Quantized Lax Equations and Their Solutions

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Quantized Lax Equations and Their Solutions

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Abstract

Integrable systems on quantum groups are investigated. The Heisenberg equations possessing the Lax form are solved in terms of the solution to the factorization problem on the corresponding quantum group.

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1 Introduction

The discovery of the inverse scattering method [12] was a real breakthrough in theory of the classical completely integrable Hamiltonian systems, which goes back to the classical papers of Euler, Lagrange, Liouville, Jacobi and others. The systematic way to construct and solve completely integrable Hamiltonian systems using the theory of Lie groups and their representations originated in the works of Kostant [17], Adler [1] and Symes [31]; it was further developed by many other authors. The invention of Lie-Poisson groups by Drinfeld [4] made it possible to develop the general concepts underlying the theory of classical integrable systems and the integration of the corresponding equations of motions [25]. We refer the reader to review papers [23], [26] and to the books [8], [20] most related to our discussion. One of the main general results in the theory is the construction of integrable Hamiltonian systems possessing a Lie (Lie-Poisson in general) group of symmetries and the expression of their solutions in terms of the solution to the factorization problem on this group [25]. The concrete classes of models differ by the type of symmetry group and by the type of factorization. Within this approach, the fundamental methods (inverse scattering method, algebro-geometric methods of solution) and the fundamental notions of the soliton theory, such as τ-function [15] and Baker-Akhieser function [6], [24], found their unifying and natural group-theoretical explanation.

The theory of integrable models of quantum mechanics and quantum field theory also made a remarkable progress within the quantum version of the inverse scattering method, which goes back to the seminal Bethe ansatz for solving the Heisenberg spin chain. We refer the reader, for a review of related topics, to the books [10], [2] or to the papers [7], [18], [33]. This development made it possible to introduce quantum groups [5], [14] algebraic objects, playing in the quantum case a role analogous to that of the Lie groups in the classical theory. However, we were still missing (with the exception of the quantum integrable systems with discrete time evolution [21]) the quantum analogue of the factorization theorem for the solution to the Heisenberg equations of motion of a quantum integrable system. However, we have to mention the remarkable paper [19] in this relation.

This paper is an attempt to formulate a quantum version of the factorization theorem. As in the classical case, there is a simple direct proof, and also a more conceptual proof which gives a generalization of the classical construction (based on symplectic reduction) due to Semenov-Tian-Shansky [25] and to which we devote the main text. We hope that this construction has an interest of its own. It could be, for example, interesting in relation with the Bethe ansatz (Proposition 4), and it might be useful for a proper formulation of the quantum version of the τ-function.

We prefer to describe the direct proof in Appendix 1.

Section 2 contains the construction of a quantum dynamical system on a dual quasi-triangular Hopf algebra $F$, with the Hamiltonian $h$ taken as an arbitrary co-commutative function on $F$, and gives its Lax pair formulation. Section 3 introduces a larger quantum dynamical system on the corresponding Heisenberg double $D_H$ with a very simple time evolution, such that the original quantum dynamical
system (under some additional assumptions) can be identified with a reduction of it. Section 4 contains our main result concerning the solution of the quantum Lax equation of Section 2. Section 5 gives the formulation of our results in a form suitable for integrable quantum chains or (after performing the continuous limit) integrable quantum field theories. Appendix 1 is devoted to a direct proof of our main theorem. Finally in Appendix 2, we give a possible formulation of the factorization problem in the case of factorizable Hopf algebras.

The paper is written for physicists. So, for example, we are working formally with a notion of a dual Hopf algebra, which would need more detailed specification in the infinite-dimensional case. Further, all algebraic tensor products used in the paper would have to be properly completed in the infinite-dimensional case. Correspondingly we do not discuss the precise sense in which the universal elements, such as R-matrix, T-matrix, etc., exist. Apart from this, all constructions of the paper are still valid. For these subtleties and for more information about quantum groups we refer the reader to the existing monographs on the subject (e.g. [3]).

