \rangle \right\} $$

The set

$$ S = \left\{ y \in Y; \forall u \in T_y Y \ T\eta(u) = 0 \Rightarrow \langle u, dG \rangle = 0 \right\} $$

is called the critical set and the set

$$ C = \left\{ p \in P; \exists y \in Y_p \forall u \in T_y Y \ T\eta(u) = 0 \Rightarrow \langle u, dG \rangle = 0 \right\} $$

is called the constraint set. The sets $D$, $S$ and $C$ are related by

$$ C = \tau_P(D) = \eta(S). $$

6
The algorithm described in Section 5 produces a sequence of sets

$$C^0, C^1, \ldots, C^k, \ldots,$$

where

$$C^0 = C$$

and

$$C^k = \left\{ p \in P; \ p \in C^{k-1} \text{ and } \exists y \in Y, \forall u \in \Gamma_y T \eta(u) \in T^*_p C^{k-1} \Rightarrow \langle u, dG \rangle = 0 \right\}$$

for $k > 0$. The terminal constraint set $\overline{C}$ is the set

$$\overline{C} = \left\{ p \in P; \ \exists y \in Y, \forall u \in \Gamma_y T \eta(u) \in T^*_p \overline{C} \Rightarrow \langle u, dG \rangle = 0 \right\}. \quad (39)$$

The terminal critical set $\overline{S}$ and the terminal equation $\overline{D}$ are the sets

$$\overline{S} = \left\{ y \in Y; \ p = \eta(y) \in \overline{C} \text{ and } \exists y \in Y, \forall u \in \Gamma_y T \eta(u) \in T^*_p \overline{C} \Rightarrow \langle u, dG \rangle = 0 \right\}.$$  \quad (41)

and

$$\overline{D} = \left\{ w \in TP; \ p = \tau_p(w) \in \overline{C} \text{ and } \exists y \in Y, \forall u \in \Gamma_y T (\eta(u) \wedge w, \omega) = \langle u, dG \rangle \right\}. \quad (42)$$

We have the relation

$$\overline{D} = D \cap T \overline{C}. \quad (43)$$

This version of the extraction algorithm should be applied to a Dirac system after it has been converted to generalized Dirac systems. This is more efficient than the direct application of procedure used in [3]. If $C^1 = C^0$, then the Dirac system is integrable.

9 Examples.

Example 1. It is possible to give the dynamics of a non-relativistic particle of mass $m$ and charge $e$ a gauge independent form in a four-dimensional configuration space analogous to the five-dimensional space of Kaluza. Let $\xi: N \to M$ be a principal fibration with the additive group of real numbers as structure group and a manifold $M$ of dimension 3 as base. Let $X: N \to T N$ be the fundamental vector field. The structure group action is a one-parameter group of differentiable automorphisms of $N$. The field $X$ is the generator of this action. Let $g$ be a pseudoriemannian metric of signature 2 such that $\langle X, g \circ X \rangle = 1$ and $d_X g = 0$. A submanifold $C \subset T^* N$ is defined by

$$C = \left\{ r \in T^* N; \ \langle X, r \rangle = \epsilon \right\}. \quad (44)$$

The dynamics of the particle is a Dirac system generated by the Hamiltonian

$$H: C \to \mathbb{R}$$

$$: r \mapsto \frac{1}{2m} (||r||^2 + \epsilon^2) - \epsilon V(\xi(\pi_N(r))), \quad (45)$$

where $V: M \to \mathbb{R}$ is the scalar potential of the electric field. This Dirac system is integrable. The submanifold $C$ is coisotropic because its codimension is equal to 1. Leaves of the characteristic distribution are orbits of the group action of the structure group lifted to the cotangent bundle $T^* N$. Invariance of the metric implies that the Hamiltonian is constant on leaves of the characteristic foliation of $C$ (see [4]).
Example 2. Let $V$ be an Euclidean vector space with a metric tensor $g: V \to V^*$. The implicit differential equation \[ D = \{(x,p,\dot{x},\dot{p}) \in V \times V^* \times V \times V^*; \ x \neq 0, \ p = k(\|x\|^2g(\dot{x}) - \langle x, g(x)g(\dot{x}) \rangle), \ \dot{p} = k(\|\dot{x}\|^2g(x) - \langle \dot{x}, g(x)g(\dot{x}) \rangle)\} \] (46)
is a Dirac system generated by the Hamiltonian
\[ H: C \to \mathbb{R} \]
\[ : (x,p) \mapsto \frac{\|p\|^2}{2k\|x\|^4} \] (47)
defined on the constraint set
\[ C = \{(x,p) \in V \times V^*; \ x \neq 0, \langle x, p \rangle = 0\}. \] (48)
This system can be presented as a generalized Dirac system generated by the Morse family
\[ \overset{\circ}{V} \times V^* \times \mathbb{R} \xrightarrow{G} \mathbb{R} \]
\[ \eta \]
\[ \overset{\circ}{V} \times V^* \] (49)
where
\[ \overset{\circ}{V} = \{x \in V; \ x \neq 0\} \] (50)
and $G$ is the function
\[ G: \overset{\circ}{V} \times V^* \times \mathbb{R} \to \mathbb{R} \]
\[ : (x,p,\lambda) \mapsto \frac{\|p\|^2}{2k\|x\|^4} + \lambda\langle x, p \rangle \] (51)
and
\[ \eta: \overset{\circ}{V} \times V^* \times \mathbb{R} \to \overset{\circ}{V} \times V^* \]
\[ : (x,p,\lambda) \mapsto (x,p) \] (52)
is the canonical projection.

