Minimal Orbits of Metrics and Elliptic Operators

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Abstract
The group of diffeomorphisms of a compact manifold acts isometrically on the space of Riemannian metrics with its $L^2$ metric. Following [1], [15], we define minimal orbits for this action by a zeta function regularization. We show that odd dimensional isotropy irreducible homogeneous spaces give rise to minimal orbits, and find a flat two torus giving a stable minimal orbit. We also define an infinite dimensional family of elliptic operators on a bundle over a manifold $M$ with an action by automorphisms of the bundle. The orbits are parametrized by the metrics on $M$. In odd dimensions, all orbits are minimal if the cohomology of the elliptic complex vanishes. In this case, the determinant of an associated elliptic operator is a smooth invariant of $M$. This invariant is defined for some classes of 3-manifolds. It is similar to analytic torsion, and has a combinatorial analogue.

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1 Introduction

Let $X$ be a Riemannian manifold with an isometric action of a Lie group. If $X$ is finite dimensional, it follows from Hsiang’s theorem [7] that orbits of minimal volume among all nearby orbits of the same type are in fact minimal submanifolds of $X$. Gauge theory provides an infinite dimensional analogue of this situation, where $X$ is the space of connections on a principal bundle over a compact manifold $M$, and the Lie group is the gauge group. In [8], [15], minimal orbits were defined in this context by a zeta function regularization and examples of minimal orbits were given. Zeta functions enter the discussion since in finite dimensions the first variation formula computes the variation of the determinant of the metric on a submanifold; in infinite dimensions this determinant is formally the determinant of a Laplacian type operator and is defined by Ray-Singer/zeta function regularization.

The regularization actually computes $\text{Tr} \, \mathcal{N} \mathcal{I} \mathcal{I}$, the component of the trace of the second fundamental form in the direction $\mathcal{N}$; an orbit is minimal if $\text{Tr} \, \mathcal{N} \mathcal{I} \mathcal{I} = 0$ for all $\mathcal{N}$. The regularization in [15] had the disadvantage of being finite only for certain orbits. In [1], a term was added to the regularization which guarantees that the new regularized definition of $\text{Tr} \, \mathcal{N} \mathcal{I} \mathcal{I}$ is always finite. This counterterm is zero in finite dimensions, so both regularizations generalize the usual notion of $\text{Tr} \, \mathcal{N} \mathcal{I} \mathcal{I}$.

In this paper we treat two new cases of infinite dimensional Riemannian manifolds with isometric actions of infinite dimensional groups. In the first case, the manifold is $\mathcal{M}$, the space of Riemannian metrics on a fixed compact manifold $X$, and the group $\mathcal{D}$ is the space of diffeomorphisms of $X$. It turns out that this setup is technically more difficult to handle than the gauge theory case, as here the group carries no natural metric. The resulting Laplacians used to define the regularization thus depend on a fixed choice of metric on $X$, and these nonnatural Laplacians must be related to the natural Laplacians that have appeared previously in discussions of $\mathcal{M}$. The theory also becomes more complicated when orbits of varying type occur. The main results are as follows (Theorems 3.1, 3.2, 3.3, 3.5).

**Theorem 1.1**

(i) In odd dimensions, the orbit of the volume one $G$-invariant metric on a isotropy irreducible homogeneous space $G/H$ is minimal within the space of volume one metrics on $G/H$.

(ii) The orbits of the flat 2-tori of volume one associated to the the points $(0,1)$ and $(1/2, \sqrt{3}/2)$ in the upper half plane are minimal within the space of all flat tori of volume one. The orbit associated to $(1/2, \sqrt{3}/2)$ is a stable minimal orbit.

(iii) An orbit of isolated diffeomorphism type is minimal.

Note that the isotropy irreducible homogeneous spaces include the symmetric spaces, but many more examples exist. These spaces are minimal orbits of infinite dimension and codimension. Part (iii) is an easy corollary of Hsiang’s theorem in finite dimensions, but is nontrivial in infinite dimensions.
The other case treated is an infinite dimensional family OP of elliptic operators on a bundle $S$ over a compact manifold $M$. These operators are of the form $s^{-1}D_{g}s$, where $D_{g}$ is an elliptic operator associated to the metric $g$ on $M$, and $s \in \text{Aut}_{g}(S)$ is a $g$-orthogonal automorphism of $S$. A group $\text{Aut}(S)$ isomorphic to $\text{Aut}_{g}(S)$ acts on an element of OP by further conjugation, and this action can be made isometric with respect to a natural metric on $\text{OP} \approx M \times \text{Aut}(S)$ which is not a product metric. Thus this case is a combination of the gauge theory and metric cases. We give two examples of such operators: the de Rham operator $D_{g} = d + \delta$ acting on $\Lambda^*T^*M$, and the torsion operator $D_{g} = d\varphi + \delta\varphi$ acting on forms with values in a flat bundle associated to a representation of $\pi_{1}(M)$.

Here it turns out that all orbits are minimal if dim $M$ is odd and the kernel of $D_{g}$ on $\text{End}(S)$ is trivial for one $g$. As expected, minimality is related to the variation of the volume element, which is formally the determinant of an operator, in this case $D_{g}^*D_{g}$.

**Theorem 1.2** (cf. Theorem 4.2) Let $\zeta(s)$ be the zeta function for $D_{g}^*D_{g}$, where $D_{g}$ acts on skew-symmetric endomorphisms of $S$. If $M$ is a closed, oriented, odd dimensional manifold, and $\text{ker } D_{g} = 0$ for one metric on $M$, then $\zeta'(0)$ is independent of the metric $g$.

Thus $\zeta'(0)$ is an invariant of the smooth structure of such manifolds. In the second case, this invariant is defined for certain acyclic representations of $\pi_{1}(M)$, and such representations exist for some known classes of 3-manifolds. Like analytic torsion, this invariant has a natural combinatorial analogue; we do not know if these quantities are equal.

The case of 3-manifolds gives the strongest results. For the classes of 3-manifolds mentioned above, $\zeta'(0)$ is an invariant for each $i$, where $\zeta_{i}(s)$ is the zeta function for the Laplacian on $\text{End}(S)$-valued $i$-forms (Theorem 4.5). We also produce invariants of 3-manifolds associated to representations of the fundamental group that are either irreducible or isolated (Theorem 4.6).

The paper is organized as follows. In §2 the first variation formula in finite dimensions is rederived in terms of the zeta function of a finite dimensional transformation which is the analogue of the Laplacians appearing in infinite dimensions. This serves as motivation for the later sections. We also discuss the effect of varying the metric on the submanifold, which is an unnecessary complication in finite dimensions but is forced upon us in infinite dimensions.

In §3 we handle the case of $\mathcal{M}$. §3.1 gives the general theory in infinite dimensions, §3.2 computes minimal orbits of flat 2-tori, §3.3 treats the case of orbits of varying type and discusses isotropy irreducible homogeneous spaces, and §3.4 gathers some local computations of the Laplacians used. §4 treats the de Rham and torsion operators.

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In this section we rederive the first variation formula for an immersed submanifold $M$ of a Riemannian manifold $(\overline{M}, \overline{g})$ in terms of the eigenvalues of a finite dimensional operator. Each step in this calculation has counterparts in the usual derivation (cf. [12, Ch. 1]). Afterwards, we modify the first variation formula to motivate some calculations in infinite dimensions.

Let $i: M \to \overline{M}$ be the immersion and set $L = L_x = d_i: T_x M \to T_i(x)\overline{M}$. We fix a Riemannian metric $g$ on $M$. To be consistent with the notation in the rest of the paper, we set $\overline{\Delta} = \overline{\Delta}_x = L^* L: T_x M \to T_x M$. (To be strictly consistent, we should relabel $L^*$ as $T^*$. There exists an orthonormal basis $\{\phi_i\}$ of $T_x M$ consisting of eigenfunctions of $\overline{\Delta}$, i.e. $\overline{\Delta} \phi_i = \lambda_i \phi_i$. Since $i$ is an immersion, $\lambda_i > 0$. If we let $X_i = L \phi_i/\sqrt{\lambda_i}$, then $\{X_i\}$ is an orthonormal basis of $T_i(x)\overline{M}$. For a fixed $x \in M$, we may extend $\{\phi_i\}$ near $x$ so that $\overline{\Delta} \phi_i(y) = \lambda_i(y) \phi_i(y)$ for all $y$ in the neighborhood $U$ of $x$.

Take a variation $F: M \times (\pm \epsilon, \epsilon) \to \overline{M}$ with variation vector field $N_x = dF_{(x,0)}(\partial_\alpha)$, where $\alpha$ is the parameter for $(\pm \epsilon, \epsilon)$. We assume $N \perp i(M)$. (Strictly speaking, $N \in \Gamma(i^* T\overline{M})$, but near $x$ we may write $N \in \Gamma(T\overline{M})$.) Let $X^\perp$ denote the projection of a vector $X \in T\overline{M}$ into the normal bundle to $d_i(TM)$ (which is locally defined) in $T\overline{M}$. Then $\text{Tr}_N II$, the component of the trace of the second fundamental form at $x$ in the direction of $N = N_x$ is by definition

$$\text{Tr}_N II = \langle (\nabla_{X^\perp} X_i)^\perp, N \rangle_{\overline{g}},$$

where $\nabla$ is the Levi-Civita connection on $\overline{M}$ and we are using summation convention. Here we omit mentioning the point $x$ in $\text{Tr}_N II$. Using $\langle X, N \rangle = 0$, we get

$$\text{Tr}_N II = \langle \nabla_{X^\perp} X_i, N \rangle = \perp \langle X_i, \nabla_X N \rangle$$

$$= \perp \langle \frac{L \phi_i}{\sqrt{\lambda_i}}, \nabla_{L \phi_i/\sqrt{\lambda_i}} N \rangle = \perp \frac{1}{\lambda_i} \langle L \phi_i, \nabla_{L \phi_i} N \rangle.$$

Here and from now on all inner products are with respect to $\overline{g}$ unless otherwise noted. We now extend $\phi_i, N$ to vector fields on $U \times (\pm \epsilon, \epsilon)$, $F(U \times (\pm \epsilon, \epsilon))$ respectively by trivially setting $\phi_i(y, \alpha) \overset{\text{def}}{=} \phi_i(y, 0)$ and setting $N_{F_{(x,0)}} = dF_{(x,0)}(\partial_\alpha)$. $L$ also extends to the operator $dF: TU \times (\pm \epsilon, \epsilon) \to T\overline{M}$. Thus

$$\text{Tr}_N II = \perp \frac{1}{\lambda_i} \langle L \phi_i, \nabla_N L \phi_i \rangle \perp \frac{1}{\lambda_i} \langle L \phi_i, [L \phi_i, N] \rangle.$$
The last term vanishes, since \([L\phi_i, N] = [dF(\phi_i), dF(\partial_\alpha)] = dF[\phi_i, \partial_\alpha] = dF(0) = 0\), and so
\[
\text{Tr}_N \Pi = \frac{1}{\lambda_i} \langle L\phi_i, \nabla_N L\phi_i \rangle. \tag{2.1}
\]

**Remark:** Let \(G = \overline{\Delta}^{-1}\) be the “Green’s operator” for \(\overline{\Delta}\). Then (2.1) becomes
\[
\text{Tr}_N \Pi = \frac{1}{\lambda_i} \langle LG\phi_i, \nabla_N L\phi_i \rangle
= \frac{1}{\lambda_i} \text{Tr} (GL^*\nabla_N L)
= \left. \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\overline{\Delta}^* L^*\nabla_N L})dt \right|_{t=1}
= \left. \int_0^\infty t^s \text{Tr}(e^{-t\overline{\Delta}^* L^*\nabla_N L})dt \right|_{s=0}. \tag{2.2}
\]

Let \(\zeta(s) = \sum \lambda_i^{-s}\) be the zeta function of \(\overline{\Delta}\). Then \(\zeta(0) = \dim M\), and so the variation of \(\zeta(0)\) in the direction \(N\) satisfies \(\delta_N \zeta(0) = 0\). Thus we may rewrite \(\text{Tr}_N \Pi\) as
\[
\text{Tr}_N \Pi = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\overline{\Delta}^* L^*\nabla_N L})dt \bigg|_{s=1} \frac{\delta_N \zeta(0)}{2(s+1)} \bigg|_{s=1}. \tag{2.3}
\]

Following [1], [15], we will use (2.3) as the regularization of \(\text{Tr}_N \Pi\) in infinite dimensions. (In [15], the next to last line in (2.2) was used as the regularization, and the importance of the last term was shown in [1].)

Continuing with the derivation of the first variation formula, we set \(\overline{g}_\alpha\) to be the restriction of \(\overline{g}\) to \(T_{F(x, \alpha)} (M \times \{\alpha\})\), and let
\[
L_\alpha = \left. dF_{[x, \alpha]} \right|_{\tau_M \times 0}.
\]
By (2.1), we have
\[
\text{Tr}_N \Pi = \frac{1}{\lambda_i} \langle L\phi_i, \nabla_N L\phi_i \rangle
= \frac{1}{2\lambda_i} \langle L\phi_i, L\phi_i \rangle
= \frac{1}{2\lambda_i} \left. \frac{d}{d\alpha} \right|_{\alpha=0} \langle L_\alpha \phi_i, L_\alpha \phi_i \rangle_{\overline{g}_\alpha}
= \frac{1}{2\lambda_i} \left. \frac{d}{d\alpha} \right|_{\alpha=0} \langle \Delta_\alpha \phi_i, \phi_i \rangle_{\overline{g}} \tag{2.4}
\]
where \(\delta_\alpha \overline{\Delta}\) denotes the variation of \(\overline{\Delta}\) in the direction \(\alpha\). This expression is independent of the extension of the orthonormal basis \(\{\phi_i\}\) on \(U\) to an orthonormal basis on \(U \times (\perp \epsilon, \epsilon)\). So extend \(\{\phi_i\}\) to \(\{\phi_i(y, \alpha)\}\) on \(U \times (\perp \epsilon, \epsilon)\) so that \(\Delta_\alpha \phi_i(\alpha) = \lambda_i(\alpha) \phi_i(\alpha)\) (dropping \(y\) from the notation). Note that this is not the same \(\phi(y, \alpha)\) as before.
Then for $\delta = \delta_0$ and $\dot{\lambda}_i = (d/d\alpha)|_{\alpha=0}\lambda_i$, the formula $(\delta \Sigma_\alpha)\phi_i + \Sigma_\alpha(\delta \phi_i) = \dot{\lambda}_i \phi_i + \lambda_i \delta \phi_i$ at $\alpha = 0$ yields

$$
\langle (\delta \Sigma_\alpha)\phi_i, \phi_i \rangle_g = \langle \delta \phi_i, \Sigma_\phi_i \rangle_g + \langle \dot{\lambda}_i \phi_i, \phi_i \rangle_g + \langle \lambda_i \delta \phi_i, \phi_i \rangle_g \\
= \langle \lambda_i \delta \phi_i, \phi_i \rangle_g + \dot{\lambda}_i + \lambda_i \delta \phi_i, \phi_i \rangle_g \\
= \dot{\lambda}_i.
$$

Combining (2.4) and (2.5) gives

$$
\text{Tr}_N \Pi = \frac{1}{2} \sum_i \frac{\dot{\lambda}_i}{\lambda_i}.
$$

We remark that there may be trouble defining $\dot{\lambda}_i$ where an eigenvalue bifurcates, but this difficulty disappears when we sum over $i$, so the computation above is valid.

For $\zeta(s) = \sum_i (\lambda_i)^{-s}$, it is easy to check from (2.6) that

$$
\text{Tr}_N \Pi = \frac{1}{2} \delta \zeta'(0).
$$

An infinite dimensional analogue is given in (3.10).

