Harmonic Maps between Spheres

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Vienna, Preprint ESI 251 (1995)  
August 23, 1995
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Abstract

We prove the existence of two countable families of harmonic maps from $S^k$ to $S^k$ for $3 \leq k \leq 6$. We also study the stability and the limiting behaviour of these maps and we explain why the solutions disappear for $k \geq 7$.

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1 Introduction

Harmonic maps between spheres have been studied by mathematicians since a long time (see [1] and references therein) so it came as a surprise when recently one of the authors has found two new families of harmonic maps of degree zero and one between 3-dimensional spheres [2]. Each of these families comprises a countable sequence of maps which are labelled by a certain nodal number. The purpose of this paper is to generalize the result of [2] to maps between higher dimensional spheres. We shall show that smooth maps \( f : S^k \to S^k \) contain two countable families of harmonic representatives of degree zero and one, when \( 3 \leq k \leq 6 \), and that for \( k \geq 7 \) these solutions disappear.

The plan of the paper is as follows. In Section 2 we derive the harmonic map equation. The main existence result for \( 3 \leq k \leq 6 \) is proven in Section 3. Section 4 is devoted to the proof of non-existence for \( k \geq 7 \). In Section 5 we analyze the stability of our harmonic maps, and finally in Section 6 the limiting behaviour of solutions is discussed.

2 Preliminaries

Let \( f : M \to N \) be a smooth map between Riemannian manifolds \((M, g)\) and \((N, h)\). The energy of the map \( f \) is defined by

\[
E(f) = \frac{1}{2} \int_M h_{AB}(f) \frac{\partial f^A}{\partial x^i} \frac{\partial f^B}{\partial x^j} g^{ij} \, dV_M,
\]

where \( x^i \) denote local coordinates in \( M \) and \( f^A \) denote local coordinates of the point \( f(x) \) in \( N \). The critical points of this energy functional are called harmonic maps (in physical literature the model defined by (1) is sometimes called the \( \sigma \)-model).

In this paper we consider the case when both the domain and the target space are \( k \)-dimensional unit spheres: \( M = N = S^k \), where \( k \geq 3 \). Let \((\psi, \vartheta)\) be polar coordinates on \( M = S^k \) centered at the north pole, in which the metric is \( ds^2 = d\psi^2 + \sin^2\psi \, ds^2|_{S^{k-1}} \). Let \((\Psi, \Theta)\) be similar coordinates on \( N \). We restrict our attention to maps which are equivariant with respect to the action of the \( SO(k) \) group, i.e., we assume that
\( \Psi = f(\psi), \Theta = \vartheta. \) Then the energy functional (1) reduces to
\[
E(f) = \frac{1}{2} V(S^{k-1}) \int_0^\pi \left( f'^2 + (k - 1) \frac{\sin^2 f}{\sin^2 \psi} \right) \sin^{k-1} \psi \, d\psi, \tag{2}
\]
and the associated Euler-Lagrange equation is
\[
\frac{1}{\sin^{k-1} \psi} (\sin^{k-1} \psi f')' - \frac{k - 1}{2} \frac{\sin 2f}{\sin^2 \psi} = 0. \tag{3}
\]
The rest of this paper is devoted to the study of solutions of Eq.(3) and their properties.

**Definition 1.** A \( C^2 \)-solution of Eq.(3) is called regular if both its energy and energy density are finite, i.e.
\[
E(f) < \infty \quad \text{and} \quad \rho(f) = f'^2 + (k - 1) \frac{\sin^2 f}{\sin^2 \psi} < \infty \quad \text{for all} \quad \psi \in [0, \pi]. \tag{4}
\]
By inspection of Eq.(3) we can make the following observations. First, the problem has the \( Z_2 \) symmetry: if \( f \) is a solution of Eq.(3), so is its antipodal reflection \( \bar{f} = \pi - f \).
Second, there are two trivial constant regular solutions: \( f_0 = 0 \) and \( \bar{f}_0 = \pi \) for which the energy (2) attains the global minimum \( E_0 = 0 \). Geometrically, they correspond to maps into the north and south pole, respectively. Another constant solution is \( f_e = \pi / 2 \), which geometrically is a map into the equator. Note that although this map is not regular, its energy \( E(f_e) \) is finite.

