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Breathers in Nonlinear Lattices

S. Aubry

The principle of anticontinuous limit is applied to arrays of coupled nonlinear oscillators in any dimension. Under hypothesis which are generically fulfilled and providing their coupling be not too strong, it is proven that this dynamical system exhibits spatially exponentially localized and time periodic solutions called "breathers". There also exist time periodic solutions ("multibreathers") corresponding to (nonlinear) superposition of breathers with arbitrary spatial distribution. Many of them can support an "elastic distortion" of the relative phase between the breathers and generate new time periodic solutions carrying locally an energy flow with possibly vortices (in two and more dimensions). These results extend for array of weakly coupled rotators where the breather solution named "rotobreather" then correspond to a single rotator rotating while all the others oscillate. Another extension can be done for a single quantum electron coupled to a classical array of anharmonic oscillators. For an electronic transfer integral not too large, there exist time periodic dynamical solutions ("polarobreather") where both the electronic wave function and a lattice vibration are localized and bound one with each other.

Adiabatic Curvature and the $S$-Matrix

J.E. Avron

This is joint work with L. Sadun. We study the relation of the adiabatic curvature associated to scattering states and the scattering matrix. We show that there cannot be any formula relating the two locally. However, the Chern numbers, which give the total curvature, can be computed from the $S$-matrix, by integrating an appropriate 3-form. We show that level crossings of the on-shell $S$-Matrix can be assigned an index so that the Chern number of the scattering states is the sum of the indices, and we construct an example
which is the natural scattering analog of Berry’s spin 1/2 Hamiltonian.

**Bloch Solutions of Harper Equation**

V. Buslaev

The talk is based on a series of works joint with A. Fedotov. The set of the solutions of Harper equation is a module over the ring of $\hbar$-periodic functions. Using this algebraic structure one can introduce a notion of Bloch solution which is a natural generalization of the standard notion of Bloch solution. Under some restrictive conditions one can prove existence of such solutions. It guarantees that the equation is not on the spectrum. The second idea was called monodromization. It is connected with a proper generalization of the notion of the classical monodromy matrix and for Harper equation can be considered as a precise realization of Wilkinson’s renormalization idea. It allows to extend essentially the conditions of the existence of the Bloch solutions. For small $\hbar$ one can characterize the spectrum as a set quite precisely and can describe the spectrum as an almost explicit Cantor set of zero Lebesgue measure.

**Anomalous Transport in a Simple Kicked Classical and Quantal Hamiltonian System**

S. De Bievre

(This is work in collaboration with G. Forni.) We study the asymptotic behaviour of the kicked Hamiltonian map $(x_0, v_0) \in S^1 \times R \rightarrow (x_1 = x_0 + \omega, v_1 = v_0 + f(x_0)) \in S^1 \times R$ for irrational rotation numbers $\omega$ and of its quantum equivalent, which has singular spectrum. As a typical result, we show that, if the Fourier coefficients $f_n$ of $f$ satisfy $f_n \sim |n|^{-\nu}$ ($\nu > 1/2$), then, for suitable Liouville $\omega$, and for all $\epsilon$, there exists a constant $C_\epsilon$ so that \( \|< v_m >\| \geq C_\epsilon m^{1+\epsilon} \). Here $< \cdot >$ denotes time average. In addition, for any irrational $\omega$ and all $\epsilon$ there exists a sequence $m_k \in N$ and a constant $C_\epsilon$ so that \( \|< v_m >\| \leq C_\epsilon m_k^{-\nu + \epsilon} \).
On Embedded Eigenvalues

R. del Rio

It is known that if the spectrum of a Sturm-Liouville problem is a perfect set, then there exists an uncountable set in the spectrum where it is not possible to add eigenvalues by means of local perturbations to the potential. It will be shown that this is not true if we just require the spectrum to contain a perfect set.