Now let $\{\phi_i^*\}$ be the frame of $T^*_x M$ dual to $\{\phi_i\}$, and let dvol be the volume form for $i(M)$ at $i(x)$. Then

$$
\langle \text{Tr}_N \Pi, \text{dvol} \rangle = \frac{1}{2} \sum_i \frac{\dot{\lambda}_i}{\lambda_i} \det \frac{1}{2}((L \phi_i, L \phi_j)) \phi_i^* \wedge \ldots \wedge \phi_n^* \\
= \frac{1}{2} \sum_i \frac{\dot{\lambda}_i}{\lambda_i} \det \frac{1}{2}((\Sigma \phi_i, \phi_j)) \phi_i^* \wedge \ldots \wedge \phi_n^* \\
= \frac{1}{2} \sum_i \frac{\dot{\lambda}_i}{\lambda_i} (\prod_i \lambda_i)^{1/2} \phi_i^* \wedge \ldots \wedge \phi_n^* \\
= \frac{1}{2} \sum_i (\prod_i \lambda_i)^{-1/2} (\dot{\lambda}_1 \lambda_2 \ldots \lambda_n + \ldots + \lambda_1 \ldots \lambda_n) \phi_i^* \wedge \ldots \wedge \phi_n^* \\
= N((\prod_i \lambda_i)^{1/2}) \phi_i^* \wedge \ldots \wedge \phi_n^*.
$$

Also, the volume form at $F(x, \alpha)$ is given by

$$
\det \frac{1}{2} ((L_\alpha \phi_i, L_\alpha \phi_j)) \phi_i^* \wedge \ldots \wedge \phi_n^* = (\prod_i \lambda_i(\alpha))^{1/2} \phi_i^* \wedge \ldots \wedge \phi_n^* = N \det \frac{1}{2} (L_\alpha \phi_j) \phi_i^* \wedge \ldots \wedge \phi_n^*.
$$

(cf. [12, p. 8]). Combining (2.7) and (2.8) gives

$$
\langle \text{Tr} \Pi, N \rangle \text{dvol} = \langle \text{Tr}_N \Pi, \text{dvol} \rangle = N(\text{dvol}).
$$
This is the first variation formula, which is usually written in the global form

$$\perp \int_M \langle \text{Tr} \ II, N \rangle \ dvol = N \left( \int_M dvol \right).$$

We now discuss the effect of allowing the metric $g_\alpha$ on $M \times \{\alpha\}$ to vary with $\alpha$. Of course, there is no need for this complication in finite dimensions, but it cannot be avoided in the next section.

So put the metric $g_\alpha \oplus d\alpha^2$ on $M \times (\perp \epsilon, \epsilon)$, where $g_0 = g$. By (2.4), we have

$$\text{Tr}_N II = \frac{1}{2\lambda_i} \left. \frac{d}{d\alpha} \right|_{\alpha=0} \langle L_\alpha \phi_i, L_\alpha \phi_i \rangle_{g_\alpha}$$

$$= \frac{1}{2\lambda_i} \left. \frac{d}{d\alpha} \right|_{\alpha=0} \langle \Delta_\alpha \phi_i, \phi_i \rangle_{g_\alpha}.$$

Here $\Delta_\alpha = L_\alpha^* L_\alpha$, where $L_\alpha^*$ is now defined by

$$\langle L_\alpha \phi_i, \psi \rangle_{g_\alpha} = \langle \phi, L_\alpha^* \psi \rangle_{g_\alpha}.$$

Thus

$$\text{Tr}_N II = \frac{1}{2\lambda_i} \langle \delta \Delta_\alpha \phi_i, \phi_i \rangle_{g} \perp \frac{1}{2\lambda_i} N_{ab} \langle \Delta_\alpha \phi_i, \phi_i \rangle_{g_\alpha}$$

where $N_{ab} = \left. \frac{d}{d\alpha} \right|_{\alpha=0} g_\alpha$. Thus we can write $\delta_N$ for $\delta = \delta_\alpha$. Using $\Delta_\alpha = \lambda_i \phi_i = \overline{\Delta}_\alpha$ at $\alpha = 0$, we get

$$\text{Tr}_N II = \frac{1}{2} \sum_i \langle \delta \Delta_\alpha \phi_i, \phi_i \rangle_{g} \perp \frac{1}{2} \sum_i N_{ab} \phi_i^a \phi_i^b$$

$$= \frac{1}{2\Gamma(s)} \left. \frac{d}{ds} \right|_{s=1} \int_0^\infty t^{s-1} \text{Tr}_g (\delta_N \Delta_\alpha \cdot e^{-t}) dt$$

$$= \frac{1}{2} \sum_i g_{ik} g^{sc} N_{ca} \phi_i^a \phi_i^b.$$

Now set

$$\zeta_N(s) = \frac{1}{\Gamma(s)} \left. \frac{d}{ds} \right|_{s=1} \int_0^\infty t^{s-1} \text{Tr}_g (\delta_N \Delta_\alpha \cdot e^{-t}) dt$$

$$\overline{\zeta}_N(s) = \frac{1}{\Gamma(s)} \left. \frac{d}{ds} \right|_{s=1} \int_0^\infty t^{s-1} \text{Tr}_g (\delta_N \overline{\Delta}_\alpha \cdot e^{-t}) dt$$

The calculation above gives

$$\text{Tr}_N II = \frac{1}{2} (\zeta_N(1) + \text{Tr}(\overline{\Delta})).$$
where \((\bar{N}\phi)^s = g^{s} N_{\alpha\beta} \phi^\alpha\) - i.e. \(\bar{N}\) lowers an index by \(N\) and raises an index by \(g\). Similarly, repeating the calculation above starting at (2.4) but now using \(\Sigma_\alpha\), we obtain
\[
\text{Tr}_N \Pi = \frac{1}{2} \zeta_N(1).
\]
Thus \(\zeta_N(1) = \zeta_N(1) + \text{Tr}(\bar{N})\).

Finally,
\[
\text{Tr}(\bar{N}) = \text{Tr}(\bar{N}\Delta^{-s}) \bigg|_{s=0} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\bar{N} e^{-t\Delta}) \, dt \bigg|_{s=0}.
\]
Thus
\[
\text{Tr}_N \Pi = \zeta_N(1) = \zeta_N(1) + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\bar{N} e^{-t\Delta}) \, dt \bigg|_{s=0}.
\]
This formula is the finite dimensional analogue of (3.16), with the operator \(B\) in (3.16) a slightly more complicated version of \(\bar{N}\) and the added difficulty that \(\ker \Delta\) need not vanish. While it seems unnatural in finite dimensions to work with \(\zeta_N(1)\), it turns out to be the natural choice in infinite dimensions.

## 3 Metrics and diffeomorphisms

In this section we will apply the first variation formula of §2 to the infinite dimensional situation of the orbits of the diffeomorphism group of a compact manifold \(M\) within the space of Riemannian metrics on \(M\). In particular, we will define what it means for an orbit to be minimal within the space of metrics, and relate this minimality to the determinant of a Laplacian-type operator. This is similar to the gauge theory case considered in [8], [15] and the general theory in [1], but has extra complications arising from the lack of a natural metric on the gauge group. We also produce several examples of minimal orbits.

In §3.1, we set up the general theory when all orbits have the same diffeomorphism type. We will apply this to find minimal orbits of flat 2-tori in §3.2. For other examples of minimal orbits, we need to treat the case of orbits of varying type. This is done in §3.3. Finally, in the last subsection we collect some local calculations.

### 3.1 Global theory

A. Fix a compact \(n\)-manifold \(M\). Let \(\mathcal{M}\) denote the space of smooth Riemannian metrics on \(M\), and let \(\mathcal{D}\) denote the group of smooth diffeomorphisms of \(M\). \(\mathcal{D}\) acts on \(\mathcal{M}\) by pullback: if \(\psi \in \mathcal{D}, g \in \mathcal{M}\), then \(\psi \cdot g = \psi^*g\). If we impose standard Sobolev norms on \(\mathcal{M}, \mathcal{D}\), then \(\mathcal{M}\) becomes a Banach manifold and \(\mathcal{D}\) an ILH Lie group [17], and the action of \(\mathcal{D}\) on \(\mathcal{M}\) is as differentiable as desired. \(\mathcal{D}\) is also a group
before Sobolev norms are imposed, and once the norms are chosen, composition of diffeomorphisms produces a diffeomorphism also as differentiable as desired. We will assume that the choice of norms has been made.

Fix a metric $g_0$ on $M$. The orbit $O_{g_0}$ through $g_0$ is diffeomorphic to $\mathcal{D}/S_{g_0}$, where $S_{g_0}$ is the stabilizer of $g_0$. As in finite dimensions, it would be natural to assume that $S_{g_0} = \{id\}$, so that the map $\psi \mapsto \psi^*g_0$ is an immersion of $\mathcal{D}$ in $\mathcal{M}$. To insure that all orbits are of the same diffeomorphism type, we will assume instead that the dimension of $S_\sigma$ is constant for all $\sigma$ near $g_0$. To make the analogy with §1, we need Riemannian metrics on $\mathcal{M}, \mathcal{D}$. Now $\mathcal{M}$ comes with the standard $L^2$ inner product. Namely, $\mathcal{M}$ is an open cone in $\Gamma(S^2T^*M)$, the space of (sections of the) symmetric two-tensors on $M$, and the inner product of $h, k \in T_{g_0}\mathcal{M}$ is given by

$$\langle h, k \rangle_{g_0} = \int_M (g_0)^{ik} (g_0)^{lj} h_{ij} k_{kl} \, \text{dvol}_{g_0},$$

where we follow the convention of writing a global integral in terms of a locally defined integrand. Here of course $g_0 = (g_0)_{ij} \, dx^i \otimes dx^j$ locally, and similarly for $h, k$, with $(g_0)^{ij}$ the inverse matrix to $(g_0)_{ij}$. (In contrast to the gauge theory case, where $\mathcal{M}$ is replaced by a space of connections, this metric is not flat.) $\mathcal{D}$ acts on $\mathcal{M}$ via isometries; the geometry of $\mathcal{M}$ and the quotient space $\mathcal{M}/\mathcal{D}$ is treated in [4], [5].

To put a metric on $T_{g_0}\mathcal{D}$, it is sufficient to put an inner product on $T_{id}\mathcal{D}$ and then left translate it to all of $\mathcal{D}$. However, $T_{id}\mathcal{D} = \Gamma(TM)$ has no natural metric, although once $g_0$ is chosen it has the $L^2$ metric

$$\langle X, Y \rangle_{g_0} = \int_M (g_0)^{ij} X_i Y_j \, \text{dvol}_{g_0},$$

for $X = X^i \partial_i, Y = Y^i \partial_i$. We will also call this metric on $\mathcal{D}$ just $g_0$.

We now proceed as in finite dimensions. We consider a variation $F: \mathcal{D} \times (\perp, \epsilon) \to \mathcal{M}$ with $F(\psi, 0) = \psi^*g_0$. We put the product metric $g_0 + d\alpha^2$ on $\mathcal{D} \times (\perp, \epsilon)$ and set $L_\alpha = dF(id, \alpha): \Gamma(TM) \to \Gamma(S^2T^*M)$. At the point $g_\alpha = F(id, \alpha)$, define $L_\alpha^*, L_\alpha$ by

$$\langle L_\alpha^* \omega, \eta \rangle_{g_\alpha} = \langle \omega, L_\alpha^* \eta \rangle_{g_0} = \langle \omega, L_\alpha \eta \rangle_{g_\alpha}$$

for $\omega \in \Gamma(TM), \eta \in \Gamma(S^2T^*M)$. Set $\Sigma_\alpha = \nabla_\alpha L_\alpha, \Delta_\alpha = L_\alpha^* L_\alpha$. Of course $\Sigma_0 = \Delta_0$.

Note that since we must use the product metric as in finite dimensions, we cannot use the natural operator $\Delta_\alpha$, but are forced to use the non-natural $\Sigma_\alpha$. Our assumption on the stabilizer is equivalent to assuming that $\dim \ker \Sigma_\alpha$ is independent of $\alpha$.

Following [1], [15], we now define minimal orbits of metrics by means of (2.3). We let $\{\phi_i\}$ be a $g_0$-orthonormal basis of $L^2(TM)$ satisfying $\Sigma_\alpha \phi_i = \Delta \phi_i = \lambda_i \phi_i$. As we will see in Corollary 3.2, $\Sigma_\alpha$ is elliptic, so such a basis exists for all $\alpha$, and by standard techniques can be chosen to depend smoothly on $\alpha$. We set the zeta function of $\Sigma_0$ to be

$$\zeta(s) = \sum_{\lambda_i \neq 0} \lambda_i^{-s}.$$  (3.1)

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We similarly define $\zeta_0$ for $\Delta_\alpha$. This converges for $\text{Re}(s)$ sufficiently large, and has a meromorphic continuation to all of $\mathbb{C}$ with a regular value at zero; this follows in a well known way from the ellipticity of $\Delta$ and the subsequent asymptotic expansion of its heat kernel. Note that in contrast to the finite dimensional case, the kernel of $\Delta$ need not be trivial. However, by ellipticity the dimension of the kernel of $L$ and hence of $\Delta$ is finite, and is independent of $g \in \mathcal{O}_{g_0}$, since $L_{id} = (d\psi)^{-1}L_{\psi}d\psi$, for $\psi: \mathcal{D} \to \mathcal{D}$ acting by left multiplication. It follows that $\mathcal{O}_{g_0}$ is always a submanifold of $\mathcal{M}$, and that $L^*_\psi L_{\psi}$ is isospectral to $L^*_{id} L_{id}$; in particular, $\zeta'(0)$ for the zeta functions associated to these operators is constant along orbits.

**Definition:** The component of the trace of the second fundamental form in the direction $N$ for the orbit of a metric $g_0$ is defined to be

$$
\text{Tr}_N \Pi = \lim_{s \to 1} \left[ -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(e^{-t\Delta}L^*\nabla_N L) \, dt \right] \frac{\delta_N \zeta(0)}{2(s-1)}. 
$$

(3.2)

An orbit $\mathcal{O}_{g_0}$ is **minimal** if $\text{Tr}_N \Pi = 0$ for all normal vectors $N$ at $g_0$.

**Remarks:** Here $\nabla$ is the Levi-Civita connection for the $L^2$ metric on $\mathcal{M}$, defined as usual by

$$
2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle \perp Z \langle X, Y \rangle + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle \perp \langle Y, Z \rangle, X
$$

(cf. [4]). The term $\nabla_N L$ in (3.2) equals $(d/d\alpha)|_{\alpha=0}L_\alpha$ in a frame in which $\nabla_N = \delta_N$ (i.e. $L$ is varying). Since we are taking the trace at $g_0$, we may replace $\Delta$ by $\Delta$ in the integral. However, $\zeta_\alpha(s) = \zeta_0(s)$, the zeta function for $\Delta_\alpha$, only at $\alpha = 0$, so we cannot replace $\delta_N \zeta(0)$ by $\delta_N \zeta(0)$. Note that (3.20) shows that $\zeta_\alpha(0)$ is smooth in $\alpha$ under our assumption, so $\delta_N \zeta(0)$ makes sense. As is shown below in (3.10), the last term in (3.2) subtracts off a possible pole from the first term, so we can also write

$$
\text{Tr}_N \Pi = F.P. \left[ -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(e^{-t\Delta}L^*\nabla_N L) dt \right]_{s=1},
$$

(3.3)

where $F.P.$ denotes the finite part. Note that since $\mathcal{D}$ acts isometrically, $\text{Tr}_N \Pi = 0$ for all normal vectors at $g_0$ if the same is true at any $g \in \mathcal{O}_{g_0}$. This is clear for orbits of isometric actions in finite dimensions, and can be checked by directly examining the right hand side of (3.2); an easier proof will be given below.