To the best of our knowledge the only nonconstant regular solution of Eq.(3) known in literature is the identity map \( f_1 = \psi \) (and the corresponding antipodal map \( \bar{f}_1 = \pi - \psi \)). Do there exist other solutions? As we shall see the answer depends on the dimension \( k \). In the next section we shall prove that for \( 3 \leq k \leq 6 \), Eq.(3) has two countable families of regular solutions of degree zero and of degree one, which will be referred to as even and odd solutions respectively. For even solutions \( f(\psi) = f(\pi - \psi) \), while for odd solutions \( f(\psi) = \pi - f(\pi - \psi) \). An element of each family \( \{f_n\} \) is labelled by a nodal number \( n \) defined by \( n = \dim f_n^{-1}(\pi / 2) \) (so \( n \) is even for even solutions and odd for odd solutions). Even solutions may be viewed as "excitations" of the polar map \( f_0 \), while odd solutions are "excitations" of the identity map \( f_1 \). By \( Z_2 \)-symmetry for each family \( \{f_n\} \) there exists its antipodal reflection \( \{\bar{f}_n\} \). In Section 4 we show that all the excitations disappear for \( k \geq 7 \).
3 Proof of existence

In order to study solutions of Eq.(3) it is convenient to introduce new dependent and independent variables:

\[ x = \ln(\tan(\psi/2)) \quad \text{and} \quad h = f - \frac{\pi}{2}. \] (5)

The range of \( x \) is from \(-\infty\) (for \( \psi = 0 \)) to \(+\infty\) (for \( \psi = \pi \)). In terms of these variables Eq.(3) has the form of the pendulum equation with variable damping

\[ h'' - (k - 2) \tanh x \, h' + \frac{k - 1}{2} \sin 2h = 0. \] (6)

Under the reflection \( x \to -x \) even solutions are symmetric \( h(x) = h(-x) \), while odd solutions are antisymmetric \( h(x) = -h(-x) \), hence it is sufficient to consider Eq.(6) for \( x \geq 0 \). The formal power series expansion near \( x = 0 \) for odd solutions is

\[ h(x) = bx + O(x^3), \] (7)

and for even solutions

\[ h(x) = d + O(x^2). \] (8)

Since \( x = 0 \) is a regular point of Eq.(6), we are guaranteed that these series have nonzero radii of convergence and therefore determine one parameter families of local solutions near \( x = 0 \) which are analytic in \( b \) and \( d \). Solutions starting with initial data (7) or (8) will be called \( b \)-orbits or \( d \)-orbits, respectively. Without loss of generality both \( b \) and \( d \) are assumed to be nonnegative; by the reflection symmetry all results have analogous statements for \( b \to -b \) and \( d \to -d \).

Now we shall analyze the behaviour of \( b \) and \( d \)-orbits. To this end the following elementary a priori global estimates are very useful.

**Lemma 1:** \( b \)-orbits and \( d \)-orbits exist (i.e., \( h \) and \( h' \) are finite) for all \( x \geq 0 \).

**Proof:** Let us rewrite Eq.(6) in the form

\[ \left( \frac{1}{\cosh^{k-2} x} h' \right)' = -\frac{k - 1}{2 \cosh^{k-2} x} \sin 2h. \] (9)
Integrating this from zero to some $x > 0$ yields

$$\frac{1}{\cosh^{k-2} x} h'(x) = h'(0) - \frac{k-1}{2} \int_0^x \frac{dx}{\cosh^{k-2} x} \sin 2h.$$  \hspace{1cm} (10)

Thus $h'(x)$ is finite for any $x \geq 0$ and therefore $h(x)$ is also finite.

**Definition 2:** The region $\mathcal{H}$ is defined by

$$\mathcal{H} = \{(h, h', x) : |h| < \frac{\pi}{2}, x > 0, (h, h') \neq (0, 0)\}.$$ \hspace{1cm} (11)

**Lemma 2:** An orbit which leaves $\mathcal{H}$ goes monotonically to $\pm \infty$.

**Proof:** Define a function

$$W(h, h') = \frac{1}{2} h'^2 + \frac{k-1}{2} \sin^2 h.$$ \hspace{1cm} (12)

Using Eq.(6) we have

$$\frac{dW}{dx} = (k-2) \tanh x h'^2,$$ \hspace{1cm} (13)

hence the function $W$ is monotonically increasing for $x > 0$. Suppose that $h(x_0) = \pi/2$ and $h'(x_0) > 0$ for some $x_0 > 0$ ($h'(x_0) = 0$ would imply $h(x) \equiv \pi/2$). Then, for $x > x_0$

$$W(x) - W(x_0) = \frac{1}{2} [h'^2(x) - h'^2(x_0)] + \frac{k-1}{2} (\sin^2 h(x) - 1) > 0,$$ \hspace{1cm} (14)

which implies that $h'(x) > h'(x_0)$ for $x > x_0$. The analogous argument can be made for an orbit leaving $\mathcal{H}$ via $h = -\pi/2$.

