K–Theory of Noncommutative Lattices

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Abstract

Noncommutative lattices have been recently used as finite topological approximations in quantum physical models. As a first step in the construction of bundles and characteristic classes over such noncommutative spaces, we shall study their K-theory. We shall do it algebraically, by studying the algebraic K-theory of the associated algebras of ‘continuous functions’ which turn out to be noncommutative approximately finite dimensional (AF) C*-algebras. We also work out several examples.
1 Introduction

Topological lattices, namely finite topological spaces, were introduced in [1] (with the name posets or partially ordered sets) as finite topological approximations to ‘continuum’ spaces. Their ability to capture some of the topological information of the space they approximate has been the motivation for their use in quantum theories [2, 3, 4]. The idea is to construct alternative lattice theories which are able to describe also some topologically non trivial configurations. An example of a promising result in this direction is the construction of \( \theta \)-states for particles on the poset approximations to a circle. In a suitable limit these states give the usual \( \theta \)-states for particles on the circle [2, 3]. Also, there is in progress work on gauge field theories [4].

In [3] it was observed that a poset \( P \) is truly a noncommutative space (in fact a noncommutative lattice) since it can be described as the structure space (space of irreducible representations) of a noncommutative \( C^* \)-algebra \( \mathcal{A} \). Therefore, such an algebra is the analogue of the commutative algebra of complex valued continuous functions defined on any Hausdorff topological space. The algebra \( \mathcal{A} \) can be thought of as an algebra of operator valued functions over \( P \). The use of these algebras leads to noncommutative geometry [5] as the natural tool to construct “geometric” structures on noncommutative lattices. It turns out that the algebras which are relevant are approximately finite dimensional (AF) postliminal algebras. There are in general several algebras which are associated with the same \( P \) and several ways of constructing any such an algebra [6]. In this paper we will use a diagrammatic method due to Bratteli [7]. Although this method gives only one \( \mathcal{A} \) among all possibilities, it will be enough for our purposes.

In this paper, as a first step in the construction of bundles and characteristic classes over noncommutative lattices, we shall study the \( K \)-theory of such noncommutative lattices. We shall do it algebraically, by studying the algebraic \( K \)-theory of the associated algebras. Algebraically, the analogue of vector bundles over a noncommutative lattice \( P \) are projective modules of finite type over the corresponding algebra \( \mathcal{A} \). The \( K \)-theory group of \( \mathcal{A} \) classifies (stable) equivalence classes of such projective modules. For the class of algebras we are interested in, Bratteli diagrams allow to explicitly construct these groups.

The paper is organized as follows. In Section 2 we shall recall the construction of topological lattices as approximation spaces of ‘continuum’ topological spaces. Section 3 is devoted to the description of the noncommutative algebras associated with the noncommutative lattices, and the description of the Bratteli diagrams. Several examples are worked out. Section 4 treats the notion of projective modules and the \( K \)-theory. Again, several examples are presented.
2 Topological Lattices

We shall describe how to construct a topological lattice, namely a topological space consisting of a finite (or in general a countable) number of points from any covering of a 'continuum' topological space \( M \). The idea is simply to identify any two points of \( M \) which cannot be separated or distinguished by the sets in the covering \([1]\).

Let us suppose that we are given a covering \( \mathcal{U} = \{O_\lambda\} \) of \( M \) which is a topology for the latter. Given any two points \( x \) and \( y \) in \( M \), we shall call them equivalent, \( x \sim y \), if every set \( O_\lambda \) containing either point \( x \) or \( y \) contains the other too,

\[
x \sim y \quad \text{if and only if} \quad x \in O_\lambda \Leftrightarrow y \in O_\lambda \quad \forall \ O_\lambda .
\]  

(2.1)

Then \( \sim \) is clearly an equivalence relation. We replace \( M \) by \( P(M) =: M/\sim \) and endow this space with the quotient topology. If \( \pi : M \to P(M) \) is the natural projection, then a set in \( P(M) \) is declared to be open if and only if its inverse image for \( \pi \) is open in \( M \). This topology is the coarsest one making \( \pi \) continuous.

When \( M \) is compact, the number of sets \( O_\lambda \) in \( \mathcal{U} \) can be taken to be finite so that \( P(M) \) is an approximation to \( M \) by a finite number of points. When \( M \) is not compact we take it locally compact so that each point has a neighborhood intersected by only finitely many \( O_\lambda \) and \( P(M) \) is a “finitary” (countable) approximation to \( M \) \([1]\). If \( P(M) \) has \( N \) points, we also denote it by \( P_N(M) \).

\[\text{Figure 1: An open cover for the circle } S^1 \text{ and the resulting discrete space } P_4(S^1).\]

Let us illustrate these considerations for a cover of \( M = S^1 \) by four open sets \( O_1, \cdots, O_4 \), with \( O_{1,2} \subseteq O_3 \cap O_4 \), as in Fig. 1. The map \( \pi : S^1 \to P_4(S^1) \) is given by

\[
O_1 \to x_1 , \quad O_3 \setminus (O_2 \cap O_4) \to x_3 , \\
O_2 \to x_2 , \quad O_4 \setminus (O_2 \cap O_4) \to x_4 ,
\]  

(2.2)

and a basis for the quotient topology for \( P_4(S^1) \) is provided by

\[
\{x_1\} , \ \{x_2\} , \ \{x_1, x_2, x_3\} , \ \{x_1, x_2, x_4\} .
\]  

(2.3)
Notice that our assumptions allow us to isolate points in certain sets of the form \( O_3 \setminus [O_3 \cap O_2] \) which may not be open. So, there are in general points in \( P(M) \) coming from sets which are not open in \( M \) and therefore are not open in the quotient topology. As a consequence, the space \( P(M) \) is, in general, neither Hausdorff nor \( T_1 \). However, it can be shown [1] that it is a \( T_0 \) space. This means that given any two points, there exists an open set which contains only one of the points and not the other but there is not necessarily an open set containing only the second point and not the first one. For example, given the points \( x_1 \) and \( x_3 \) of \( P_4(S^1) \), the open set \( \{x_1\} \) contains \( x_1 \) and not \( x_3 \), but there is no open set containing \( x_3 \) and not \( x_1 \). Moreover, the points \( x_1 \) and \( x_2 \) are open while only the points \( x_3 \) and \( x_4 \) are closed.

An equivalent and useful way to read off the topology of \( P(M) \) is obtained by introducing in \( P(M) \) a partial order \( \preceq \) [8]. We declare that:

\[
x \preceq y \quad \text{if every open set containing } y \text{ contains also } x .
\]

The space \( P(M) \) then becomes a partially ordered set (or a poset). We shall also write \( x \prec y \) to indicate that \( x \preceq y \) and \( x \neq y \).

Any poset can be represented by a Hasse diagram constructed by arranging its points at different levels and connecting them using the following rules:

1) if \( x \prec y \), then \( x \) is at a lower level than \( y \);

2) if \( x \prec y \) and there is no \( z \) such that \( x \prec z \prec y \), then \( x \) is at the level immediately below \( y \) and these two points are connected by a line called a link.

In the language of partially ordered sets, the smallest open set \( O_x \) containing a point \( x \in P(M) \) consists of all \( y \)'s preceding \( x \): \( O_x = \{ y \in P(M) : y \preceq x \} \). In the Hasse diagram, it consists of \( x \) and all points we encounter as we travel along links from \( x \) to the bottom. The open sets \( \{O_x, x \in P(M)\} \) are a basis for the topology of \( P(M) \) which can therefore be inferred directly from the Hasse diagram.

