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ABSTRACT

We reformulate the Hamiltonian approach to lattice gauge theories such that, at the classical level, the gauge group does not act canonically, but instead as a Poisson-Lie group. At the quantum level, it then gets promoted to a quantum group gauge symmetry. The theory depends on two parameters - the deformation parameter $\lambda$ and the lattice spacing $a$. We show that the system of Kogut and Susskind is recovered when $\lambda \to 0$, while QCD is recovered in the continuum limit (for any $\lambda$). We thus have the possibility of having a two parameter regularization of QCD.
1 Introduction

There has been some interest recently in developing a q-deformed Yang-Mills theory.[1] One motivation for this activity is the possibility of breaking the standard gauge symmetry of Yang-Mills theories without the introduction of Higgs fields. The attempts discussed previously appear to be rather involved. They often require the development of differential structures on quantum groups. In another approach, which is the one we shall report on here, one may q-deform gauge theories on the lattice - and then take the continuum limit. However, as we shall show, this leads to a negative result. That is, after applying the procedure we recover ordinary gauge theory in the continuum limit. Thus rather than a q-deformed Yang-Mills theory, we have the possibility of a two parameter regularization of QCD, the two parameters being the deformation parameter (which we denote by $\lambda$) and the lattice spacing $a$. This situation may also be of interest, since it could then happen that certain physical quantities may converge faster in the two-parameter space than they do for the case of a single parameter $a$.

Our approach is to start with the Hamiltonian formulation of lattice gauge theories due to Kogut and Susskind.[2] The dynamics for that system is given in terms of rigid rotators located at the links of the lattice. The Poisson structure which is taken for the rotators is the usual one, i.e., it is written on the cotangent bundle of the relevant group. Recently, an alternative Poisson structure (and Hamiltonian) for the rigid rotator was found.[3] (Also see [4].) In the new formulation, rotations are not canonically implemented, but rather they are Lie-Poisson symmetries.[5] Since Lie-Poisson symmetries are known to be promoted to quantum group symmetries after quantization, [6] the rotation group is deformed to a quantum group. The lattice analogue of the rotation symmetry is gauge symmetry, and thus
if we utilize this alternative Poisson structure to describe the rigid rotators on the lattice, we obtain a quantum group gauge symmetry upon quantization.

In Sec. 2 we review the standard Hamiltonian formulation of lattice gauge theories. For simplicity we shall limit our discussion to $SU(2)$ gauge theories in the absence of fermions. Then the Hamiltonian dynamics can be written on a product space consisting of rigid rotators, modulo the space of $SU(2)$ gauge transformations. As stated above, to each such rotator one associates the standard Poisson structure, which is written on the cotangent bundle of $SU(2)$. Physically, a rotator is attached to each link on the lattice, while gauge transformations correspond to points on the lattice. The Hamiltonian for the theory is required to be gauge invariant and reduce to the QCD Hamiltonian in the limit of zero lattice spacing, while gauge transformations are required to be canonical in the classical theory and are generated by Gauss law constraints.

We next review the alternative classical Hamiltonian formulation of the rigid rotator in Sec. 3. There the six dimensional phase space is taken to be the $SL(2,C)$ group manifold. As in the standard formulation, the Hamiltonian is invariant under left and right $SU(2)$ transformations (which contain ordinary rotations). Once again, these transformations do not correspond to canonical symmetries, but rather they are Lie-Poisson symmetries. In the system given here we introduce a parameter $\lambda$ (the ‘deformation parameter’), and the standard Hamiltonian formalism for the rotator is recovered in the limit $\lambda \to 0$.

The alternative Hamiltonian formulation of the rigid rotator is applied to lattice gauge theories in Sec. 4. We thereby deform the Kogut Susskind Hamiltonian dynamics in a way which preserves the $SU(2)$ gauge symmetry of the classical theory- but which replaces canonical symmetry transformations by Lie-Poisson transformations. As in the Kogut Susskind
system, rotator degrees of freedom are assigned to each link on the lattice and gauge transformations are associated with each point on the lattice. However, now we shall describe the rotator degrees of freedom in terms of $SL(2, C)$ variables. In addition to the parameter $a$ denoting the lattice spacing, our theory possesses the deformation parameter $\lambda$ and we require that the Kogut Susskind Hamiltonian formalism is recovered in the limit $\lambda \to 0$. We shall show that our system yields the correct continuum limit of $SU(2)$ gauge theory even when the deformation parameter is different from zero.

In section 5 we make some preliminary remarks about the quantization of this system. When Lie-Poisson symmetries are present in the classical theory the standard practice is to apply the method of deformation quantization [7]. Fixing the quantum dynamics using the method of deformation quantization requires writing down a star product on the space of classical observables. This is in general a difficult task and shall not be attempted here. Instead, we shall only demand, as is usually done, that the $SU(2)$ Poisson Lie group symmetry of the classical theory gets replaced upon quantization by an $SU_q(2)$ quantum group symmetry[6]. Since an $SU_q(2)$ matrix is attached at each point on the lattice, we in fact end up with an $SU_q(2)$ gauge symmetry. The quantum analogues of the $SL(2, C)$ matrices are quantum double matrices which are then associated with each link on the lattice. The commutation relations for the latter matrices are required to be covariant under $SU_q(2)$ gauge transformations, while the quantum Hamiltonian is invariant.

After this work was completed we learned of a series of papers by S.A. Frolov[8] where a similar system is discussed. It differs from ours in the nature of gauge transformations. For us, all link variables transform in an identical fashion at any particular site, as in the spirit of Kogut and Susskind. As a result of this it is easy to write down the Wilson loop operators, but difficult to give an explicit expression for the gauge generator (i.e., the Gauss
law) in terms of the link variables. (So far we have been unable to do it.) On the other hand, Frolov simply postulates a Gauss law. From it one then obtains a complicated gauge transformation rule for the link variables (where links attached to same vertex transform differently), and hence a complicated form for the Wilson loop operators.