We now show that the right hand side of (3.2) is always finite. We have

$$
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(e^{-t\Delta}L^*\nabla_N L) dt
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \langle L^*\nabla_N L_{\phi_i}, e^{-t\Delta} \phi_i \rangle_{g_0} dt
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \langle e^{-t\Delta} \phi_i, \nabla_N L_{\phi_i} \rangle_{g_0} dt
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \langle e^{-t\Delta} \phi_i, \nabla_N L_{\phi_i} \rangle_{g_0} dt.
$$
\[
\begin{align*}
&= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \langle L^* \nabla N L \phi_i, \phi_i \rangle_{g_0} \, dt \\
&= \left. \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \langle \nabla N L_\alpha \phi_i, \nabla N L_\alpha \phi_i \rangle_{g_0} \, dt \right|_{\alpha=0} \\
&= \left. \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \frac{d}{d\alpha} \langle L_\alpha \phi_i, L_\alpha \phi_i \rangle_{g_0} \, dt \right|_{\alpha=0} \\
&= \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \frac{d}{d\alpha} \langle \delta L_\alpha \phi_i, \phi_i \rangle_{g_0} \, dt \\
&= \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0} (\delta \nabla e^{-t\Delta}) \, dt \\
&= \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0} (\delta \nabla e^{-t\Delta}) \, dt.
\end{align*}
\]

where

\[
\bar{\zeta}_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0} (\delta \nabla e^{-t\Delta}) \, dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0} (\delta \nabla e^{-t\Delta}) \, dt.
\]

Of course, this computation should be read as being valid for \(\text{Re}(s)\) sufficiently large, where the convergence of the integrals is easily established. For example, consider the term \(\text{Tr}_{g_0} (\delta \nabla e^{-t\Delta})\) in the last equation. The operator \(\delta \nabla e^{-t\Delta}\) has kernel \((\delta \nabla e^{-t\Delta})_{x} e(t, x, y)\), where \(e\) is the kernel of \(e^{-t\Delta}\), and so has a good asymptotic expansion as \(t \to 0\) (cf. \S2.1). Breaking the integral \(\int_0^\infty\) into \(\int_0^1 + \int_1^\infty\) and plugging in the asymptotic expansion into the first integral shows that this integral exists near zero for \(\text{Re}(s)\) sufficiently large. Also, since \(\nabla = \Delta\) at \(g_0\), \(\text{Tr}(\delta \nabla e^{-t\Delta}) = \text{Tr}(\delta \nabla) \text{Tr}(e^{-t\Delta})\). Now the kernel of \(Le^{-t\Delta}\) has exponential decay as \(t \to \infty\), since \(\nabla = \ker L\), and hence so does the kernel of \(\delta \nabla \text{Tr}(e^{-t\Delta})\). On \(\ker L\), \(\langle L \delta \nabla \cdot e^{-t\Delta} \phi, \phi \rangle = \langle \delta \nabla L \cdot e^{-t\Delta} \phi, L \phi \rangle = 0\), and so \(\text{Tr}_{g_0}(\delta \nabla \text{Tr}(e^{-t\Delta})\) also has exponential decay as \(t \to \infty\). Thus the integral exists at infinity. (By (2.4), (3.5), the definition (3.2) of the regularized trace agrees with the definition in [1, (3.5)].)

To proceed with the proof of the finiteness of (3.2), we note that by the Mellin transform

\[
\bar{\zeta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0} (e^{-t\Delta} \perp P) \, dt,
\]

where \(P\) is the orthogonal projection of \(L^*(TM, g_0)\) onto the kernel of \(\nabla\); adding in this projection makes the integral finite near infinity.

**Lemma 3.1** (cf. [15, Lemma 5.5]) For all \(s \in \mathbb{C}\),

\[
(s \perp 1) \bar{\zeta}_N(s) = \perp \delta_N \bar{\zeta}(s \perp 1) \tag{3.6}
\]
At poles of $\zeta_N(s)$, this equation is to be interpreted as saying that the poles of $\zeta_N(s)$ coincide with the poles of $\zeta(s)$ shifted by one.

**Proof.** By the uniqueness of the meromorphic continuation of $\zeta_N(s)$ and $\zeta(s)$, it suffices to prove the equation for $\Re(s) > 0$. Under the assumption that $\dim \ker \Delta$ is constant, we get

$$
(s \perp 1)\zeta_N(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}((\delta_N \Delta e^{-t\Delta}))
$$

$$
= \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \text{Tr}(\perp t(\delta_N \Delta e^{-t\Delta}))
$$

$$
= \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_N \text{Tr}(e^{-t\Delta})
$$

$$
= \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_N \text{Tr}(e^{-t\Delta} \perp P)
$$

Here we have used (2.5) to write

$$
\perp t \cdot \text{Tr}_{\delta}(\delta_N \Delta \cdot e^{-t\Delta}) = \perp t \sum_i e^{-\lambda_i t} \langle \delta_N \Delta \phi_i, \phi_i \rangle = \perp t \sum_i e^{-\lambda_i t} \lambda_i = \delta_N(\sum_i e^{-\lambda_i t}) = \delta_N \text{Tr}_{\delta_0}(e^{-t\Delta}).
$$

(3.8)

Combining this Lemma with (3.2), (3.4), we get

$$
\text{Tr}_{N \perp} = \frac{1}{2} \lim_{s \to 1} \left[ \frac{\perp \delta_N \zeta(s \perp 1)}{s \perp 1} + \frac{\delta_N \zeta(0)}{s \perp 1} \right]
$$

$$
= \frac{1}{2} \lim_{s \to 0} \left[ \frac{\perp \delta_N \zeta(0) + s \zeta'(0) + O(s^2)}{s} + \frac{\delta_N \zeta(0)}{s} \right]
$$

(3.9)

In summary, by (3.3), (3.5), (2.7), we have

$$
\text{Tr}_{N \perp} = \frac{1}{2} \perp P. \zeta_N(1) = \frac{1}{2} \delta_N \zeta'(0).
$$

(3.10)

It is standard to relate the right hand side of (3.10) to the regularized volume element for $O_{ga}$ at $g_a$. For $\phi_i$ fixed to be independent of $\alpha$, the volume element to the orbit at $g_a$ is formally

$$
\sqrt{\det(\langle L_{\alpha} \phi_i, L_{\alpha} \phi_i \rangle_{ga})} \phi_i^1 \wedge \phi_i^2 \wedge \ldots = \sqrt{\det(\langle \Delta_{\alpha} \phi_i, \phi_i \rangle_{ga})} \phi_i^1 \wedge \phi_i^2 \wedge \ldots
$$

(3.11)

$$
\sqrt{\det(\langle \Delta_{\alpha} \phi_i, \phi_i \rangle_{ga})} \phi_i^1 \wedge \phi_i^2 \wedge \ldots
$$

(3.12)
Since the \{\phi_i\} are \(g_0\)-orthonormal (and not \(g_\alpha\)-orthonormal), it is heuristically plausible that the expression under the square root in (3.11) should give the determinant of \(\Delta_\alpha\), whereas the corresponding term in (3.12) should not be thought of as \(\det \Delta_\alpha\). Using the Ray-Singer regularization of the determinant of a Laplacian type operator, we define (the "Hodge star" of) the volume element to \(O_{\phi_a}\) to be the nonnatural \(\exp(\int \frac{1}{2\pi} \zeta_\alpha(0))\), where \(\zeta_\alpha\) is the zeta function for \(\Delta_\alpha\). In particular, (3.10) shows that an orbit is minimal iff it is minimal among all nearby orbits, provided we assume that all nearby orbits are of the same type (i.e., all nearby orbits have \(\text{dim ker } L_\alpha = \text{dim ker } L_a\), or equivalently are diffeomorphic to \(O_{\phi_a}\). The point here is that the zeta function behaves discontinuously in \(\alpha\) if the dimension of the kernel jumps, so our analysis breaks down.) As in [15, Thm. 5.14], we interpret this as an infinite dimensional analogue of Hsiang’s theorem in finite dimensions, which reduces the search for minimal orbits to checking variations only through orbits, and not through arbitrary submanifolds.

B. In order to produce examples of minimal orbits, we need to compare \(\delta_N \zeta'(0)\) with the more natural \(\delta_N \zeta'(0)\) for two reasons: \(\zeta'(0)\), although notoriously difficult to compute, can be handled in some special cases (cf. Theorem 3.2), and Bleecker’s theorem about critical metrics applies to natural Lagrangians (cf. Theorem 3.5). Here \(\zeta(s), \zeta_\alpha(s)\) are defined in the usual way from the nonzero eigenvalues of \(\Delta, \Delta_\alpha\).

Looking back at (3.4), we get

\[
\zeta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{i} e^{-\lambda_i t} \frac{d}{d \alpha} \langle L \phi_i, L \phi_i \rangle_{g_0} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{i} e^{-\lambda_i t} \frac{d}{d \alpha} \langle \Delta_a \phi_i, \phi_i \rangle_{g_\alpha} dt.
\]

Denote \(g_0\) just by \(g\). Since \(\frac{d}{d \alpha}\bigg|_{\alpha=0} g_\alpha = N\), we have

\[
\frac{d}{d \alpha} \bigg|_{\alpha=0} \langle \Delta_a \phi_i, \phi_i \rangle_{g_\alpha} = \frac{d}{d \alpha} \bigg|_{\alpha=0} \int_M (g_\alpha)_{ab} \phi^s_i (\Delta_a \phi_i)^b d\text{vol}(g_\alpha)
= \int_M N_{ab} \phi^s_i (\Delta_0 \phi_i)^b d\text{vol}(g_0) + \int_M g_{ab} \phi^s_i ((\delta \Delta_0) \phi_i)^b d\text{vol}(g_0)
+ \int_M g_{ab} \phi^s_i ((\Delta_0 \phi_i)^b (\text{tr}_g N) d\text{vol}(g_0) )
= \int_M [g_{ab} g^s \rho c N_{ca} \phi^a_i (\Delta_0 \phi_i)^b + (\delta \Delta_0) \phi_i, \phi_i \rangle_{g_0}
+ (\text{tr}(N) \phi_i, \Delta_0 \phi_i \rangle_{g_0}] d\text{vol}(g_0).
\]

Define the 0th order operator \(A\) on \(\Gamma(TM)\) by \(A: \phi^s \partial_a \rightarrow g^s c N_{ca} \phi^a \partial_a\). Note that \(A\) lowers an index by \(N\) and raises an index by \(g = g_0\).

Thus

\[
\zeta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{i} e^{-\lambda_i t} \{ \langle A \phi_i, \Delta_0 \phi_i \rangle_{g_0} + \langle \delta \Delta_0 \phi_i, \phi_i \rangle_{g_0}\}
\]

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Because $g^* N_{a} = N_{a}^*$ is a self-adjoint transformation of $TM$, $A$ is self-adjoint. Explicitly, we have

$$\langle A \phi, \psi \rangle = \int g_{b a} g^* N_{a} \phi^a \psi^b \, d\text{vol}(g_0) = \int g_{a b} \phi^a (N^a \delta_b^c \psi^c) \, d\text{vol}(g_0),$$

so

$$(A^* \psi)^a = N^a \delta_b \psi^b = N^a \psi^b = (A \psi)^a.$$  

Thus if we set

$$\zeta_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\delta \Delta e^{-t \Delta}) \, dt,$$

(3.14) gives

$$\bar{\zeta}_N(s) = \zeta_N(s) + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(B \Delta e^{-t \Delta}) \, dt,$$

where $B$ is the $0^{th}$ order operator on $TM$ given by

$$B \phi = A \phi + \text{tr}_{g_0} N \cdot \phi.$$  

It is easy to extract from (3.13) that $B$ is characterized by

$$\frac{d}{ds} \bigg|_{s=0} \langle \phi, \phi \rangle_{g_0} = \langle B \phi, \phi \rangle_{g_0}. \quad \text{(3.15)}$$

Using $(\partial_i + \Delta)e^{-i\Delta} = 0$ gives

$$\bar{\zeta}_N(s) = \zeta_N(s) \downarrow \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(B \partial_i e^{-t \Delta}) \, dt$$

$$= \zeta_N(s) + \frac{s}{\Gamma(s)} \int_0^\infty t^{s-2} \text{Tr}(B (e^{-t \Delta} \perp P)) \, dt$$

$$= \zeta_N(s) + \frac{1}{\Gamma(s+1)} \int_0^\infty t^{s-2} \text{Tr}(B e^{-t \Delta} \perp BP) \, dt. \quad \text{(3.16)}$$

Recall that $P$ denotes projection onto the kernel of $\overline{\Delta} = \Delta$; this term is added to make the integrals converge at infinity so that the integration by parts is valid. By the remarks after (3.2) and (3.5),

$$\text{Tr}_N \Pi = \frac{1}{2} F.P. \bar{\zeta}_N(1) = \frac{1}{2} F.P. \left( \zeta_N(1) + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(B e^{-t \Delta} \perp BP) \, dt \right) \bigg|_{s=0}. \quad \text{(3.17)}$$
We now analyze the last term in (3.17). Let $e^{-t\Delta}$ have kernel $e(t, x, y) \in \Gamma(T_x M \otimes T_y M)$ with asymptotic expansion

$$
e(t, x, x) \sim \sum_{k=0}^{\infty} t^{k-(n/2)} a_k(x, x) \quad \text{as } t \downarrow 0$$

($n = \dim M$). Then $Be^{-t\Delta}$ has kernel $B_x e(t, x, y)$ where $B_x$ means $B$ acting in the $x$-variable. Thus

$$B_x e(t, x, y) \bigg|_{x=y} \sim \sum_{k} t^{k-n/2} B_x a_k(x, y) \bigg|_{x=y},$$

and so for $N >> 0$,

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr}(Be^{-t\Delta} \perp BP) dt \bigg|_{s=0}$$

$$= \frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} \left( \sum_{k=0}^{N} t^{k-(n/2)} \int_M \text{Tr} B_x a_k \text{ dvol}(g_0) \right) + O(t^N \perp \text{Tr}(BP)) \bigg|_{s=0}$$

$$+ \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} \text{Tr}(Be^{-t\Delta} \perp BP) dt \bigg|_{s=0}$$

$$= \frac{1}{\Gamma(s)} \sum_{k=0}^{N} \int_M \text{Tr} B_x a_k \text{ dvol}(g_0) + O(t^N t^{-n/2+s} \perp \frac{\text{Tr}(BP)}{s}) \bigg|_{s=0} + 0$$

(3.18)

$$= \begin{cases} 
\int_M \text{Tr} B_x a_{n/2} \text{ dvol}(g_0) \perp \text{Tr}(BP) & \text{n even} \\
\perp \text{Tr}(BP) & \text{n odd}
\end{cases}$$

In particular, the last term in (3.17) is always finite, so

$$\text{Tr}_N II = \frac{1}{2} F.P. \overline{\zeta}_N(1) = \frac{1}{2} \left[F.P. \zeta_N(1) + \int_M \text{Tr} B_x a_{n/2} \text{ dvol}(g_0) \perp \text{Tr}(BP)\right],$$

with the understanding that the integral is zero in odd dimensions. As in (2.7), this shows that

$$\text{Tr}_N II = \frac{1}{2} [\delta_N \zeta'(0) + \int_M \text{Tr} B_x a_{n/2} \text{ dvol}(g_0) \perp \text{Tr}(BP)].$$

(3.19)

To sum up, in odd dimensions the nonlocal quantities $\delta_N \overline{\zeta}(0), \delta_N \zeta'(0)$ differ only by $\text{Tr}(BP)$, and in even dimensions they differ by this term and the integral of a local expression.