Pure Point Spectrum
for Finite-Difference Quasiperiodic Operators and Applications

E. Dinaburg

The talk considers ergodic families of long-range finite-difference operators with quasiperiodic coefficients decaying exponentially. For such families, Anderson localization takes place almost surely with respect to a parameter. The result is applied to a problem of splitting of Landau levels in a small periodic field.

Continuous and Discrete Schrödinger Operators on Graphs

P. Exner

We extend the known correspondence between the Kronig-Penney model and certain Jacobi matrices to a wide class of Schrödinger operators on graphs. In general, the coefficients of the resulting difference operators depend on the spectral parameter. We illustrate the result on examples of rectangular lattices and comb-shaped graphs.

Nature of the Spectrum of Harper Equation

A. Fedotov

The talk can be considered as a continuation of previous Buslaev’s talk and also is based on our joint works. The Bloch solutions described in the previous talk exist out of the spectrum and create a new notion, notion of the complex Lyapunov exponent. Its imaginary part is related with the classical Lyapunov
exponent. The boundary values of its real part are related with the integrated density of states. The integrated density of states is equivalent to the spectral measure of the operator and is a continuous function. Therefore the spectrum of Harper equation is singular continuous. Using the notion of Bloch solution one can prove directly that a whole class of equations similar to Harper equation does not have point spectrum.

**Singular Spectra in a Physically Realistic Model for Bloch Electrons in Magnetic Fields**

T. Geisel

Bloch electrons in magnetic fields represent one of the major physical examples providing singular quantum mechanical spectra. The widely used single-band descriptions (e.g., in terms of Harper's equation) leads to classically integrable Hamiltonians, whereas the real classical limit exhibits chaotic behaviour. They thus fail in particular in lateral surface superlattices where at present experimentalists try to detect the Hofstadter butterfly. A new model is derived, which is reminiscent of Harper's equation, but is exact under the most general conditions and exhibits chaos in the classical limit. The spectrum and its fingerprints in the Hall- and magnetoresistance are calculated in the regime of overlapping Landau bands.

**Continuous Selection of Eigenvalues**

A. Gordon

Some recent results in the rank one perturbation theory ([G], [dRMS]) and in the theory of almost periodic Schrödinger operators [JS] turn out to be particular cases of one general fact concerning families of self-adjoint operators $A(s)$: if $A(s) = A + B(s)$, where $B(s)$ is bounded and depends continuously on a point $s$ of a complete $\sigma$-compact metric space $S$, and if $A(s)$ has some eigenvalues for a sufficiently rich (in the Baire category sense) subset of $S$, then there exists an eigenvalue $E(s)$ of $A(s)$ which is continuous in $s$ on some open subset of $S$. This statement is applicable to other special cases as well.

Some Dynamical Implications of Multifractal Energy Spectra

I. Guarneri

Some theoretical estimates and empirical results illustrating the role of fractal spectral dimensions in wavepacket dynamics are discussed. Theoretical estimates mainly concern the asymptotic growth of the “effective dimension” of orbits, and of the “entropic” width of wavepackets. Numerical results concern the quantum dynamics of model systems such as the Kicked Harper model, the Kicked Spin model, and others.

Some Recent Results for Random Operators

P.D. Hislop

This talk will concentrate on localization properties of random families of self-adjoint operators at the band edges and in unperturbed spectral gaps. Examples of such operators include random perturbations of Landau and periodic Schrödinger operators, and operators describing wave propagation in randomly perturbed, periodic media.