2 Review of the Kogut Susskind Hamiltonian Formalism

As stated earlier, we limit our discussion to $SU(2)$ gauge theories in the absence of fermions, and the Hamiltonian formulation of lattice gauge theories\cite{2} is then described solely in terms of rigid rotators. In the standard formulation the phase space for a single rotator is spanned by angular momentum variables $j_a$ and an $SU(2)$ matrix $u$ with Poisson brackets given by

$$
\{j_a, j_b\} = \epsilon_{abc} j_c ,
$$

$$
\{j_a, u\} = \frac{i}{2} \sigma_a u ,
$$

with the Poisson brackets of matrix elements of $u$ with themselves being zero. $\sigma_a$ are Pauli matrices. The brackets for $u$ and $j_a$ are preserved under the following transformation:

$$
uu
\begin{align*}
\{j_a, j_b\} &= \epsilon_{abc} j_c , \\
\{j_a, u\} &= \frac{i}{2} \sigma_a u ,
\end{align*}
$$

where $v_L$ and $v_R$ are independent $SU(2)$ matrices. It easily follows that (3) and (4) are canonical transformations.

Physically, we are to suppose that a rotator [with Poisson brackets given by (1) and (2)] is attached to each link on a three dimensional cubic lattice, while gauge transformations
[analogous to those in (3) and (4) parametrized by \( v_L \) and \( v_R \)] are associated with the points on the lattice. We make this statement more precise below.

Following ref. [2], an arbitrary point on a lattice will be denoted by a vector \( \vec{r} \), while a link connecting point \( \vec{r} \) to a neighboring point in the direction \( \hat{m} \) is denoted by \((\vec{r}, \hat{m})\). Here \( \hat{m} \) can be one of the six unit vectors running along the lattice. For each link \((\vec{r}, \hat{m})\) we have the phase space variables

\[
u(\vec{r}, \hat{m}) \in SU(2) \quad \text{and} \quad j_\xi(\vec{r}, \hat{m}) .
\]

The inverse of \(u(\vec{r}, \hat{m})\) is obtained by traversing the link \((\vec{r}, \hat{m})\) in the \(-\hat{m}\) direction, i.e.

\[
u(\vec{r}, \hat{m})^{-1} = \nu(\vec{r} + a\hat{m}, -\hat{m}) .
\]

The phase space for \(SU(2)\) lattice gauge theory consists of all variables (5). Their nonvanishing Poisson brackets are given by

\[
\{ j_\xi(\vec{r}, \hat{m}), j_\zeta(\vec{r}, \hat{m}) \} = \epsilon_{abc} j_\zeta(\vec{r}, \hat{m}) \\
\{ j_\xi(\vec{r}, \hat{m}), \nu(\vec{r}, \hat{m}) \} = i\sigma_\xi \nu(\vec{r}, \hat{m}) ,
\]

for all links \((\vec{r}, \hat{m})\).

The Poisson brackets (7) and (8) must yield the standard Poisson structure for Yang-Mills theory in the continuum limit, i.e.

\[
\{ A^a_i(\vec{x}), E^b_j(\vec{y}) \} = \delta_{ij} \delta_{ab} \delta^3(\vec{x} - \vec{y}) ,
\]

where \( A^a_i(\vec{x}) \) are Yang-Mills potentials and \( E^b_j(\vec{y}) \) are the electric field strengths. In order to recover this bracket from (7) and (8), one interprets \(u(\vec{r}, \hat{m})\) according to

\[
u(\vec{r}, \hat{m}) = \exp(iag\sigma_\xi \frac{1}{2} A^\xi_i(\vec{r})\hat{m}_i) .
\]
Then \( u(\vec{r}, \hat{m}) \) goes to \( 1 + i a q \frac{e}{2} A^a_i(\vec{r}) \hat{m}^i \) when \( a \to 0 \). \( j_a(\vec{r}, \hat{m}) \) is interpreted as the line integral of the electric field along the link \((\vec{r}, \hat{m})\). Thus

\[
j_a(\vec{r}, \hat{m}) = -\frac{a^2}{g} E_i^a(\vec{r}) \hat{m}^i. \tag{11}\]

Using \((8)\) we then get

\[
\{ A^a_i(\vec{r}), E^b_j(\vec{r}) \} \to \frac{1}{a^2} \delta_{ij} \delta_{ab}, \quad \text{as } a \to 0, \tag{12}\]

which agrees with \((9)\). [From \((7)\) the Poisson brackets of \( E(\vec{r}) \) with itself go like \( 1/a^2 \) which as a density distribution vanishes in the continuum limit.]

Gauge transformations are associated with points on the lattice. At the point \( \vec{r} \) we define

\[
v(\vec{r}) \in SU(2). \]

As we are in the Hamiltonian formulation of the theory the gauge transformations are time independent. Gauge transformations on the phase space variables \( u(\vec{r}, \hat{m}) \) and \( j_a(\vec{r}, \hat{m}) \) correspond to

\[
u(\vec{r}, \hat{m}) \to v(\vec{r})^\dagger u(\vec{r}, \hat{m}) v(\vec{r} + a \hat{m}), \tag{13}\]

\[
j_a(\vec{r}, \hat{m}) \sigma_a \to v(\vec{r})^\dagger j_a(\vec{r}, \hat{m}) \sigma_a v(\vec{r}), \tag{14}\]

The transformation \((13)\) is consistent with \((6)\). Upon comparing with \((3)\) and \((4)\) it is evident that \((13)\) and \((14)\) are canonical transformations.

The next task is to write down the Hamiltonian. The requirements are that it be gauge invariant and also that it reduces to the standard field theory Hamiltonian in the limit of zero lattice spacing. Concerning the first requirement, an obvious set of gauge invariant quantities
are the kinetic energies of the rotators. Indeed, the sum of all such kinetic energies is one ingredient $H_0$ in the Kogut Susskind Hamiltonian,

$$H_0 = \frac{g^2}{2a} \sum_{\vec{r}, \vec{m} > 0} j_\nu(\vec{r}, \vec{m}) j_\nu(\vec{r}, \vec{m})$$  \hspace{1cm} (15)$$

where $\vec{m} > 0$ indicates that the sum is only over positive directions of $\vec{m} > 0$. The constants $g$ and $a$ represent the gauge coupling and lattice spacing, respectively. The Hamiltonian (15) cannot be the full story since it leads to a trivial system with all rotators noninteracting. Furthermore, in the limit of zero lattice spacing, $H_0$ only gives the electric field contribution to the QCD Hamiltonian. It is well known that the magnetic field contribution can come from Wilson loop variables constructed on the lattice. For this we let $\Gamma(\vec{r}, \vec{m}, \vec{n})$ denote a square plaquette connecting the point $\vec{r}$ to its nearest neighbors in the lattice $\vec{m}$ and $\vec{n}$ directions and then denote the associated Wilson loop by $W(\Gamma(\vec{r}, \vec{m}, \vec{n}))$. It is given by

$$W(\Gamma(\vec{r}, \vec{m}, \vec{n})) = \text{Tr} u(\vec{r}, \vec{m}) u(\vec{r} + a\vec{m}, \vec{n}) u(\vec{r} + a\vec{n}, \vec{m})^\dagger u(\vec{r}, \vec{n})^\dagger.$$  \hspace{1cm} (16)$$