2 Quantum Lax pairs

The starting point of the following investigation is the quasi-triangular Hopf algebra \( U \) and its dual Hopf algebra \( F = U^* \).

We shall use the standard notation: \( m, \Delta, S \) and \( \varepsilon \) for product, coproduct, antipode and co-unit, respectively, in both \( U \) and \( F \), and also the notation \( \sum x_{(1)} \otimes x_{(2)} \) for the result of coproduct \( \Delta \) applied to \( x \), in \( U \) or \( F \). We start from the commutation relation [9]

\[
R_{12} T_1 T_2 = T_2 T_1 R_{12},
\]

where \( R \in U \otimes U \) is the universal R-matrix and \( T \) is the universal element in \( U \otimes F \) (sometimes called universal T-matrix).

In the following we will always use the notation like

\[
T_1 T_2 \equiv T_{13} T_{23},
\]

so that, for instance, (1) means an equality in \( U \otimes U \otimes F \) and the indices 1 and 2 refer to the different copies of \( U \) in this triple tensor product.

We want to study the following quantum dynamical system on the quantum group \( F \). The Hamiltonian \( h \) is taken to be a co-commutative element in \( F \); it holds

\[
\Delta(h) = \sigma \Delta(h),
\]

where \( \sigma \) is the flip operation. The set of all such elements form a commutative subalgebra in \( F \) [5]. The quantum dynamics is given by the following Heisenberg equations of motion:

\[
i \dot{T}_2 = [h, T_2] = \langle h \otimes id, T_1 T_2 - T_2 T_1 \rangle
\]
where we used the commutation relations on $F$ with the universal elements $R^+_2 = R_{24}$ and $R^- = R_{12}^{-1}$ and the co-commutativity of $h$, and where the time derivative applies to the second tensor-factor of $T \in U \otimes F$ belonging to $F$. Therefore we can state the following proposition, generalizing the discussion of [19]:

**Proposition 1.** The Heisenberg equations can be written in the Lax form

\[
\begin{align*}
    i \dot{T} & = [M^\pm, T], \quad \text{where} \\
    M^\pm_T & = \langle h \otimes \text{id}, ((1 - (R^\pm_{12})^{-1})T_1) \rangle.
\end{align*}
\]

In order to construct a solution for this set of equations we shall consider in the next sections a quantized version of the construction by Semenov-Tian-Shansky in [25].

### 3 Dynamics on the quantum Heisenberg double

The quantum Heisenberg double $D_H$ (quantum cotangent bundle of $F$) is a smash product algebra $D_H = U \bowtie F$ [32] defined with the help of the left action of $U$ on $F$.

We shall use the following description of $D_H$ [27], [35]:

**Proposition 2.** The Heisenberg double $D_H$ is defined by the following relations

\[
\begin{align*}
    R_{12} T_1 T_2 & = T_2 T_1 R_{12}, \\
    R_{21} L^+_1 L^+_2 & = L^+_2 L^+_1 R_{21}, \\
    R_{21} L^-_1 L^-_2 & = L^-_2 L^-_1 R_{21}, \\
    L^\pm_1 T_2 & = T_2 R^\pm_{12} L^\pm_1.
\end{align*}
\]

The universal T-matrix $T$ as well as universal L-matrices $L^\pm = R^\pm$ are understood as elements of $U \otimes D_H$ in the above equalities.

We shall introduce one more element of $U \otimes D_H$, denoted as $Y$ and defined as

\[
Y = L^+_1 (L^-)^{-1}.
\]

Now let us consider the quantum dynamical system on the Heisenberg double $D_H$ with the Hamiltonian $\mathcal{H}$ chosen to be a Casimir of $U \subset D_H$ of the form [11]

\[
\mathcal{H} = \text{Tr}^v (Y^{-1}_{12} D_1),
\]

where $D \in U$ is defined, with the help of the universal R-matrix $R = \sum R^{(1)} \otimes R^{(2)}$, as $D = \sum R^{(1)} S(R^{(2)})$ and the superscript $v$ indicates the trace in the first factor of $U \otimes U \subset U \otimes D_H$ evaluated in an arbitrary representation $v$ of $U$.