Singular Continuous Diffraction Spectrum

for a ‘Structure Intermediate Between Quasiperiodic and Random’

A. Hof

Aubry, Godrèche and Luck (Europhys. Lett. 4 (1987) 639–643, J. Stat. Phys. 51 (1988) 1033–1074) have considered a model of atoms on the line defined by an irrational rotation $\alpha$ on the circle $T = R/Z$ in which the positions $u_n$ of the atoms are given by

$$u_n - u_{n-1} = 1 + \xi \mathbb{1}_{[0,\beta)}(n\alpha),$$

where $0 < \beta < 1$ and $u_0, \xi \in R$ are parameters. They have shown by a combination of scaling arguments and numerical work that for $\beta = \frac{1}{2}$, $\alpha = \tau^{-2}$ ($\tau = (\sqrt{3} + 1)/2$) and irrational $\xi$ the ‘structure factor’

$$S := \lim_{L \to \infty} (2L)^{-1} \left| \sum_{x_k \in [-L,L]} e^{-2\pi i \xi x_k} \right|^2$$

is a purely singular continuous measure (apart from a delta function at 0). We prove that $S$ is purely singular continuous apart from the delta function.
at 0 for every irrational $\xi$, every $\beta$ and generic $\alpha$. We also prove that $S$ is continuous (has no delta functions apart from the one at 0) for all irrational $\alpha$, Lebesgue-a.e. $\beta$—depending on $\alpha$—and all $\xi$ such that $1/\xi$ is not rationally dependent on $\alpha$ (this includes the parameters $\beta = \frac{1}{2}$, $\alpha = \tau^{-2}$ considered by Aubry et al).

**Phonons and Electrons in Quasiperiodic Systems**

T. Janssen

The character of spectra and wave functions of quasiperiodic Schrödinger operators has been investigated mainly on one-dimensional models. Some properties seem to be rather universal for all kinds of quasiperiodic structures, others seem to depend more on finer details. We want to discuss the relation between certain properties of phonons and electrons in quasiperiodic systems and their structure. For 2D and 3D systems there are practically no rigorous results. For these an analysis is given of numerical calculations. Several characteristics have been found in experiments.

**The $K$-Theoretical Gap Labelling of Particles Moving in a Self-Similar Tiling**

J. Kellendonk

The $C^*$-algebra of observables for a particle moving in a tiling may be obtained from the groupoid associated to the tiling. A survey over this construction will be given. The group of coinvariants of the groupoid can be explicitly computed for tilings which allow for an invertible substitution (deflation) such as the Penrose tilings. It is related to the $K_0$-group of the algebra of observables and the result yields in particular the set of possible gap labels for Schrödinger operators describing the particle motion.

**Lifshitz' Tails for Random Schrödinger Operators with a Constant Magnetic Field**

W. Kirsch

This is joint work with K. Broderix, D. Hundertmark and H. Leschke. We consider a Schrödinger operator with a constant magnetic field and a random
potential in 2 dimensions. The random potential is Poissonian with a single site potential \( f \) decaying polynomially at infinity. We prove that the density of states of such operators decays exponentially fast near the bottom of the spectrum (= Lifshitz' tails), the Lifshitz exponent however is different from the case of vanishing magnetic field.

Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators and Jacobi Matrices with Slowly Decreasing Potentials

A. Kiselev

We prove that for any one-dimensional Schrödinger operator with potential \( V(x) \) satisfying decay condition \( |V(x)| \leq Cx^{-3\alpha-\epsilon} \), the absolutely continuous spectrum fills the whole positive semi-axis. The description of the set in \( \mathbb{R}^+ \) on which the singular part of the spectral measure might be supported is also given. Analogous results hold for Jacobi matrices.

Localization of Acoustic and Electromagnetic Waves in Random Media

A. Klein

Localization of classical waves has attracted a lot of attention in recent years. The nature of this phenomenon is the same as for electron waves, namely multiple scattering. I consider acoustic and electromagnetic waves which are described by the self-adjoint operators \( A = -\nabla \cdot \frac{1}{\varepsilon(x)} \nabla \) on \( L^2(\mathbb{R}^d) \) and \( M = \nabla \times \frac{1}{\varepsilon(x)} \nabla \times \) on \( L^2(\mathbb{R}^3, \mathbb{C}^3) \), respectively, where for acoustic waves \( \varepsilon(x) \) is the position dependent mass density of the medium, and for electromagnetic waves \( \varepsilon(x) \) is the position dependent dielectric constant.