From (13) it is clear that (16) is gauge invariant for all $\vec{r}$, $\vec{m}$ and $\vec{n}$. It is also clear that terms like (16) introduce nontrivial interactions between the rotators. Upon writing $u(\vec{r}, \vec{m})$ according to (10), it has been shown that $W(\Gamma(\vec{r}, \vec{m}, \vec{n}))$ yields the usual magnetic field contribution to the action [associated with the plaquette $\Gamma(\vec{r}, \vec{m}, \vec{n})$] upon taking the continuum limit $a \to 0$. More specifically, upon expanding to fourth order in $a$ we get that

$$W(\Gamma(\vec{r}, \vec{m}, \vec{n})) \to \text{Tr} \left( \mathbb{I} - \frac{1}{2} g^2 a^4 F_{\vec{m}\vec{n}}(\vec{r})^2 \right), \quad \text{as } a \to 0,$$  \hspace{1cm} (17)$$

where $F_{\vec{m}\vec{n}}(\vec{r}) = F^{\vec{z}}_{ij}(\vec{r}) \frac{\gamma_2}{2} \vec{m}; \vec{n}, F_{ij}^{\vec{z}} = \epsilon_{ijk} B_k^{\vec{z}}$ being the magnetic field strength tensor. We can finally express the Kogut Susskind Hamiltonian according to

$$H = H_0 + H_1, \quad H_1 = \frac{1}{2ag^2} \sum_{\vec{r}, \vec{m}, \vec{n}} \left( W(\Gamma(\vec{r}, \vec{m}, \vec{n})) + W(\Gamma(\vec{r}, \vec{m}, \vec{n}))^* - 4 \right).$$  \hspace{1cm} (18)$$
The sum in $H_1$ is over all plaquettes. The coefficients in $H_0$ and $H_1$ were chosen so that $H$ has the usual continuum limit, i.e.

$$H \rightarrow \frac{1}{2} \sum p \, a^2 \left( E_i^a (\vec{r}) E_i^a (\vec{r}) + B_i^a (\vec{r}) B_i^a (\vec{r}) \right) , \quad \text{as } a \rightarrow 0 ,$$

(19)

A final ingredient in this system (which turns out to be a source of difficulty for us when we deform the system) is that the gauge symmetry (13) and (14) implies the existence of first class constraints. These constraints $G_a (\vec{r}) \approx 0$ are defined at points on the lattice and they generate the gauge symmetry. Thus their nonvanishing Poisson brackets may be given by:

$$\{ G_a (\vec{r}) , j_b (\vec{r} , \hat{m}) \} = \epsilon_{abc} \, j_c (\vec{r} , \hat{m}) , \quad \text{(20)}$$

$$\{ G_a (\vec{r}) , u (\vec{r} , \hat{m}) \} = \frac{i}{2} \sigma_a \, u (\vec{r} , \hat{m}) . \quad \text{(21)}$$

A solution for $G_a (\vec{r})$ is

$$G_a (\vec{r}) = \sum_m j_a (\vec{r} , \hat{m}) , \quad \text{(22)}$$

the sum being over all links connected to the lattice point $\vec{r}$. In the continuum limit this gives the usual Gauss law constraint. In the quantum theory we must impose that the operator corresponding to $G_a (\vec{r})$ annihilates all physical states.

3 Alternative Hamiltonian formulation of the Rigid Rotator

We next review the alternative classical Hamiltonian formulation of the rotator given in ref. [3]. There it was shown that the six dimensional phase space describing a rigid rotator
could be spanned by the set \( \{ d^{(-)} \} \) of \( 2 \times 2 \) complex unimodular matrices which constitute ‘classical double variables’. Thus the phase space is \( SL(2, C) \).

The Hamiltonian of ref. [3] was a function of only \( \text{Tr}d^{(-)}d^{(-)^\dagger} \). More specifically, it was given by

\[
H_{r,cl}(\lambda) = \frac{1}{2\lambda^2} (\text{Tr} \ d^{(-)}d^{(-)^\dagger} - 2),
\]

where \( \lambda \) is a constant which plays the role of a deformation parameter. This Hamiltonian is invariant under left and right \( SU(2) \) transformations:

\[
d^{(-)} \to v_L^\dagger d^{(-)} v_R, \quad v_L, v_R \in SU(2),
\]

and these transformations are analogous to separate rotations of the body and space axes of a rigid rotator.

Although the Hamiltonian is invariant under (24), these transformations do not correspond to canonical symmetries due to a nontrivial quadratic Poisson structure which was assumed for \( d^{(-)} \). Using tensor product notation the Poisson brackets can be compactly written as follows:

\[
\{ d^{(-)}_1, d^{(-)}_2 \} = -d^{(-)}_1 d^{(-)}_2 r - r^\dagger d^{(-)}_1 d^{(-)}_2.
\]

Here \( d^{(-)}_1, d^{(-)}_2 \) and \( r \) denote \( 4 \times 4 \) matrices with \( d^{(-)}_1 = d^{(-)} \otimes I \), \( d^{(-)}_2 = I \otimes d^{(-)} \) and \( r \) given by

\[
r = \frac{i\lambda}{4} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

The brackets (25) are skew symmetric. Furthermore, the \( r \)-matrix (26) is known to satisfy the classical Yang-Baxter relations which insures that the Jacobi identity holds for (25).
It is clear that the transformations (24) do not preserve the Poisson brackets and hence they are not canonical symmetries. They are instead Lie-Poisson symmetries. For this we now associate a certain Poisson structure to the $SU(2)$ matrices $v_L$ and $v_R$ involved in the transformations (24). These Poisson brackets are chosen to be compatible with those of the observables $d(-)$ [cf. eq. (25)] along with the transformation (24). They are given by

\[
\{ v_A, v_A \} = [ r, v_A v_A ], \quad A = L, R
\] (27)

\[
\{ v_L, v_R \} = 0,
\] (28)

\[
\{ v_A, d(-) \} = 0,
\] (29)

where $v_A = v_A \otimes \mathbb{I}$, and $v_A = \mathbb{I} \otimes v_A$. To show compatibility we note that the left hand side of (25) transforms to

\[
\{ v_L^\dagger d(-) v_R, v_L^\dagger d(-) v_R \},
\] (30)

which we now can compute using (25) and (27-29). We get

\[
- v_L^\dagger d(-) v_R v_L^\dagger d(-) v_R - r^\dagger v_L^\dagger d(-) v_R v_L^\dagger d(-) v_R .
\] (31)

Using (24), the right hand side of (25) also transforms to (31), thus showing that the brackets (27-29) for $v_A$ are compatible with (25) and the transformation (24). Consequently (24) is a Lie-Poisson transformation.