**Corollary:** There are no regular solutions of Eq.(6) of the homotopy degree bigger than one.

**Lemma 3:** If an orbit stays in $\mathcal{H}$ for all $x \geq 0$, and $h(x)$ has a finite number of zeros, then $h' \to 0$ and $h \to \pm \pi/2$ as $x \to \infty$. 
Proof: First note that in \((\because)\) a solution \(h\) has no positive minima nor negative maxima, as follows immediately from Eq. (6). Thus for sufficiently large \(x\) the solution \(h(x)\) is monotonic (since by assumption it cannot oscillate around zero at infinity) and therefore has a limit as \(x \to \infty\). This implies that \(h'(\infty) = 0\), hence by Eq. (6) \(h(\infty) = \pm \pi/2\) or 0. To complete the proof it remains to show that the case \(h(\infty) = 0\) is impossible. To prove this suppose that \(h(\infty) = 0\) and \(x_0\) is the last extremum of \(h\). Without loss of generality one may assume that \(h\) has a maximum at \(x_0\), hence \(h(x) \geq 0\) for all \(x \geq x_0\). Now, multiply Eq. (9) by \(\cos h\) and integrate by parts from \(x_0\) to infinity. This yields

\[
\int_{x_0}^{\infty} \frac{dx}{\cosh^{k-2} x} \sin h \left( h'' + (k - 1) \cos^2 h \right) = 0,
\]

which is a contradiction unless \(h \equiv 0\).

Remark: Actually, Lemma 3 is true without assuming that \(h\) has a finite number of zeros, however then the proof is more complicated (and that stronger result is not needed for the proof of existence).

In what follows a \(b\) or \(d\)-orbit is called a connecting orbit if its endpoint at infinity is \((h = \pm \pi/2, h' = 0)\). These points are the saddle-type critical points of the asymptotic \((x \to \infty)\) autonomous equation

\[
h'' - (k - 2)h' + \frac{k - 1}{2} \sin 2h = 0.
\]

It is easy to show that the connecting orbits approach the saddles along the stable manifolds (see [2] for the proof)

\[
\pm h(x) = -\frac{\pi}{2} + c e^{-x} + O(e^{-3x}).
\]

Note that an orbit ending at the saddle-type critical point cannot be generic so connecting orbits can exist only for a discrete set of initial values \(b\) and \(d\).

Now, having established the above elementary properties of solutions, we can proceed with the proof of existence. The main idea, taken from the work [3,4] of Smoller,
Wasserman and Yau, is to apply the shooting method. After [3] we introduce the following notation. Let $x_e(b)$ be the smallest $x > 0$ at which the $b$-orbit exits? For any $b$-orbit define $\theta(x, b)$ by

$$\theta(0, b) = -\frac{\pi}{2} \quad \text{and} \quad \theta(x, b) = \arctan \left( \frac{h'(x, b)}{h(x, b)} \right) \quad \text{for } x > 0. \quad (18)$$

The rotation number, $\Omega(b)$, of the $b$-orbit is given by

$$\Omega(b) = -\frac{1}{\pi} (\theta(x_e(b), b) - \theta(0, b)). \quad (19)$$

For $d$-orbits the definitions are analogous except that $\theta(0, d) = 0$. Note that for connecting orbits $\Omega = n/2$ where $n$ is the nodal number of the solution $h$, i.e. number of zeros of $h(x)$ for $x \in (-\infty, \infty)$. Below the solutions of Eq. (6) corresponding to connecting orbits will be denoted by $h_n$.

The main result of this paper, that is the existence of two infinite families of harmonic maps between spheres, will follow from

**Theorem 1:** If $3 \leq k \leq 6$, then for each odd $n$ there exists a connecting $b_n$-orbit with $\Omega(b_n) = n/2$, and for each even $n$ there exists a connecting $d_n$-orbit with $\Omega(d_n) = n/2$.

For concreteness we shall present the proof of Theorem 1 for odd solutions (the proof of existence of even solutions is analogous). The following technical proposition is crucial for the argument.

**Proposition 1:** Assume $3 \leq k \leq 6$. Given any $N > 0$, there is an $\epsilon > 0$, such that if $0 < b < \epsilon$, then $\Omega(b) > N$.

Proposition 1 is the analogue of Proposition 3.5 in [4] and its proof follows closely the proof of that proposition. Proposition 1 says that $b$-orbits with sufficiently small $b$ have an arbitrarily large rotation number. This is a sort of compactness property guaranteeing that $b$-orbits with bounded rotation have $b$ greater than some positive constant.
**Proof:** In two steps. The first step is to show that a $b$-orbit stays close to the origin $(h, h') = (0, 0)$ for bounded time provided that $b$ is sufficiently small. Define a distance function $\rho(x, b)$ by

$$\rho^2(x, b) = h^2(x, b) + h'^2(x, b).$$

(20)

**Lemma 4:** Given any $T > 0$ and $\eta > 0$, there exists an $\epsilon(\eta, T)$ such that if $b < \epsilon$ then $\rho(x, b) < \eta$ for $x \leq T$.