For \( P_4(S^1) \), the partial order reads

\[
x_1 \prec x_3 , \ x_1 \prec x_4 , \ x_2 \prec x_3 , \ x_2 \prec x_4 . \quad (2.4)
\]

The corresponding Hasse diagram is shown in Fig. 2. In this Figure, the smallest open set containing, for instance, \( x_3 \) is \( \{x_1, x_2, x_3\} \) just as in (2.3).

Other examples of interest are the posets for the two-dimensional sphere \( S^2 \). Fig. 3 shows the Hasse diagram for the poset \( P_6(S^2) \) derived in [1]. A basis for the topology is given by

\[
\{x_1\} , \ \{x_2\} , \ \{x_1, x_2, x_3\} , \ \{x_1, x_2, x_4\} , \\
\{x_1, x_2, x_3, x_4, x_5\} , \ \{x_1, x_2, x_3, x_4, x_6\} . \quad (2.5)
\]

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Finally, we mention that one of the most remarkable properties of a poset is its ability to accurately reproduce some of the homotopy groups of the space it approximates. For example, as for $S^1$, $\pi_1(P_N(S^1)) = \mathbb{Z}$ whenever $N \geq 4$ [1]. Similarly, as for $S^2$, $\pi_1(P_\alpha(S^2)) = \{0\}$ and $\pi_2(P_\alpha(S^2)) = \mathbb{Z}$. It is this fact that makes posets capable to capture some global topological information relevant for quantum physics such as $\theta$-states [2, 3]. Furthermore, the topological space being approximated can be recovered by considering a sequence of finer and finer coverings, the appropriated framework being that of projective systems of topological spaces. We refer to [1, 9] for details.

Figure 2: The Hasse diagram for $P_4(S^1)$.

Figure 3: The Hasse diagram for the poset $P_\alpha(S^2)$. 
3 Noncommutative Lattices

From the algebraic point of view, any poset $P$ can be described in terms of some noncommutative $C^*$-algebra $\mathcal{A}$ which plays the role of the algebra of continuous functions on $P$. The space $P$ can be identified with the space $\hat{\mathcal{A}} = \text{Prim}\mathcal{A}$ of primitive ideals of $\mathcal{A}$ endowed with the Jacobson topology, an ideal of $\mathcal{A}$ being called primitive if it is the kernel of an irreducible representation of $\mathcal{A}$. This fact will make any poset a truly noncommutative space [5], hence also the name \textit{noncommutative lattice}. The Jacobson topology just reduces to the partial order defined by inclusion of ideals. Given any two ideals $I_1, I_2 \in \hat{\mathcal{A}}$, $I_1 \preceq I_2$, if and only if $I_1 \subseteq I_2$.

The class of algebras which are relevant is that of approximately finite dimensional (AF) postliminal algebras. As we shall explain in more details in Section 3.1, AF means that an algebra can be approximated in norm by direct sums of finite dimensional matrix algebras. As for postliminal, we refer to [10, 11] for the exact definition. For what concern this paper, it just refers to the fact that irreducible (unitary) representations of $\mathcal{A}$ are completely characterized by their kernels, so that any poset (noncommutative lattice) can also be thought of as the set of classes of irreducible representations.

There are several ways to associate an algebra to a poset and they are described in [6]. For the study of the bundle theory and of the associated $K$-theory, the one derived by Bratteli [7] is the most useful one. Bratteli’s method is based on a diagrammatic representation of AF algebras which we shall describe in the rest of this section.

3.1 The Algebra of a Poset and Bratteli Diagrams

A $C^*$-algebra $\mathcal{A}$ is said to be \textit{approximately finite dimensional} (AF) if $\mathcal{A}$ has a unit $I$ and there exists an increasing sequence

$$
A_1 \overset{I_1}{\hookrightarrow} A_2 \overset{I_2}{\hookrightarrow} A_3 \overset{I_3}{\hookrightarrow} \cdots \overset{I_{n-1}}{\hookrightarrow} A_n \overset{I_n}{\hookrightarrow} \cdots
$$

(3.1)

of finite dimensional $C^*$-subalgebras of $\mathcal{A}$, such that $\mathcal{A}$ is the norm closure of $\bigcup_n A_n$. Here the maps $I_n$ are injective *-homomorphisms. This allow to write easily the norm. As a set, $\bigcup_n A_n$ is made of coherent sequences,

$$
\bigcup_n A_n = \{ a = (a_n)_{n \in \mathbb{N}} , a_n \in A_n \mid \exists N_0 : a_{n+1} = I_n(a_n), \forall n > N_0 \}.
$$

(3.2)

Now the sequence $(|a_n|)_{n \in \mathbb{N}}$, is eventually decreasing, since $|a_{n+1}| \leq |a_n|$ (the maps $I_n$ are norm decreasing) and therefore convergent. One writes for the norm

$$
||(a_n)\| = \lim_{n \to \infty} ||a_n||_{A_n}.
$$

(3.3)

Since the maps $I_n$ are injective, the expression (3.3) gives directly a true norm and not simply a seminorm and there is no need to quotient out the zero norm elements. In a
more sophisticated parlance, \( \mathcal{A} \) is the inductive (or direct) limit of the sequence \( \{ \mathcal{A}_n, I_n \}_N \) [11, 12].

Each subalgebra \( \mathcal{A}_n \), being a finite dimensional \( C^* \)-algebra, is a matrix algebra and therefore can be written as a direct sum

\[
\mathcal{A}_n = \bigoplus_{k=1}^{n} M^{(k)}(d_k, \mathbb{C}),
\]

where \( M^{(k)}(d_k, \mathbb{C}) \) is the algebra of \( d_k \times d_k \) matrices with complex coefficients. Given any two such matrix algebras \( \mathcal{A}_1 = \bigoplus_{j=1}^{n_1} M^{(1)}(d_j, \mathbb{C}) \) and \( \mathcal{A}_2 = \bigoplus_{k=1}^{n_2} M^{(2)}(d_k, \mathbb{C}) \) with \( \mathcal{A}_1 \subset \mathcal{A}_2 \), one can always choose suitable bases in \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) such that we can identify \( \mathcal{A}_1 \) with a subalgebra of \( \mathcal{A}_2 \) of the following form

\[
\mathcal{A}_1 \simeq \bigoplus_{k=1}^{n_2} \left( \bigoplus_{j=1}^{n_1} N_{kj} M^{(1)}(d_j, \mathbb{C}) \right).
\]

Here we identify \( \bigoplus_{j=1}^{n_1} N_{kj} M^{(1)}(d_j, \mathbb{C}) \) with a subalgebra of \( M^{(2)}(d_k, \mathbb{C}) \) and the coefficients \( N_{kj} \) are nonnegative integers satisfying the condition

\[
\sum_{j=1}^{n_1} N_{kj} d_j = d_k.
\]

We can represent \( \mathcal{A}_1 \), \( \mathcal{A}_2 \) and the embedding \( \mathcal{A}_1 \subset \mathcal{A}_2 \) by means of a diagram (the Bratteli diagram), which can be constructed out of the dimensions \( d_j \) (\( j = 1, \ldots, n_1 \)) and \( d_k \) (\( k = 1, \ldots, n_2 \)) of the diagonal blocks of the two algebras and the numbers \( N_{kj} \) that describe the embedding. To construct the diagram we draw two horizontal rows of vertices, the top (bottom) one representing \( \mathcal{A}_1 \) (\( \mathcal{A}_2 \)) and consisting of \( n_1 \) (\( n_2 \)) vertices labeled by \( d_1, \ldots, d_{n_1} \) \( (d_1, \ldots, d_{n_2}) \). Then for each \( j = 1, \ldots, n_1 \) and \( k = 1, \ldots, n_2 \), draw \( N_{kj} \) edges between \( d_j \) and \( d_k \). We will write \( d_j^{(K)} \backslash d_k^{(K+1)} \) to denote the fact that \( M^{(K)}(d_j^{(K)}, \mathbb{C}) \) is embedded in \( M^{(K+1)}(d_k^{(K+1)}, \mathbb{C}) \) with multiplicity \( p \). The Bratteli diagram of the AF algebra \( \mathcal{A} \) will be denoted by \( D(\mathcal{A}) \).