The Heisenberg equations on $D_H$ take the form

\[
\begin{align*}
    \dot{Y} & = 0, \\
    i \dot{T} & = T \xi^H,
\end{align*}
\]
with
\[ \xi^H = \text{Tr}_1^c(R_{12}^{-1} Y_{1}^{-1} R_{21}^{-1} D_1 - R_{12}^{-1} Y_{1}^{-1} R_{12} D_1). \] (8)

Again the time derivative in (7) applies to the second tensor-factors of \( T \) and \( Y \) belonging to \( D_H \). Since \( \xi^H \in U \otimes U \subset U \otimes D_H \) is evidently time-independent, these Heisenberg equations are solved trivially
\[ Y(t) = Y(0), \quad T(t) = T(0) \exp(-it\xi^H). \] (9)

In the following we will assume that \( U \) itself is a quantum double \( D(U_-) \) of some Hopf algebra \( U_\cdot \).

Therefore we have
\[ U = U_- \otimes U_+ , \quad \text{with } U_+ = U_-^{op}\Delta \]
as a linear space and coalgebra. Similarly, we can write
\[ F = F_- \otimes F_+ , \quad \text{with } F_\pm = U_\pm^* \]
as a linear space and an algebra. Correspondingly we have \( R \in U_- \otimes U_+ \) and the universal element \( T \) factorizes in \( U \otimes F \) as [9]
\[ T = AZ , \quad \text{with } A \in U_- \otimes F_- , \quad \text{and } Z \in U_+ \otimes F_+ . \] (10)

The commutation relations of the elements \( Y \) and \( Z \) assumed as elements in \( U \otimes D_H \) play a crucial role in the following. They are given by the following lemma.

**Lemma 1.** The elements \( Y \) and \( Z \) commute in the following way
\[ R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12} , \]
\[ R_{12} Z_1 Z_2 = Z_2 Z_1 R_{12} , \]
\[ Z_1 Y_1 Z_2 = Z_2 Z_1 Y_1 R_{12} . \] (11)

**Proof.** Only the last assertion is non-trivial. We shall omit the details of the proof of this relation, which follows immediately from the discussion of [16] (all arguments given there we need are valid also in the general situation of the present paper), if we keep in mind the difference in the decomposition of the universal T-matrix used there and the decomposition (10). The resulting difference is that the element \( Q \) used in [16] does not appear in the commutation relations at all.

In order to make contact with the quantum dynamical system described in Section 2, we need the following proposition.

**Proposition 3.** There exists an embedding of \( F \hookrightarrow D_H \), which is an algebra homomorphism, given by
\[ \Tilde{T} = Z Y^{-1} Z^{-1} . \] (12)
This means that the relation

\[ R_{12} T_1 T_2 = T_2 T_1 R_{12} \]  

holds in \( U \otimes H_D \). We shall use the symbol \( \hat{F} \) for the image of this embedding.

**Proof.** The proof is straightforward using the commutation relations (11).

This embedding of the original quantum group \( F \) in the Heisenberg double will be used later on to project down a solution (9) to the Heisenberg equations (7) in \( H_D \) to a solution of the Lax equation (4) on the original quantum phase space \( F \).

For doing this the following identification of the Hamiltonians of the corresponding systems is important. Our starting Hamiltonian \( h \) on the quantum group \( F \) of Section 2 was supposed to be a co-commutative element in \( F \). In the case when \( F \) as its own left comodule decomposes to a direct sum of all its irreducible comodules (a coarse form of the Peter-Weyl theorem) the most general co-commutative element \( h \) is of the form

\[ h = \text{Tr}^v T, \]  

where the trace in the first factor of \( T \in U \otimes F \) is taken in an appropriate representation \( v \) of \( U \). For simplicity, we shall assume in the following our Hamiltonian \( h \) to be exactly of this type.