We next proceed with the Hamilton equations of motion. For this, the Poisson brackets (25) are insufficient because the Hamiltonian for the system involves $d(-)^\dagger$, as well as $d(-)$.
We therefore need to know the Poisson brackets of \( d(-)^\dagger \) with \( d(-) \). From [3] we have

\[
\{ d(-)_1, d(+)_2 \} = -d(-)_1 d(+)_2 r - r d(-)_1 d(+)_2 ,
\]

(32)

where \( d(\pm) = d(-)^\dagger \). The variables \( d(\pm) \) along with the Poisson brackets (25) and (32) (which are consistent with the Jacobi identity) define the \textit{classical double}. Using eqs. (23), (25) and (32) one obtains the following equations of motion. They were found to be:

\[
\frac{d}{dt} d(-)^{-1} = i \frac{i}{2\lambda} [d(-) d(-)^\dagger]_{tt},
\]

(33)

where \([A]_{t\ell}\) denotes the traceless part of \(2 \times 2\) matrix \(A\), i.e., \([A]_{t\ell} = A - \frac{1}{2} \text{Tr}(A) \times \text{I} \).

To make the connection with the isotropic rigid rotator system, we decompose the \( SL(2, \mathbb{C}) \) matrix \( d(-) \) into the product of an element of the \( SU(2) \) subgroup and an element of the Borel subgroup \( SB(2, \mathbb{C}) \). We do this as follows:

\[
d(-) = \ell(-) u , \quad u \in SU(2) , \quad \ell(-) = \begin{pmatrix} x_0 & 0 \\ x_- & x_0^{-1} \end{pmatrix} \in SB(2, \mathbb{C}) ,
\]

(34)

where \(x_0\) is real and \(x_-\) is complex. The map from \( SU(2) \times SB(2, \mathbb{C}) \) to \( SL(2, \mathbb{C}) \) is two-to-one. The Poisson brackets (25) for the classical double variables \( d(-) \) are recovered with the following choice of brackets for \( u \) and \( \ell(-) \):

\[
\{ u_{12} , u_{12} \} = [ r_{12} , u_{12} ] ,
\]

(35)

\[
\{ \ell(-)_1 , \ell(-)_2 \} = -[ r_{12} , \ell(-)_1 \ell(-)_2 ]
\]

(36)

\[
\{ \ell(-)_1 , u_{12} \} = -\ell(-)_1 r_{12} u .
\]

(37)
Upon substituting (34) into the equation of motion (33) we get

$$\frac{d}{dt} u^\dagger u - i\frac{1}{2\lambda} [\ell(-)^\dagger \ell(-)]_{\mu\ell} = -\ell(-)^{-1} \frac{d}{dt} \ell(-).$$

(38)

The left hand side of eq. (38) is traceless and antihermitean, while the right hand side is an element of the $SB(2,C)$ Lie algebra. It follows that the left and right hand sides must separately vanish, leading to:

$$\frac{d}{dt} u^\dagger u = \frac{i}{2} J_a \sigma_a, \quad \text{where} \quad J_a = \frac{1}{2\lambda} \text{Tr} \, \ell(-)^\dagger \ell(-) \sigma_a,$$

(39)

along with

$$\frac{d}{dt} \ell(-) = 0.$$  

(40)

$J_a$ in eq. (39) can now be interpreted as the physical angular momenta of the isotropic rotator. Since they are functions of only $\ell(-)$ and $\ell(-)^\dagger$, they are conserved. Now if we associate the $SU(2)$ matrix $u$ with the orientation of the rotator (or the transformation between space and body axes), then eq. (39) gives the desired result, namely, that the rigid body undergoes a uniform precession.

The above Hamiltonian formulation of the isotropic rotator is a deformation of the usual Hamiltonian formulation, where as we stated earlier, $\lambda$ plays the role of the deformation parameter. The usual Hamiltonian formulation is recovered when $\lambda$ goes to zero, which we refer to as the 'canonical limit'. For this to happen we must first show how to express the matrix $\ell(-)$ in terms of the canonical angular momentum variables $j_a$ [as opposed to the variables $J_a$ which do not obey the canonical Poisson bracket relations (1)]. We write

$$\ell(-) = \exp (i\lambda e^a j_a),$$

(41)

where $e^a$ are generators of $SB(2,C)$. They can be expressed in terms of Pauli matrices $\sigma_a$.
as follows:

\[ e^a = \frac{1}{2}(i\sigma_a + \epsilon_{a\hat{b}\hat{c}}\sigma_{\hat{b}}). \]  

(Actually, for the purpose of taking the canonical limit, we only need that relation \( (41) \) holds up to second order in \( \lambda \).) Eq. \( (41) \) is equivalent to assigning \( x_0 \) and \( x_- \) in \( (34) \) according to

\[ x_0 = \exp\left(-\frac{\lambda j_3}{2}\right), \quad x_- = -2 \frac{j_1 + ij_2}{j_3}\text{sinh}\left(\frac{\lambda j_3}{2}\right). \]  

Now to lowest order in \( \lambda \), the Hamiltonian \((23)\) reduces to

\[ H_{rot}(\lambda \to 0) = \frac{1}{2} j_a j_a, \]  

while, using eq. \((34)\), the Poisson brackets \((35-37)\) go to

\[ \{j_a,j_b\}_{\lambda \to 0} = \epsilon_{abc} j_c, \]  

\[ \{j_a,u\}_{\lambda \to 0} = \frac{i}{2}\sigma_a u, \]  

\[ \{u_1,u_2\}_{\lambda \to 0} = 0. \]

which agrees with eqs. \((1)\) and \((2)\). We thus arrive at the canonical description of an isotropic rotator with the moment of inertia set equal to one. The Hamiltonian \((23)\) and the Poisson brackets \((35-37)\) therefore define a one parameter deformation of the canonical description of the isotropic rigid rotator. When \( \lambda \neq 0 \), the chiral transformations are Lie-Poisson symmetries. They reduce to canonical symmetries when \( \lambda \to 0 \). In that limit, the variables \( u \) and \( j_a \) transform in the usual way, i.e. as in \((3)\) and \((4)\).

