Remarks on Virasoro and Kac–Moody algebras

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1 Generalities

The algebra of vector fields on a circle plays an important role in connection with bosonic string. A central extension of it gives the celebrated Virasoro algebra. It also provides us with a good example of infinite dimensional Lie algebras which is manageable and for which representation theory is well understood [GO 1986], [GF 1968], [Wi 1988]. From this point of view, having different realizations of this abstract algebra may be quite useful for applications, for instance to look for extensions of loop algebras [PS 1986], [Mi 1989] which form another important class of infinite dimensional Lie algebras.

1.1 Virasoro algebra

The Virasoro algebra is a central extension of the Witt algebra $W$, which is defined in a basis $\{X_n : n \in \mathbb{Z}\}$ by the commutation rules

$$[X_n, X_m] = (m - n)X_{n+m}.$$ 

The standard realization in vector fields is given by the (complex) vector fields on the circle defined by

$$X_n = -ie^{int} \frac{\partial}{\partial t}.$$ 

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In other words, the Witt algebra $W$ is spanned by the functions $F_n(t) = -ie^{int}$ with the Lie bracket $[F,G] = FG' - F'G$. A realization in vector fields on the real line may be also written in the real form

$$X_n = e^{nx} \frac{\partial}{\partial x}.$$  

The Virasoro algebra is the central extension of $W$ given by the cocycle

$$\omega(X_n, X_m) = \frac{c}{12} n(n^2 - 1) \delta_{n,-m},$$

i.e. the commutation rules are

$$[X_n, X_m] = (m - n)X_{n+m} + \frac{c}{12} n(n^2 - 1)C,$$

with $C$ being the central element. The above cocycle is cohomologically equivalent to the cocycle

$$\omega_0(X_n, X_m) = n^3 \delta_{n,-m}$$

which, in the realization by functions $F_n = -ie^{int}$ on the circle, can be written as

$$\omega_0(F_n, F_m) = \frac{i}{2\pi} \int_0^{2\pi} F'_n(t)F''_m(t)dt.$$  

Integrating by parts, we can also write it in a visible skew-symmetric form

$$\omega_0(F, G) = \frac{i}{4\pi} \int_0^{2\pi} (F'G'' - F''G')dt.$$  

By looking at previous formulae it is immediate to notice that a general Witt algebra may be written in the form

$$[X_a, X_b] = (b - a)X_{a+b},$$

with $a, b$ being arbitrary real numbers, so that it is graded by reals. Having now any subgroup $\Gamma$ of reals, we can consider the subalgebra $W(\Gamma)$ spanned by elements indexed by $\Gamma$. In particular, $W(\mathbb{Z})$ is the standard Witt algebra. Also the cocycle $\omega_0(X_a, X_b) = a^3 \delta_{a,-b}$ makes sense for all algebras $W(\Gamma)$. Such algebras are isomorphic if and only if $\Gamma' = c\Gamma$ for a constant $c$, and we will have an infinite family of non-isomorphic Virasoro algebras. In particular,
choosing $a_1, \ldots, a_n \in \mathbb{R}$ not rationally related (linearly independent over $\mathbb{Q}$) and generating the group $\Gamma$ by them, we obtain a Virasoro algebra with the Witt part realized in vector fields on an $n$-dimensional torus as we will show in Remark 3. We can easily provide a realization of $W(\mathbb{R})$ in terms of vector fields. Defining namely $f_a = e^{iax}$ and

$$X_a = -if_a \frac{\partial}{\partial x},$$

we find at once $[X_a, X_b] = (b - a)X_{a+b}$. As for the cocycle we can take the one geometrically expressed by

$$(*) \quad \omega(X_a, X_b) = \lim_{t \to \infty} \frac{i}{2t} \int_0^t (f'_a(x)f''_b(x) - f''_a(x)f'_b(x))dx$$

which gives us $\omega(X_a, X_b) = a^3 \delta_{a,-b}$. If we are not on the line but on an open interval, say, $(0, T)$, we can write the cocycle in the form

$$(**) \quad \omega(X_a, X_b) = \lim_{k \to \infty} \frac{i}{2Tk^3} \int_0^T (f'_{ka}(x)f''_{kb}(x) - f''_{ka}(x)f'_{kb}(x))dx.$$

In the previous construction we can replace the one dimensional manifolds with any manifold $M$. Let us denote by $X_0$ a vector field on a manifold $M$. It is easy to see that the solutions $\gamma$ of the eigenvalue problem on $\mathfrak{g}(M, \mathbb{C})$, $L_{X_0} f = i c \lambda f,$ form a subgroup $\Gamma \subset \mathbb{R}$. Here $c$ is a quantity which is independent of $\lambda$ and may be a nowhere vanishing function of the constants of motion, so after a redefinition of $X_0$ we may consider $c = 1$. We denote by $f^k_\lambda$ the eigenfunctions corresponding to the eigenvalue $\lambda$. We define $X^k_m = -i f^k_m X_0$ and get

$$[X^k_n, X^r_m] = (m - n) X^{(k+r)}_{m+n}.$$

At the moment we do not bother on how $(s)$ is related with $(k)$ and $(r)$. The set of vector fields associated with the eigenfunctions of a given eigenvalue constitute an abelian algebra. The ratio of any two eigenfunctions belonging to the same eigenvalue is a constant of the motion for $X_0$.