**Proposition 4.** For any representation \( v \) of \( U \) the equality

\[ \text{Tr}_1^v(Y_1^{-1} D_1) = \text{Tr}_1^v(Z_1 Y_1^{-1} Z_1^{-1}) = \text{Tr}_1^v(\tilde{T}_1) \]  

holds in \( D_H \). Roughly speaking the embedding (12) sends the trace of \( \tilde{T} \) in any representation of \( U \) to the quantum trace of \( Y \) in the same representation, and we can identify the Hamiltonian \( h \) with the reduction to the \( \hat{F} \) of the Hamiltonian \( \mathcal{H} \).

**Proof.** It holds in any representation \( v \) of \( U \) that

\[ \text{Tr}_1(\hat{R}_{12}^{-1} D_1) = 1. \]  

Here and in the rest of the proof we assume that both copies of \( U \) to which indices 1 and 2 refer are taken in the representation \( v \).

From the third relation in (11) we get

\[ Y_1^{-1} Z_1^{-1} Z_2 P_{12} = Z_2 \hat{R}_{12}^{-1} Y_2^{-1} Z_2^{-1}, \]  

where \( P_{12} \) is the permutation operator in the representation \( v \). Now taking the quantum trace of this equation and using (16) we obtain

\[ \text{Tr}_1^v Y_1^{-1} Z_1^{-1} Z_2 P_{12} D_1 = Z_2 Y_2 Z_2^{-1}. \]  

Taking now the usual trace in the second tensor-factor (and renaming the tensor-factors) yields the desired identity

\[ \text{Tr}_1^v(Y_1^{-1} D_1) = \text{Tr}_1^v(Z_1 Y_1^{-1} Z_1^{-1}). \]
4 Solution to the Lax equation

Now we can return to our dynamical system, on \( T \in U \otimes D_H \) governed by the Hamiltonian \( \mathcal{H} \) of the form (6), of Section 3. There, we constructed the solution (9) to the equations of motion of this quantum dynamical system; we showed that there exists an embedding of the original quantum group \( F \) in the Heisenberg double \( D_H \), such that the Hamiltonians of the corresponding systems coincide after this embedding. In this chapter we are going to use it to obtain a solution to the quantized Lax equations (4) on the quantum group \( F \).

Let us denote as \( g(t) \) the following element
\[
g(t) = Z(0) \exp(-it\xi^H)Z(0)^{-1} \in U \otimes D_H.\]

Now, the time evolution on \( D_H \) is an algebra homomorphism, and so the decomposition of \( T(t) \) in \( U \otimes F(t) \) in the same form as in (10) makes sense:
\[
T(t) = \Lambda(t)Z(t),
\]
with \( \Lambda \in U_- \otimes F_-(t) \),
and \( Z \in U_+ \otimes F_+(t) \).\]

So as a consequence of (9) the element \( g(t) \) can be expressed as
\[
g(t) = \Lambda(0)^{-1}T(t)Z(0)^{-1},
\]
\[
= \Lambda(0)^{-1}\Lambda(t)Z(t)Z(0)^{-1}.\]

This gives us a decomposition of \( g(t) \in U \otimes D_H \), with
\[
g(t) = g_{-}(t)g_{+}(t) , \quad g_{\pm} \in U_\pm \otimes D_H ;
\]
\[
g_{-}(t) = \Lambda(0)^{-1}\Lambda(t) , \quad g_{+}(t) = Z(t)Z(0)^{-1}.\]

We will now show that \( g(t) \) and its factors \( g_{\pm}(t) \) are actually elements of \( U \otimes \tilde{F} \subset U \otimes D_H \).