Finally, we discuss the usual volume fixing conventions. As is clear from Lemma 3.2, under a scaling of the metric $g \mapsto \lambda^2 g$, we have $\Delta \mapsto \lambda^{-2} \Delta$. This implies that $\zeta'(0) \mapsto \zeta'(0) + 2 \log \lambda \cdot \zeta(0)$. As in (3.18),

$$\zeta(0) = \begin{cases} 
\int_M \text{Tr} a_{n/2} \text{ dvol}(g_0) \perp \dim \ker \Delta & \text{n even} \\
\perp \dim \ker \Delta & \text{n odd}
\end{cases}$$

(3.20)
Thus $\zeta'(0)$ is not scale invariant unless $n$ is odd and we are in the “generic” case
where $\Delta = 0$, which corresponds to $M$ admitting no one-parameter family of isometries.
So in general, we must restrict attention to infinitesimally volume preserving variations of the metric,
and to those directions $N$ with $\int_M \text{tr}(N) d\text{vol}(g_0) = 0$. These
directions need not be normal to $\mathcal{O}_{g_0}$. However, writing $N = N^{T} + N^{v}$ in its tangent-
and normal components, we have $\delta_{N^{T}} \zeta'(0) = 0$ and so $\delta_{N^{V}} \zeta'(0) = \delta_{N^{V}} \zeta'(0)$. Thus we
will restrict attention to normal variations which are projections of infinitesimally
volume preserving variations $N$, and we still have that an orbit is minimal (among orbits with such variation vector field) iff
$\delta_{N} \zeta'(0) = 0$. The easiest way to arrange this is to restrict attention to $\mathcal{M}_k$, the set of metrics on $M$ of fixed volume $k$; $\mathcal{M}_k$ is
a codimension one submanifold of $\mathcal{M}$.

There are topological conditions which force all orbits to be of generic type. Of
course, the diffeomorphism type of an orbit $\mathcal{O}_{g_0}$ is determined by the stabilizer $S_{g_0} = \ker L$. This is the space of infinitesimal isometries of $g_0$, so $\dim \ker L$ equals the
dimension of the space of isometries of $g_0$. If $\hat{A}(M) \neq 0$, then as noted in [9, p. 59], by a result of Atiyah-Hirzebruch $M$ does not admit a circle action, much less a
nondiscrete Lie group of isometries. Moreover, if $p_{1}(M) = 0$, then there is an infinite
sequence of characteristic numbers which are obstructions to $M$ admitting a circle action [14].

3.2 Minimal flat tori

We will now determine two minimal orbits of flat 2-tori of fixed volume and show
that one orbit is a stable minimum. This proceeds in two steps: first showing that we may use the natural $\zeta'(0)$ to compute $\text{Tr}_{N} \Pi$, and then using the action of $SL(2, \mathbb{Z})$ on
the space of tori to find critical metrics for $\zeta'(0)$. Finally, work of Montgomery [16] determines the flat metric for which $\zeta'(0)$ is minimal.

As we will see, the dimension of $\ker \overline{\zeta}$ is independent of the flat torus. This
implies that we can use Definition (3.2) to compute $\text{Tr}_{N} \Pi$, since $\overline{\zeta}(0)$ is a smooth
function of the tori. By (3.19), $\text{Tr}_{N} \Pi = (1/2)[\delta_{N} \zeta'(0) \perp \text{Tr}(BP)]$ provided

$$\int_{M} \text{tr} B_{x} a_{1} \ d\text{vol}(g_0) = 0$$

whenever $g_0$ is a flat metric on a torus. Of course the variation direction $N$ contained
in the definition of $B$ must be infinitesimally volume preserving—i.e.

$$\int_{M} \text{tr}_{g_0} \ N \ d\text{vol}(g_0) = 0.$$ 

Note that in fact $\text{tr}_{g_0} N = 0$; i.e. $N$ is volume element preserving. For if the
torus is given by the lattice spanned by $(1,0), (a,b)$, then the volume form for the
coordinate chart $x \leftrightarrow x + ay, y \leftrightarrow by$ is $bdx \wedge dy$ and the volume of the torus is of
course $b$. Thus the condition $\delta_{N}b = 0$ is equivalent to both volume preserving and
volume form preserving.

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**Proposition 3.1** On an \( n \)-manifold \( M \), \( \text{Tr}(Be^{-\Delta}) \) has an asymptotic expansion

\[
\text{Tr}(Be^{-\Delta}) \sim \sum_{k=0}^{\infty} \frac{t^{k-(n/2)}}{k!} \int_M b_k(x) \text{dvol}.
\]

On flat even dimensional manifolds, \( b_{n/2}(x) \equiv 0 \).

Of course, \( b_k(x) = B_x a_k(x, y)|_{x=y} \), so the existence of the asymptotic expansion will follow from that for \( e^{-t\Delta} \); this in turn is immediate from the ellipticity of \( \Delta \), which we will show in §3.4 along with the proof of Prop. 3.1.

**Corollary 3.1** For volume preserving variations of flat 2-tori, \( \text{Tr}_N \mathbb{II} = (1/2)(\delta_N \zeta'(0) \perp \text{Tr}(BP)) \).

**Proof:** It is standard that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(Be^{-\Delta}) \, dt \bigg|_{t=0} = \int_M b_m(x) \, \text{dvol}
\]
on a \( 2m \)-manifold. By the Proposition, \( b_1 \) vanishes on a flat torus. \( \qed \)

We next show that \( \text{Tr}(BP) = 0 \). Complex structures on \( S^1 \times S^1 \) are parametrized by the upper half plane \( \mathbb{H} \), where the point \( \tau = (a, b) \) corresponds to the torus associated to the lattice generated by \( (1, 0), (a, b) \). Two structures are isomorphic iff the lattices differ by an element of \( SL(2, \mathbb{Z}) \) or a homothety. Conversely, a flat torus comes from a lattice and so gives rise to a complex structure. However, a homothety of a lattice gives rise to a flat torus with a scaled metric, so these tori are not isometric. Thus tori of fixed volume are in one-to-one correspondence with \( \mathbb{H}/SL(2, \mathbb{Z}) \).

For the associated torus \( T^2_{(a, b)} \) with coordinate chart \( [0, 1] \times [0, 1] \rightarrow T_a, (x, y) \mapsto (x + ay, by) \), the metric takes the form

\[
(g_{ij}) = \begin{pmatrix}
1 & a \\
a & a^2 + b^2
\end{pmatrix}
\]
since \( \partial_x = \tilde{i}, \partial_y = a\tilde{i} + b\tilde{j} \), where \( \{\tilde{i}, \tilde{j}\} \) are the standard basis of \( \mathbb{R}^2 \). To show that \( \text{Tr}(BP) = 0 \) for all tangent vectors \( N \) at \( (a, b) \), it suffices to consider the cases where \( N \) is a vertical vector or a horizontal vector. (We avoid naming these vectors to avoid confusion with \( \partial_x, \partial_y \) above.) Consider first a horizontal vector \( N \). When we vary the metrics in the \( a \) direction, the variation two-tensor for the metrics is

\[
N = \begin{pmatrix}
0 & 1 \\
1 & 2a
\end{pmatrix}
\]

(The trace of \( N \) is nonzero because we are not working in an orthonormal frame at a point.) It is easily seen that the kernel of \( \Delta \) is two dimensional on \( T^2_{(a, b)} \), since
the group of isometries of $T_\Delta$ consists of translations and possibly a discrete group of rotations. In particular, $\{\partial_x, \partial_y\}$ span ker $\Delta$. The $L^2$ inner products of this basis are given by
\[
\langle \partial_x, \partial_x \rangle = b, \quad \langle \partial_x, \partial_y \rangle = ab, \quad \langle \partial_y, \partial_y \rangle = (a^2 + b^2)b.
\]
(For example,
\[
\langle \partial_x, \partial_y \rangle = \int_{[0,1] \times [0,1]} g_{ij}(\partial_x)^i(\partial_y)^j \, \text{dvol} = \int_{[0,1] \times [0,1]} g_{12} b \, \text{dxdy} = ab.
\]
Thus an orthonormal basis of ker $\Delta$ is given by the group of isometries of $T_\Delta$. For example,
\[
\text{Tr}(\text{BP}) = \frac{1}{b} \langle B \partial_x, \partial_x \rangle + \frac{1}{b^3} \langle B (\partial_y \perp a \partial_x), \partial_y \perp a \partial_x \rangle
\]
\[
= \frac{1}{b} \int_{[0,1] \times [0,1]} g_{11} [g^{i\ell} N_{\alpha\alpha} \delta^1 \alpha] \, \text{dvol}
\]
\[
+ \frac{1}{b^3} \int_{[0,1] \times [0,1]} g_{12} [g^{i\ell} N_{\alpha\alpha} \delta^2 \alpha \perp a (g^{i\ell} N_{\alpha\alpha} \delta^1 \alpha)] \, \text{dvol}
\]
\[
- \frac{a}{b^3} \int_{[0,1] \times [0,1]} g_{11} [g^{i\ell} N_{\alpha\alpha} \delta^2 \alpha \perp a (g^{i\ell} N_{\alpha\alpha} \delta^1 \alpha)] \, \text{dvol}
\]
\[
= \frac{1}{b} \int_{[0,1] \times [0,1]} N_{11} + \frac{1}{b^3} \int_{[0,1] \times [0,1]} (N_{22} \perp a N_{11}) \perp \frac{a}{b^3} \int_{[0,1] \times [0,1]} (N_{12} \perp a N_{11})
\]
\[
= 0
\]
since $N_{11} = 0$.

For a vertical vector $N$, we not only alter $b$ but also rescale the torus to fix the volume. Thus for $t \in [0, \epsilon]$ we consider the torus with chart given by $(x, y) \mapsto ((1 \perp t)^{-1}(x + ay), (1 \perp t)b)$. This is the torus of volume $b$ associated to the point $(a, (1 \perp t)^{-2}b)$. (We use $(1 \perp t)^2$ rather than $1 \perp t$ to avoid square roots in the calculation.) Now \( \partial_x = ((1 \perp t)^{-1}, 0), \partial_y = ((1 \perp t)^{-1} a, (1 \perp t)b) \), so
\[
(g_{ij}) = \begin{pmatrix}
(1 \perp t)^{-1} & (1 \perp t)^{-2}a \\
(1 \perp t)^{-2}a & (1 \perp t)^{-2}a^2 + (1 \perp t)^2b^2
\end{pmatrix}
\]
Thus
\[
N = \begin{pmatrix}
2 & 2a \\
2a & 2a^2 + 2b^2
\end{pmatrix}
\]
and
\[
\text{Tr}(\text{BP}) = \frac{1}{b} \langle B \partial_x, \partial_x \rangle + \frac{1}{b^3} \langle B (\partial_y \perp a \partial_x), \partial_y \perp a \partial_x \rangle
\]
\[
= \frac{1}{b} \int_{[0,1] \times [0,1]} N_{11} + \frac{1}{b^3} \int_{[0,1] \times [0,1]} (N_{22} \perp a N_{11}) \perp \frac{a}{b^3} \int_{[0,1] \times [0,1]} (N_{12} \perp a N_{11})
\]
\[
= 0
\]
These cancellations indicate that a second proof that $\text{Tr}(BP) = 0$ can be obtained by mimicking the proof in Theorem 3.5 below, replacing $G$ by $\mathbb{R}^2$, $G/H$ by the torus, and using the $G$ invariance of the flat metric. We leave this to the reader. In any case, we find that for volume preserving variations of flat tori,

$$\text{Tr}_N \Pi = \frac{1}{2} \delta_N' \zeta(0). \quad (3.21)$$

This natural equation determines two flat tori whose orbits are minimal.

**Theorem 3.1** The orbits of the flat tori associated to the lattices $(1,0),(0,1)$ and $(1,0)$, $(1/2, \sqrt{3}/2)$ are minimal within the space of all flat metrics of fixed volume.

**Proof:** The points $(0,1),(1/2, \sqrt{3}/2)$ are the only points in the upper half plane with nontrivial stabilizer for the action of $SL(2, \mathbb{Z})$, and the stabilizer subgroups are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3$ at these points, respectively [24, Ch. VII]. The differential of the action of a generator of the stabilizer groups therefore acts via rotation of $\pi, 2\pi/3$ at the two fixed points. Since the action is by isometries, it must take the gradient vector of $\zeta'(0)$ to itself. Thus the gradient vector must vanish at these two points—i.e. $\delta_N \zeta'(0) = 0$.

The proof above is a (trivial) example of Palais’ symmetric criticality principle; Hsiang’s theorem is a nontrivial example [18].

We can obtain more information about the orbit at $(1/2, \sqrt{3}/2)$ by computing $\zeta(s)$ in terms of the Epstein zeta function for the dual lattice of the torus. Recall that the dual lattice $L^*$ to a lattice $L$ in $\mathbb{R}^2$ is given by the set of $x^* \in \mathbb{R}^2$ such that $(x^*, x) \in \mathbb{Z}$ for all $x \in L$. It is shown in [2, Ch. III]. that the spectrum (with multiplicity) of the Laplacian $\Delta^0 = \perp \sum_{i=1}^2 (\partial/\partial x_i)^2$ on functions on the torus is given by $\{4\pi^2 |x|^2 : x^* \in L^*\}$. If $L^*$ is spanned by $\bar{a}^* = (1,0), \bar{b}^* = (b_1,b_2)$, then for $x^* = m\bar{a}^* + nb^*$, $|x^*|^2 = m^2 + 2b_1 mn = (b_1^2 + b_2^2)n^2 = f(m,n)$. Thus

$$\zeta_{\Delta^0}(s) = \left(\frac{1}{4\pi^2}\right)^s \sum_{m,n \in \mathbb{Z} \atop (m,n) \neq (0,0)} f(m,n)^{-s} = \left(\frac{1}{4\pi^2}\right)^s \zeta_{E,L^*}(s),$$

where the last term is by definition the Epstein zeta function of the lattice $L^*$.

**Proposition 3.2** Let $\zeta_E(s) = \zeta_{E,L^*}(s)$ denote the Epstein zeta function associated to the lattice $L^*$. Then the zeta function for $\Delta$ for the torus $T^2_F$ associated to the lattice $L$ satisfies

$$\zeta(s) = (1 + 2^{-s})(4\pi^2)^{-s} \zeta_E(s). \quad (3.22)$$

**Proof:** The eigenfunctions for $\Delta^0$ are given by

$$f_{x^*}(y) = e^{2\pi (x^*, y)}$$
for \( x^* \in L^* \) [2, Ch. III.B]. Thus any function \( f \in L^2(T^*_L) \) can be expressed as
\[
f(y) = \sum_{x^* \in L^*} \alpha \cdot x^* \cdot f_{x^*}(y).
\]

A smooth 1-form \( u = \sum_{i=1}^{2} u_i(x)dx^i \) has the decomposition
\[
\begin{cases}
u_1(y) = \sum_{x^* \in L^*} \alpha \cdot x^* \cdot f_{x^*}(y) \\
u_2(y) = \sum_{x^* \in L^*} b_{x^*} f_{x^*}(y)
\end{cases}
\]

By the local expression for \( \Delta \) given in Corollary 3.2, the eigenvalue equation for \( \Delta \) becomes
\[
(\Delta u)_j = \lambda \left[ \sum_{i=1}^{2} \frac{\partial^2 u_j}{\partial x^i_i} + \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{2} \frac{\partial u_i}{\partial x^i} \right) \right] = \lambda u_j.
\]

Substituting (3.23) into (3.24) yields
\[
\begin{align*}
\sum_{x^* \in L^*} 4\pi^2 (|x^*|^2 \alpha_a + \alpha \beta b_{x^*}) f_{x^*} &= \lambda \sum_{x^* \in L^*} \alpha \cdot x^* \cdot f_{x^*} \\
\sum_{x^* \in L^*} 4\pi^2 (|x^*|^2 b_{x^*} + \alpha \beta a_{x^*} + \beta^2 b_{x^*}) f_{x^*} &= \lambda \sum_{x^* \in L^*} b_{x^*} f_{x^*},
\end{align*}
\]

where \( x^* = (\alpha, \beta) \). Setting \( \frac{\lambda}{4\pi^2} = \mu \), we find that the eigenvalue \( \lambda \) satisfies
\[
\begin{vmatrix}
|x^*|^2 & \alpha^2 & \alpha \beta \\
\alpha \beta & |x^*|^2 & \beta^2
\end{vmatrix} = 0
\]

Thus we have
\[
\mu = |x^*|^2, \ 2|x^*|^2.
\]

Note that the zero eigenforms of \( \Delta \) form a two dimensional space, agreeing with our earlier computation.