4 Deformation of \( SU(2) \) lattice gauge theory

We are now ready to deform the Kogut Susskind Hamiltonian dynamics in a way which preserves the \( SU(2) \) gauge symmetry of the classical Hamiltonian function - but which
replaces the canonical symmetry transformations into Lie-Poisson transformations. As before we assign rotator degrees of freedom to each link on the lattice. However, now we shall describe the rotator degrees of freedom in terms of classical double variables

\[ d^{(-)}(\vec{r}, \hat{m}) \in SL(2, C), \]  

(48)

rather than the variables \( j_s(\vec{r}, \hat{m}) \) and \( u(\vec{r}, \hat{m}) \).

To each classical double variable \( d^{(-)}(\vec{r}, \hat{m}) \) we assign the Poisson structure given by (25) and (32). Thus we replace the set of Poisson brackets (7) and (8) by

\[
\left\{ d^{(-)}_1(\vec{r}, \hat{m}), d^{(-)}_2(\vec{r}, \hat{m}) \right\} = -d^{(-)}_1(\vec{r}, \hat{m}) d^{(-)}_2(\vec{r}, \hat{m}) r - r^\dagger d^{(-)}_1(\vec{r}, \hat{m}) d^{(-)}_2(\vec{r}, \hat{m}), \]  

(49)

\[
\left\{ d^{(-)}_1(\vec{r}, \hat{m}), d^{(+)}_2(\vec{r}, \hat{m}) \right\} = -d^{(-)}_1(\vec{r}, \hat{m}) d^{(+)}_2(\vec{r}, \hat{m}) r - r^\dagger d^{(-)}_1(\vec{r}, \hat{m}) d^{(+)}_2(\vec{r}, \hat{m}), \]  

(50)

for all links \((\vec{r}, \hat{m})\), and where

\[ d^{(+)}(\vec{r}, \hat{m}) = d^{(-)}(\vec{r}, \hat{m})^\dagger. \]  

(51)

[In analogy to (6), we obtain the inverse of \( d^{(-)}(\vec{r}, \hat{m}) \) by traversing the link \((\vec{r}, \hat{m})\) in the \(-\hat{m}\) direction, i.e.

\[ d^{(-)}(\vec{r}, \hat{m})^{-1} = d^{(-)}(\vec{r} + a\hat{m}, -\hat{m}). \]  

(52)

In (49) and (50), we once again resort to tensor product notation with \( d^{(\pm)}_1(\vec{r}, \hat{m}) = d^{(\pm)}(\vec{r}, \hat{m}) \otimes 1\), \( d^{(\pm)}_2(\vec{r}, \hat{m}) = 1 \otimes d^{(\pm)}(\vec{r}, \hat{m}) \) and the \( r \)-matrix is given by (26). Since the \( r \)-matrix depends on \( \lambda \) we are introducing a new parameter (in addition to \( a \)) to the lattice theory. We shall require that the Kogut Susskind Hamiltonian formalism is recovered in the
limit $\lambda \to 0$. This is in addition to the requirement that the Hamiltonian function be gauge invariant.

With regard to gauge invariance, for us gauge transformations are associated with points on the lattice, just as was the case in the Kogut Susskind formalism. Thus at the point $\vec{r}$ we once again define $v(\vec{r}) \in SU(2)$. Now in analogy to (24), we uniquely define gauge transformations of the phase space variables $d^{(\pm)}(\vec{r}, \hat{m})$ according to

$$d^{(\pm)}(\vec{r}, \hat{m}) \to v(\vec{r})^\dagger d^{(\pm)}(\vec{r}, \hat{m}) v(\vec{r} + a\hat{m}) \quad (53)$$

As was true for a single rotator, such transformations are not canonical. Instead they are Lie Poisson. In this regard, we attach a product Poisson structure to the gauge degrees of freedom $v(\vec{r})$,

$$\left\{ v_1(\vec{r}), v_2(\vec{r}) \right\} = [ r , v_1(\vec{r}) v_2(\vec{r}) ] , \quad (54)$$

This Poisson structure is compatible with the brackets (49) and (50), along with the transformations (53). Hence (53) are Lie-Poisson transformations.

From (23) we already know how to write down the gauge invariant deformation of the first term $H_0$ in the Kogut Susskind Hamiltonian. It is just

$$H_0(\lambda) = \frac{g^2}{2a\lambda^2} \sum_{\vec{r},m>0} \left( \text{Tr} \ d^{(-)}(\vec{r}, \hat{m}) \ d^{(-)}(\vec{r}, \hat{m})^\dagger - 2 \right) . \quad (55)$$

We can make a decomposition of the $SL(2,C)$ matrix $d^{(-)}(\vec{r}, \hat{m})$ in terms of $SU(2)$ and $SB(2,C)$ subgroups in an identical manner to what was done in the previous section, i.e.

$$d^{(-)}(\vec{r}, \hat{m}) = \ell^{(-)}(\vec{r}, \hat{m}) \ u(\vec{r}, \hat{m}) , \quad u(\vec{r}, \hat{m}) \in SU(2) , \quad \ell^{(-)}(\vec{r}, \hat{m}) \in SB(2,C) . \quad (56)$$

The Poisson brackets (49) for the classical double variables $d^{(-)}(\vec{r}, \hat{m})$ are recovered with
the following choice of brackets for $u(\vec{r}, \hat{m})$ and $\ell(-)(\vec{r}, \hat{m})$:

$$\{ u_1(\vec{r}, \hat{m}), u_2(\vec{r}, \hat{m}) \} = [ r_1, u_1(\vec{r}, \hat{m}) u_2(\vec{r}, \hat{m}) ] ,$$  

(57)

$$\{ \ell_1(-)(\vec{r}, \hat{m}), \ell_2(-)(\vec{r}, \hat{m}) \} = - [ r_1, \ell_1(-)(\vec{r}, \hat{m}) \ell_2(-)(\vec{r}, \hat{m}) ]$$  

(58)

$$\{ \ell_1(-)(\vec{r}, \hat{m}), u_2(\vec{r}, \hat{m}) \} = - \ell_1(-)(\vec{r}, \hat{m}) r_1 u_2(\vec{r}, \hat{m}) .$$  

(59)

Furthermore, we make the $SB(2,C)$ matrix depend on $\lambda$ as in the previous section, i.e.

$$\ell(-)(\vec{r}, \hat{m}) = \exp \left( i \lambda \epsilon^a j_a(\vec{r}, \hat{m}) \right) .$$  

(60)

It then follows that $H_0(\lambda)$ is a deformation of $H_0$, i.e. in analogy to (44), we have

$$H_0(\lambda \to 0) = H_0 .$$

Also the transformation (53) reduce to canonical transformations (13) and (14) in the limit.