Let us define
\[
M = R_{-}(h_{[1]}) \otimes h_{[2]} - R_{+}(h_{[1]}) \otimes h_{[2]} \in U \otimes \tilde{F},\]

with
\[
R_{+}(x) = \sum \langle x, R^{(1)} \rangle R^{(2)} ,
\]
\[
R_{-}(x) = \sum S(R^{(1)}) \langle x, R^{(3)} \rangle ,\]

and demonstrate that
\[
g(t) = \exp(-itM(0)) \in U \otimes \tilde{F}.\]

That this is really true follows from the definition of \( \xi^H \) (3), co-commutativity of \( h \) and the following chain of identities:
\[
\text{Tr}_1^\nu(R^{-1}_{12} Y^{-1}_{1} (R^\pm_{12})^{-1} D_1) = Z_2^{-1} Z_1 \text{Tr}_1^\nu(R^{-1}_{12} Y^{-1}_{1} (R^\pm_{12})^{-1} D_1) ,\]
where we used successively the third and the second relations of (11) and the relation (18).

Let us mention that \( M = M^+ - M^- \), with \( M^\pm \) the elements of \( U \otimes \tilde{F} \) entering the Lax equation (4). From the equality \( \xi^H = Z^{-1}MZ \) that we just proved, and from the time independence of \( \xi^H \), we have

\[
M(t) = g_+(t)M(0)g_+(t)^{-1}.
\]

Writing now

\[
g_+(t) = Z(t)Z(0)^{-1} = \exp(-it(1 \otimes \mathcal{H}))Z(0)\exp(it(1 \otimes \mathcal{H}))Z(0)^{-1}
\]

and using the last equality (with a = sign) in (26) and Proposition 4 we get immediately

\[
g_+(t) = \exp(-it(1 \otimes h)\exp(-it(M^+(0) - 1 \otimes h)).
\]

For \( g_- (t) \) we get similarly

\[
g_-(t) = \exp(-it(1 \otimes h - M^- (0)))\exp(it(1 \otimes h)),
\]

which follows from (28) and (45) (in Appendix 1). This shows that indeed \( g^\pm \in U_\pm \otimes \tilde{F} \), as we claimed. Moreover \( g^\pm \) are the unique solutions of the equations

\[
i \dot{g}_+ = M^+ g_+.
\]

and

\[
i \dot{g}_- = -g_- M^-,
\]

with initial condition \( g^\pm (0) = 1 \). Starting now from:

\[
\check{T}(t) = Z(t)Y^{-1}(0)Z(t)^{-1},
\]

we arrive at the main result of this paper:

**Theorem 1.** Let \( U \) be a quasi-triangular Hopf algebra and let \( F \) be its dual Hopf algebra. Let \( g(t) \) be given by (25), with the Hamiltonian \( h \), taken to be any co-commutative element of \( F \), and let \( U_\pm \) denote the ranges of the mappings \( R_\pm \) (24). Then \( g(t) \) can be factorized:

\[
g(t) = g_-(t)g_+(t),
\]

\( g^\pm (t) \in U_\pm \otimes F \) given by (28), (29). Moreover \( g^\pm (t) \) are the unique solutions of equations (30), (31), with initial conditions \( g^\pm (0) = 1 \). The element \( T(t) \in U \otimes F \), given by

\[
T(t) = g_+(t)T(0)g_+(t)^{-1} = g_-(t)^{-1}T(0)g_-(t),
\]

8
solves the quantum Lax equation (4). In the case of factorizable $U$ we can interpret (33) as a well-formulated factorization problem in $U \otimes F$ (see Appendix 2).

Although we proved here Theorem 1 only in the special case of $U$ being a quantum double and the Hamiltonian $h$ being of the form (14), we formulated it more generally. We shall give a simple direct proof of Theorem 1 in full generality in Appendix 1.

The second equality in (34) is due to fact that $g(t)$ commutes with $T(0)$ in $U \otimes F$, which is easily seen, e.g. from (4).

To specify completely our quantum dynamical system, we have to choose a representation $\pi$ of the quantum group $F$. The algebra of quantum observables will be the image $\pi(F)$ of $F$ in the chosen representation. The time evolution of an observable $\pi(a), a \in F$, will then be given by $\pi(a)(t) = \pi((a \otimes \text{id}, T(t)))$.