**Proof:** From Eqs.(20) and (6) we obtain

$$\rho' = (k - 2) \tanh x h'^2 + hh' - (k - 1)hh' \frac{\sin 2h}{2h} \leq (k - 2) h'^2 + k |hh'| \leq A \rho^2,$$

(21)

where $A = 3k/2 - 2$, because $h'^2 \leq \rho^2$ and $2|hh'| \leq \rho^2$. Multiplying Eq.(21) by $1/\rho^2$ and integrating from 0 to $x$ yields

$$\rho(x, b) \leq e^{Ax} \rho(0, b),$$

(22)

so for any $T > 0$ and $\eta > 0$, if $b = \rho(0, b) < e^{-AT} \eta$ then $\rho(x, b) < \eta$ for $x \leq T$. This concludes the proof of Lemma 4.

The second step is to show that $\theta'(x, b)$ is uniformly bounded from above by a negative constant provided that $\rho(x, b)$ is close to zero. This is the only place in the proof where the parameter $k$ enters in a nontrivial way. From Eqs.(18) and (6), we have

$$\theta = \frac{hh'' - h'^2}{h^2 + h'^2} = -\frac{h'^2}{h^2 + h'^2} + \frac{h}{h^2 + h'^2} [(k - 2) \tanh x h' - \frac{k - 1}{2} \sin 2h]$$

$$= -\sin^2 \theta + \frac{k - 2}{2} \tanh x \sin 2\theta - (k - 1) \cos^2 \theta \frac{\sin 2h}{2h}. $$

(23)

This can written as

$$\theta' = -\frac{k}{2} + \frac{k - 2}{2} (|\sin 2\theta| - \cos 2\theta) + \delta,$$

(24)

where

$$\delta = \frac{k - 2}{2} (\tanh x \sin 2\theta - |\sin 2\theta|) + (k - 1) \cos^2 \theta (1 - \frac{\sin 2h}{2h}).$$

(25)

The function $g(\theta) = |\sin 2\theta| - \cos 2\theta$ has a maximum equal to $\sqrt{2}$, hence

$$\theta' \leq \frac{k - (4 + 2\sqrt{2})}{2(1 + \sqrt{2})} + \delta.$$

(26)
By Lemma 4, for any \( T > 0 \), by choosing sufficiently small \( b \), we can make the second term on the r.h.s of (25) arbitrarily small for \( x \leq T \). Then, for \( k < k^* = 4 + 2\sqrt{2} \), the r.h.s. of (26) is negative i.e. \( \theta' \) is uniformly bounded from above by a certain negative constant \( c \) (depending on \( k \)): \( \theta' < c \) for \( x \leq T \). This completes the second step.

The rest of the proof of Proposition 1 follows from integrating \( \theta' \) from zero to \( T = N \pi / |c| \). Namely,

\[
\theta(T) - \theta(0) = \int_0^T \theta'(s)ds < -N \pi,
\]

but \( x_+(b) > T \), so \( \Omega(b) > -\frac{1}{\pi}[\theta(T) - \theta(0)] > N \) which ends the proof of Proposition 1.

**Remark.** Proposition 1 fails for \( k \geq 7 \) because the critical point \( (h = 0, h' = 0) \) of the linearized asymptotic equation

\[
h'' - (k - 2)h' + (k - 1)h = 0
\]

changes its character at \( k^* = 4 + 2\sqrt{2} \) from the unstable spiral for \( k < k^* \) to the unstable node for \( k > k^* \).

Now we return to the proof of Theorem 1. The proof amounts to construct connecting orbits inductively.

**Step 1:** Construction of the first orbit\(^1\).

Define a set 

\[
S_1 = \{ b : b\text{-orbit exits } \theta = 0 \text{ with } \Omega(b) \leq \frac{1}{2} \}.
\]

To see that the set \( S_1 \) is not empty consider the monotonically increasing function \( W(x) = \frac{b}{2}h'^2 + \frac{k-1}{2} \sin^2 h \). For \( b > \sqrt{2} \), \( W(0) > 1 \) hence \( W(x) > 1 \) for \( x > 0 \), which implies that \( h'(x) \) is strictly positive for all \( x > 0 \), and therefore such a \( b \)-orbit exits \( \theta \) through \( h = \pi/2 \) with \( \Omega(b) < 1/2 \).