As a very simple example, let us consider the subalgebra \( \mathcal{A} \) of the algebra \( \mathcal{B}(\mathcal{H}) \) of bounded operators on an infinite dimensional (separable) Hilbert space \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), given in the following manner. Let \( \mathcal{P}_j \) be the projection operators on \( \mathcal{H}_j \), \( j = 1, 2 \) and \( \mathcal{K}(\mathcal{H}) \) the algebra of compact operators on \( \mathcal{H} \). Then, consider the algebra

\[
\mathcal{A}(\bigvee) = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2.
\]

The use of the symbol \( \mathcal{A}(\bigvee) \) will be clear later. This \( C^* \)-algebra can be obtained as the direct limit of the following sequence of finite dimensional algebras:

\[
\begin{align*}
\mathcal{A}_0 &= M(1, \mathbb{C}) \\
\mathcal{A}_1 &= M(1, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\end{align*}
\]
\[
\mathcal{A}_2 = M(1, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\mathcal{A}_3 = M(1, \mathbb{C}) \oplus M(4, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\vdots \\
\mathcal{A}_n = M(1, \mathbb{C}) \oplus M(2n - 2, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\vdots
\]

where, for \( n \geq 1 \), \( \mathcal{A}_n \) is embedded in \( \mathcal{A}_{n+1} \) as the subalgebra \( M(1, \mathbb{C}) \oplus (M(1, \mathbb{C}) \oplus M(2n - 2, \mathbb{C}) \oplus M(1, \mathbb{C})) \oplus M(1, \mathbb{C}) \):

\[
a_n = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & m_{(2n-2)\times(2n-2)} & 0 \\
0 & 0 & \lambda_2
\end{bmatrix} \rightarrow \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & m_{(2n-2)\times(2n-2)} & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & \lambda_2
\end{bmatrix}.
\]

Figure 4: The Bratteli diagram of the algebra \( \mathcal{A}(\vee) \).

The corresponding Bratteli diagram is in Fig. 4.

3.2 From Bratteli Diagrams to Posets

Bratteli diagrams are useful not only because one can very easily read the finite approximations of an AF algebra \( \mathcal{A} \) out of them, but also because it is possible to identify its ideals and decide which ones are primitive. As we have mentioned before, the topology is given by constructing a poset whose partial order is provided by inclusion of ideal.
In other words, both $\text{Prim}(\mathcal{A})$ and its topology can be determined from the Bratteli diagram of $\mathcal{A}$.

One has the following very important result whose proof is given in [13].

**Proposition 3.1**

1. There is a one-to-one correspondence between the ideals $\mathcal{I}$ of $\mathcal{A}$ and the subsets $
abla = \nabla_\mathcal{I}$ of the Bratteli diagram satisfying the following two properties:
   
i) if $M^{(n)}(d_k, \mathcal{C}) \in \nabla$ and $M^{(n)}(d_k, \mathcal{C}) \searrow M^{(n+1)}(d_j, \mathcal{C})$ then necessarily $M^{(n+1)}(d_j, \mathcal{C})$ belongs to $\nabla$;
   
   ii) if the condition $M^{(n)}(d_k, \mathcal{C}) \searrow M^{(n+1)}(d_j, \mathcal{C})$ implies that $M^{(n+1)}(d_j, \mathcal{C}) \in \nabla$ for all $j = \{1, 2, \ldots, N_{n+1}\}$ then $M^{(n)}(d_k, \mathcal{C})$ belongs to $\nabla$.

2. An ideal $\mathcal{I}$ of $\mathcal{A}$ is primitive if and only if its diagram $\nabla_\mathcal{I}$ satisfies:
   
   iii) $\forall n$ there exists an $M^{(m)}(d_j, \mathcal{C})$, with $m > n$, not belonging to $\nabla$ such that, for all $k \in \{1, 2, \ldots, N_n\}$ with $M^{(n)}(d_k, \mathcal{C})$ not in $\nabla$, one can find a sequence $M^{(n)}(d_k, \mathcal{C}) \searrow M^{(n+1)}(d_\alpha, \mathcal{C}) \searrow M^{(n+2)}(d_\beta, \mathcal{C}) \searrow \ldots \searrow M^{(m)}(d_j, \mathcal{C})$.

**Remarks.**

1. The whole $\mathcal{A}$ is an ideal which, by definition, is not primitive since the trivial representation $\mathcal{A} \rightarrow 0$ is not irreducible.

2. The ideal $\{0\} \subset \mathcal{A}$ is primitive if and only if $\mathcal{A}$ has one irreducible faithful representation. This can be understood from the Bratteli diagram in the following way. The set $\{0\}$ is not a subdiagram of $\mathcal{D}(\mathcal{A})$, being represented by the element 0 of the matrix algebra of each finite level, so that there is at least one element $a \in \mathcal{A}$ not belonging to the ideal $\{0\}$ at any level. Thus to check if $\{0\}$ is primitive, i.e. to check property (iii) above, we have to see whether we can connect all the points at a level $n$ to a single point at a level $m > n$. For example this is the case for the diagram of Fig. 4. Later, we shall construct examples in which $\{0\}$ is not a primitive ideal.

Proposition 3.1 above allows to understand the topological properties of $\text{Prim}(\mathcal{A})$ at once. This becomes particularly simple if the algebra admits only a finite number of nonequivalent irreducible representations. In this case $\text{Prim}(\mathcal{A})$ is a $T_0$-topological space with only a finite number of points, hence a finite poset $\mathcal{P}$. To reconstruct the latter we just need to draw the Bratteli diagram $\mathcal{D}(\mathcal{A})$ and find the subdiagrams that, according to properties (i, ii, iii), correspond to primitive ideals. Then $\mathcal{P}$ has as many
points as the number of primitive ideals and the partial order relation in $P$ is simply
given by the inclusion relations that exist among the primitive ideals.

Again, consider, as an example, the diagram of Fig. 4. The corresponding AF
algebra $\mathcal{A}$ in (3.7) contains only three nontrivial ideals, whose diagrams are represented
in Fig. 5(a,b,c). In this picture the points belonging to the ideals are marked with
a “★”. It is not difficult to check that only $I_1$ and $I_2$ are primitive ideals, since $I_3$
does not satisfy property $(iii)$ above. Clearly, $\{0\}$ belongs to both $I_1$ and $I_2$ so that $Prim(\mathcal{A})$ is the $\vee$ poset of Fig. 6.