In this lecture I report on my work with A. Figotin on acoustic and electromagnetic waves in random media obtained by random perturbations of a periodic medium. The properties of the medium are described by the position dependent quantity \( \varepsilon(x) \). We assume that in the periodic medium the acoustic and electromagnetic operators have a gap in the spectrum, and prove that random perturbations of the periodic medium create exponentially localized eigenstates in a vicinity of the edges of the gap. (More precisely, we show that, in a vicinity of the edges of the gap, the random operators have pure point spectrum with exponentially decaying eigenfunctions.)
Localization for Some Continuous Random Schrödinger Operators

F. Klopp

In this talk, we will be concerned with the spectrum of random Schrödinger operators acting on $L^2(\mathbb{R}^d)$; they will be chosen of the following type

$$H = H_0 + \sum_{x \in \mathbb{Z}^d} t_x V_x.$$

Here $H_0 = -\Delta + W$ where $W$ is a $\mathbb{Z}^d$-periodic potential. $V$ is an exponentially localized function; $V_x$ denotes the function $V$ shifted at the point $x$. The $(t_x)_{x \in \mathbb{Z}^d}$ are i.i.d. random variables which are assumed to have a nice probability density.

We essentially study two cases: unbounded and bounded random variables. In the first case we use the decay of the random variables at infinity to prove that, for sufficiently negative energies, the almost sure spectrum of $H$ is pure point, exponentially localized.

In the second case, using a Lifshits tail assumption, we prove that the lower edge of the spectrum is exponentially localized.

Both of these results are obtained under weak assumptions on $V$; in particular we do not need any sign assumptions on the potential $V$. The main step is a new proof of the Wegner estimate that works without sign assumptions on $V$ for energies below the spectrum of $H_0$. To prove a Wegner estimate inside any gap of the spectrum of $H_0$, one can use the same technique if one assumes that the potential $V$ and the random variables $(t_x)_{x \in \mathbb{Z}^d}$ keep a constant sign.

Singular Continuous Spectrum

for Koopman, Cocycle and Schrödinger Operators on $L^2(X)$

O. Knill

We will talk about singular continuous spectrum of unitary Koopman operators $U : f \mapsto f(T)$ on $L^2(X)$ attached to a measure preserving transformation $T$ of a probability space $(X, m)$, cocycle operators $U : f \mapsto a f(T)$ with a unitary multiplication operator $a$ and electromagnetic Schrödinger operators $L = \sum_{i=1}^d U_i + U_i^*$ on $L^2(X)$, where the $U_i$ are cocycle operators $U_i f = a_i f(T_i)$ belonging to $d$ commuting transformations $T_i$ and unitary multiplication operators $a_i$. The electromagnetic field attached to $L$ is the operator 2-form $F_{ij} = U_i U_j^* U_j U_i^*$. A special case is the Mathieu operator given by $U_1 f(x) = f(x + \alpha)$, and $U_2(x) = e^{ix} f(x)$ which has the constant magnetic field $e^{ix}$. 

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Possible Critical States in 2d Disordered Systems

M. Kohmoto

We study noninteracting electrons on the square lattice in a magnetic field. We focus on the nature of the state at the center of the band in several kinds of disorder: onsite, hopping amplitude and magnetic field. The results of multifractal analysis will be presented.

Power-Law Subordinacy
and Dimensional Hausdorff Properties of Singular Spectra

Y. Last

This is joint work with S. Jitomirskaya. We present an extension of the Gilbert-Pearson theory of subordinacy, which relates dimensional Hausdorff spectral properties of one-dimensional Schrödinger operators to the behavior of solutions of the corresponding Schrödinger equation. We use this theory to analyze the dimensional Hausdorff properties for several examples having singular-continuous spectrum, including sparse barrier potentials, the almost Mathieu operator and the Fibonacci Hamiltonian.