Now, let \( b_1 = \inf S_1 \). Proposition 1 guarantees that \( b_1 > 0 \). The \( b_1 \)-orbit cannot exit \( \theta = 0 \) via \( h = \pi/2 \) because the same would be true for nearby orbits with \( b < b_1 \), violating the

\(^1\)Although the first orbit is known explicitly, it is helpful to see how its existence follows from the general argument.
definition of \( b_1 \). Thus the \( b_1 \)-orbit stays in \( ? \) for all \( x > 0 \), which implies by Lemma 3 that it is a connecting orbit, hence \( \Omega(b_1) = 1/2 \). It should be emphasized that this construction does not guarantee the uniqueness of the \( b_1 \)-orbit. However, we know from Proposition 1 that the set of initial values \( b \) with \( \Omega(b) = 1/2 \) is bounded from below by a positive constant, that is

\[
\tilde{b}_1 = \inf \{ b : b \text{-orbit is a connecting orbit with } \Omega(b) = \frac{1}{2} \} > 0.
\] (30)

**Step 2:** Construction of the second orbit.

Let us take \( b \) slightly smaller than \( \tilde{b}_1 \). Then, by the definition of \( \tilde{b}_1 \) we have \( \Omega(b) > 1/2 \). The idea (see Proposition 3.4 in [3]) is to show that the \( b \)-orbit does not make another rotation, i.e. \( \Omega(b) < 3/2 \).

**Lemma 5:** If \( \tilde{b}_1 - \epsilon < b < \tilde{b}_1 \) for sufficiently small \( \epsilon \), then the \( b \)-orbit exits \( ? \) transversally through \( h = -\pi/2 \) with \( 1/2 < \Omega(b) < 3/2 \).

**Proof:** Let \( A \) and \( B \) be respectively the points at which the \( b \)-orbit exits the first and the second quadrant of the \((h, h')\) plane, i.e. \( h' (x_A, b) = 0 \) and \( h(x_B, b) = 0 \). The idea (following the proof of Proposition 4.8 in [3]) is to show that the function \( W(x) = \frac{1}{2} h'^2 + \frac{k-1}{2} \sin^2 h \) is greater than \((k - 1)/2\) at \( B \) provided that the point \( A \) is close to the fixed point \((h = \pi/2, h' = 0)\). Since \( W \) is increasing along the orbit, this will imply that the orbit cannot cross the hyperplane \( h' = 0 \) for \( x > x_B \). From Eq.(13) we have (using the notation \( h_A = h(x_A) \))

\[
W(x_B) - W(x_A) = (k - 2) \int_{x_A}^{x_B} \tanh x \, h'^2 \, dx > (k - 2) \tanh x_A \int_{h_A}^{h_B} h' \, dh .
\] (31)

In order to estimate the last integral notice that for \( x > x_A \)

\[
W(x) - W(x_A) = \frac{1}{2} h'^2(x) + \frac{k-1}{2} [\sin^2 h(x) - \sin^2 h(x_A)] \geq 0 ,
\] (32)

hence

\[
|h'(x)| \geq \sqrt{k - 1} \sqrt{\sin^2 h(x_A) - \sin^2 h(x)} .
\] (33)
Therefore
\[ W(x_B) > W(x_A) + (k - 2)\sqrt{k - 1} \tanh x_A \int_0^{h_A} \sqrt{\sin^2 h_A - \sin^2 h} \, dh. \] (34)

The r.h.s. of this inequality is an increasing function of \( x_A \) and \( h_A \). By continuous dependence of solutions on initial data by choosing \( b \) sufficiently close to \( \bar{b}_1 \) (i.e. sufficiently small \( \epsilon \)) we can have \( h_A \) arbitrarily close to \( \pi/2 \) for an arbitrarily large \( x_A \). It is easy to check by the straightforward computation that for, say, \( x_A > 1 \) and \( \pi/3 < h_A < \pi/2 \), the r.h.s. of the inequality (34) is greater than \( (k - 1)/2 \), hence \( h' < 0 \) for \( x > x_A \). This concludes the proof of Lemma 5.

In view of Lemma 5 the set
\[ S_3 = \{ b : b\text{-orbit exits ? via } h = -\frac{\pi}{2} \text{ with } \Omega(b) \leq \frac{3}{2} \}, \quad (35) \]
is not empty, so we can repeat the argument of step 1. As before, define \( b_3 = \inf S_3 \). By Proposition 1 \( b_3 \) is strictly positive. The \( b_3 \)-orbit cannot exit ? via \( h = -\pi/2 \) because the same would be true for nearby orbits with \( b < b_3 \), violating the definition of \( b_3 \). Thus the \( b_3 \)-orbit is a connecting orbit with \( \Omega(b_3) = 3/2 \). The subsequent connecting orbits are obtained by the repetition of the above construction.

4 Non-existence of excitations for \( k \geq 7 \)

In this Section we consider the case \( k \geq 7 \). We prove that in this case Eq.(6) has no connecting orbits with the rotation number \( \Omega > 1/2 \). The idea is first to use a simple comparison result to bound the rotation number \( \Omega \) from above by the rotation number of the linearized equation, and then compute the latter explicitly.