![Diagram](image)

**Figure 5:** The three ideals of the algebra $\mathcal{A}(\vee)$.

![Diagram](image)

**Figure 6:** The poset $\vee$ as the primitive spectrum of the algebra $\mathcal{A}(\vee)$.  

3.3 The Bratteli Diagram of a Poset

Under some rather mild hypotheses which are always verified in the cases of posets, it is possible to reverse the construction of previous section and thus construct an AF algebra starting from a poset. Such a reconstruction rests on another result by Bratteli [13], which specifies the class of topological spaces which are the primitive ideal spaces of AF algebras. Here, by using the techniques of [13] we shall explain how to explicitly find an AF algebra \( \mathcal{A} \) (or rather its Bratteli diagram \( \mathcal{D}(\mathcal{A}) \)) whose primitive ideal space is a given finite poset \( P \). First we will give the general construction and then discuss several examples.

Let \( \{K_1, K_2, K_3, \ldots \} \) be the collection of all closed sets in the poset \( P \), with \( K_1 = P \). To construct the \( n \)-th level of the Bratteli diagram \( \mathcal{D}(\mathcal{A}) \), we consider the subcollection \( \mathcal{K}_n = \{K_1, K_2, \ldots, K_n\} \) and denote with \( \mathcal{K}'_n \) the smallest collection of (closed) sets in \( P \) containing \( \mathcal{K}_n \) which is closed under union and intersection. The collection \( \mathcal{K}_n \) determines a partition of the space \( P \) by taking intersections and complements of the sets \( K_j \in \mathcal{K}_n \) \( (j = 1, \ldots, n) \). We denote with \( Y(n,1), Y(n,2), \ldots, Y(n,k_n) \) the minimal sets of such partition. Also, we write \( F(n,j) \) for the smallest set in the subcollection \( \mathcal{K}'_n \) which contains \( Y(n,j) \). Then, the diagram \( \mathcal{D}(\mathcal{A}) \) is constructed as follows:

1. the \( n \)-th level of \( \mathcal{D}(\mathcal{A}) \) has \( k_n \) points, one for each set \( Y(n,k), k = 1, \ldots, k_n; \)
2. at the level \( n \) of the diagram, the point which corresponds to \( Y(n,i) \) is linked to the point at the level \( n+1 \) corresponding to \( Y(n+1,j) \) if and only if \( Y(n,i) \cap F(n+1,j) \neq \emptyset \). In this case, the multiplicity of the embedding is always 1.

**Example 1.** To illustrate this construction, let us consider again the \( \sqcup \) poset of Fig. (6), \( P = \{x_1, x_2, x_3\} \). This topological space contains four closed sets:

\[
K_1 = \{x_1, x_2, x_3\}, K_2 = \{x_2\}, K_3 = \{x_3\}, K_4 = \{x_2, x_3\} = K_2 \cup K_3. \tag{3.10}
\]

Thus it is not difficult to check that:

\[
\begin{align*}
\mathcal{K}_1 &= \{K_1\} & \mathcal{K}'_1 &= \{K_1\} & Y(1,1) &= \{x_1, x_2, x_3\} & F(1,1) &= K_1 \\
\mathcal{K}_2 &= \{K_1, K_2\} & \mathcal{K}'_2 &= \{K_1, K_2\} & Y(2,1) &= \{x_2\} & F(2,1) &= K_2 \\
& & & & Y(2,2) &= \{x_1, x_3\} & F(2,2) &= K_1 \\
\mathcal{K}_3 &= \{K_1, K_2, K_3\} & \mathcal{K}'_3 &= \{K_1, K_2, K_3, K_4\} & Y(3,1) &= \{x_2\} & F(3,1) &= K_2 \\
& & & & Y(3,2) &= \{x_1\} & F(3,2) &= K_1 \\
& & & & Y(3,3) &= \{x_3\} & F(3,3) &= K_3 \\
\mathcal{K}_4 &= \{K_1, K_2, K_3, K_4\} & \mathcal{K}'_4 &= \{K_1, K_2, K_3, K_4\} & Y(4,1) &= \{x_2\} & F(4,1) &= K_2 \\
& & & & Y(4,2) &= \{x_1\} & F(4,2) &= K_1 \\
& & & & Y(4,3) &= \{x_3\} & F(4,2) &= K_3
\end{align*}
\tag{3.11}
\]

\]
Since $P$ has only a finite number of points and hence a finite number of closed sets, the partition of $P$ repeats itself after a certain level ($n = 3$ in this case). Fig. 7 shows

![Bratteli diagram](image)

Figure 7: The Bratteli diagram associated with the poset $\emptyset$. Here $nj$ stands for $Y_{nj}$.

the corresponding diagram, obtained through rules (1) and (2) above. Recalling then that the first matrix algebra that gives origin to an AF algebra is $\mathbb{C}$ and using the fact that all the embeddings have multiplicity one, we eventually obtain the sequence of finite dimensional algebras shown by the Bratteli diagram of Fig. 4. As we have said previously, such a diagram corresponds to the AF algebra $A(\emptyset) = \mathbb{C}P_1 + K(H) + \mathbb{C}P_2$.

It is a general fact that a Bratteli diagram describing any (finite) poset ‘stabilizes’ after a certain level $n_0$, namely it repeats itself. From that level, both the number of points, as well as the embeddings from one level to the next one, do not change. As we shall mention later, for the construction of the $K$-theory groups, we only need the stable part of the Bratteli diagram and we shall only construct this part in the remaining examples.

Now, the Bratteli diagram stabilizes at the level $n_0$ if the family $\mathcal{K}_{n_0}$ of closed sets we choose is such that it determines a partition of the poset which distinguishes each point of the poset itself. In particular, this is the case if we choose $n_0$ in such a manner that $\mathcal{K}_{n_0}$ contains all closed set. Then, each $Y(n_0, j)$ will contain a single point of the poset and $F(n_0 + 1, j)$ will be the smallest closet set containing $Y(n_0, j)$.

**Example 2.** The poset $\|\|$ of Fig. 8.
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8}
\caption{The poset $|\cdot|$.}
\end{figure}

Here $n_0 = 4$ and the stable partition is given by
\begin{align}
Y(n_0,1) &= \{x_3\} & F(n_0 + 1, 1) &= \{x_3\} \\
Y(n_0,2) &= \{x_1\} & F(n_0 + 1, 2) &= \{x_1, x_3, x_4\} \\
Y(n_0,3) &= \{x_2\} & F(n_0 + 1, 3) &= \{x_2, x_4\} \\
Y(n_0,4) &= \{x_4\} & F(n_0 + 1, 4) &= \{x_4\}.
\end{align}

The corresponding Bratteli diagram is in Fig. 9. The set $\{0\}$ is not an ideal. The algebra limit $\mathcal{A}(\mathcal{I})$ turns out to be a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ with $\mathcal{H}_i$, $i = 1, 2, 3$ infinite dimensional Hilbert spaces [6],
\begin{equation}
\mathcal{A}(\mathcal{I}) = \mathcal{P}(\mathcal{H}_1) \oplus \mathcal{P}(\mathcal{H}_2 \oplus \mathcal{H}_3) \oplus \mathcal{K}(\mathcal{H}_2 \oplus \mathcal{H}_3) \oplus \mathcal{K}(\mathcal{H}_3).
\end{equation}

Here, $\mathcal{K}$ denotes compact operators and $\mathcal{P}$ orthogonal projection.