It is also straightforward to write down the gauge invariant deformation of the second term $H_1$ in the Kogut Susskind Hamiltonian. For this we construct a set of Wilson loop variables $W^\lambda(\Gamma(\vec{r}, \hat{m}, \hat{n}))$ using $\ell(-)(\vec{r}, \hat{m})$,

$$W^\lambda(\Gamma(\vec{r}, \hat{m}, \hat{n})) = \text{Tr} \ d(-)(\vec{r}, \hat{m}) d(-)(\vec{r} + a \hat{m}, \hat{n}) d(\mp)(\vec{r} + a \hat{n}, \hat{m}) \ d(\mp)(\vec{r}, \hat{n}) .$$  

(61)

From (53) it easily follows that $W^\lambda(\Gamma(\vec{r}, \hat{m}, \hat{n}))$ is gauge invariant. In this regard (61) is not unique, as we can arbitrarily replace the different factors $d(\pm)$ with $d(\mp)$ in the formula for $W^\lambda(\Gamma(\vec{r}, \hat{m}, \hat{n}))$. On the other hand we shall show that for the choice (61) we recover the correct continuum limit ($a \to 0$) of $SU(2)$ gauge theory. Concerning the canonical limit $\lambda \to 0$, upon using (56) and (60), we get that $d(\pm)(\vec{r}, \hat{m}) \to u(\vec{r}, \hat{m})$ and hence

$$W^{\lambda \to 0}(\Gamma(\vec{r}, \hat{m}, \hat{n})) = W(\Gamma(\vec{r}, \hat{m}, \hat{n})) .$$
Thus $W^\lambda(\Gamma_{(\vec{r},\vec{m},\vec{n})})$ is a gauge invariant deformation of $W(\Gamma_{(\vec{r},\vec{m},\vec{n})})$. We can now write down the gauge invariant deformation of the Kogut Susskind Hamiltonian. It is

$$H(\lambda) = H_0(\lambda) + H_1(\lambda) , \quad H_1(\lambda) = \frac{1}{ag^2} \sum_{\Box} \left( W^\lambda(\Gamma_{(\vec{r},\vec{m},\vec{n})}) + W^\lambda(\Gamma_{(\vec{r},\vec{m},\vec{n})})^* - 4 \right). \quad (62)$$

The sum in $H_1$ is over all plaquettes.

The final ingredient in this system is the analogue of the Gauss law constraints $G_\lambda^\alpha(\vec{r}) \approx 0$. They generate the gauge symmetry (53) and thus their non vanishing Poisson brackets should be of the form:

$$\{ G_1^\lambda(\vec{r}), \frac{d}{2}(\vec{r}, \hat{m}) \} = X(\vec{r}) \frac{d}{2}(\vec{r}, \hat{m}). \quad (63)$$

We want that $G_1^\lambda(\vec{r}) \to G_\alpha(\vec{r})\frac{a}{2}$ when $\lambda \to 0$. We have been unable to find an explicit solution for $G_1^\lambda(\vec{r})$ expressed in terms of $\ell(\vec{r}, \hat{m})$ and $u(\vec{r}, \hat{m})$ which is consistent with these requirements.

Above we have shown that the canonical limit ($\lambda \to 0$) of our model is the system of Kogut Susskind. We now show that the continuum limit ($a \to 0$) of our model is standard $SU(2)$ gauge theory (for any value of $\lambda$). In this regard, we once again write $u(\vec{r}, \hat{m})$ in terms of Yang-Mills potentials $A_\alpha^a(\vec{x})$ according to (10). In addition, using (11) and (41) we get that

$$\ell(\vec{r}, \hat{m}) = \exp \left( -i\lambda \frac{g^2}{a} e^a E_i^a(\vec{r}) \hat{m}_i \right). \quad (64)$$

Now from the Poisson bracket (59) we get that

$$\{ E_i^a(\vec{r}), A_j^b(\vec{r}) \} e^a \otimes \sigma^b \to -\frac{2}{\lambda a^3} \delta_{ij} r , \quad (65)$$

when $a \to 0$. Because the classical $r$–matrix can be written $r = \frac{\lambda}{2} e^a \otimes \sigma^a$, we recover the canonical Poisson brackets (12) for $SU(2)$ gauge theory. [These are the only nonvanishing
Poisson brackets, as \{E, E\} and \{A, A\} go like \(a^n, \ n > -3\), which as a density distribution vanishes in the continuum limit.

It remains to take the continuum limit of the Hamiltonain \(H(\lambda), \) eq. (55). In this regard we note that the term \(H_0(\lambda)\) can be written

\[
H_0(\lambda) = \frac{g^2}{2a \lambda^2} \sum_{\vec{r}, \hat{m} > 0} \left( \text{Tr} \ e^{\lambda(\vec{r}, \hat{m})} e^{\lambda^{-1}(\vec{r}, \hat{m})^\dagger} - 2 \right) .
\]  