Quantum Intermittency in Almost-Periodic Lattice Systems
Derived from their Spectral Properties

G. Mantica

Jacobi matrices associated with singular measures in the family of Iterated Functions Systems can be constructed employing a Stieltjes technique which is immune from the ill-conditioning affecting classical polynomial sampling. These matrices pass some numerical tests for almost-periodicity. We investigate numerically and theoretically the (quantum) time dynamics of the systems so constructed: we compute the exact asymptotic behaviour of wavefunction coefficients, and we derive an approximate relation between the long-time asymptotic of the moments of the (lattice) position operator and spectral multi-fractal quantities.
Unbounded Point Spectrum
for Some Continuous Random Schrödinger Type Operators

S. Molchanov

One of the popular physical hypotheses says that the random Schrödinger operator in $d$-dimensional space, where $d$ is greater or equal than 3, has continuous spectrum for all very large energies (and arbitrary coupling constants). The objective of the talk is to discuss two classes of continuous random Hamiltonians with the unbounded point spectrum. One of these classes is related with the random network structures in $d$-dimensional space with $d$ greater or equal than 3 (like system of the bonds of $d$-dimensional lattice with random “boundary” conditions at the vertices). The second class represents the random operators with “non-local” potentials (sums of rank-one projectors with random amplitudes). Analysis of the second type operators has some common points with the well-known results by A. Figotin, L. Pastur, B. Simon on so-called “Maryland model”.

Surface Waves

L. Pastur

The talk is going to be partially a review and is devoted to the spectral analysis of the continuous and discrete Schrödinger operator whose potential (random or almost periodic) is supported on a subspace (a plane, a line) and the Laplace operator with analogous boundary conditions.

Asymptotics for the Schrödinger Equation:
Wave Functions and Forms

D.B. Pearson

Results are presented relating the asymptotics of quadratic expressions in solutions of the Schrödinger equation, and spectral theory. In particular, methods used by physicists to determine eigenvalues through the use of Milne’s equation and related second and third order differential equations may be generalised and applied to problems involving dense point or singular continuous spectrum. The asymptotics of solutions corresponding to a complex boundary condition will also be discussed, as well as the properties of related families of transfer matrices.
Introduction to Singular Continuous Spectrum

B. Simon

This is a tutorial/review type of lecture, mostly aimed at people who are NOT experts in mathematical spectral theory. It will include a quick introduction to spectral theory in general and to singular continuous spectra in particular, and a short historical review on operators with singular continuous spectrum.

Topics in Singular Continuous Spectra for Schrödinger Operators

B. Simon

The talk will have two parts. First, I will discuss joint work with Gordon, Jitomirskaya and Last that proves that the almost Mathieu operator at self dual coupling has purely singular continuous spectrum. This will depend on a suitable version of duality. Second, I will discuss joint work with Last on stability of the ac spectrum that, in particular, proves for almost periodic models that the ac spectrum is everywhere (rather than merely almost everywhere) constant.

Singular Continuous Spectrum for the Schrödinger Equation with Sturmian and Substitutional Potentials

A. Sütő

Jacobi matrices with diagonals given by Sturmian or substitutive sequences often have a pure singular continuous spectrum on a zero measure set. In 30 minutes we resume the main ingredients of the proof of this result.

A Semi-Classical Approach to Anderson Localization and Related Spectral Problems of the Magnetic Schrödinger Operator

W.M. Wang

We study the operator

\[ P = \left( \frac{1}{i} \frac{\partial}{\partial x} + B y \right)^2 + \left( \frac{1}{i} \frac{\partial}{\partial y} - B x \right)^2 + \sum_{i \in \mathbb{Z}^2} \alpha_i v_i \]

on \( L^2(\mathbb{R}^2) \), where \( B > 0 \) is a constant. \( \{\alpha_i\}_{i \in \mathbb{Z}^2} \) is a family of independently,
identically distributed random variables with distribution density \(g\), \(v_i(x) = v(x - i)\), where \(v \in C_0^\infty(\mathbb{R}^2; \mathbb{R})\). Away from the Landau levels, in the large \(B\) limit and under suitable assumptions on \(g\), we obtained the following results.