5 Lax equations for quantum chains

In this section we will discuss how the above result modifies in the case of a quantum spin chain. The algebra of observables $F^{\otimes N}$ for the chain consists of $N$ independent copies $F^n$, $n = 1, 2, ..., N$, of the dual Hopf algebra $F$ of a quasi-triangular Hopf algebra $U$. We also see $N + 1 \equiv 1$. We will denote as $L^n \in U \otimes F^n$ the copy of the universal T-matrix corresponding to the site $n$. We reserve the character $T$ for the quantum monodromy matrix

$$T = L^1...L^N \in U \otimes F^{\otimes N}. $$

Then we have the following relations in $U \otimes F^{\otimes N}$

$$R_{12}L^i_1L^i_2 = L^i_2L^i_1R_{12},$$
$$L^i_1L^j_2 = L^j_2L^i_1, \quad i \neq j. \quad (35)$$

The quantum monodromy matrix satisfies

$$R_{12}T_1T_2 = T_2T_1R_{12} \quad (36)$$

and for the partial products

$$\psi^n = L^1...L^{n-1}, \quad \psi^1 = 1, $$

we obtain

$$R_{12}\psi^n_1\psi^n_2 = \psi^n_2\psi^n_1R_{12}. \quad (37)$$

We will choose our Hamiltonian $h \in F^{\otimes N}$ as any element of $F^{\otimes N}$ of the form

$$h = \langle(H \otimes \text{id}), T \rangle, \quad (38)$$

with co-commutative $H \in F$. Again such elements form a commutative subalgebra in $F^{\otimes N}$. 

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Proposition 5. The Lax equations for the site \( n \) have the following form:

\[
\begin{align*}
    i\dot{L}^n &= M^\pm L^n - L^n M^{(n+1)} \quad \text{where} \\
    M_\pm^n &= \langle H \otimes \text{id}, (1 - (R_1^\pm)^{-1})(\psi^n)\rangle^{-1} T_1 \psi^n. 
\end{align*}
\]

This can be easily shown using the commutation relations (35), (37) and co-
commutativity of \( H \) in the same way as in Proposition 1.

Lax pair of Proposition 5 formalizes concrete examples of Lax pairs known for
particular integrable quantum chains or integrable field theoretical models [13], [29],
[28], [19], [34], [36].

To avoid a cumbersome notation we introduce again a notation similar to that
in the previous section:

\[
\begin{align*}
    \hat{M} &= R_- (H_{(1)}) \otimes H_{(2)} - R_+ (H_{(1)}) \otimes H_{(2)} \in U \otimes F \\
    M &= \langle \hat{M} \otimes \text{id}, \text{id} \otimes T \rangle \in U \otimes F^{\otimes N}. 
\end{align*}
\]

The following modification of Theorem 1 can be proved analogically as in the
previous sections. The twisted Heisenberg double of ref. [27] should be used for
this. However, there is also a direct proof using Theorem 1.

Theorem 2. Let \( U \) be a quasi-triangular Hopf algebra, \( F \) its dual Hopf algebra.
Let us assume a quantum chain system as described above with the Hamiltonian
\( h \) given in (38), where \( H \) is any co-commutative element of \( F \). Then the elements
\( g^n(t) \in U \otimes F^{\otimes N}:
\]

\[
g^n(t) = (\psi^n(0))^{-1} \exp(-itM(0))\psi^n(0), \quad (40)
\]

can be decomposed as

\[
g^n(t) = g^n_- (t) g^n_+ (t), \quad (41)
\]

with \( g^n_{\pm} (t) \in U_{\pm} \otimes F^{\otimes N} \) given by

\[
\begin{align*}
    g^n_- (t) &= \exp(-it(1 \otimes h) \exp(-it(M^+ n(0) - 1 \otimes h))), \\
    g^n_+ (t) &= \exp(-it(1 \otimes h - M^- n(0)) \exp(it(1 \otimes h))). 
\end{align*}
\]