It is clear that we have obtained a generalization of the Witt algebra because now eigenvalues can have infinite degeneracy. However, if it makes
sense to consider powers $f^a$ with $a \in \mathbb{R}$ or belonging to some other subgroup of $\mathbb{R}$, we can extract a subalgebra of the kind considered on the one dimensional manifolds.

Let us take an orbit $\gamma$ for the vector field $X_0$. Now we can define a cocycle for $W$ which allows for the construction of a central extension, i.e. a Virasoro algebra.

First of all, let us assume that for a subgroup $\Gamma \subset \mathbb{R}$ we have eigenfunctions $\{f_a : a \in \Gamma\}$, $L_{X_0}(f_a) = if_a$, with $f_a f_b = f_{a+b}$ and that the time parametrizing the orbit runs for the infinite future (the same we can do for the infinite past): 

$$t : (0, +\infty) \rightarrow \gamma.$$ 

Then we can proceed as in the case of the real line, i.e. to define the basis of the Witt algebra by $X_a = -if_a X_0$ and the cocycle by

$$\omega(X_a, X_b) = \lim_{s \to +\infty} \frac{i}{2s} \int_0^s (f_a'(t)f_b''(t) - f_a''(t)f_b'(t))dt.$$ 

Indeed, since on the orbit with the time parametrization $X_0 = \frac{\partial}{\partial t}$, we have $f_a(t) = c_a e^{iat}$ with $c_a c_b = c_{a+b}$ and

$$\omega(X_a, X_b) = \lim_{s \to +\infty} \frac{c_{a+b}}{2s} (a - b)ab \int_0^s e^{i(a+b)t}dt = c_0 a^3 \delta_{a, -b}.$$ 

Of course, if the orbit $\gamma$ is periodic with the period $T$, the above reduces to

$$\omega(X_a, X_b) = \frac{i}{T} \int_0^T f_a'(t)f_b''(t)dt = i \int_{\gamma} L_{X_0}(f_a)L_{X_0}^2(f_b)\theta,$$

with $\theta$ being the normalized $X_0$-invariant 1-form on $\gamma$, which is the classical form of the Virasoro cocycle.

If the time parametrizing the orbit runs only over an interval (vector field $X_0$ is not complete), we can use the formula similar to (**)..

Our construction associates a realization of Virasoro algebra for any part of an orbit of $X_0$ on $M$. These various realizations are parametrized by the the constants of the motion for $X_0$.

Remark 1. We notice that we might have started with any vector field $X$ on $M$ and construct a Lie algebra structure on $\mathfrak{g}(M)$ by setting $[f, g] =
$fL_{X}g - gL_{X}f$. This is known as a Jacobi structure associated with $X$. On each closed orbit $\gamma$ of $X$ we consider the Gelfand-Fuks cocycle

$$\omega(f, g) = \lambda \int_{\gamma} ((L_{X}f) d(L_{X}g) - (L_{X}g) d(L_{X}f)) \theta,$$

for $\theta$ being the canonical 1-form on $\gamma$, providing us with a central extension depending on $\gamma$.

Note that in many cases the manifold $M$ is a complex one and the vector field $X_{0}$ is a meromorphic vector field, i.e it has the form

$$X_{0} = a^{i}(z) \frac{\partial}{\partial z^{i}},$$

where $z^{i}$ are local complex coordinates on $M$ and functions $a^{i}(z)$ are meromorphic functions.

The simplest example is the case of Riemann sphere, which may be considered as stereographic projection of standard sphere $S^{2}$ in three-dimensional space. In this case we have

$$X_{0} = z \frac{d}{dz}$$

and

$$X_{m} = z^{m+1} \frac{d}{dz} = z^{m} X_{0}. $$

The case of $n$-dimensional torus will be considered in Remark 3 below. Our construction is valid also for Riemann surfaces of higher genus, but it is different from construction of Krichever and Novikov [KN 1987ab].