1. We prove that the spectrum is pure point with (at least) exponentially decaying eigenfunctions. The rate of decay is shown to be proportional to \(\sqrt{B}\).
2. We prove that the density of states is a \(C^\infty\) function.
3. We obtain an asymptotic expansion (in \(1/B\)) of the density of states. If we further assume that \(v \geq 0\) and that the support of \(v_i\) intersect so that \(\sum_{i \in \mathbb{Z}} v_i \geq s > 0\), then the expansion in 3. holds for all energies in the spectrum for the density of states measure considered as a distribution. These three results provide a rather clear picture of the spectral behaviour of \(P\) away from the Landau levels. The talk will focus mainly on the first result.

**Renormalisation of the Harper Hamiltonian: Generalised Wannier Functions and the First Order Correction**

M. Wilkinson

I will discuss some new developments of a renormalisation scheme for Harper's equation. This equation is

\[
\psi_{n+1} + \psi_{n-1} + 2 \cos(2\pi \beta n + \delta)\psi_n = E\psi_n
\]

and it can also be written in the form \(2(\cos \hat{p} + \cos \hat{x})|\psi\rangle = E|\psi\rangle\), with \(\hbar = 2\pi \beta\).

In the neighbourhood of a rational value of \(\beta\), \(\beta_0 = p/q\), the spectrum can be divided into \(q\) 'bands', each of which is described by an effective Hamiltonian \(H'(\hat{x}', \hat{p}')\), and commutator \([\hat{x}', \hat{p}'] = i\hbar'\). To lowest order, the effective Hamiltonian is obtained by a Peierls quantisation of the dispersion relation for the rational spectrum, and \(\hbar'\) is given by a complex formula which includes the quantised Hall conductance integer \(M\). Because the new effective Hamiltonian is similar to the original one (a periodic function of \(\hat{x}\) and \(\hat{p}\)), this procedure is a renormalisation group transformation acting on both the Hamiltonian and on \(\hbar\). This result was described in *J.Phys.*, **A20**, 4337, (1987).

I will describe a re-formulation of the original calculation which uses a set of generalised Wannier functions \(|\phi_\mu\rangle, \mu = 1, \ldots, N\) as a basis set for the band. The number of generalised Wannier states, \(N\) is related to the Chern integer \(M\) by \(qM + pN = 1\). One advantage of using the Wannier function basis is that it can be shown that the new effective Hamiltonian preserves the phase space symmetry of the original Hamiltonian. The generalised Wannier states
have a surprisingly complicated transformation under fourfold rotations, involving both Fourier transformations and ‘squeezing’. This work is discussed in *J. Phys.*, A27, 8123, (1994).

The Wannier function approach facilitates the calculation of corrections to the effective Hamiltonian as a power series in $\hbar'$. The first order correction at the band edges was obtained by Bellissard and Rammal, and is sometimes known as the ‘Wilkinson-Rammal formula’. The general formula involves a Peierls quantisation of the Wilkinson-Rammal expression, and an additional term which vanishes at the band edges. The new term is not gauge invariant: it depends on the choice of phase of the rational Bloch states. Gauge transformations of the Bloch states can be related to canonical transformations of the effective Hamiltonian. This work is a collaboration with Ritchie Kay; a paper is in preparation.

**Gauge Fields, Mode Mixing and Potential Distributions in Quantum Waveguides**

W. Zwerger

It is shown that the elimination of transverse motion in a quantum waveguide of general shape leads to a nonabelian gauge field for the longitudinal motion. By a gauge transformation the field may be eliminated at the expense of introducing an off-diagonal energy matrix which describes the mixing of different modes. Using this method we discuss the local potential distribution with finite transport currents in simple waveguides, e.g., a quantum point contact.