**Lemma 6:** Let \( h_L \) satisfy the linearized equation
\[ h''_L - (k - 2) \tanh x h'_L + (k - 1) h_L = 0 \] (36)
with the initial conditions (7) or (8), and let \( \Omega_L \) be the associated rotation number, defined as in (19). Then, \( \Omega(b) < \Omega_L(b) \) and \( \Omega(d) < \Omega_L(d) \).
Proof: In analogy to (23), we have
\[ \theta'_L = -\sin^2 \theta + \frac{k - 2}{2} \tanh x \sin 2\theta_L - (k - 1) \cos^2 \theta_L. \] (37)
Define \( u = \theta - \theta_L \). Subtracting (37) from (23), after simple manipulations, we get
\[ u' = su + g \] (38)
where
\[ s = (k - 2) \frac{\sin^2 \theta - \sin^2 \theta_L}{\theta - \theta_L} + \frac{k - 2}{2} \tanh x \frac{\sin 2\theta - \sin 2\theta_L}{\theta - \theta_L}, \] (39)
and
\[ g = (k - 1) (1 - \frac{\sin 2h}{2h}) \cos^2 \theta. \] (40)
Since \( g \geq 0 \) and \( s \) is a uniformly bounded continuous function, it follows from (38) that \( u(x) \geq 0 \) for \( x \geq 0 \), completing the proof.

Now we need to show that \( \Omega_L < 1/2 \). The straightforward method is to solve Eq.(36) explicitly in terms of hypergeometric functions and then compute \( \Omega_L \). There is however a simpler way. Let us change the variable \( y = \sinh(x) \). Then Eq.(36) becomes
\[(1 + y^2)h''_L(y) - (k - 3)y h'_L(y) + (k - 1)h_L = 0. \] (41)
Differentiating this equation twice yields (using the notation \( v = h''_L(y) \))
\[(1 + y^2)v'' + (7 - k) y v' + (7 - k)v = 0. \] (42)
From this equation one can see why the value \( k = 7 \) is critical. For even solutions of Eq.(41) \( v(0) < 0 \) (assuming that \( d = h_L(0) > 0 \)), so if \( k > 7 \), \( v(x) < 0 \) for all \( x \geq 0 \). Similarly, for odd solutions \( v(0) = 0 \), so if \( k > 7 \), \( v(x) < 0 \) (assuming that \( b = h'_L(0) > 0 \)) for all \( x \geq 0 \). Finally, for \( k = 7 \) Eq.(41) can be easily solved
\[ h_L = d (1 - 3y^2) \quad \text{or} \quad h_L = b (y - \frac{1}{3} y^3). \] (43)
Summarizing, for \( k \geq 7 \), the solution \( h_L \) has exactly one zero for \( x > 0 \), which implies that \( \Omega_L < 1/2 \). Using Lemma 6 we conclude that for \( k \geq 7 \) there are no nontrivial even solutions of Eq.(6) and \( h_1 \) is the only odd solution (assuming it is unique in the nodal class \( n = 1 \)).
5 Stability

Now we consider the linear stability of harmonic maps described above. The Hessian of the energy functional at \( h \) is

\[
\delta^2 E(h)(\xi, \xi) = \frac{1}{2} V(S^{k-1}) \int_{-\infty}^{\infty} \left( \xi'^2 - (k - 1) \cos(2h) \xi^2 \right) \frac{dx}{\cosh^{k-2} x},
\]

which leads to an eigenvalue problem

\[
- \cosh^k x \left( \frac{1}{\cosh^{k-2} x} \xi'' - (k - 1) \cosh^2 x \cos(2h) \xi \right) = \lambda \xi.
\]

Let \( \xi^n_j \) (and resp. \( \lambda^n_j \)) denote the \( j \)th (where \( j = 1, 2, ... \)) eigenfunction (eigenvalue) around the regular solution \( h_n \). Since the solutions \( h_n \) are not known analytically (except for \( h_1 \)), the eigenmode equation (45) can only be solved numerically. Without doing so we can estimate the number of negative eigenvalues as follows. Consider a perturbation \( \xi_{\text{conf}} \) induced by the conformal Killing vector field on \( S^k \), \( \mathcal{K} = \sin \psi \frac{\partial}{\partial \psi} \). In terms of the \( x \)-coordinate, \( K \) is the generator of translations \( K = \frac{\partial}{\partial x} \) (in fact this is the defining property of the coordinate \( x \)), hence

\[
\xi_{\text{conf}} \equiv \mathcal{K} h = h'(x).
\]

It is easy to check by differentiating Eq.(6) that \( \xi_{\text{conf}} \) satisfies Eq.(45) with \( \lambda_{\text{conf}} = (2 - k) \). Since by construction \( \xi_{\text{conf}} \) associated with the solution \( h_n \) has \( n - 1 \) nodes, it follows that \( h_n \) has \( n - 1 \) negative eigenvalues lying below \( \lambda_{\text{conf}}[5] \).

Conjecture: The solution \( h_n \) (considered as a critical point of the reduced energy functional) has exactly \( n \) negative eigenvalues.