Moreover \( g^n_\pm \) are the unique solutions of equations (30), (31) (all entries indexed by
\( n \)), with initial conditions \( g^n_{\pm} (0) = 1 \). The elements \( L^n (t) \in U \otimes F^{\otimes N} 
\]

\[
L^n (t) = g^n_- (t) L^n (0) (g^n_+ (t))^{-1} = (g^n_- (t))^{-1} L^n (0) g^n_+ (t) 
\]

(43)
solve the chain Lax equations (39). In the case of factorizable \( U \), elements \( g^n_{\pm} \) can be
thought of as a solution to the factorization problem for \( g \) as formulated in Appendix
2.

Proof. Following the same reasoning as led to Proposition 1, we can establish
that the Heisenberg equations of motion for entries of the quantum monodromy
matrices

\[
T^n = (\psi^n)^{-1} T \psi^n = L^n \ldots L^N L^{1 \ldots N^{-1}},
\]

9
for chains obtained from the original one by a shift \((1, \ldots, N) \mapsto (n, \ldots, N, 1, \ldots, n-1)\), are precisely of the form \((4)\), with \(M^\pm = M^\pm_n\). So the time evolution of the quantum monodromy matrix \(T^n\) is given by Theorem 1, with \(g(t) = \exp(-it(M^+n(0) - M^-n(0)))\), which means that all elements \(\exp(-it(M^+n(0) - M^-n(0))) \in U \otimes F^{\otimes N}\) can be decomposed as claimed.

It remains only to show that

\[
\exp(-it(M^+n(0) - M^-n(0))) = (\psi^n(0))^{-1}\exp(-itM(0))\psi^n(0),
\]

This is, however, a consequence of the co-commutativity of \(H\) and the following equality:

\[
\psi_1^n(R^{\pm}_{12})^{-1}(\psi_2^n)^{-1}T_1 = (\psi_2^n)^{-1}(R^{\pm}_{12})^{-1}T_1\psi_2^n,
\]

which easily follows from (35), (37).

The rest is trivial.

In this paper we did not mention the dressing symmetries of the quantum integrable systems at all. However, dressing symmetries can be introduced in a way completely analogous to the classical case (for the classical case see [25]). This aspect of the theory of quantum integrable systems will be discussed elsewhere.

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Appendix 1: Direct proof of Theorem 1

As in the classical case there is a simple direct proof of Theorem 1 and hence also of Theorem 2, which as it follows from the discussion of the preceding section, is a simple consequence of the Theorem 1.

Let \(U\) be any quasi-triangular Hopf algebra, \(F\) its dual Hopf algebra and \(U_{\pm}\) the range of the maps \(R_{\pm}\) (24). First of all let us mention that \(g_{\pm} \in U_{\pm} \otimes F\), given by (28) and (29):

\[
g_+(t) = \exp(-it(1 \otimes h)\exp(-it(M^+(0) - 1 \otimes h)),
g_-(t) = \exp(-it(1 \otimes h - M^-(0)))\exp(it(1 \otimes h)).
\]

solve the equations (30), (31) with initial condition \(g_{\pm}(0) = 1\), for any co-commutative Hamiltonian \(h\). This is easily checked by a direct computation. As an immediate consequence we find that \(T(t)\) given by (34) solve the Lax equations (4).

Now we shall show that the elements \(R_-(h_{(1)}) \otimes h_{(2)} \in U_- \otimes F\) and \(R_+(h_{(1)}) \otimes h_{(2)} \in U_+ \otimes F\) commute. Let us compute

\[
R_{20}^{-1}R_{12}T_1 = R_{20}^{-1}R_{12}T_0T_1 = R_{20}^{-1}R_{12}R_{10}T_1T_0R_{10}^{-1}
\]
Dualizing the first and last term in the above chain of equalities, which take place in \( U \otimes U \otimes U \otimes F \), in the components 0 and 1 with \( h \otimes h \in F \otimes F \), and using the co-commutativity of the Hamiltonian \( h \), we have

\[
[R_- (h_{(1)}) \otimes h_{(2)}, R_+ (h_{(1)}) \otimes h_{(2)}] = 0.
\]

(45)

This shows that

\[
g_- (t) g_+ (t) = \exp (-i t (M^+ (0) - M^- (0))).
\]

(46)

So we have proved Theorem 1 directly.