In conclusion, the zeta function of \( \Delta \) is given by
\[
\zeta(s) = \sum_{x^* \in L^* - \{(0,0)\}} \left( \frac{1}{|x^*|^2 4\pi^2} \right)^s + \left( \frac{1}{8\pi^2 |x^*|^2} \right)^s = \left( 1 + \frac{1}{2^s} \right) (4\pi^2)^{-s} \zeta_E(s)
\]

The value of \( \zeta_E'(0) \) is given in terms of the Dedekind eta function [11, Ch. 20]. However, it seems difficult to determine lattices for which \( \zeta_E'(0) \) is critical this way.
Since the volume element is formally given by the (square root of the) determinant of $\nabla$, by (3.10), (3.21), it is natural to measure the stability of a minimal orbit by the second variation $\delta_{N,M}^2 \det \Delta$. For flat tori, by (3.10), (3.19), $\zeta(0)$ and $\zeta'(0)$ differ by a constant, so we may measure stability by $\delta_{N,M}^2 \det \Delta$. We will say that a minimal orbit is stable if $\delta_{N,M}^2 \zeta'(0) \geq 0$ for all $N, M$.

**Theorem 3.2** The orbit of the flat torus associated to $(1/2, \sqrt{3}/2)$ is stable within the space of flat tori of fixed volume.

**Proof:** Set $\xi(s) = \zeta_E(s) \Gamma(s)(2\pi)^{-s}$. By [16, p. 75], $\xi(s)$ has a minimum at the torus associated to $(1/2, \sqrt{3}/2)$ for all $s \in (0, 1/2)$. (This uses the fact that $L^* = L$ for this lattice.) Thus $0 \leq \delta_{N,M}^2 \xi(s)$ for all $s \in (0, 1/2)$. Substituting (3.22) for $\zeta_E(s)$ gives

$$0 \leq \frac{(4\pi)^s}{2^s + 1} \delta_{N,M}^2 (\xi(s) \Gamma(s)) = \frac{(4\pi)^s}{2^s + 1} (\delta_{N,M}^2 \zeta(0) + s \delta_{N,M}^2 \zeta'(0) + O(s^2))(\frac{1}{s} + 1 + O(s^2)) = \frac{(4\pi)^s}{2^s + 1} (\delta_{N,M}^2 \zeta'(0) + O(s)).$$

Here we have used $\zeta(0) = \dim \ker \Delta = \pm 2$ for all flat tori. Letting $s$ go to zero gives $0 \leq \delta_{N,M}^2 \zeta'(0)$.

The same argument for first variations gives another proof that this torus gives a minimal orbit.

**3.3 Orbits of varying type**

While the case of generic orbits (and more generally families of orbits of fixed type) treated in §3.1 is easiest to handle, minimal orbits often occur outside these cases. In particular, in finite dimensions, it is an easy corollary of Hsiang’s theorem that orbits of isolated diffeomorphism type are minimal submanifolds. The proof uses the exponential map, which may not be available in our context. We now discuss how to handle orbits of varying type in infinite dimensions. The main results are that orbits of isolated type are minimal (Thm. 3.3) and that isotropy irreducible homogeneous spaces with invariant metrics are minimal (Thm. 3.5).

Let $\overline{\Lambda} = \overline{\Lambda}_0$ be the first nonzero eigenvalue of $\overline{\Lambda}_0$. There is a neighborhood of 0 in $(T_{g_0} O_{\overline{g}_0})^-$ such that $\overline{\Lambda}/2$ is not in the spectrum of $\overline{\Lambda}_{g_0+\alpha}$ for all $N \in U$. For $g_\alpha = g_0 + \alpha N$ ($N \in U, \alpha \in [0, 1]$), let $\overline{P} = \overline{P}_\alpha$ be $g_\alpha$-orthogonal projection into the sum of the eigenspaces of $\overline{\Lambda}_\alpha$ with eigenvalues less than $\overline{\Lambda}/2$. Following [15, (5.17)], for $T > 1$ set

$$\overline{\zeta}_T(s) = \overline{\zeta}_{T,\alpha}(s) = \frac{1}{\Gamma(s)} \int_0^T t^{-1} \text{Tr}_{g_\alpha} (e^{-t \overline{\Lambda}_0}) dt + \frac{1}{\Gamma(s)} \int_T^\infty t^{-1} \text{Tr}_{g_\alpha} (e^{-t \overline{\Lambda}_0 (\overline{P} - \overline{P}_\alpha)}) dt.$$
Both integrals are now smooth functions of $\alpha$. Note that $\zeta_T(s) = \zeta(s)$ if $\dim \ker \Delta$ is constant near $g_0$.

We have

$$
\delta_N \zeta_T(s \downarrow 1) = \delta_N \zeta_T(s \downarrow 1) + \frac{1}{\Gamma(s \downarrow 1)} \int_T^\infty t^{s-2} \delta_N \text{Tr}(e^{-t\Delta}) \, dt
$$

$$
= \frac{1}{\Gamma(s \downarrow 1)} \int_0^\infty t^{s-2} \delta_N \text{Tr}(e^{-t\Delta}) \, dt
$$

$$
+ \frac{1}{\Gamma(s \downarrow 1)} \int_T^\infty t^{s-2} \delta_N \text{Tr}(e^{-t(\Delta - F)}) \, dt
$$

$$
= \frac{1}{\Gamma(s \downarrow 1)} \int_0^\infty t^{s-2} \delta_N \text{Tr}(e^{-t\Delta}) \, dt
$$

Here we have used the part of (3.7) which does not assume that the kernel has constant dimension, and we recall that $\delta_N \text{Tr}(e^{-t\Delta})$ has exponential decay at infinity by the remarks after (3.5). Thus

$$
\delta_N \zeta_T(s \downarrow 1) = \downarrow(s \downarrow 1) \zeta_N(s) \downarrow + \frac{1}{\Gamma(s \downarrow 1)} \int_T^\infty t^{s-2} \delta_N \text{Tr}_g (F e^{-t\Delta}) \, dt. \tag{3.25}
$$

Moreover, the last term in $\zeta_T(s)$ is zero at $s = 0$, while plugging in the asymptotics for $\text{Tr}(e^{-t\Delta})$ into the first integral yields

$$
\zeta_T(0) = \zeta(0) + \dim \ker \Delta = \begin{cases} 
\int_M \text{tr} \, a_{\alpha/2} & n \text{ even} \\
0 & n \text{ odd}
\end{cases}
$$

We now extend the definition of $\text{Tr}_N \Pi$ for orbits of arbitrary type.

**Definition:**

$$
\text{Tr}_N \Pi = \lim_{s \rightarrow 1} \left[ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_g (e^{-t\Delta} L^* \nabla_N L) dt \downarrow + \frac{\delta_N \zeta_T(0)}{2(s \downarrow 1)} \right]
$$

$$
= \lim_{s \rightarrow 1} \left[ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_g (e^{-t\Delta} L^* \nabla_N L) dt + \frac{\delta_N (\zeta(0) + \dim \ker \Delta)}{2(s \downarrow 1)} \right].
$$
As before, we may replace $\overline{\Delta}$ with $\Delta$. Since (3.4) is unchanged, by (3.25) we get

$$\text{Tr}_{N} II = \frac{1}{2} \lim_{t \to -1} \left[ \frac{1}{s+1} \delta_{N} \zeta_{T}(s+1) \int_{T}^{t} t^{-\frac{1}{2}} \delta_{N} \text{Tr}(\mathcal{P} e^{-i\overline{\Delta}})dt + \frac{1}{s+1} \delta_{N} \zeta_{T}(0) \right]$$

$$= \frac{1}{2} \delta_{N} \zeta_{T}(0) \int_{T}^{\infty} t^{-1} \delta_{N} \text{Tr}(\mathcal{P} e^{-i\overline{\Delta}})dt.$$  \hspace{1cm} (3.26)

(It is shown in [15, p.200] that $\delta_{N} \text{Tr}(\mathcal{P} e^{-i\overline{\Delta}})$ has exponential decay at infinity.)

**Theorem 3.3** An orbit $O_{g_{0}}$ of isolated diffeomorphism type is minimal.

**Proof:** The isometric action of the stabilizer $S_{g_{0}}$ on $\mathcal{M}$ induces an action on $T_{g_{0}}O_{g_{0}}$, still given by $\phi \cdot \nu = \phi^{*} \nu$, which is easily seen to be unitary. Thus $S_{g_{0}}$ acts unitarily on $X = (T_{g_{0}}O_{g_{0}})^{-}$.

Since $\mathcal{M}$ is compact, $S_{g_{0}}$ is a compact Lie group, so $X$ splits into a sum of finite dimensional irreducible representations $X_{i}$ of $S_{g_{0}}$. On each piece we can define $\text{Tr} II |_{X_{i}} = \sum_{j}(\text{Tr}_{N_{j}} II)N_{j}$, where $\{N_{j}\}$ is an orthonormal basis of $X_{i}$. $\text{Tr} II |_{X_{i}}$ is of course independent of the choice of this basis, so for all $\phi \in S_{g_{0}}$, $\text{Tr} II |_{X_{i}} = \sum_{j}(\text{Tr}_{\phi(N_{j})} II)\phi(N_{j})$.

But by (3.26) $\text{Tr}_{N} II = \text{Tr}_{\phi(N)} II$, since $\overline{\Delta}$ stays isospectral under the action of $\phi$ (cf. [15, (5.23)]). Thus $\text{Tr} II |_{X_{i}}$ is fixed by $S_{g_{0}}$: i.e. $\text{Tr} II |_{X_{i}} = \phi^{*}(\text{Tr} II |_{X_{i}})$.

If $O_{g_{0}}$ is not minimal, then $\text{Tr}_{N_{0}} II \neq 0$ for some $N_{0} \in X$, and hence on some neighborhood of $N_{0}$ in $X$. The vector space spanned by the $\{N_{j}\}$, $i = 1,2,\ldots$, is dense in $X$, and so $\text{Tr} II |_{X_{i}} \neq 0$ for some $i$. (If we knew that $N \mapsto \text{Tr}_{N} II$ were continuous in $N$, then from $\text{Tr}_{N_{0}} II \neq 0$ we could directly conclude $\text{Tr} II |_{X_{i}} \neq 0$ for some $i$.)

Set $\text{Tr} II |_{X_{i}} = A$. We now claim that the orbits $O_{g_{0} + \epsilon A}$ are diffeomorphic to $O_{g_{0}}$ for all $\epsilon > 0$; this contradicts the isolation of $O_{g_{0}}$. Note that for all $\phi \in D$

$$d\phi(A) = \frac{d}{d\alpha} \bigg|_{\alpha = 0} \phi^{*}(g_{0} + \alpha A) = \frac{d}{d\alpha} \bigg|_{\alpha = 0} \phi^{*}g_{0} + \alpha \phi^{*}A = \phi^{*}A,$$

so $\phi^{*}(g_{0} + \epsilon A) = \phi^{*}g_{0} + \epsilon d\phi(A)$. Thus $O_{g_{0} + \epsilon A} = O_{\phi^{*}(g_{0} + \epsilon A)}$. Also, since $d\phi(A) = A$, we have $\phi^{*}(g_{0} + \epsilon A) = g_{0} + \epsilon A$ iff $\phi \in S_{g_{0}}$. Thus $O_{g_{0} + \epsilon A} \approx D/S_{g_{0}} \approx O_{g_{0}}$. 

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Theorem 3.4 The orbits of the following metrics are minimal within the space of metrics of fixed volume:

(i) the standard metric $g_0$ on $S^n$;
(ii) the standard metric $g_1$ on $\mathbb{R}^n$.

Consider the product metric $ds^2 \oplus g_0$ on $S^1 \times S^n$, where $f_{s^1} ds^2 = 2\pi$. Consider the space $\mathcal{M}'$ of all metrics on $S^1 \times S^n$ except diffeomorphism orbits of metrics $eds^2 \oplus e^{1/n}g_0$, for $e > 0$. Then the orbit of $ds^2 \oplus g_0$ is minimal within $\mathcal{M}'$. The same statement holds for $ds^2 \oplus g_1$ on $S^1 \times \mathbb{R}^n$.

Proof: $S^n$ and $\mathbb{R}^n$ with their standard metrics are the only compact $n$-manifolds of fixed volume with isometry group of dimension $n(n + 1)/2$ [9, Thm. 3.1]. Thus these orbits are isolated. $S^1 \times S^n$, $S^1 \times \mathbb{R}^n$, with metrics $ads^2 \oplus \beta g_0$, $ads^2 \oplus \beta g_1$, are the only compact $n + 1$-manifolds with isometry groups of dimension $\frac{1}{2}n(n + 1) + 1$ [9, Thm. 3.3]. If we set $\alpha = \beta = 1$ and exclude other metrics of the same volume, then these orbits are minimal.

We now produce a much larger list of minimal orbits (including the standard metrics on symmetric spaces) by replacing the nonnatural $\zeta_T(s)$ with its natural analogue.

Let $\lambda_0$ be the first nonzero eigenvalue of $\Delta_0$, and let $P = P_\alpha$ denote $g_\alpha$-orthogonal projection into the eigenspaces of $\Delta_0$ lying below $\lambda_0/2$. Set

$$
\zeta_T(s) = \zeta_{T, \alpha}(s) = \frac{1}{\Gamma(s)} \int_0^T t^{s-1} \text{Tr}_{\alpha}(e^{-t\Delta_0}) dt + \frac{1}{\Gamma(s)} \int_T^{\infty} t^{s-1} \text{Tr}_{\alpha}(e^{-t(\Delta_0 - P_\alpha)}) dt
$$

For $\alpha$ close to zero, both terms on the right hand side are smooth in $\alpha$. By (3.16) and (3.25), we get

$$
\delta_N \zeta_T(s \perp 1) = \perp(s \perp 1) \zeta_N(s) \perp \frac{1}{\Gamma(s \perp 1)} \int_T^{\infty} t^{s-2} \delta_N \text{Tr}_{\alpha}(P_\alpha e^{-t\Delta_0}) dt
$$

$$
= \perp(s \perp 1)[\zeta_N(s) \perp \frac{1}{\Gamma(s \perp 1)} \int_0^{\infty} t^{s-2} \text{Tr}_{\alpha}(Be^{-t\Delta_0} \perp BP) dt]
$$

$$
\perp \frac{1}{\Gamma(s \perp 1)} \int_T^{\infty} t^{s-2} \delta_N \text{Tr}_{\alpha}(P_\alpha e^{-t\Delta_0}) dt
$$

$$
= \delta_N \zeta_T(s \perp 1) + \frac{s \perp 1}{\Gamma(s \perp 1)} \int_0^{\infty} t^{s-2} \text{Tr}_{\alpha}(Be^{-t\Delta_0} \perp BP) dt
$$

$$
+ \frac{1}{\Gamma(s \perp 1)} \int_T^{\infty} t^{s-2} \delta_N[\text{Tr}_{\alpha}(P_\alpha e^{-t\Delta_0}) \perp \text{Tr}_{\alpha}(P_\alpha e^{-t\Delta_0})] dt.
$$

Thus by (3.26) (and omitting the $\alpha'$s),

$$
\text{Tr}_N \Pi = \frac{1}{2} \lim_{s \to 1} \left[ \frac{1}{s \perp 1} \frac{\delta_N \zeta_T(s \perp 1)}{\Gamma(s \perp 1)} + \frac{1}{\Gamma(s \perp 1)} \int_0^{\infty} t^{s-2} \text{Tr}_{\alpha}(Be^{-t\Delta_0} \perp BP) dt
$$

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Substituting this in to \( t^{-1} \delta_N [T_{g_0} (P e^{-t \Delta}) \perp T_{g_0} (P e^{-t \Delta})] dt \) yields

\[
\frac{\delta_N \zeta_T(0)}{s \perp 1} \quad (3.27)
\]

We now make the third term in the above limit more natural. We have

\[
\frac{\delta_N \zeta_T(0)}{s \perp 1} = \frac{1}{s \perp 1} \left[ \frac{\delta_N \zeta_T(s \perp 1)}{s \perp 1} + \frac{1}{\Gamma(s \perp 1)} \int_T^\infty t^{s-2} \delta_N Tr_{g_0} (P e^{-t \Delta}) dt \right]_{s=1}
\]

Now we know \( \frac{1}{\Gamma(s \perp 1)} \int_0^\infty t^{s-1} Tr (B e^{-t \Delta} \perp BP) dt \) is finite, so

\[
\frac{1}{\Gamma(s \perp 1)} \int_0^\infty t^{s-1} Tr (B e^{-t \Delta} \perp BP) dt \bigg|_{s=1} = 0
\]

Thus by (3.25) for \( \zeta_T(s) \),

\[
\frac{\delta_N \zeta_T(0)}{s \perp 1} = \frac{1}{s \perp 1} \left[ \frac{\delta_N \zeta_T(s \perp 1)}{s \perp 1} + \frac{1}{\Gamma(s \perp 1)} \int_T^\infty t^{s-2} \delta_N Tr_{g_0} (P e^{-t \Delta}) dt \right]_{s=1}
\]

Substituting this into (3.27) yields

\[
TrNI = \lim_{s \to 1} \frac{1}{2} \left[ \frac{\delta_N \zeta_T(s \perp 1)}{s \perp 1} + \frac{\delta_N \zeta_T(0)}{s \perp 1} \right]
\]

Plugging in the Taylor series for \( \zeta_T(s \perp 1) \) near \( s = 1 \) as before gives

\[
TrNI = \frac{1}{2} \delta_N \zeta_T(0) \quad (3.28)
\]
Recall that a homogeneous space $G/H$ is called isotropy irreducible if the linearized isotropy representation of the identity component of $H$ on $T_{[1,0]}(G/H)$ is irreducible. A complete list of simply connected examples other than symmetric spaces was given by Manturov, Wolf and Kraemer, see e.g. [10],[25].