The system is not integrable. The integrable part $\overline{D}$ is obtained through the following steps:
\[ C^0 = \left\{(x,p) \in \overset{\circ}{V} \times V^*; \langle x, p \rangle = 0\right\}, \] (53)
\[ TC^0 = \left\{(x,p,\delta x,\delta p) \in \overset{\circ}{V} \times V^* \times V \times V^*; \langle x, p \rangle = 0, \langle \delta x, p \rangle + \langle x, \delta p \rangle = 0\right\}. \] (54)
\[ T^*C^0 = \left\{ (x, p, \delta x, \delta p) \in \mathbb{V} \times \mathbb{V}^* \times \mathbb{V} \times \mathbb{V}^*; (x, p) = 0, \right. \\
\left. \delta x = \frac{\langle \delta x, g(x) \rangle}{\|x\|^2} x, \delta p = -\frac{\langle \delta x, g(x) \rangle}{\|x\|^2} p \right\}, \tag{55} \]

\[ C^1 = \left\{ (x, p) \in \mathbb{V} \times \mathbb{V}^*; p = 0 \right\} \tag{56} \]

\[ TC^1 = \left\{ (x, p, \delta x, \delta p) \in \mathbb{V} \times \mathbb{V}^* \times \mathbb{V} \times \mathbb{V}^*; (x, p) = 0, \delta p = 0 \right\}. \tag{57} \]

\[ T^*C^1 = TC^1, \]

\[ C^2 = C^1 = \overline{C}, \tag{59} \]

and

\[ \overline{D} = \left\{ (x, p, \dot{x}, \dot{p}) \in \mathbb{V} \times \mathbb{V}^* \times \mathbb{V} \times \mathbb{V}^*; (x, p) = 0, \dot{x} = \frac{\dot{x}, g(x)}{\|x\|^2} x, \dot{p} = 0 \right\}. \tag{60} \]

Solutions are curves of the form

\[ \gamma: \mathbb{R} \to \mathbb{V} \times \mathbb{V}^* \]

\[ : t \mapsto (f(t)u, 0), \tag{61} \]

where \( u \in \mathbb{V} \) and \( f \) is a function such that \( f(t) \neq 0 \).

**Example 3.** Let \((M, V, \mu)\) be the affine space-time of special relativity with the Minkowski metric \(g: V \to V^*\). Dynamics of a free particle of mass \(m\) is the generalized Dirac system

\[ D = \left\{ (x, p, \dot{x}, \dot{p}) \in M \times \mathbb{V}^* \times \mathbb{V} \times \mathbb{V}^*; (g^{-1}(p), p) > 0, \dot{p} = 0, \exists \lambda \in \mathbb{R}_+, \dot{x} = \frac{\lambda}{m} g^{-1}(p) \right\} \]

\[ = \left\{ (x, p, \dot{x}, \dot{p}) \in M \times \mathbb{V}^* \times \mathbb{V} \times \mathbb{V}^*; (\dot{x}, g(\dot{x})) > 0, p = \frac{m}{\|\dot{x}\|^2} \frac{m}{\|\dot{x}\|^2} \right\} \tag{62} \]

generated by the Morse family

\[ \begin{array}{ccc}
M \times K^+ \times \mathbb{R}_+ & \xrightarrow{G} & \mathbb{R} \\
\eta \downarrow & & \downarrow \\
M \times K^+ & \xrightarrow{\eta} & M \times K^+ 
\end{array} \tag{63} \]

where

\[ K^+ = \{ p \in V^*; (g^{-1}(p), p) > 0 \}, \tag{64} \]

\[ \eta: M \times K^+ \times \mathbb{R}_+ \to M \times K^+ \]

\[ : (x, p, \lambda) \mapsto (x, p) \tag{65} \]
is the canonical projection and

\[ G: M \times K_+ \times \mathbb{R}_+ \to \mathbb{R} \]
\[ : (x, p, \lambda) \mapsto \lambda (\|p\| - m). \]  \hspace{1cm} (66)