We can give now a simple example of a family of realizations of Virasoro algebras on $\mathbb{R}^{2} - \{0\}$. On $\mathbb{R}^{2} - \{0\}$ we consider

$$X_{0} = -ih(r) \frac{\partial}{\partial \varphi}, \quad h(r) \neq 0, \quad r \in \mathbb{R}^{+}.$$

The eigenvalue problem

$$L_{X_{0}}f = c \lambda f$$

has solutions for $\lambda = m, \quad c = h(r), \quad m \in \mathbb{Z}$, with $f_{m} = \alpha(r) e^{im\varphi}$, $\alpha(r)$ is any function of $r$, thus the eigenvalue has infinite degeneracy. We have

$$[f_{m}, f_{n}] = \alpha(r) \beta(r) e^{i(m+n)\varphi} h(r) (n - m).$$
Let us consider the Gelfand-Fuks co cycle. We have
\[
\omega(f_m, f_n) = \mu \int_{r=r_0} (L_{X_0} f_m) d(L_{X_0} f_n) - (L_{X_0} f_n) d(L_{X_0} f_m)) \theta
= i \mu \hbar^2 (r_0) \alpha(r_0) \beta(r_0) m^3 2\pi \delta_{m+n,0}.
\]
(1)

Notice that \( l(r) \) is in the kernel of \( \omega \) for any function \( l(r) \).
To extract a Virasoro algebra, we redefine \( X_0 \) so that \( c = 1 \), and select \( f_1 = \alpha(r)e^{i\omega} \), \( f_m = (f_1)^m \) to get \([f_m, f_n] = (n - m)f_{m+n}\). The use of the Gelfand-Fuks cocycle provides us with Virasoro algebra.

**Remark 2.** As for the Witt algebra, we notice that on \( \mathbb{R}^2 - \{0\} \) we can consider a different choice for \( X_0 \), for instance
\[
X_0 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.
\]

Solutions for
\[
L_{X_0} f = \lambda f, \quad \lambda \in \mathbb{Z},
\]
are provided by homogeneous monomials, thus every monomial \( f_m^{k,m-k} = x^k y^{m-k} \) solves
\[
L_{X_0} f = mf
\]
and we can write the commutator as
\[
[f_m^{k,m-k}, f_n^{r,n-r}] = (n - m)f_{m+n}^{k+r,m+n-(k+r)}.
\]
As eigenfunctions \( f_\alpha \), with \( L_{X_0} (f_\alpha) = i\alpha f_\alpha \), we may take for instance
\[
f_\alpha = ((xy)^2)^{i\alpha}.
\]

Then, we get the Witt algebra \( W(\mathbb{R}) \) putting \( X_a = -i f_a X_0 \). This time \( X_0 \) does not have periodic orbits, therefore taking any trajectory, for instance \( x = y > 0 \), we find that
\[
\omega(X_a, X_b) = \lim_{i \to +\infty} \frac{i}{2\ln i} \int_1^i [L_{X_0}(x^{i\alpha}), L_{X_0}(x^{i\beta})] \frac{dx}{x}
\]
gives us a cocycle.
**Remark 3.** We can apply the construction to the case of the irrational flow on the $n$-dimensional torus, now

$$X_0 = \sum_{j} a_j \frac{\partial}{\partial x_j} \quad 0 \leq x_i \leq 2\pi.$$ 

The eigenfunctions of this vector field are

$$Y^{(l_1, \ldots, l_n)} = e^{i l_1 x_1} \cdots e^{i l_n x_n}$$

with eigenvalues $i \sum a_j l_j$; they are dense on $\mathbb{R}$. We can use the grading given by the dense subgroup $\Gamma = \{a \in \mathbb{R} : a = a_j l_j, \ l_j \in \mathbb{Z}\}$ and denote $Y^{(l_1, \ldots, l_n)}$ by $Y_a$ getting a general Witt algebra $W(\Gamma)$. In this case $X_0$ has no nontrivial constants of the motion, therefore eigenvalues are not degenerate.

The Lie algebra structure is given by

$$[F, G] = F L_{X_0} G - G L_{X_0} F.$$ 

In the present case we obtain

$$[Y_a, Y_b] = (b - a)Y_{a+b}, \ a, b \in \Gamma$$

and the extension

$$\omega(Y_a, Y_b) = a^3 \delta_{a,-b}$$

is obtained via the formula (*) on a dense orbit. It is easy to see that this is a cocycle which can be also written as

$$\omega(F, G) = \int_T [L_{X_0} F, L_{X_0} G] \mu,$$

where $\mu = dx_1 \wedge \ldots \wedge dx_n$ is an invariant measure on the torus: $L_{X_0} \mu = 0$.

Up to now we have realized the Witt algebra as a subalgebra of vector fields on $M$. One may try to realize the Virasoro algebra in terms of vector fields on $M$. We shall show that the Virasoro algebra cannot be realized in vector fields on a finite-dimensional manifold.