**Theorem 3.5** Let $M = G/H$ be an odd dimensional simply connected isotropy irreducible homogeneous space with its $G$-invariant metric $g_0$ of volume 1. Then $O_{g_0}$ is minimal within the space of all volume one metrics.

**Proof:** In odd dimensions, we have by (3.18) and (3.28)

$$
\text{Tr}_{N} II = \frac{1}{2} \delta_{N} \zeta''_{\tau}(0) + \frac{1}{2} \text{Tr}(BP) \pm \frac{1}{2} \int_{T}^{\infty} t^{-1} \delta_{N} \text{Tr}(P e^{-t \Delta}) dt.
$$

Now $\zeta_{\tau, \alpha}(s)$ is a natural Lagrangian in the sense that it depends naturally on the metric $g_0$, except for the term $P_{\alpha}$, which depends on the nonnatural $\lambda_{0}$. However, thinking of $\lambda_{0}$ as just a universal constant shows that $\zeta_{\tau, \alpha}(s)$ is a smooth natural Lagrangian for metrics $g_{\alpha}$ near $g_{0}$. (It will fail to be smooth for metrics on manifolds with $\lambda_{0}/2$ in the spectrum of $\Delta$.) This is enough to apply Bleecker’s theorem [3] that invariant metrics on isotropy irreducible homogeneous spaces are critical for natural Lagrangians. (In brief, the gradient vector for this Lagrangian at $g_{0}$ will be a $G$-invariant symmetric two-tensor on $G/H$ and so by hypothesis will be a multiple of the metric. Since we consider metrics of fixed volume, this multiple must be zero.) We conclude that $\delta_{N} \zeta''_{\tau}(0) = 0$ for all $N \in (T_{g_{0}}O_{g_{0}})^{-}$. Thus for these $N$,

$$
\text{Tr}_{N} II = \frac{1}{2} \text{Tr}(BP) \pm \frac{1}{2} \lim_{T \to \infty} \int_{T}^{\infty} \delta_{N} \text{Tr}(P e^{-t \Delta}) dt = \frac{1}{2} \text{Tr}(BP).
$$

We now show that $\text{Tr}(BP) = 0$. By [25, Thm. 17.1], the identity component of the isometries of $G/H$ is given by multiplication of cosets by elements of $G$. (For $S^{7} = \text{Spin}(7)/G_{2} = SO(8)/SO(7)$, we choose $G = SO(8)$.) If $\{p_{i}\}$ is an orthonormal basis of $\mathfrak{g}$, the Lie algebra of $G$, then $\text{ker} \: \Delta$ is spanned by $\{P_{i}\}$, where $(P_{i})_{\mathfrak{g}H} = (d/dt)|_{t=0}(\exp_{\mathfrak{g}} t p_{i})gH$. This basis is $L^{2}$-orthonormal, as

$$
\langle P_{i}, P_{j} \rangle = \int_{G/H} \langle (d/dt)|_{t=0}(\exp_{\mathfrak{g}} t p_{i})gH, (d/dt)|_{t=0}(\exp_{\mathfrak{g}} t p_{j})gH \rangle_{\mathfrak{g}H} \text{dvol} = \int_{G/H} \langle p_{i}, p_{j} \rangle_{\mathfrak{g}} \text{dvol} = \delta_{ij}
$$

by the $G$-invariance of the metric.

$\text{Tr}(BP)$ has two terms, one of which is the trace of the pointwise multiplication by $\text{tr}(N)$. This terms contributes

$$
\int_{G/H} \sum_{i} \langle \text{tr}(N) P_{i}, P_{i} \rangle \text{dvol} = (\text{dim} \: \mathfrak{g}) \int_{G/H} \text{tr}(N) \text{dvol} = 0.
$$
since $N$ is infinitesimally volume preserving. The second term $A$ contributes
\[
\int_{G/H} \sum_i g_{ln} g_{ic} N_c a \phi_i \phi_i^m \; d\text{vol},
\]
if $P_i = \phi_i \phi_i$ locally. At a point we can of course take $\partial a = P_a$, in which case $g_{ij} = \delta_{ij}, P_i = \delta_i^2$ and the integrand becomes $\sum_i N_i i = \text{tr}(N)$. Thus the second term also contributes zero.

**Remark:** From the proof we see that $0 = \text{Tr}_N II = \frac{1}{2} \int_0^\infty t^{-1} \delta_N \text{Tr}(P e^{-\Delta} $, which implies that $\delta_N \text{Tr}(P e^{-\Delta}) = 0$. Thus $g_0$ is critical even for this nonnatural Lagrangian.

### 3.4 Local Computations

In this subsection we will produce the asymptotic expansion for $e^{-it\Delta}, e^{-it\Delta}$. Of course the existence of the asymptotic expansion is immediate once we check that $\Delta, \Sigma$ are elliptic (Corollary 3.2).

So fix a metric $g_0$ with associated Levi-Civita connection $\nabla$ and Ricci curvature tensor $R_{ij} dx^i \otimes dx^j$; we raise and lower indices using $g_0$. Pick $u = u^a \partial_i \in \Gamma(TM), \omega = \omega_{ab} dx^a \otimes dx^b \in \Gamma(S^2 T^* M)$. Recall that we compute $\nabla, \Sigma$ with respect to another metric $g$. Set $\mu_0 = \sqrt{\text{det} g_0}, \mu = \sqrt{\text{det} g}.$

**Lemma 3.2** In local coordinates we have
\[
(L u)_{ab} = (\nabla_a u)_b + (\nabla_b u)_a
\]
\[
(L^* \omega)^a = \pm 2 \nabla^b \omega^a_b
\]
\[
(\Sigma u)^a = 2 (g_0)^{ab} \frac{\mu}{\mu_0} \nabla^b \omega^a_{cb} + 2 (g_0)^{ab} \frac{\mu}{\mu_0} R^c_{ab} u^c.
\]

**Proof:** For completeness, we include a proof of the well known first statement [9, 112]. Given a vector field $X$, let $\mathcal{L}_X$ denote Lie derivative. Define a derivation on tensors by $A_X = \mathcal{L}_X \perp \nabla_X$; on vector fields we have $A_X Y = \perp \nabla_X Y$. Let $\phi_t$ be a family of diffeomorphisms of $M$ with $\phi_0 = \text{Id}, (d/dt)|_{t=0} \phi_t = u$. Since the metric is parallel, we get
\[
(L u)_{ab} = \left( \frac{d}{dt} \bigg|_{t=0} \phi_t g \right) (\partial_a, \partial_b) = (\mathcal{L}_a g)(\partial_a, \partial_b)
\]
\[
= (A_{a} g)(\partial_a, \partial_b)
\]
\[
= A_a (g(\partial_a, \partial_b)) \perp g(\partial_a \partial_a, \partial_b) \perp g(\partial_a, A_a \partial_b)
\]
\[
= g(\nabla_a u, \partial_b) + g(\partial_a, \nabla_b u),
\]

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since $A_u$ vanishes on functions. The last line equals
\[(\nabla_a u)^i g_{ik} + (\nabla_i u)^k g_{ia} = (\nabla_a u)_b + (\nabla_b u)_a.\]

For the second equation, we have
\[
\langle Lu, \omega \rangle = \int_M (((\nabla_a u)_b + (\nabla_b u)_a) \partial_a \otimes \partial_b, \omega_{cd} \partial_c \otimes \partial_d) \, d\mathrm{vol} \\
= \int_M ((\nabla_a u)_b + (\nabla_b u)_a) \omega^{ab} \, d\mathrm{vol} \\
= \frac{1}{2} \int_M u_b \nabla_a \omega^{ab} \, d\mathrm{vol} \\
= \frac{1}{2} \int_M u_c \nabla_a (\omega^c_b g_{ba}) \, d\mathrm{vol} \\
= \frac{1}{2} \int_M u_c g_{ba} \nabla_a \omega^c_b \, d\mathrm{vol} \\
= \frac{1}{2} \langle u, g_{ba} \nabla_a \omega^c_b \rangle.
\]

Thus $(L^* \omega)^a = \frac{1}{2} \nabla^b \omega^a_b$.

For the third equation, starting as above we get
\[
\langle Lu, \omega \rangle = \int_M (((\nabla_a u)_b + (\nabla_b u)_a) \omega_{cd} g^{ac} g^{bd} \, d\mu \\
= \frac{1}{2} \int_M u_a g^{ac} \nabla_c \omega_{lb} g^{bd} \, d\mu \\
= \frac{1}{2} \int_M u^b g^{ac} \nabla_c \omega_{lb} \, d\mu \\
= \frac{1}{2} \int_M u^b (g_0)_l (g_0)^w g^{oc} \nabla_c \omega_{lw} \frac{\mu}{\mu_0} \, d\mu_0 \\
= \langle u, \frac{1}{2} (g_0)^w (g_0)^l g^{oc} \nabla_c \omega_{lw} \frac{\mu}{\mu_0} \rangle.
\]

Thus
\[ (T^* \omega)^a = \frac{1}{2} (g_0)^w (g_0)^l \frac{\mu}{\mu_0} \nabla^d \omega_{lw}. \]

For the fourth equation, we compute
\[
(\Delta u)^a = \frac{1}{2} \nabla^b ((L u)^a)_b = \frac{1}{2} \nabla^b (g^{ae} g_{bd} \nabla_c u^d + g^{ae} g_{cd} \nabla_b u^d) \\
= \frac{1}{2} \nabla^b (g_{bd} \nabla^a u^d + \nabla_b u^a) = \frac{1}{2} (\nabla^b \nabla_b u^a + g_{bd} \nabla^a \nabla_b u^d) \\
= \frac{1}{2} (\nabla^b \nabla_b u^a + R^b_a u^d) = \frac{1}{2} (\nabla^b \nabla_b u^a + \nabla^a \nabla_b u^b + R^a_b u^b).
\]

We leave the proof of the last statement to the reader.

The following is a straightforward consequence of the Lemma.
**Corollary 3.2** The symbol of $\Delta$ is given by

$$\sigma(\Delta) = \perp 2(p_0(x, \xi) + p_1(x, \xi) + p_2(x, \xi))$$

where $p_i$ $(i = 0, 1, 2)$ is homogeneous of degree $2 - i$ in $\xi$ and is given by

$$p_0(x, \xi) = [\xi^2 \delta_j^k + \xi^k \xi_j]$$

$$p_1(x, \xi) = i [3g^{tm} \Gamma_{ij}^k \xi_m + g^{tm} \Gamma_{im}^t \xi_j \delta_j^k + g^{tm} \Gamma_{mj}^k \xi_i + g^{mk} \Gamma_{jm}^t \xi_j]$$

$$p_2(x, \xi) = \perp [g^{tm} \delta_j \Gamma_{mj}^k + g^{tj} \delta_j \Gamma_{mj}^k \perp g^{mt} \Gamma_{ij}^m \Gamma_{mj}^k + g^{mk} \Gamma_{jm}^t + R^{k}] \cdot$$

In particular, $\Delta$ is elliptic. Thus $\Delta = \Delta_\alpha$ is elliptic for $\alpha$ close to zero.

Note that the principal symbol of $\Delta$ is not scalar as for usual Laplacians.

**Proof of Proposition 3.1:** By the ellipticity of $\Delta$ and [6, Lemma 1.7.1], $\text{Tr}(e^{-it\Delta})$ has an asymptotic expansion $\sum_k A_k t^k (n/2)$, and hence so does $\text{Tr}(Be^{-it\Delta})$. Moreover, the coefficients $A_k$ are integrals $\int_M a_k$ of polynomials in the jets of the symbol of $\Delta$. Since these polynomials are independent of coordinates, by a standard argument the $a_k$ must be polynomials of curvature expressions and their covariant derivatives (and constants). In particular, the asymptotic expansion of $\text{Tr}(\text{tr}(N) e^{-it\Delta})$ is just $\sum_k B_k t^k (n/2)$, where $B_k = \int_M \text{tr}(N) a_k$. For the term $b_{n/2}$, a homogeneity count shows that no constant terms appear. Similarly, for the term $A$ in $B$ given before (3.16), the asymptotic expansion of $\text{Tr}(A e^{-it\Delta})$ has coefficients which are integrals of expressions involving $N$ and curvature terms. The important point here is that no derivatives of $N$ occur [6, Lemma 1.7.7]. Thus if $R$ denotes a generic curvature term, $b_{n/2}$ will also contain terms of the form $\text{tr}(N)R$, as well as terms given by contracting indices in $N$ against indices in $R$ and then contracting all remaining indices.

---

### 4 Invariants in odd dimensions

In this section we consider families of elliptic operators with the action of an infinite dimensional group. While the operators we define are a somewhat artificial combination of gauge theory and Riemannian geometry, we do obtain large families of minimal orbits (Theorem 4.2), in contrast to the discrete minimal orbits produced in the last section. In some cases this produces invariants of odd dimensional compact manifolds.
4.1 General theory

We consider a family $\mathcal{A}$ of operators $D$ acting between sections of two vector bundles over a compact manifold $M$. Assume there is an action of an infinite dimensional Lie group $G$ on $\mathcal{A}$ with the following property: if $i = i_D: G \to \mathcal{A}$ maps $G$ to the orbit of $D$, then $L = di : \Gamma(E) \to \Gamma(F)$ is an injective elliptic operator acting on sections of two (other) vector bundles.

Gauge theory provides the basic non-example of such a family. $\mathcal{A}$ is the space of connections on a fixed bundle, $G$ is the gauge group, and for a connection $A$, we have the covariant derivative $D_A : \Lambda^0(\text{End}(E)) \to \Lambda^1(\text{End}(E))$. Of course $D_A$ cannot be elliptic as the dimensions of these two bundles differ; it is the ellipticity of $D_A^* D_A$ that is crucial in gauge theory.