Then using (64) we find that \(H_0(\lambda)\) yields the electric field energy of \(SU(2)\) gauge theory

\[
H_0(\lambda) \to \frac{1}{2} \sum \vec{r} a^3 E_i^z(\vec{r}) E_i^z(\vec{r}) ,
\]  

as \(a \to 0\). Furthermore, \(H_1(\lambda)\) in eq. (62) yields the magnetic field energy of \(SU(2)\) gauge theory but this requires some algebra to prove. To proceed we use (10) and (64) to write the link variables \(d^{(\lambda)}(\vec{r}, \hat{m})\) according to

\[
d^{(\lambda)}(\vec{r}, \hat{m}) = \exp \left( -i \lambda \frac{a^2}{g} e^z E_i^z(\vec{r}) \hat{m} \right) \exp \left( i a g \frac{\sigma^z}{2} A_i^z(\vec{r}) \right), \quad \hat{m} > 0,
\]  

where here we find it more convenient to evaluate \(A\) and \(E\) at the central point \(\vec{r}_c\) of the link \((\vec{r}, \hat{m})\). We shall assume that (68) is valid for \(\hat{m} > 0\). Due to (52) it cannot then also be valid for \(\hat{m} < 0\). Instead, from (52) we get that

\[
d^{(\lambda)}(\vec{r}, \hat{m}) = d^{(\lambda)}(\vec{r} + a \hat{m}, -\hat{m})^{-1} = \exp \left( i a g \frac{\sigma^z}{2} A_i^z(\vec{r}) \hat{m} \right) \exp \left( -i \lambda \frac{a^2}{g} e^z E_i^z(\vec{r}) \hat{m} \right), \quad \hat{m} < 0 .
\]  

We now consider the plaquette \(\Gamma_0\) in the \(1 - 2\) plane centered at \(\vec{x} = (x_1, x_2)\). Upon substituting (68) and (69) into (61) and taking \(\vec{r} = (x_1 - \frac{a}{2}, x_2 - \frac{a}{2}), \hat{m} = (1, 0)\) and \(\hat{n} = (0, 1)\), we get the following expression for \(W^\lambda(\Gamma_0)\):

\[
W^\lambda(\Gamma_0) = \text{Tr} \exp \left( -i \lambda \frac{a^2}{g} e^z E_i^z(x_1, x_2 - a/2) \right) \exp \left( i a g \frac{\sigma^z}{2} A_i^z(x_1, x_2 - a/2) \right)
\]
\begin{align}
&\times \exp \left( -i \frac{a^2}{g} e^a E^a_2 (x_1 + a/2, x_2) \right) \exp \left( i a g \frac{\sigma^a}{2} A^a_2 (x_1 + a/2, x_2) \right) \\
&\times \exp \left( -i a g \frac{\sigma^a}{2} A^a_1 (x_1, x_2 + a/2) \right) \exp \left( i \frac{a^2}{g} e^a E^a_1 (x_1, x_2 + a/2) \right) \\
&\times \exp \left( -i a g \frac{\sigma^a}{2} A^a_2 (x_1 - a/2, x_2) \right) \exp \left( i \frac{a^2}{g} e^a E^a_2 (x_1 - a/2, x_2) \right), \quad (70)
\end{align}

where we used \( d^{(\pm)}(\vec{r}, \vec{n})^\dagger = d^{(-)}(\vec{r} + a \hat{n}, -\hat{n}) \) which follows from (51) and (52). After expanding this expression to fourth order in \( a \), some work shows that we recover the Kogut-Susskind result, i.e.

\[
W^\lambda (\Gamma_0) \to \text{Tr} \left( \mathbb{1} - \frac{1}{2} g^2 a^A (F_{12}(\vec{x}))^2 \right), \quad \text{as} \; a \to 0,
\]

which is the same as (17). Hence \( H_1(\lambda) \) yields the magnetic field energy

\[
H_1(\lambda) \to \frac{1}{2} \sum_{\vec{r}} a^3 B^\xi_\gamma (\vec{r}) B^\gamma_\xi (\vec{r}), \quad \text{as} \; a \to 0,
\]

and the total Hamiltonian \( H(\lambda) \) reduces to that of \( SU(2) \) Yang-Mills theory in the continuum limit.

## 5 Quanitzation

We now consider the quantization of the lattice theory discussed in the previous section. When Lie-Poisson symmetries are present in the classical theory the standard practice is to apply the method of deformation quantization [7], where we do not identify the quantum mechanical commutation relations with \( i\hbar \) times the corresponding classical Poisson brackets, but only demand that they agree in the limit \( \hbar \to 0 \). Also we do not identify the quantum Hamiltonian \( H(\lambda) \) function (with classical variables replaced by their corresponding quantum operators) with the classical Hamiltonian function \( H(\lambda) \). We instead only demand that \( H(\lambda) \) reduces to \( H(\lambda) \) in the limit \( \hbar \to 0 \).
Fixing the quantum dynamics using the method of deformation quantization requires writing down a star product on the space of classical observables. This is generally a difficult task and shall not be attempted here. Instead, we shall only demand that the $SU(2)$ Poisson Lie group gauge symmetry which is present in the classical theory gets replaced upon quantization by a gauge symmetry which is associated with the quantum group $SU_q(2)$ [6]. The latter can be defined in terms of $2 \times 2$ matrices $\{T\}$ whose matrix elements $T_{ij}$ are not $c$-numbers. Rather they satisfy the commutation relations:

\[ RT_{12} T_{21} = T_{21} T_{12} R, \]

with $T = T \otimes 1_2$, $T = 1_1 \otimes T$ and $R$, the quantum $R$-matrix, given by

\[ R = q^{-1/2} \begin{pmatrix} q & 1 \\ q - q^{-1} & 1 \end{pmatrix}, \]

where $q$ is a $c$-number. In addition to (73), $T$ satisfies a unitarity condition $T^\dagger T = 1_2$ and also a deformed unimodularity condition $det_q T = 1$, where $det_q T = T_{11}T_{22} - qT_{12}T_{21}$. The latter constraint is possible because $det_q T$ so defined commutes with all matrix elements $T_{ij}$. $R$ satisfies the quantum Yang-Baxter equation. If we set $q = e^{\pm \lambda/2}$, then in the limit $\hbar \to 0$ $R$ tends to $1_2 - i\hbar r + O(\hbar^2)$, and consequently (73) reduces to

\[ [T_{12}, T_{12}] = i\hbar [r, T_{12} T_{12}] + O(\hbar^2). \]

We thereby recover the algebra (27) of the $SU(2)$ matrices $v_A$ parametrizing the Lie-Poisson symmetries.
In quantizing the lattice theory described in the previous section, we would like to define an $SU_q(2)$ matrix $T(\vec{r})$ at each point $\vec{r}$ on the lattice. Then the quantum analogue of the Poisson brackets (54) is

$$R \ T (\vec{r}) \ T (\vec{r}) = T (\vec{r}) \ T (\vec{r}) R \ ,$$

and we assume that $SU_q(2)$ matrices at different points commute.