### Appendix 2: Factorization problem

Here we make an attempt to formulate a quantum analogue of the factorization problem from the classical case [25] in the case where \( U \) is a factorizable Hopf algebra [22]. Similarly to [22], we can give in this case an equivalent description of the algebra structure of the tensor product \( U \otimes F \). We shall omit details.

The claim is that as a linear space \( U \otimes F = F(-) \otimes F(+) \), where \( F(\pm) \) are subalgebras of \( U \otimes F \), both as algebras isomorphic to \( F \). They are embedded into \( U \otimes F \) via the the following algebra morphisms:

\[
R_\pm : F \rightarrow U \otimes F,
\]

\[
x \mapsto R_\pm (x_{(1)}) \otimes x_{(2)},
\]

where \( R_\pm \) are given by (24). This vector space isomorphism can be made into an algebra isomorphism if the commutation relations between the elements of the two copies \( F(\pm) \) of \( F \) are introduced through

\[
(1 \otimes x)(y \otimes 1) = \langle R_{21}^{-1}, y_{(1)} \otimes x_{(1)} \rangle y_{(2)} \otimes x_{(2)} \langle R_{21}, y_{(3)} \otimes x_{(3)} \rangle,
\]

(47)

so that, as an algebra, \( U \otimes F \) is isomorphic to a bicrossproduct of two copies of \( F \).

This means that any element \( \alpha \in U \otimes F \) can be expressed as

\[
\alpha = \sum \alpha_i \alpha_{i^+},
\]

(48)

with all \( \alpha_i \) lying in the range of the map \( R_- \) and all \( \alpha_{i^+} \) lying in the range of the map \( R_+ \), respectively. All \( \alpha_{i^\pm} \) are given unambiguously. It may happen that some particular \( \alpha \in U \otimes F \), if expressed in this way, is a simple product of two factors

\[
\alpha = \alpha_\alpha \alpha_\alpha^+,
\]

(49)

\( \alpha_\alpha^+ \) being the image under \( R_+ \) of a (unique) invertible element \( x \in F \) and \( \alpha_\alpha \) being image under \( R_- \) of the inverse \( x^{-1} \) of the same element \( x \). If this is the case, we shall refer to the unique elements

\[
\alpha_- = \alpha_\alpha (1 \otimes x),
\]
as to the solution of the factorization problem for \( \alpha \in U \otimes F \).

Clearly the elements \( g_\pm \) and \( g^n_\pm \) from Theorems 1 and 2 are, in the case of the factorizable \( U \), solutions to the factorization problem for \( g \) and \( g^n \), respectively.

Finally we have to note that in concrete examples it is possible to give an alternative characterization of the factorization of elements \( g \) or \( g^n(t) \). We shall discuss this very briefly for typical example when our starting Hopf algebra is the quantum double of a Yangian \( Y \). Other cases are similar. Let \( T_\lambda \) be the automorphism of \( Y \) of [5]: \( U = D(Y) \). We shall use the same notation \( T_\lambda, \lambda \in \mathbb{C} \) for its extension (via duality) to the full double. Then the decomposition of \((T_\lambda \otimes id)g^n(t)\),

\[
(T_\lambda \otimes id)g^n(t) = (T_\lambda \otimes id)g^n_\pm(t)(T_\lambda \otimes id)g^n_\pm(t),
\]

is uniquely determined by the assumption that \((T_\lambda \otimes id)g^n_\pm(t)\) are regular as functions of \( \lambda \) in \( \mathbb{C}P_1 \setminus \{\infty\} \) and \( \mathbb{C}P_1 \setminus \{0\} \), respectively, and \((T_\lambda \otimes id)g^n_\pm = 1\).

References