Moving further away from gauge theory, we assume that for each metric $g$ on $M$, there is a natural operator $D_g : \Gamma(S) \to \Gamma(S)$ for some bundle $S$ over $M$ which carries a Riemannian or Hermitian metric associated to $g$, and we consider the family of operators $\mathcal{A} = \text{OP} = \{ s^{-1} D_g s : s \in \text{Aut}(S) \}$, where $\text{Aut}(S) = \text{Aut}_g(S)$ is the $g$-orthogonal (or $g$-hermitian) automorphisms of $S$. A group $G = \text{Aut}(S)$ isomorphic to $\text{Aut}_g(S)$ for each $g$ acts on $\text{OP}$ by further conjugation: $s' = s'^I : s^{-1} D_g s \mapsto (s')^{-1} s^{-1} D_g s s'$. In our examples below, $L = di$ takes $\Gamma(\text{End}_s(S))$ to itself, where $\text{End}_s(S)$ denotes the skew-symmetric (or skew-hermitian) endomorphisms of $S$; in the notation above, $E = F = \text{End}_s(S)$.

As a basic example, consider $D_g = d + \delta : \Lambda^*(T^*M) \to \Lambda^*(T^*M)$. At $g$,

$$L t(\phi) = [d + \delta, t](\phi) = (d + \delta)(t\phi) \perp t((d + \delta)\phi),$$

for $\phi \in \Lambda^*(T^*M), t \in \text{End}_s(\Lambda^*(T^*M))$, which we will check is an elliptic operator on $\text{End}_s(\Lambda^*(T^*M))$. (Here and from now on we won’t distinguish between a bundle and its sections.) However, it is not clear that $\ker L = 0$, so this is not a bona fide example.

The operator $L = [D_g, \cdot]$, which we will abbreviate to $D_g$ or just $D$, acts on all of $\text{End}(S)$ and obeys the usual calculus rules.

**Lemma 4.1**  
(i) For $\phi \in \Gamma(S), s \in \text{End}_s(S)$, $D(s \phi) = (Ds)\phi + sD\phi$.
(ii) For $s, t \in \text{End}_s(S)$, $D(s \circ t) = Ds \circ t + s \circ Dt$.
(iii) For $t \in \text{Aut}(S)$, $Dt^{-1} = \perp t^{-1} \circ Dt \circ t^{-1}$.

**Proof:** (i) is immediate. For (ii), we have

$$(Ds)(t\phi) + s(Dt)(\phi) = D(st\phi) \perp sD(t\phi) + s[D(t\phi) \perp tD\phi] = D(st\phi) \perp stD\phi = (D(st))(\phi).$$

For (iii), we note that $0 = D(\text{Id}(\phi)) \perp \text{Id}(D\phi) = (D(\text{Id}))(\phi)$. Thus

$$0 = D(\text{Id}) = D(t \circ t^{-1}) = Dt \circ t^{-1} + t \circ Dt^{-1}$$
and so \(Dt^{-1} = \bot t^{-1} \circ Dt \circ t^{-1} \). 

For all our examples, the symbol of \(D_g^2\) will be given by \(\xi \mapsto |\xi|^2 \cdot \text{Id}\) for cotangent vectors \(\xi \neq 0\), and we will have \(L = [D_g, \cdot]\). In this case, we show that \(\text{OP}\) splits as the product of \(\mathcal{M}\) and \(\text{Aut}(S)\). Define

\[
\alpha: \text{OP} \to \mathcal{M} \text{ by } \alpha(s^{-1} D_g s) = g.
\]

**Proposition 4.1** The map \(\alpha\) is well defined with fiber

\[
\alpha^{-1}(g) = \frac{\text{Aut}(S)}{\ker D_g|_{\text{Aut}(S)}}.
\]

In the denominator on the right hand side, \(D_g\) is acting on \(\text{End}_g(S)\). This fibration is singular if the dimension of \(\ker D_g\) is not locally constant in \(g\).

**Proof:** To show \(\alpha\) is well defined, we must check that \(s^{-1} D_g s = t^{-1} D_g t\) implies \(g = \tilde{g}\). If \(s^{-1} D_g s \phi = t^{-1} D_g t \phi\) for all \(\phi \in \Gamma(S)\), then replacing \(\phi\) by \(t^{-1} \phi\) gives \(D_g s t^{-1} \phi = s t^{-1} D_g \phi\). Thus \((st^{-1})^{-1} D_g (st^{-1})^{-1} \phi = D_g \phi\), so by the last Lemma,

\[
D_g \phi + (st^{-1})^{-1} (D_g (st^{-1})) \phi = D_g \phi
\]

This implies that the leading order symbols of \(D_g\) and \(D_{\tilde{g}}\) agree, as do the leading order symbols of \(D_g^2, D_{\tilde{g}}^2\). Since the symbol determines the metric in our cases, we have \(g = \tilde{g}\).

For the fiber \(\alpha^{-1}(g)\), we first show that \(st^{-1} \in \ker D_g\) implies \(s^{-1} D_g s = t^{-1} D_g t\). By the calculation above, it suffices to show that \((st^{-1})^{-1} D (st^{-1}) \phi = D \phi\) for all \(\phi \in \Gamma(S)\). This follows from

\[
(st^{-1})^{-1} D (st^{-1}) \phi = D \phi + (st^{-1})^{-1} (D(st^{-1})) \phi = D \phi
\]

since \(D(st^{-1}) = 0\).

Conversely, if \(s^{-1} D_g s = t^{-1} Dt\), then by (4.1) with \(\tilde{g} = g\), we get \((st^{-1})^{-1} (D(st^{-1})) \phi = 0\) for all \(\phi \in \Gamma(S)\), so \(D(st^{-1}) = 0\).

Thus under our assumption that \(L = 0\), we get the following result.

**Theorem 4.1** \(\text{OP}\) is topologically the product of \(\mathcal{M}\) and \(\text{Aut}(S)\).

Although \(\text{OP}\) is a product with coordinates \((g, s) \mapsto s^{-1} D_g s\), its natural metric depends on \(g\) and so is not a product metric. For the tangent space \(T_{(g, \text{Id})} \text{OP} = T_g \mathcal{M} \oplus T_{\text{Id}} \text{Aut}_g(S) = \Gamma(S^2 T^*M \oplus \text{End}_g(S))\) has the product of the metric on \(S^2 T^*M\) induced by \(g\) and the metric on \(\text{End}_g(S) \subset \text{End}(S)\) induced by \(g\). The latter gives a left invariant metric on \(\text{Aut}_g(S)\) and so a metric on \(T_{(g, r)} \text{OP}\).
Lemma 4.2 The action of $\text{Aut}(S)$ on $\text{OP}$ is isometric.

It is understood here that $\text{Aut}(S)$ acts via the action of $\text{Aut}_g(S)$ on the fiber over $D_g$.

Proof: First, pick $h \in T_g \mathcal{M}$ at $(g,s) \leftrightarrow s^{-1}D_g s$, so $h$ is the tangent vector to the family $s^{-1}D_g + th$. For $s_0 \in \text{Aut}(S)$, $(s_0)_*(h)$ is the tangent vector to the family $s_0^{-1}s^{-1}D_g + th; s_0$. This tangent vector corresponds to $h = (h,0)$ under the isomorphism $T_{s^{-1}}D_{s_0} \text{OP} \cong T_g \text{MET} \oplus T_{s_0} \text{Aut}(s)$. Thus $(s_0)_*h = h$ (where $h$ on either side of this equation is considered as a vector in $T_{s^{-1}}D_g \text{OP}$, $T_{s^{-1}}D_g s_0 \text{OP}$, respectively). Therefore $s_0$ acts isometrically on $T_g \mathcal{M}$ directions.

The action of $(s_0)_*$ in $T_s \text{Aut}_g(S)$ directions is isometric because the metric in these directions is $\text{Aut}_g(S)$-invariant. Since $(s_0)_*$ preserves metric and automorphism directions, it preserves their orthogonality. This suffices to prove the lemma. 

The operators $D_g = L$ acting on $\text{End}_s(S)$ extend naturally to operators $L_s = dF_{s,0}$ at each point $s \in \text{Aut}(S)$ as in §1. If $s$ acts on $\text{Aut}(S)$ by right multiplication and on $\text{OP}$ by conjugation (with differential $\text{Ad}(s)$), then it is straightforward to check that

$$\text{Ad}(s) \circ L_{\text{id}} = L_s \circ ds,$$

where $L_{\text{id}} = D_g$ at a fixed metric $g$. Since $\text{Ad}(s)$ and $ds$ act by isometries, the last equation implies that $L_s^* L_s$ is isometric to $L_g^* L_g$. Thus the zeta functions for $L_s^* L_s$ are constant along the orbits of $\text{Aut}(S)$ in $\text{OP}$. In particular, $\zeta'(0)$ is constant along orbits.

We are now in the setup of the previous section, as we have the action of an infinite dimensional group $\text{Aut}(S)$ on an infinite dimensional Riemannian manifold $\text{OP}$, but the group has no natural metric. Since we are assuming that the kernel of $L$ is zero, we can use the theory in §3.1, the only difference being that the operator $B$ becomes a new 0th order operator defined by (3.15) involving $N_{ab}$, $\text{tr} N_{g_{ij}}$, since we are taking inner products on $\text{End}_s(S)$. Defining operators $D_g^\flat, D_g^\flat : \Gamma(\text{End}_s(S)) \rightarrow \Gamma(\text{End}_s(S))$ analogously to $T_s, L_s^*$, we let $\zeta(s)$ be the zeta function for $\Delta_g = D_g^* D_g$ and similarly define $\overline{\zeta}(s)$. Equation (3.19) becomes

$$\text{Tr}_N \Pi = \left\{ \begin{array}{ll} \frac{1}{2} \delta_N \zeta'(0) + \int_M \text{tr } B_a a_{n/2} \, d\text{vol} & \text{even} \\ \frac{1}{2} \delta_N \zeta'(0) & \text{odd} \end{array} \right. \quad (4.2)$$

since the operator $P$ vanishes by assumption.

We now consider this equation in odd dimensions. In this case, $\overline{\zeta}(0) = \text{dim ker } \overline{\nabla} = \text{dim ker } D = 0$, so $\delta_N \overline{\zeta}(0) = 0$. Combining (3.2) with (4.2) gives

$$\frac{1}{2} \delta_N \zeta'(0) = \text{Tr}_N \Pi = - \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(e^{-t\Delta} L^* \nabla_N L) \, dt \bigg|_{s=1} \quad (4.3)$$

$$= - \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(L^* \nabla_N L e^{-t\Delta}) \, dt \bigg|_{s=1}.$$
Recall that here $L = [D_g, \cdot]$ acts on $L^2(\text{End}_s(S))$. The last term in (4.3) equals
\[
\text{Tr}_{g_0}(L^* \nabla_N L e^{-t \Delta}) = \sum_i \langle L^* \nabla_N L e^{-t \Delta} \phi_i, \phi_i \rangle_{g_0}
\]
\[
= \sum_i \langle \nabla_N L e^{-t \Delta} \phi_i, L \phi_i \rangle_{g_0}
\]
\[
= \sum_i \langle \nabla_N L e^{-t \Delta} \lambda_i \phi_i, L \phi_i \rangle_{g_0} \quad (4.4)
\]
\[
= \sum_i \langle \nabla_N L e^{-t \Delta} L^* \phi_i, \frac{L \phi_i}{\sqrt{\lambda_i}} \rangle_{g_0}
\]
\[
= \text{Tr}_{g_0}(\nabla_N L e^{-t \Delta} L^*)
\]
Here $\nabla_N L e^{-t \Delta} L^*: L^2(\text{End}_s(S)) \to T_{g_0} \text{OP} \supset L^2(\text{End}_s(S))$, so using $\text{Tr}$ in the last line is a slight abuse of notation. More precisely, this calculation can be rewritten in terms of heat kernels. For example, the first, fourth and final term in (4.4) become
\[
\int_M \left. \text{Tr}_{g_0}(\langle L^* \nabla_N L \rangle_x e^{L^* L}(t, x, y) \right|_{y=x}
\]
\[
= \int_M \left. \text{Tr}_{g_0}(\langle \nabla_N L \rangle_x e^{-\lambda x} \phi_i(x) \otimes \frac{L \phi_i(y)}{\sqrt{\lambda_i}} \right|_{y=x}
\]
\[
= \int_M \left. \text{Tr}_{g_0}(\langle \nabla_N L \rangle_x (L^*)_y e^{L^* L}(t, x, y) \right|_{y=x}.
\]
In any case, by (4.3) and (4.4), and using $\Delta = L^* L$, we get
\[
\frac{1}{2} \delta_N \zeta'(0) = -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(\nabla_N L e^{-t \Delta} L^*) dt \bigg|_{t=1}
\]
\[
= -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(\nabla_N L L^* e^{-t L^*}) dt \bigg|_{t=1} \quad (4.5)
\]
\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{g_0}(\nabla_N \partial_t (e^{-t L^*} \perp P_1) dt \bigg|_{t=1}.
\]
In the last step, we have used $(\nabla_N L)L^* e^{-t \Delta} = \nabla_N (L^* L^*) e^{-t \Delta}$, since only $L$ and not $L^*, e^{-t \Delta}$ are varying in the direction $N$. Also, $P_1$ denotes the orthogonal projection onto the kernel of $L^*$; this term is added to make the last integral decay exponentially at infinity. Again, the trace in the last integral means
\[
\int_M \left. \text{Tr}(\langle \nabla_N \rangle_x \partial_t (e^{L^* L}(t, x, y) \perp \sum_i \omega_i(x) \otimes \omega_i(y)) \right|_{y=x},
\]
where $\{\omega_i\}$ is an orthonormal basis of $\ker L^*$. The term involving $P_1$ vanishes:
\[
\text{Tr}(\nabla_N P_1) = \sum_i \langle \nabla_N \omega_i, \omega_i \rangle_{g_0} = \frac{1}{2} N \sum_i \langle \omega_i, \omega_i \rangle
\]
\[
= \frac{1}{2} N (\dim \ker L^*) = 0.
\]
Continuing with (4.5), we have
\[
\frac{1}{2} \frac{\delta_N \zeta'(0)}{\Gamma(s)} = \frac{s-1}{\Gamma(s)} \int_0^\infty t^{s-2} \left. \text{Tr}(\nabla_N e^{-t L L^*}) dt \right|_{s=1} = \frac{s}{\Gamma(s+1)} \int_0^\infty t^{s-1} \left. \text{Tr}(\nabla_N e^{-t L L^*}) dt \right|_{s=0}.
\]
(4.6)
Now $\nabla_N e^{-t L L^*}$ has kernel $(\nabla_N)_x \epsilon_{LL^*}(t, x, y)$. If
\[
\epsilon_{LL^*}(t, x, y) = e^{-r^2(x,y)/4} \left( \sum_{k=0}^N a_k(x, y)t^{k-n/2} + O(t^N) \right)
\]
is the asymptotic expansion of $\epsilon_{LL^*}$ for $x$ close to $y$ as $t \to 0$, then
\[
\left. (\nabla_N)_x \epsilon_{LL^*}(t, x, y) \right|_{y=x} \sim \sum_{k=0}^\infty (\nabla_N)_x a_k(x, y) \left| \left. t^{k-n/2} \right|_{y=x}.
\]
(4.8)
This follows from applying $(\nabla_N)_x$ to both sides of (4.7) and using $dr^2 = O(r)$.

Break the last integral in (4.6) into $\int_0^1 + \int_1^\infty$ and plug the right hand side of (4.8) into $\int_1^t$. As in (3.18), this yields
\[
\left. \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\nabla_N e^{-t L L^*}) dt \right|_{s=0} = \left\{ \begin{array}{ll} \frac{1}{M} \text{Tr}((\nabla_N)_x a_{n/2}(x, y)) |_{y=x} & \text{dim } M \text{ even} \\ 0 & \text{dim } M \text{ odd}. \end{array} \right.
\]
(4.9)
This equation with (4.6) shows that $\zeta'(0)$ is critical for all normal directions in odd dimensions; equivalently every orbit is minimal. Since we know that $\zeta'(0)$ is constant on orbits, we have the following theorem:

**Theorem 4.2** Let $M$ be an odd dimensional manifold with a bundle $S$ with a metric naturally associated to a Riemannian metric $g$ on $M$. For OP and Aut($S$) defined as above, assume that the infinitesimal action $L = [D_g, \cdot]$ is an invertible elliptic operator on $\text{End}_S(S)$. Then $\zeta'(0)$ for $L^* L$ acting on $L^2(\text{End}_S(S))$ is independent of the choice of metric on $M$, and so $\zeta'(0)$ is a smooth invariant of $M$. Equivalently, every orbit of Aut($S$) in OP is minimal.