With regards to the $SL(2, \mathbb{C})$ matrix $d^{(-)}$, we replace it by the $2 \times 2$ matrix $D^{(-)}$ having operators as matrix elements. The quantum analogues of the Poisson brackets (25) and (32) are [3]

$$R^{(+)} \ D^{(-)} \ D^{(-)} = D^{(-)} \ D^{(-)} \ R \ ,$$

$$R^{(-)} \ D^{(-)} \ D^{(+)} = D^{(+)} \ D^{(-)} \ R \ ,$$

where $D^{(+)} = D^{(-)^{-1}}$, $R^{(+)} = R^T$ (the superscript $T$ denoting transpose) and $R^{(-)} = R^{-1}$. The matrices $D^{(\pm)}$ along with the commutation relations (77) and (78) define the quantum double. In the limit $\hbar \to 0$, $R^{(+)}$ and $R^{(-)}$ tend to $1 + i\hbar r^T + O(\hbar^2)$ and $1 - i\hbar r + O(\hbar^2)$, respectively, and the algebra given in (25) and (32) is recovered from (77) and (78) to first order in $\hbar$. In addition, the commutation relations (77) and (78) are covariant under left and right $SU_q(2)$ transformations[3]:

$$D^{(-)} \to T^\dagger_L \ D^{(-)} \ T_R \ , \quad T_L, T_R \in SU(2) \ .$$

Here both $T_L$ and $T_R$ satisfy commutation relations (73) and we assume that matrix elements of $T_L$ commute with matrix elements of $T_R$. 

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In the lattice theory, we assign a matrix $D(-)(\vec{r}, \hat{m})$ to each link $(\vec{r}, \hat{m})$ on the lattice. Then the quantum analogues of the Poisson brackets (49) and (50) are

$$R_{1}^{(+)}D_{1}(\vec{r}, \hat{m})D_{2}(\vec{r}, \hat{m}) = D_{2}(\vec{r}, \hat{m})D_{1}(\vec{r}, \hat{m})R_{1}, \tag{80}$$

$$R_{1}^{(-)}D_{1}(\vec{r}, \hat{m})D_{2}(\vec{r}, \hat{m}) = D_{2}(\vec{r}, \hat{m})D_{1}(\vec{r}, \hat{m})R_{1}, \tag{81}$$

where $D^{(\pm)}(\vec{r}, \hat{m}) = (D^{(-)}(\vec{r}, \hat{m}))^{-1}$, and we assume that $D$ matrices associated with different links commute. Now gauge transformations on the quantum double variables are given by

$$D^{(-)}(\vec{r}, \hat{m}) \rightarrow T(\vec{r})^\dagger D^{(-)}(\vec{r}, \hat{m})T(\vec{r} + \hat{a}\hat{m}), \tag{82}$$

and as they are of the same form as (79) they preserve the commutation relations [eqs. (80) and (81)]. The commutation relations for $D^{(\pm)}(\vec{r}, \hat{m})$ are therefore covariant under $SU_q(2)$ gauge transformations.

We next must write down the quantum analogues of $H_0(\lambda)$ and $H_1(\lambda)$. Our requirements are that these terms are invariant under $SU_q(2)$ gauge transformations (82) and also that they reduce to $H_0(\lambda)$ and $H_1(\lambda)$ when $\hbar \rightarrow 0$.

We begin with $H_0(\lambda)$. It represents the sum of kinetic energies of the rotators. Its quantum analogue $H_0(\lambda)$ is known[3]. To write it we need to introduce the “quantum” trace $\text{Tr}_q$. [9] $\text{Tr}_q$ of a $2 \times 2$ matrix $M = [M_{ij}]$ is defined according to

$$\text{Tr}_q M = qM_{11} + q^{-1}M_{22}. \tag{83}$$

Unlike the usual trace, $\text{Tr}_q$ does not have the general property of invariance under cyclic permutations. It does however serve as an “adjoint invariant” for $SU_q(2)$. By this we mean the following:

$$\text{Tr}_q T^{-1}MT = \text{Tr}_q M, \tag{84}$$
where $T$ satisfies the commutation relations (73) and we assume that matrix elements of $T$ commute with those of $M$. The relation (84) can be explicitly verified using the $2 \times 2$ representations for $T$. From (84) it follows that

$$\text{Tr}_q D^{(-)}(\hat{\mathbf{r}}, \hat{\mathbf{m}}) D^{(-)}(\hat{\mathbf{r}}, \hat{\mathbf{m}})^\dagger$$

is invariant under $SU_q(2)$ gauge transformations (82). Then a possible choice for $H_0(\lambda)$ is

$$H_0(\lambda) = \frac{g^2}{2a\lambda^2} \sum_{\hat{\mathbf{r}}, \hat{\mathbf{m}} > 0} \left( \text{Tr}_q D^{(-)}(\hat{\mathbf{r}}, \hat{\mathbf{m}}) D^{(-)}(\hat{\mathbf{r}}, \hat{\mathbf{m}})^\dagger - 2 \right),$$

(85)
as it reduces to (55) when $\hbar \to 0$.

Using the quantum trace it is also easy to write down a quantum analogue of the Wilson loop variables $W^\lambda(\Gamma_{(\hat{\mathbf{r}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})})$ defined in (61). We write

$$W^\lambda(\Gamma_{(\hat{\mathbf{r}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})}) = \text{Tr}_q D^{(-)}(\hat{\mathbf{r}}, \hat{\mathbf{m}}) D^{(-)}(\hat{\mathbf{r}} + a\hat{\mathbf{m}}, \hat{\mathbf{n}}) D^{(-)}(\hat{\mathbf{r}} + a\hat{\mathbf{m}} + a\hat{\mathbf{n}}, -\hat{\mathbf{m}}) D^{(-)}(\hat{\mathbf{r}} + a\hat{\mathbf{n}}, -\hat{\mathbf{m}}).$$

(86)

From (84) it follows that $W^\lambda(\Gamma_{(\hat{\mathbf{r}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})})$ is invariant under $SU_q(2)$ gauge transformations (82). The quantum version of the Hamiltonian (62) is then

$$H(\lambda) = H_0(\lambda) + H_1(\lambda), \quad H_1(\lambda) = \frac{1}{a g^2} \sum_{\mathbf{r}, \mathbf{m}, \mathbf{n}} \left( W^\lambda(\Gamma_{(\mathbf{r}, \mathbf{m}, \mathbf{n})}) + W^\lambda(\Gamma_{(\mathbf{r}, \mathbf{m}, \mathbf{n})})^\dagger - 4 \right).$$

(87)
The sum in $H_1$ is again over all plaquettes.

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