**Example 3.** The poset $P_4(S^1)$ for the one-dimensional sphere in Fig. 1. Here $n_0 = 4$
and the stable partition is given by

\[ Y(n_0, 1) = \{x_3\} \quad F(n_0 + 1, 1) = \{x_3\} \]
\[ Y(n_0, 2) = \{x_1\} \quad F(n_0 + 1, 2) = \{x_1, x_3, x_4\} \]
\[ Y(n_0, 3) = \{x_2\} \quad F(n_0 + 1, 3) = \{x_2, x_3, x_4\} \]
\[ Y(n_0, 4) = \{x_4\} \quad F(n_0 + 1, 4) = \{x_4\} \]  

(3.14)

The corresponding Bratteli diagram is in Fig. 10. The set \( \{0\} \) is not an ideal. The

Figure 10: The stable part of the Bratteli diagram for the circle poset \( P_4(S^1) \).

algebra limit \( \mathcal{A}(P_4(S^1)) \) turns out to be a subalgebra of bounded operators on the
Hilbert space \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_4 \), with \( \mathcal{H}_i, i = 1, \ldots, 4 \) infinite dimensional Hilbert
spaces [6],

\[ \mathcal{A}(P_4(S^1)) = \mathcal{C}\mathcal{P}(\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{C}\mathcal{P}(\mathcal{H}_2 \oplus \mathcal{H}_3) \oplus \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2) \oplus \mathcal{K}(\mathcal{H}_3 \oplus \mathcal{H}_4). \]  

(3.15)

Here \( \mathcal{K} \) denotes compact operators and \( \mathcal{P} \) orthogonal projection.

**Example 4.** The poset \( P_6(S^2) \) for the two-dimensional sphere in Fig. 3. Here \( n_0 = 6 \)
and the stable partition is given by

\[ Y(n_0, 1) = \{x_3\} \quad F(n_0 + 1, 1) = \{x_3\} \]
\[ Y(n_0, 2) = \{x_3\} \quad F(n_0 + 1, 2) = \{x_3, x_5, x_6\} \]
\[ Y(n_0, 3) = \{x_1\} \quad F(n_0 + 1, 3) = \{x_1, x_3, x_4, x_5, x_6\} \]
\[ Y(n_0, 4) = \{x_2\} \quad F(n_0 + 1, 4) = \{x_2, x_3, x_4, x_5, x_6\} \]
\[ Y(n_0, 5) = \{x_4\} \quad F(n_0 + 1, 5) = \{x_4, x_5, x_6\} \]
\[ Y(n_0, 6) = \{x_6\} \quad F(n_0 + 1, 6) = \{x_6\} \]  

(3.16)

The corresponding Bratteli diagram is in Fig. 11. The set \( \{0\} \) is not an ideal. The
algebra limit \( \mathcal{A}(P_6(S^2)) \) turns out to be a subalgebra of bounded operators on the
Hilbert space \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_8 \) with \( \mathcal{H}_i, i = 1, \ldots, 8 \) infinite dimensional Hilbert
spaces [6],

\[ \mathcal{A}(P_6(S^2)) = \mathcal{C}\mathcal{P}(\mathcal{H}_5 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_8 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_3)) \]
Figure 11: The stable part of the Bratteli diagram for the sphere poset $P_6(S^2)$.

\[ \oplus \Phi P (\mathcal{H}_6 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_7 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_3)) \]
\[ \oplus \lbrack \mathcal{K} (\mathcal{H}_5 \oplus \mathcal{H}_6) \otimes \Phi P (\mathcal{H}_1 \oplus \mathcal{H}_4) \rbrack \oplus \lbrack \mathcal{K} (\mathcal{H}_7 \oplus \mathcal{H}_8) \otimes \Phi P (\mathcal{H}_2 \oplus \mathcal{H}_3) \rbrack \]
\[ \oplus \mathcal{K} [\mathcal{H}_1 \oplus (\mathcal{H}_5 \oplus \mathcal{H}_6) \oplus \mathcal{H}_2 \oplus (\mathcal{H}_7 \oplus \mathcal{H}_8)] \oplus \mathcal{K} [\mathcal{H}_3 \otimes (\mathcal{H}_7 \oplus \mathcal{H}_8) \oplus \mathcal{H}_4 \otimes (\mathcal{H}_5 \oplus \mathcal{H}_6)] \]

(3.17)

Here $\mathcal{K}$ denotes compact operators and $P$ orthogonal projection.

4 Projective modules of finite type and $K$-theory

Given an algebra $\mathcal{A}$ playing the rôle of the algebra of continuous functions on some noncommutative space, the analogue of vector bundles is provided by the notion of projective module of finite type (or finite projective module) over $\mathcal{A}$. Indeed, by the Serre-Swan theorem [14], locally trivial, finite-dimensional complex vector bundles over a compact Hausdorff space $M$ are in one to one correspondence with finite projective modules over the algebra $\mathcal{A} = C(M)$. To the vector bundle $E$ one associates the $C(M)$-module $\mathcal{E} = \Gamma (M, E)$ of continuous sections of $E$. Conversely, if $\mathcal{E}$ is a finite projective modules over $C(M)$, the fiber $E_m$ of the associated bundle $E$ over the point $m \in M$ is

\[ E_m = \mathcal{E} / \mathcal{E} I_m , \]

(4.1)
where the ideal $\mathcal{I}_m \subset \mathcal{C}(M)$, corresponding to the point $m \in M$, is given by \[ (4.2) \]

As we shall see, isomorphism and stable isomorphism have a meaning in the context of finite projective modules over a $C^*$-algebra $\mathcal{A}$ and the group $K_0(\mathcal{A})$ will be the group of (stable) isomorphism classes of finite projective (right) modules over $\mathcal{A}$.

Given any finite projective right module $\mathcal{E}$ over $\mathcal{A}$, there exists an integer $N$ together with a projector $p \in \mathcal{M}(N, \mathcal{A})$ ($N \times N$ matrices with entries in $\mathcal{A}$), $p^2 = p$, and an isomorphism of $\mathcal{E}$ with the right $\mathcal{A}$-module \[ (4.3) \]

In fact, we shall restrict to the class of hermitian modules which correspond to projectors $p$ obeying the additional condition $p^* = p$, the operation $^*$ being the composition of the $^*$ operation in the algebra $\mathcal{A}$ with usual matrix transposition.

We shall now give few fundamentals of the $K$-theory of $C^*$-algebras having in mind mainly AF algebras \cite{12}. Two projectors $p, q \in \mathcal{M}(N, \mathcal{A})$ are equivalent if there exists a matrix $u \in \mathcal{M}(N, \mathcal{A})$ such that $p = u^*u$ and $q = uu^*$. In order to be able to add equivalence classes of projectors, one considers all finite matrix algebras over $\mathcal{A}$ at the same time, by considering $\mathcal{M}(\infty, \mathcal{A})$ which is the non complete $^*$-algebra obtained as inductive limit of finite matrices \footnote{The completion of $\mathcal{M}(\infty, \mathcal{A})$ is $\mathcal{A} \otimes K$, with $K$ the algebra of compact operators on $l_2$. The algebra $\mathcal{A} \otimes K$ is also called the stabilization of $\mathcal{A}$.

\[ (4.4) \]

Now, two projectors $p, q \in \mathcal{M}(\infty, \mathcal{A})$ are said to be equivalent, $p \sim q$, when there exists an $u \in \mathcal{M}(\infty, \mathcal{A})$ such that $p = u^*u$ and $q = uu^*$. The set $V(\mathcal{A})$ of equivalence classes $[\cdot]$ is made an abelian semigroup by defining an addition by

\[ (4.5) \]

The groups $K_0(\mathcal{A})$ is the universal canonical group associated with the semigroup $V(\mathcal{A})$ and may be defined as

\[ (4.6) \]
The extra \([r]\) is necessary to get transitivity and make \(\sim\) an equivalence relation. This is the reason why one is classifying only stable classes. From definition (4.6), an equivalence class \(([p],[q]) \in K_0(\mathcal{A})\) can also be written as a formal difference \([p] - [q]\).