There is a dichotomy for $D_g$ in odd dimensions: either $\ker D_g = 0$ for all $g \in \mathcal{M}$, in which case $\zeta'(0)$ for $D_g^2$ is a smooth invariant of $M$, or $\ker D_g \neq 0$ for all $g \in \mathcal{M}$ (in some Sobolev topology). For given $g_0$ with $\ker D_{g_0} = 0$, let $U$ be a maximal connected neighborhood of $g_0$ for which $\ker D_g = 0$ for all $g \in U$. Since $\ker D_g = 0$ is an open condition, $U \neq \emptyset$. If $U \neq \mathcal{M}$, then $g' \in \partial U$ has $\ker D_{g'} \neq 0$, by the maximality of $U$. Take $g_i \in U$ with $g_i \to g'$. On $U$, $\zeta'(0)$ is constant by Theorem 4.2. On the other hand, the lowest eigenvalue of $D_{g_i}^2$ satisfies $\lambda_1 g_i \to \lambda_1^0 = 0$, which implies that the zeta function for $D_{g_i}^2$ satisfies $\zeta'(0) \to \infty$. Thus if $g_0$ exists, then $U = \mathcal{M}$.

In particular, in the theorem above we may assume that $L$ is invertible for one metric.
4.2 Examples

For general operators naturally associated to a metric, computing $\delta \zeta'(0)$ in metric directions is difficult. By enlarging our space of operators from (a space in 1-1 correspondence with) MET to OP, we have seen that computing $\delta \zeta'(0)$ in normal directions is possible. In this section we give some examples of operators which produce invariants.

I. The de Rham operator.

For $g \in$ MET, consider $D_g = d + \delta : \Lambda^s T^* M \to \Lambda^{s+1} T^* M$ acting on forms of mixed degree. The gauge group is $\text{Aut}([n])/(\Lambda^s T^* M)$, $g$-orthogonal, degree preserving transformations of forms. The operator $L = [d + \delta, \cdot] : L^2(\text{End}_s(\Lambda^s T^* M)) \to L^2(\text{End}_s(\Lambda^s T^* M))$ is elliptic, as it is easy to check that the symbol of $L$ at a cotangent vector $\xi$ acting on the endomorphism $\phi = \phi_I dx^I \otimes \partial_j$ is $(\wedge \xi + i_\xi)(\phi_I dx^I) \otimes \partial_j$, where $i_\xi$ is interior product with $\xi$. Since $\wedge \xi + i_\xi$ is an isomorphism of $\Lambda^s T^* M$, $L$ is elliptic.

However, $\ker L \neq 0$ in general. In particular, on the 3-torus with a fixed trivialization of the cotangent bundle $T^* T^3 \simeq T^3 \times \mathbb{R}^3$, let $s$ on 1-forms be induced by an element of $so(3)$, thought of as the Lie algebra of $SO(3)$ acting on $\mathbb{R}^3$. Extend $s$ to $k$-forms as a derivation. It is easy to check that $s$ preserves each eigenspace of $d + \delta$ acting on $\Lambda^s T^* M$, which implies that $s \in \ker D_g$.

To summarize, we have the following result.

**Theorem 4.3** Let $M$ be an odd dimensional manifolds. Assume that there exists a metric $g_0$ on $M$ such that $\ker D_{g_0} = 0$ on $\Lambda^s T^* M$. Then $\zeta'(0)$ for $D_g^* D_g$ is independent of the metric on $M$.

II. Acyclic representations of $\pi_1(M)$.

We mimic the construction of analytic torsion. Given an orthogonal or unitary representation $\rho$ of $\pi_1(M)$, we construct an associated Riemannian (or hermitian) bundle $E$ with a fixed flat connection $\nabla$. Using the flat connection and a metric on $M$, we construct exterior derivatives $d_\nabla : \Lambda^i(M; E) \to \Lambda^{i+1}(M; E)$ with adjoints $\delta_\nabla$ and associated Laplacians $\Delta_\nabla$. We then extend $d_\nabla + \delta_\nabla$ acting on all forms to $D_g : \Lambda^s(M; \text{End}_s(E)) \to \Lambda^{s+1}(M; \text{End}_s(E))$. It is straightforward to check that $D_g = [d_\nabla + \delta_\nabla, \cdot]$ and that $D_g = L$ for the action $s \mapsto s^{-1}(d_\nabla + \delta_\nabla)s$.

We assume that the associated adjoint representation $\text{ad} \rho$ of $\pi_1(M)$ on $\text{End}_s(E)$ is acyclic, i.e. $H^s(M; \text{End}_s(E)) = 0$. Of course, it is enough in odd dimensions to assume that the cohomology vanishes in even dimensions and that $D_g$ takes even forms to odd forms, but the spectral theory does not change.

For 3-manifolds, this acyclicity hypothesis is equivalent to assuming that the original representation is irreducible, and that the flat connection is isolated. There are classes of 3-manifolds for which these assumptions hold, and which therefore give
the simplest examples of perturbative Chern-Simons theory. For example, for $SU(2)$ representations the Weitzenböck formula of [13, Cor. II.2] shows that $H^1 = 0$ for any metric $g$ of positive Ricci curvature. Thus for 3-dimensional lens spaces, an irreducible $SU(2)$ representation gives acyclicity at the standard metric and hence at all metrics. Other examples include $U(1)$ representations for manifolds obtained by $(p, q)$ Dehn surgery along knots in $S^3$ (some of these manifolds also have acyclic $SU(2)$ representations), $SU(2)$ representations for Seifert-fibered homology spheres, and $SO(3)$ representations for $T^3$ and $S^1$ bundles over $T^2$ with even Euler class.

To summarize, we have:

**Theorem 4.4** Let $E$ be a flat Riemannian bundle over $M$ such that the associated representation of $\pi_1(M)$ on $\text{End}_s(E)$ is acyclic. Let $D_g : \Lambda^*(M; \text{End}_s(E)) \to \Lambda^*(M; \text{End}_s(E))$ be associated to a metric $g$ on $M$ as above. Then $\zeta'(0)$ for $D_g^2$ is independent of $g$, and hence is a smooth invariant of $M$. In particular, $\zeta'(0)$ is an invariant associated to an irreducible representation of the fundamental group of a lens space.

This result is similar to [23, Thms. 1, 2], but our operators do not satisfy the hypotheses of these theorems. Note that $\zeta'(0)$ for $D_g^2$ has a combinatorial analogue, but we do not know if these quantities are equal.

Since $\text{ad} \rho$ is an acyclic representation, its analytic torsion

$$T_{\text{ad} \rho} = \frac{1}{2} \sum_i (-1)^i \zeta_i'(0)$$

is independent of the metric on $M$. Here $\zeta_i(s)$ is the zeta function for the Laplacian $\Delta_i = d_d \delta_d + \delta_d d_d$ on $\text{End}_s(S)$-valued $i$-forms. We have also shown that

$$\zeta_{D^2}'(0) = \sum_i \zeta_i'(0)$$

is independent of the metric. In dimension three, we can use $\zeta_i(s) = \zeta_{3-i}(s)$ to rewrite these invariants as

$$T_{\text{ad} \rho} = \frac{1}{2} (-3\zeta_0'(0) + \zeta_1'(0)), \quad \zeta_{D^2}'(0) = 2(\zeta_0'(0) + \zeta_1'(0)).$$

Taking linear combinations of these invariants yields the following result.

**Theorem 4.5** Let $E$ be a flat Riemannian bundle over a 3-manifold $M$ such that the associated representation of $\pi_1(M)$ on $\text{End}_s(E)$ is acyclic. Let $\zeta_i(s)$ be the zeta function for the Laplacian on $\Lambda^i(\text{End}_s(S))$. Then $\zeta_i'(0)$ is independent of the metric on $M$, and hence is a smooth invariant of $M$.  

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We can also obtain invariants for 3-manifolds with non-acyclic representations. Let $B_i$ denote the zeroth order operator on $\Lambda^i(\text{End}_s(S))$ which measures the variation of the inner product as in (3.15). By [23, Thm. 1], [20, Thm. 7.6], for general representations the variation of the analytic torsion is given in odd dimensions by

$$\delta T_{\text{ad}}(\rho) = \frac{1}{2} \sum_i (-1)^i \text{Tr}(B_i P_i),$$

where $P_i$ is projection onto the kernel of $\Delta_i$. We also have

$$\delta \zeta_{D^2}(0) = \text{Tr}_N \Pi + \text{Tr}(BP)$$

by (3.19), (4.2). The dimension of the kernel of $D$ is independent of the metric on $M$ by Hodge theory, and it is straightforward to check that in this case the argument leading up to Theorem 4.2 remains valid. Thus we obtain

$$\delta \zeta_{D^2}(0) = \text{Tr}(BP) = \sum_i \text{Tr}(B_i P_i).$$

Summing up, we have in dimension three

$$\delta T_{\text{ad}}(\rho) = \frac{1}{2} (\perp 2 \text{Tr}(B_0 P_0) + \text{Tr}(B_1 P_1))$$

(4.10)

$$\delta \zeta_{D^2}(0) = 2 \text{Tr}(B_0 P_0) + 2 \text{Tr}(B_1 P_1).$$

Recall that $\rho$ is irreducible iff $H^0(M; E) = 0$, which is equivalent to $\text{ker} \Delta_0 = 0$ on $\Lambda^0(\text{End}_s(S))$, and that if $\rho$ is an isolated representation of $\pi_1(M)$, then $\text{ker} \Delta_1 = 0$.

**Theorem 4.6** Assume that $\rho$ is an irreducible representation of $\pi_1(M)$. Then $4T_{\text{ad}}(\rho) + \zeta_{D^2}(0)$ is independent of the metric on $M$. If $\rho$ is an isolated representation, then $2T_{\text{ad}}(\rho) + \zeta_{D^2}(0)$ is independent of the metric on $M$.

**Proof:** In the first case, we have $P_0 = 0$. The result follows from (4.10). The second statement is similar. \[\square\]

### 4.3 Remarks in even dimensions

We finish with some calculations for even dimensional manifolds. In this section, $L$ equals $[d + \delta, \cdot]$ or $[\delta_N + \delta, \cdot]$, and we assume ker $L = 0$. Looking back at (3.2), (4.2) and (4.3) and redoing the computations, we end up with

$$\frac{1}{2} \delta_N \zeta'(0) + \int_M \text{tr} B_z a_n d\text{vol} = \text{Tr}_N \Pi$$

(4.11)

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\nabla_N e^{-t L^*}) \, dt \bigg|_{s=0}$$

$$= \lim_{s \to \frac{1}{2}} \frac{\delta_N \zeta(0)}{2(s + 1)}.$$
Since the left hand side is always finite, as is the term on the second line of (4.11) by (4.9), we must have $\delta_N \zeta(0) = 0$. Recall that for fixed metric $g_0$, $\zeta(s)$ is defined for the operator $T_\alpha L_\alpha$ for nearby metrics $g_\alpha$. Thus we get the curious result that once $g_0$ is fixed, $\zeta(0)$, which is given by an integral of curvature terms in $g_0$ and $g_\alpha$ and their $g_\alpha$-covariant derivatives, is independent of nearby metrics $g_\alpha$ on an even dimensional manifold. More precisely, at $g_0$

$$\zeta_\alpha(0) = \int_M \text{tr} \, \bar{T}^0_{n/2}(x, x) \, \text{dvol},$$

where

$$c_{T_\alpha L_\alpha}(t, x, x) \sim \sum_k \bar{T}^0_k(x, x)t^{k-n/2}.$$ 

Using (3.8), we get

$$\delta_N \zeta_\alpha(0) = \frac{1}{\Gamma(s)} \left. \frac{d}{dt} \text{Tr}(\delta_N \bar{\Delta} \cdot e^{-\bar{\Delta}}) \right|_{s=0}$$

$$= \frac{1}{\text{dim}M} \text{tr} \left( (\delta_N \bar{\Delta}) \bar{T}^0_{(n/2)+1}(x, y) \right)_{y=x} \text{dvol},$$

so this last “semilocal” expression must vanish. (The coefficient $\bar{T}^0_{(n/2)+1}$ appears as in (4.8) because $\Delta r^2 = 2 + O(r)$; cf. [22] for more details).

With the aid of (4.9), (4.11) becomes

$$\frac{1}{2} \delta_N \zeta'(0) + \int_M \text{tr} \, B_x a_{n/2} \, \text{dvol} = \text{Tr}_N \Pi = \frac{1}{2} \int_M \text{tr} \, (\nabla_N)_x a_{n/2} \, \text{dvol}. \quad (4.12)$$

Thus both $\delta_N \zeta'(0)$ and $\text{Tr}_N \Pi$ localize in the sense that they are given by integrals of local expressions.

Although this is enough to give our main result in even dimensions below, (4.12) can be simplified a little more.

**Lemma 4.3** We have

$$\frac{1}{2} \int_M \text{tr} \, B_x a_{n/2} \, \text{dvol} = \int_M \text{tr} \, (\nabla_N)_x a_{n/2} \, \text{dvol}.$$ 

Thus

$$\delta_N \zeta'(0) = \frac{1}{2} \int_M \text{tr} \, B_x a_{n/2} \, \text{dvol} \quad (4.13)$$

$$\text{Tr}_N \Pi = \frac{1}{2} \int_M \text{tr} \, B_x a_{n/2} \, \text{dvol}.$$ 

**Proof:** By (3.15), we have

$$\langle \nabla_N \phi, \phi \rangle_{g_0} = \frac{1}{2} \frac{d}{d\alpha} \bigg|_{\alpha=0} \langle \phi, \phi \rangle_{g_0} = \frac{1}{2} \langle B \phi, \phi \rangle_{g_0},$$

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Thus

\[
\int_M \text{tr}((\nabla_N)_x a_{n/2}(x, y))|_{y=x} \, \text{dvol} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(\nabla_N e^{-t\Delta}) \, dt \bigg|_{s=0}
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \langle \nabla_n \phi_i, \phi_i \rangle_{g_0} \, dt \bigg|_{s=0}
\]

\[
= \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i e^{-\lambda_i t} \langle B\phi_i, \phi_i \rangle_{g_0} \, dt \bigg|_{s=0}
\]

\[
= \frac{1}{2} \int_M \text{tr} B_x a_{n/2} \, \text{dvol}
\]

\[\blacksquare\]

**Theorem 4.7** In even dimensions, every orbit of Aut(S) in OP associated to a flat metric g₀ is minimal. Equivalently, ζ'(0) is critical at flat metrics. Thus ζ'(0) is constant on families of flat metrics.

**Proof:** By [6, Lemma 1.7.7], tr Bₓa_{n/2} is given by a local expression in B and the curvature of g₀. Since g₀ is flat, we may calculate Bₓa_{n/2} in Euclidean space, where a_{n/2}(x, y) ≡ 0. The theorem follows from (4.13).

For conformally covariant operators (e.g. the Laplacian on functions on a Riemannian surface, D_{g₀}^2 on Endₓ(S)), the techniques of [19, 21] show that the variation of ζ'(0) in conformal directions is given by the integral of a local expression; this is well-known in string theory for the appropriate ∂-operator. Here we have shown that in fact the variation of ζ'(0) for D_{g₀}^2 is given by a local expression for all normal directions. Since ζ'(0) is constant on orbits, it is tempting to say that the variation of ζ'(0) is local in all metric directions. This is somewhat misleading: for a given metric direction H at g₀, its projection proj(H) into the normal space to the orbit of Aut(S) is given by a a global operator. Thus δₜHζ'(0) = δ_{proj H}ζ'(0) is given by a local curvature expression in g₀ and their B = B(proj(H))-derivatives.

**References**