To prove this conjecture it would be sufficient to show that \( \lambda_{\text{conf}} \) is the largest negative eigenvalue for each \( h_n \). We have verified this fact numerically (see Table 1) but we were able to prove it only for \( h_1 \). Actually in this case one can find the whole spectrum analytically. Namely, for \( h_1 = -\pi/2 + 2 \arctan(e^x) \) Eq.(45) is solved by

\[
\xi^1_j = \frac{1}{\cosh x} C_{j-1}^{k-1}(\psi), \quad \lambda^1_j = 2 - 2k + j(j + k - 1),
\]

(47)
where \( \psi = 2 \arctan(e^x) \) and \( C_{\frac{k+1}{2}} \) are the Gegenbauer polynomials [6]. Thus the solution \( h_1 \) has one unstable mode \( \xi_1 = \frac{1}{\cosh x} \) (because \( C_0 = 1 \)) with the eigenvalue \( \lambda_1 = (2 - k) \) (this is of course the conformal eigenmode).

Table 1: Spectrum of unstable eigenmodes for \( k = 4 \)

<table>
<thead>
<tr>
<th>n</th>
<th>( \lambda_1^n )</th>
<th>( \lambda_2^n )</th>
<th>( \lambda_3^n )</th>
<th>( \lambda_4^n )</th>
<th>( \lambda_5^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-4.35088</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-10.0270</td>
<td>-9.89580</td>
<td>-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-66.9441</td>
<td>-66.9441</td>
<td>-4.41394</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-592.759</td>
<td>-592.759</td>
<td>-10.0735</td>
<td>-9.94209</td>
<td>-2</td>
</tr>
</tbody>
</table>

It is of interest to see how the energy behaves under general (i.e., not necessarily small) conformal transformations. Given any solution \( h_n \) a conformally deformed configuration \( h_n^\alpha \) is given by

\[
h_n^\alpha(x) = h_n(x + \alpha),
\]

This one-parameter family of conformally deformed configurations defines a path in the function space which passes through the solution \( h_n \) for \( \alpha = 0 \) and goes (non-uniformly) to the vacua \( \pm \pi/2 \) for \( \alpha \to \pm \infty \). We know from the argument above that the energy along this path has a local maximum at \( \alpha = 0 \):

\[
\left. \frac{d^2 E(h_n^\alpha)}{d\alpha^2} \right|_{\alpha=0} = (2 - k) \int_{S^k} \xi_{\text{con}}^2 \, dV.
\]

This formula, first derived by Smith [7] (see also [8]), is known in the literature as the proof of non-existence of stable harmonic maps between spheres \( S^k \) for \( k \geq 3 \). Actually, for nonzero degree maps a stronger result holds, namely \( E(h_n^\alpha) \) attains a global maximum at \( \alpha = 0 \) [9].

We shall not touch here on the very interesting question of stability of the equator map \( h_\epsilon = 0 \). This problem will be studied in a separate paper dealing with the bifurcation structure at \( k = k^* \).
6 Large n behaviour

In this Section we return to the case $3 \leq k \leq 6$. We are concerned with the behaviour of solutions $h_n$ in the limit of large $n$. We shall argue that in this limit the solutions approach a limiting configuration in a self-similar manner. Although our argument will be heuristic we see no potential difficulties in converting it into a rigorous proof.

We know from Proposition 1 and the proof of Theorem 1 that the initial data $b_n$ and $d_n$ tend to zero as $n \to \infty$. Thus, by Lemma 4, the connecting orbits stay close to the origin for an arbitrarily long time $T$ if $n$ is sufficiently large. This means that for $x < T$ (region I) the solution $h_n(x)$ is well approximated by the solution of the linearized equation

$$h'' - (k - 2) \tanh x h' + (k - 1)h = 0.$$  \(50\)

Thus in this region

$$h_n(x) \approx b_n g_1(x) \quad \text{for } n \text{ odd} \quad \text{and} \quad h_n(x) \approx d_n g_2(x) \quad \text{for } n \text{ even},$$  \(51\)

where the functions $g_1(x)$ and $g_2(x)$ solve Eq.(50) with initial conditions $(g_1(0) = 0, g_1'(0) = 1)$ and $(g_2(0) = 1, g_2'(0) = 0)$, respectively. For completeness let us note that these functions can be expressed in terms of the hypergeometric functions

$$g_1(x) = \sinh x \, F\left(1 - \frac{k}{4} + \frac{\omega}{2}, 1 - \frac{k}{4} - \frac{3}{2}; -\sinh^2 x\right),$$  \(52\)

$$g_2(x) = F\left(\frac{1}{2} - \frac{k}{4} + \frac{\omega}{2}, \frac{1}{2} - \frac{k}{4} - \frac{3}{2}; -\sinh^2 x\right),$$  \(53\)

where $\omega = \frac{1}{2} \sqrt{8k - 8 - k^2}$.