The set

\[ S = \{ (x, p, \lambda) \in M \times K_+ \times \mathbb{R}_+; \|p\| = m \} \]  \hspace{1cm} (67)

is the critical set for the Morse family and the mass shell

\[ C = \{ (x, p) \in M \times K_+; \|p\| = m \} \]  \hspace{1cm} (68)

is the constraint set. We have

\[ C^0 = C, \]  \hspace{1cm} (69)

\[ TC^0 = \{ (x, p, \dot{x}, \dot{p}) \in M \times K_+ \times V \times V^*; \|p\| = m, \langle g^{-1}(p), \dot{p} \rangle = 0 \}, \]  \hspace{1cm} (70)

\[ T^* C^0 = \{ (x, p, \delta x, \delta p) \in M \times K_+ \times V \times V^*; \|p\| = m, \delta x = \frac{\delta x, \delta p}{m^2} p, \delta p = 0 \}, \]  \hspace{1cm} (71)

and

\[ C^1 = C^0. \]  \hspace{1cm} (72)

It follows that this generalized Dirac system is integrable. Solutions are oriented lines in the affine phase space \( M \times V^* \). Their orientation reflects the distinction between particles and antiparticles introduced by Stueckelberg [5] and used by Feynman [1].

**Example 4.** We analyse the dynamics of two interacting relativistic particles formulated in [8], [9]. Masses of the particles are denoted by \( m_1 \) and \( m_2 \). The interaction potential is a function \( V \) of a real positive argument.

In addition to the notation of the preceding example we introduce symbols

\[ N = \{ (x_1, x_2) \in M \times M; \langle x_2 - x_1, g(x_2 - x_1) \rangle < 0 \}, \]  \hspace{1cm} (73)

\[ P = N \times K_+ \times K_+, \]  \hspace{1cm} (74)

\[ \|x_2 - x_1\| = \sqrt{-\langle x_2 - x_1, g(x_2 - x_1) \rangle} \]  \hspace{1cm} (75)

for \((x_1, x_2) \in N\),

\[ m_1 = \sqrt{m_1^2 + V(\|x_2 - x_1\|)}, \]  \hspace{1cm} (76)

and

\[ m_2 = \sqrt{m_2^2 + V(\|x_2 - x_1\|)}. \]  \hspace{1cm} (77)

Dynamics of the particles is a generalized Dirac system

\[ D \subset M \times M \times V^* \times V^* \times V \times V \times V^* \times V^* \]  \hspace{1cm} (78)

generated by the Morse family

\[ P \times \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{G} \mathbb{R} \]
\[ \eta \]
\[ P \]  \hspace{1cm} (79)
where
\[
\eta: P \times \mathbb{R}_+ \times \mathbb{R}_+ \to P
\]
\[
: (x_1, x_2, p_1, p_2, \lambda_1, \lambda_2) \mapsto (x_1, x_2, p_1, p_2)
\] (80)
is the canonical projection and
\[
G: P \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}
\]
\[
: (x_1, x_2, p_1, p_2, \lambda_1, \lambda_2) \mapsto \lambda_1 (\|p_1\| - \overline{m}_1) + \lambda_2 (\|p_2\| - \overline{m}_2).
\] (81)
The system \(D\) is the set of vectors \((x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2)\) in \(M \times M \times V^* \times V^* \times V \times V \times V^* \times V^*\) satisfying relations
\[
\langle x_2 - x_1, g(x_2 - x_1) \rangle < 0,
\] (82)
\[
\langle \dot{x}_1, g(\dot{x}_1) \rangle > 0, \quad \langle \dot{x}_2, g(\dot{x}_2) \rangle > 0
\] (83)
\[
p_1 = \frac{\overline{m}_1 g(\dot{x}_1)}{\|\dot{x}_1\|},
\] (84)
\[
p_2 = \frac{\overline{m}_2 g(\dot{x}_2)}{\|\dot{x}_2\|},
\] (85)
\[
\dot{p}_1 = \frac{DV(\|x_2 - x_1\|)}{2\|x_2 - x_1\|} \left( \frac{\|\dot{x}_1\|}{\overline{m}_1} + \frac{\|\dot{x}_2\|}{\overline{m}_2} \right) g(x_2 - x_1),
\] (86)
\[
\dot{p}_1 + \dot{p}_2 = 0.
\] (87)
The system \(D\) is not integrable. The initial constraint set \(C^0\) is the set of covectors \((x_1, x_2, p_1, p_2) \in P\) satisfying equations
\[
\|p_1\| = \overline{m}_1,
\] (88)
\[
\|p_2\| = \overline{m}_2.
\] (89)
The tangent set \(T^*C^0\) is the set of vectors \((x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2) \in P \times V \times V^* \times V^*\) satisfying equations
\[
\|p_1\| = \overline{m}_1,
\] (90)
\[
\|p_2\| = \overline{m}_2,
\] (91)
\[
2\langle g^{-1}(p_1), \dot{p}_1 \rangle + \frac{DV(\|x_2 - x_1\|)}{\|x_2 - x_1\|} \langle \dot{x}_2 - \dot{x}_1, g(x_2 - x_1) \rangle = 0,
\] (92)
\[
2\langle g^{-1}(p_2), \dot{p}_2 \rangle + \frac{DV(\|x_2 - x_1\|)}{\|x_2 - x_1\|} \langle \dot{x}_2 - \dot{x}_1, g(x_2 - x_1) \rangle = 0.
\] (93)
The set \(T^*C^0\) is composed of vectors \((x_1, x_2, p_1, p_2, \delta x_1, \delta x_2, \delta p_1, \delta p_2) \in P \times V \times V^* \times V^*\) satisfying equations
\[
\|p_1\| = \overline{m}_1,
\] (94)
\[
\|p_2\| = \overline{m}_2.
\] (95)
\[ \delta x_1 = 2k_1 g^{-1}(p_1), \]  
\[ \delta x_2 = 2k_2 g^{-1}(p_2), \]  
\[ \delta x_3 = \frac{(k_1 + k_2)}{2\|x_2 - x_1\|} g(x_2 - x_1), \]  
\[ \delta x_4 = \frac{(k_1 + k_2)}{2\|x_2 - x_1\|} g(x_2 - x_1). \]  