There is a natural homomorphism

\[
\kappa_{\mathcal{A}} : V(\mathcal{A}) \to K_0(\mathcal{A}) , \quad \kappa_{\mathcal{A}}([p]) = [p] - [0] \tag{4.7}
\]

This map is injective if and only if the addition in \(V(\mathcal{A})\) has cancellations, namely if and only if \([p] + [r] = [q] + [r] \implies [p] = [q]\).

While for a generic \(\mathcal{A}\), the semigroup \(V(\mathcal{A})\) has no cancellations, for AF algebras this happen to be the case. By defining

\[
K_{0+}(\mathcal{A}) = : \kappa_{\mathcal{A}}(V(\mathcal{A})) , \tag{4.8}
\]

the couple \((K_0(\mathcal{A}), K_{0+}(\mathcal{A}))\) becomes, for an AF algebra \(\mathcal{A}\), an ordered group with \(K_{0+}(\mathcal{A})\) the positive cone, namely one has that

\[
\begin{align*}
K_{0+}(\mathcal{A}) &\ni 0 , \\
K_{0+}(\mathcal{A}) - K_{0+}(\mathcal{A}) &= K_0(\mathcal{A}) , \\
K_{0+}(\mathcal{A}) \cap (-K_{0+}(\mathcal{A})) &= 0 .
\end{align*} \tag{4.9}
\]

For a generic (unital) algebra the last property is not true and the couple \((K_0(\mathcal{A}), K_{0+}(\mathcal{A}))\) is not an ordered group.

There is another \(K\)-group, \(K_1\), which is constructed from unitaries or invertibles. It turns out, however, that such a group is trivial for AF algebras, namely \(K_1(\mathcal{A}) = 0\) for any AF algebras \([12]\). We shall not mention it anymore in the rest of the paper.

The construction of the \(K\)-theory of AF algebras is made easy by the following results whose proofs are in \([12]\).

**Proposition 4.1** If \(\alpha : \mathcal{A} \to \mathcal{B}\) is a homomorphism of \(C^*\)-algebras, then the induced map

\[
\alpha_* : V(\mathcal{A}) \to V(\mathcal{B}) , \quad \alpha_*([a_{ij}]) = : [\alpha(a_{ij})] , \tag{4.10}
\]

is a well defined homomorphism of semigroups. Moreover, from universality, \(\alpha_*\) extends to a group homomorphism

\[
\alpha_* = K_0(\mathcal{A}) \to K_0(\mathcal{B}) . \tag{4.11}
\]

**Proposition 4.2** If \(\mathcal{A}\) is the inductive limit of a directed system \(\{\mathcal{A}_i, \Phi_{ij}\}_I\) of \(C^*\)-algebras, then \(\{K_0(\mathcal{A}_i), \Phi_{ij}*\}_I\) is a directed system of groups and one can exchange the limits,

\[
K_0(\mathcal{A}) = K_0(\lim \mathcal{A}_i) = \lim K_0(\mathcal{A}_i) . \tag{4.12}
\]
Moreover, if \( \mathcal{A} \) is an AF algebra, then \( K_0(\mathcal{A}) \) is an ordered group with positive cone given by the limit of a directed system of semigroups

\[
K_{0+}(\mathcal{A}) = K_{0+}(\lim A_i) = \lim K_{0+}(A_i).
\] (4.13)

**Proposition 4.3** With \( k_A, k_B \) integer numbers, let \( \mathcal{A} \) and \( \mathcal{B} \) be the sum of \( k_A \) and \( k_B \) matrix algebras respectively,

\[
\mathcal{A} = M(p_1, \mathbb{C}) \oplus M(p_2, \mathbb{C}) \oplus \cdots \oplus M(p_{k_A}, \mathbb{C}),
\]
\[
\mathcal{B} = M(q_1, \mathbb{C}) \oplus M(q_2, \mathbb{C}) \oplus \cdots \oplus M(q_{k_B}, \mathbb{C}).
\] (4.14)

Then, any homomorphism \( \alpha : \mathcal{A} \to \mathcal{B} \) can be written as the direct sum of the representations \( \alpha_j : \mathcal{A} \to M(q_j, \mathbb{C}) \simeq B(\mathbb{C}^{q_j}) \), \( j = 1, \ldots, k_B \). If \( \pi_{ij} \) is the unique irreducible representation of \( M(p_i, \mathbb{C}) \) in \( B(\mathbb{C}^{q_j}) \), then \( \alpha_j \) breaks into a direct sum of the \( \pi_{ij} \). Furthermore, let \( m_{ij} \) be the non-negative integer denoting the multiplicity of \( \pi_{ij} \) in this sum. Then the induced homomorphism, \( \alpha_* = K_0(\mathcal{A}) \to K_0(\mathcal{B}) \), is given by the \( q_{k_A} \times p_{k_B} \) matrix \( (m_{ij}) \).

We finish this section by mentioning that \( K \)-theory has been proved [16] to be a complete invariant which distinguish among AF algebras if one add to the ordered group \( (K_0(\mathcal{A}), K_{0+}(\mathcal{A})) \) the notion of scale, the latter being defined for any \( C^* \)-algebra \( \mathcal{A} \) as

\[
\Sigma \mathcal{A} = \{ [p] \text{ } , p \text{ a projector in } \mathcal{A} \}.
\] (4.15)

AF algebras are completely determined, up to isomorphism, by their scaled ordered \( K_0 \) groups. The key is the fact that scale preserving isomorphisms between the ordered groups \( (K_0, K_{0+}) \) of two AF algebras are nothing but \( K \)-theoretically induced maps (4.11) of isomorphisms between the AF algebras themselves.

### 4.1 The \( K \)-theory of noncommutative lattices

The starting point to compute the ordered group \( (K_0, K_{0+}) \) for a poset is the fact that, for an AF algebra given as in 3.1, the group \( (K_0(\mathcal{A}), K_{0+}(\mathcal{A})) \) is obtained by Proposition 4.2 as the inductive limit of the sequence of groups/semigroups

\[
K_0(\mathcal{A}_1) \hookrightarrow K_0(\mathcal{A}_2) \hookrightarrow K_0(\mathcal{A}_3) \hookrightarrow \cdots
\] (4.16)

\[
K_{0+}(\mathcal{A}_1) \hookrightarrow K_{0+}(\mathcal{A}_2) \hookrightarrow K_{0+}(\mathcal{A}_3) \hookrightarrow \cdots
\] (4.17)
The inclusions

\[ T_n : K_0(\mathcal{A}_n) \hookrightarrow K_0(\mathcal{A}_{n+1}), \quad T_n : K_{0+}(\mathcal{A}_n) \hookrightarrow K_{0+}(\mathcal{A}_{n+1}), \quad (4.18) \]

are easily obtained from the corresponding inclusions \( \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1} \), being indeed their pullbacks as in (4.11). As sets we have that

\[
\begin{align*}
K_0(\mathcal{A}) &= \{ (k_n)_{n \in \mathbb{N}} : k_n \in K_0(\mathcal{A}_n) \mid \exists N_0 : k_{n+1} = T_n(k_n), \ n > N_0 \}, \\
K_{0+}(\mathcal{A}) &= \{ (k_n)_{n \in \mathbb{N}} : k_n \in K_{0+}(\mathcal{A}_n) \mid \exists N_0 : k_{n+1} = T_n(k_n), \ n > N_0 \}. 
\end{align*}
\]

(4.19) (4.20)

The structure of (abelian) group/semigroup is inherited pointwise from the addition in the groups/semigroups in the sequences (4.16), (4.17).