When $x$ is large (region III) the solution $h_n(x)$ is well approximated by the solution of the autonomous equation

$$h'' - (k - 2)h' + \frac{(k - 1)}{2} \sin 2h = 0.$$  \(54\)

Since asymptotically $\pm h_n(x) = -\pi/2 + c_n e^{-x}$, and Eq.(54) is invariant under translation $x \to x + \text{const}$, it follows that in region III we have

$$\pm h_n(x) \approx s(x - \ln c_n),$$  \(55\)
where \( s(x) \) is the solution of Eq. (54) normalized by \( s(x) = -\pi/2 + e^{-x} \) for \( x \to \infty \).

The key observation is that the regions I and III overlap over an interval (call it region II) where both approximations (51) and (55) apply. This allows us to match the asymptotic behaviour of connecting orbits at \( x = 0 \) and at \( x = \infty \). Namely, in region II the solutions are well approximated by the solutions of the linear autonomous equation

\[
h'' - (k - 2)h' + (k - 1)h = 0,
\]

hence in this region the solution is quasi-periodic

\[
g_i \approx A_i e^{\frac{\pi}{\omega}x} \sin(\omega x + \delta_i).
\]

The amplitude \( A_i \) and the phase \( \delta_i \) are independent; they depend only on whether the solution is odd \((i = 1)\) or even \((i = 2)\). In fact \( A_i \) and \( \delta_i \) can be computed from the asymptotic expansions of Eqs. (52,53) but we shall not need this information.

Now, consider two adjacent solutions \( h_{n+2} \) and \( h_n \). In region III we obtain from (55)

\[
h_{n+2}(x) \approx -h_n(x - \ln \frac{c_{n+2}}{c_n}).
\]

On the other hand in region I we have

\[
h_{n+2}(x) \approx \frac{b_{n+2}}{b_n} h_n(x) \quad \text{for } n \text{ odd} \quad \text{and} \quad h_{n+2}(x) \approx \frac{d_{n+2}}{d_n} h_n(x) \quad \text{for } n \text{ even}.
\]

Since both (58) and (59) hold in region II, using (57) one gets for odd solutions the relationship

\[
- \left( \frac{c_n}{c_{n+2}} \right)^{\frac{k-2}{2}} \sin(\omega x + \delta_1 - \omega \ln \frac{c_{n+2}}{c_n}) \approx \frac{b_{n+2}}{b_n} \sin(\omega x + \delta_1),
\]

and the analogous one for even solutions. This implies that

\[
\frac{c_{n+2}}{c_n} \approx \exp\left( \frac{\pi}{\omega} \right) \quad \text{and} \quad \frac{b_{n+2}}{b_n} \approx \frac{d_{n+2}}{d_n} \approx \exp\left( -\frac{(k-2)\pi}{2\omega} \right).
\]

These scaling relations become exact in the limit \( n \to \infty \). It follows from (61) that the amplitude of oscillations of a solution \( h_n(x) \) in region I behaves as \( \exp\left( -\frac{(k-2)\pi}{4\omega} n \right) \). Thus, for any finite \( x \) the sequence \( h_n(x) \) tends to zero as \( n \to \infty \). In this sense the equator
map $h_e = 0$ is the limiting solution as $n \to \infty$ for both families of excitations $h_n$. It is easy to show (see [2]) that the energy of the equator map

$$E_e = E(h_e) = \frac{(k - 1)^2}{2(k - 2)} V(S^k)$$

(62)

is an upper bound for the energies of regular solutions of Eq.(6). Using (61) one can also show that, as $n \to \infty$, this upper bound is approached via scaling

$$\frac{E_e - E_n}{E_e - E_{n+2}} \approx \exp\left(\frac{(k - 2)\pi^2}{\omega}\right).$$

(63)

The numerical data illustrating the limiting behaviour for large $n$ in the case $k = 4$ are displayed in Tables 2 and 3.

**Table 2: Parameters of odd solutions for $k = 4$.**

<table>
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<tr>
<th>$n$</th>
<th>$b_n$</th>
<th>$c_n$</th>
<th>$b_n/h_{n+2}$</th>
<th>$c_{n+2}/c_n$</th>
<th>$E_n/(8\pi^2/3)$</th>
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**Table 3: Parameters of even solutions for $k = 4$.**

<table>
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<tr>
<th>$n$</th>
<th>$d_n$</th>
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<th>$d_n/d_{n+2}$</th>
<th>$c_{n+2}/c_n$</th>
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Acknowledgments

We would like to thank P. Aichelburg, P. Forgacs, N. Manton, and R. Wald for discussions. P.B. is especially grateful to J. Smoller for illuminating lessons on the shooting method. We thank the Erwin Schrödinger Institute for Mathematical Physics, where this work was initiated, for hospitality. This research was supported in part by the KBN grant 2/P302/113/06.

References