and

\[ \delta p_1 + \delta p_2 = 0 \]  

with arbitrary values of the parameters \( k_1 \) and \( k_2 \).

The constraint set \( C^1 \) is the set of covectors \( (x_1, x_2, p_1, p_2) \in P \) satisfying the equation

\[ \{x_2 - x_1, p_1 + p_2\} = 0 \]  

in addition to relations \( (88) \) and \( (89) \).

The tangent set \( T C^1 \) is the set of vectors \( (x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2) \in P \times V \times V^* \times V^* \) satisfying equations

\[ \{x_2 - x_1, p_1 + p_2\} = 0 \]  

and

\[ \{x_1, x_2, p_1 + p_2\} + \{x_2 - x_1, \dot{p}_1 + \dot{p}_2\} = 0 \]  

in addition to equations \( (90), (91), (92) \) and \( (93) \).

The set \( T^* C^1 \) is composed of vectors \( (x_1, x_2, p_1, p_2, \delta x_1, \delta x_2, \delta p_1, \delta p_2) \in P \times V \times V^* \times V^* \times V^* \) satisfying equations

\[ \delta x_1 = 2k_1 g^{-1}(p_1) + k_2(x_2 - x_1), \]  
\[ \delta x_2 = 2k_2 g^{-1}(p_2) + k_3(x_2 - x_1), \]  
\[ \delta x_3 = \frac{(k_1 + k_2)}{2\|x_2 - x_1\|} g(x_2 - x_1) + k_3(p_1 + p_2), \]  

and

\[ \delta p_1 + \delta p_2 = 0 \]  

with arbitrary values of the parameters \( k_1, k_2 \) and \( k_3 \) in addition to equations \( (94) \) and \( (95) \).

The set \( C^2 \) is the same as \( C^1 \). Hence \( \overline{C} = C^1 \). The integrable part \( \overline{D} \) of \( D \) is the set of vectors \( (x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2) \in M \times M \times V^* \times V^* \times V \times V \times V^* \times V^* \) satisfying relations

\[ \{x_2 - x_1, g(x_2 - x_1)\} < 0, \]  
\[ \{x_1, g(\dot{x}_1)\} > 0, \{x_2, g(\dot{x}_2)\} > 0 \]  
\[ p_1 = \frac{m_1 g(\dot{x}_1)}{\|\dot{x}_1\|}, \]  
\[ p_2 = \frac{m_2 g(\dot{x}_2)}{\|\dot{x}_2\|}, \]  
\[ \dot{p}_1 = \frac{DV(||x_2 - x_1||)}{2\|x_2 - x_1\|} \left( \frac{\|\dot{x}_1\|}{m_1} + \frac{\|\dot{x}_2\|}{m_2} \right) g(x_2 - x_1), \]  

and

\[ \{x_2 - x_1, p_1 + p_2\} = 0. \]  

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11 References.