Furthermore, for any \( d \), the algebra of matrices \( \mathbb{M}(d, \mathbb{C}) \) has \( K \)-theory given by \( (K_0, K_{0+}) = (\mathbb{I}, \mathbb{I}_+) \), \( \mathbb{I} \) being the group of integer numbers and \( \mathbb{I}_+ \) the semigroups of natural numbers (including 0). Hence, all terms in the sequences (4.16), (4.17), are direct sums of copies of \( \mathbb{I} \) or \( \mathbb{I}_+ \).

As mentioned in Section 3.3, the Bratteli diagrams that describe (finite) posets all have the property that starting from a certain level \( n_0 \) (which is less or equal than the number of closed sets in the poset), the number of points in any diagram, as well as the embeddings from one level to the next one, does not change. This simplifies the calculation of \( (K_0, K_{0+}) \) because,

\[
\begin{align*}
K_0(\mathcal{A}_{n_0}) &= K_0(\mathcal{A}_{n_0+1}) = K_0(\mathcal{A}_{n_0+2}) = \cdots = \mathbb{I}^{\oplus k_{n_0}} , \\
K_{0+}(\mathcal{A}_{n_0}) &= K_{0+}(\mathcal{A}_{n_0+1}) = K_{0+}(\mathcal{A}_{n_0+2}) = \cdots = \mathbb{I}_+^{\oplus k_{n_0}} , 
\end{align*}
\]

(4.21) (4.22)

where \( k_{n_0} \) is the number of points in the Bratteli diagram from the level \( n_0 \) on. Furthermore, the integer valued matrices \( T_n \) in (4.18) are all equal for \( n > n_0 \). To find the group \( (K_0(\mathcal{A}), K_{0+}(\mathcal{A})) \) one has just to study the limit for \( n \to \infty \) of the inclusions

\[
\begin{align*}
T_n : \mathbb{I}^{\oplus k_{n_0}} &\hookrightarrow \mathbb{I}^{\oplus k_{n_0}}, \\
T_n : \mathbb{I}_+^{\oplus k_{n_0}} &\hookrightarrow \mathbb{I}_+^{\oplus k_{n_0}} . 
\end{align*}
\]

(4.23) (4.24)

We infer from Prop. 4.3 that for AF algebras the maps (4.23), (4.24) are always inclusions. In fact, for (finite) posets, the map (4.23) is always a bijection. As a consequence, for a poset \( P \) with \( k_{n_0} \) points in the stable part of the corresponding Bratteli diagram, and associated algebra \( \mathcal{A}_{k_{n_0}}(P) \), we shall have that

\[
K_0(P) = \mathbb{I}^{\oplus k_{n_0}} .
\]

(4.25)

The map (4.24) will not be in general a bijection.

We shall illustrate the construction of the \( K \)-groups with the Penrose Tiling AF algebra. Although this algebra is quite far from being postliminal, since there are
infinite non equivalent representations with the same kernel (the only primitive ideal),
the construction of its $K$-theory is illuminating. The corresponding Bratteli diagram
is in Fig. 12. At each level, the algebra is given by [5]

$$\mathcal{A}_n = \mathbb{M}(d_n, \mathbb{C}) \oplus \mathbb{M}(d'_n, \mathbb{C}), \quad n \geq 1,$$

with inclusion $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1},$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 \\ 0 & B \\ A \end{bmatrix}, \quad A \in \mathbb{M}(d_n, \mathbb{C}), \quad B \in \mathbb{M}(d'_n, \mathbb{C}).$$

After the second level we have then

$$K_0(\mathcal{A}_n) = \mathbb{I} \oplus \mathbb{I}, \quad K_{0+}(\mathcal{A}_n) = \mathbb{I}_+ \oplus \mathbb{I}_+.$$  \hspace{1cm} (4.28)

The inclusion (4.27) gives for the dimensions

$$d_{n+1} = d_n + d'_n,$$
$$d'_{n+1} = d_n.$$

and this gives for the inclusions (4.23), (4.24), the integer valued matrix

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \hspace{1cm} (4.30)$$

The action of the matrix (4.30) can be represented pictorially as in Fig. 13 where the
couple $(a, b)$ $(a', b')$ are both in $\mathbb{I}^2$ or $\mathbb{I}_+^2$.

Finally, for the $K$-theory we have
\[
K_0(\mathcal{A}) = \mathbf{I} \oplus \mathbf{I}.
\]

This follows immediately from the fact that the matrix \( T \) in (4.30) is invertible over the integer, its inverse being
\[
T^{-1} = \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix}.
\]

Now, from the definition of inductive limit we have that
\[
K_0(\mathcal{A}) = \{(k_n)_{n \in \mathbb{N}} : k_n \in K_0(\mathcal{A}_n) \mid \exists N_0 : k_{n+1} = T(k_n) n > N_0\}.
\]

And, \( T \) being a bijection, for any \( a_{n+1} \in K_0(\mathcal{A}_{n+1}) \), there exist an unique \( a_n \in K_0(\mathcal{A}_n) \) such that \( a_{n+1} = T a_n \). Thus, \( K_0(\mathcal{A}) = K_0(\mathcal{A}_n) = \mathbf{I} \oplus \mathbf{I} \).

2) \[
K_{0+}(\mathcal{A}) = \frac{1 + \sqrt{5}}{2} \mathbf{I}_+ \oplus \mathbf{I}_+ = \{(a, b) \in \mathbf{I}_+ \oplus \mathbf{I}_+ : \frac{1 + \sqrt{5}}{2} a + b \geq 0\}.
\]

Since \( T \) is not invertible over \( \mathbf{I}_+ \), \( K_{0+}(\mathcal{A}) \neq \mathbf{I}_+ \oplus \mathbf{I}_+ \). To construct \( K_{0+}(\mathcal{A}) \), we study the image \( T(K_{0+}(\mathcal{A})) \) in \( K_{0+}(\mathcal{A}_{n+1}) \). It is easily found to be
\[
T(K_{0+}(\mathcal{A}_n)) = \{(a_{n+1}, b_{n+1}) \in \mathbf{I}_+ \oplus \mathbf{I}_+ : a_{n+1} \geq b_{n+1}\},
\]
\[
\neq K_{0+}(\mathcal{A}_{n+1}) = \mathbf{I}_+ \oplus \mathbf{I}_+.
\]

Now, \( T \) being injective, \( T(K_{0+}(\mathcal{A}_n)) = T(\mathbf{I}_+ \oplus \mathbf{I}_+) \simeq \mathbf{I}_+ \oplus \mathbf{I}_+ \). The inclusion of \( T(K_{0+}(\mathcal{A}_n)) \) into \( K_{0+}(\mathcal{A}_{n+1}) \) is shown in Figs. 14. If we identify the subset \( T(K_{0+}(\mathcal{A}_n)) \subset K_{0+}(\mathcal{A}_{n+1}) \) with \( K_{0+}(\mathcal{A}_n) \), we can think of \( T^{-1}(K_{0+}(\mathcal{A}_{n+1})) \) as a subset of \( \mathbf{I}_+ \oplus \mathbf{I}_+ \) and of \( T^{-1}(K_{0+}(\mathcal{A}_n)) \) as the standard positive cone \( \mathbf{I}_+ \oplus \mathbf{I}_+ \). The result is then shown in Fig. 15. From definition (4.20), by going to the limit we shall have \( K_{0+}(\mathcal{A}) = \lim_{m \to \infty} T^{-m}(\mathbf{I}_+ \oplus \mathbf{I}_+) \) and the latter will be a subset of \( \mathbf{I}_+ \oplus \mathbf{I}_+ \) since \( T \) is invertible only over \( \mathbf{I}_+ \). It takes some algebra to find that the limit is \( \frac{1 + \sqrt{5}}{2} \mathbf{I}_+ \oplus \mathbf{I}_+ \), namely (4.34).

We shall now evaluate the \( K \)-theory of the posets associated with the corresponding Bratteli diagrams in Section 3.3. The strategy will be the same as the one used for the algebra of the Penrose Tiling and will consist essentially of three steps:

\[
\begin{align*}
\{ a' &= a + b \\
b' &= a
\end{align*}
\]
Figure 14: The image of $\mathbb{I}_+ \oplus \mathbb{I}_+$ under $T$.

Figure 15: The image of $\mathbb{I}_+ \oplus \mathbb{I}_+$ under $T^{-1}$.

1. Construct, as in Fig. 13 the inclusion maps $T$ in (4.23) from the stable part of the Bratteli diagram.

2. Prove that $T$ is invertible over the integer numbers. As a consequence, $K_0$ will be the direct sum of as many copies of $\mathbb{I}$ as the number of points $k_{n_0}$ in the stable level $m_0$ of the corresponding Bratteli diagram of the poset.

3. Identify the subset $T(K_0^+(A_{n_0})) \subset K_0^+(A_{n_0+1})$ with $K_0^+(A_{n_0})$; evaluate $T^{-1}(K_0^+(A_{n_0}))$; get the limit $K_0^+(A) = \lim_{m \to -\infty} T^{-m} K_0^+(A_{n_0})$.

Example 1. The $K$-theory of the poset $\mathcal{V}$. From the stable part of the corresponding Bratteli diagram in Fig. 7, we get for the inclusions (4.23), (4.24) and their inverse the
integer valued matrices
\[
T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.
\tag{4.36}
\]

Since \( T \) is invertible over \( \mathbb{Z} \), from definition (4.19), it follows that
\[
K_0(\mathcal{A}(\sqcup)) = \mathbb{Z}^3.
\tag{4.37}
\]

On the other side, with \((a, b, c) \in \mathbb{Z}_+^3\), one finds that
\[
T^{-m} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b - m(a + c) \\ c \end{pmatrix}.
\tag{4.38}
\]

While \( a, c \geq 0 \), \( b - ma - mc \) can become any negative integer provided that \( a, c \neq 0 \).

Therefore
\[
K_0^+(\mathcal{A}(\sqcup)) = \{(a, b, c) \in \mathbb{Z}_+^3 \mid \begin{array}{ll} a \in \mathbb{Z}_+ \text{ , } c \in \mathbb{Z}_+ \\ b \in \mathbb{Z} \text{ if } (a, c) \neq (0, 0) \\ b \in \mathbb{Z}_+ \text{ if } (a, c) = (0, 0) \end{array} \}.
\tag{4.39}
\]

Example 2. The \( K \)-theory of the poset \( \sqcup \). From the stable part of the corresponding Bratteli diagram in Fig. 9, we get for the inclusions (4.23), (4.24) and their inverse the integer valued matrices
\[
T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\tag{4.40}
\]

Since \( T \) is invertible over \( \mathbb{Z} \), from definition (4.19), it follows that
\[
K_0(\mathcal{A}(\sqcup)) = \mathbb{Z}^4.
\tag{4.41}
\]

On the other side, with \((a, b, c, d) \in \mathbb{Z}_+^4\), one finds that
\[
T^{-m} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b - m(a + d) \\ c - md \\ d \end{pmatrix}.
\tag{4.42}
\]

As a consequence,
\[
K_0^+(\mathcal{A}(\sqcup)) = \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid \begin{array}{ll} a \in \mathbb{Z}_+ \text{ , } d \in \mathbb{Z}_+ \\ b \in \mathbb{Z} \text{ , } c \in \mathbb{Z}_+ \text{ if } a \neq 0 \text{ , } d \neq 0 \\ b \in \mathbb{Z}_+ \text{ , } c \in \mathbb{Z}_+ \text{ if } a \neq 0 \text{ , } d = 0 \\ b \in \mathbb{Z}_+ \text{ , } c \in \mathbb{Z}_+ \text{ if } (a, c) = (0, 0) \end{array} \}.
\tag{4.43}
\]
**Example 3.** The $K$-theory of the poset $P_4(S^1)$. From the stable part of the corresponding Bratteli diagram in Fig. 10, we get for the inclusions (4.23), (4.24) and their inverse the integer valued matrices

$$
T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \\
T^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 \\
-1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(4.44)

Since $T$ is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$
K_0(A(P_4(S^1))) = \mathbb{Z}^4.
$$

(4.45)

On the other side, with $(a, b, c, d) \in \mathbb{Z}_+^4$, one finds that

$$
T^{-m} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b - m(a + d) \\ c - m(a + d) \\ d \end{pmatrix}.
$$

(4.46)

As a consequence,

$$
K_{0+}(A(P_4(S^1))) = \{(a, b, c, d) \in \mathbb{Z}^4 \mid \begin{array}{ll}
a & \in \mathbb{Z}_+ \\
b & \in \mathbb{Z} \\
c & \in \mathbb{Z} \\
d & \in \mathbb{Z}_+ \end{array} \text{ if } a \neq 0 \text{ or } d \neq 0 \}.
$$

(4.47)

Notice that as abelian group, $K_0(A(\mathbb{Z})) = K_{0+}(A(P_4(S^1))) = \mathbb{Z}^4$. But as abelian ordered group $(K_0, K_{0+})(A(\mathbb{Z})) \neq (K_0, K_{0+})(A(P_4(S^1)))$, since the positive cone $K_{0+}$ is not the same in the two groups.

**Example 4.** The $K$-theory of the poset $P_6(S^2)$. From the stable part of the corresponding Bratteli diagram in Fig. 11, we get for the inclusions (4.23), (4.24) and their inverse the integer valued matrices

$$
T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \\
T^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & -1 \\
1 & -1 & 1 & 0 & -1 & 1 \\
1 & -1 & 0 & 1 & -1 & 1 \\
-1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

(4.48)

Since $T$ is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$
K_0(A(P_4(S^1))) = \mathbb{Z}^4.
$$

(4.49)
On the other side, with \((a, b, c, d, e, f) \in \mathcal{T}^6_+\), one finds that

\[
T^{-m} \begin{pmatrix}
  a \\
  b \\
  c \\
  d \\
  e \\
  f
\end{pmatrix} = \begin{pmatrix}
  a \\
  b - m(a + f) \\
  c + m^2(a + f) - m(b + e) \\
  d + m^2(a + f) - m(b + e) \\
  e - m(a + f) \\
  f
\end{pmatrix}.
\]

(4.50)

As a consequence,

\[
K_{0+}(\mathcal{A}(P_0(S^2))) = \{(a, b, c, d, e, f) \in \mathcal{T}^6 \mid a \in \mathcal{T}_+, b \in \mathcal{T}_+ \text{ if } (a, f) = (0, 0) \}
\]

(4.51)

5 \ Final Remarks

As mentioned before, the construction of the K-theory groups for noncommutative lattices is a preliminary step in the classification and construction of bundles over them and for the theory of characteristic classes. Notably, one would like to construct non trivial bundles, like, for instance, the analogue of the monopole bundle over the lattices approximating the 2-dimensional sphere and non trivial ‘topological charges’. Work in this direction is in progress and will be reported elsewhere [17]

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