**Theorem 1** If $\mathcal{L}$ is a central extension of a Lie algebra with respect to a cocycle $\omega$ such that all cocycles from the class of $\omega$ have infinite rank, then $\mathcal{L}$ cannot be realized neither as a Lie algebra of vector fields, nor as a Lie algebra of functions with respect to a Poisson bracket on a finite-dimensional manifold.
Proof. Let $C$ be the central element and let $\mathcal{L}_0$ be a complementary subspace to $<C>$ - the linear span of $C$, i.e.

$$\mathcal{L} = \mathcal{L}_0 \oplus <C>,$$

so that $\mathcal{L}_0 \cong \mathcal{L}/<C>$ can be interpreted as the Lie algebra which is actually extended. Realizing $\mathcal{L}$ in vector fields, we can find a chart and local coordinates $(x_0,\ldots,x_n)$ such that $C = \frac{\partial}{\partial x_0}$ and for any $X \in \mathcal{L}$, $X = f_X^\theta \frac{\partial}{\partial x_0}$, and we may choose $\mathcal{L}_0$ to be actually

$$\mathcal{L}_0 = \{ X \in \mathcal{L} : f_X^0(0) = 0 \}.$$

Then, clearly, $\omega_0(X,Y) = f_{[X,Y]}^0(0)$, $X,Y \in \mathcal{L}_0$, is a cocycle in the class of $\omega$. Since $C = \frac{\partial}{\partial x_0}$ is central, $f_X^0$ do not depend on $x_0$ and

$$f_{[X,Y]}^0 = X(f_Y^0) - Y(f_X^0).$$

Therefore, for $\mathcal{L}_1 = \{ X \in \mathcal{L}_0 : X(0) = 0 \}$ being the subalgebra of $\mathcal{L}_0$ of vector fields vanishing at 0, we have $\omega_0(X,Y)(0) = 0$ for $X,Y \in \mathcal{L}_1$. Since $\mathcal{L}_1$ is at most $n$-codimensional in $\mathcal{L}_0$, the rank of $\omega_0$ is at most $n$; a contradiction. For the case of a Poisson bracket we can use a similar argument. We can choose a chart and local coordinates $(x_1,\ldots,x_n)$ such that $C(0) = 1$ and $\mathcal{L}_0 = \{ F \in \mathcal{L} : F(0) = 0 \}$. Then $\omega_0(F,G) = \{ F,G \}(0)$ is the cocycle and $\mathcal{L}_1 = \{ F \in \mathcal{L}_0 : dF(0) = 0 \}$ is a subalgebra of $\mathcal{L}_0$ of codimension at most $n$ such that

$$\omega(F,G) = 0$$

for $F,G \in \mathcal{L}_1$. \qed

**Corollary 1** The Virasoro algebra has neither a realization in terms of vector fields, nor in terms of a Poisson bracket on a finite-dimensional manifold.

**Proof.** It is sufficient to show that the cocycle in the class of Virasoro are of infinite rank. Assume that $\omega(x_n,x_m) = n^3 \delta_{n,-m} - \phi([x_n,x_m])$, for some functional $\phi \in W^*$, is of finite rank, i.e. the kernel $L$ of $\omega$ is a finite codimensional subalgebra in $W$. Realizing the Witt algebra with vector fields on the circle and using the general algebraic result of [Gr 1993, Theorem 4] describing finite codimensional subalgebras in derivation algebras of associative commutative rings, we get that $L$ contains a finite codimensional
submodule, which means in terms of our trigonometric polynomials algebra that we have an associative ideal \( I \) in \( L \) of finite codimension. Hence

\[
\frac{i}{2\pi} \int_0^{2\pi} (F(t)e^{int})'(G(t)e^{-int})'' dt = \phi([F(t)e^{int}, G(t)e^{-int}])
\]

for all \( n \in \mathbb{Z} \) and \( F, G \in I \). It is easy to see that on both sides we get polynomials with respect to \( n \). Since in the left hand side we have

\[
n^3 \int_0^{2\pi} F(t)G(t)dt + ...
\]

and we have no term with \( n^3 \) on the right, it must be

\[
\int_0^{2\pi} F(t)G(t)dt = 0
\]

for all \( F, G \in I \), which is impossible.

\[ \square \]

1.2 Kac-Moody algebras

Affine Kac-Moody algebras are extensions of current (loop) algebras. We assume now that on \( M \), along with the vector field \( X_0 \), we also have vector fields \( X_1, X_2, \ldots, X_n \) closing a Lie algebra \( g \), i.e.,

\[
[X_i, X_j] = C_{ij}^k X_k
\]

and

\[
L_{X_0} f_m = i m f_m.
\]

We further require \([X_0, X_j] = 0 \), \( \forall j \in \{1, \ldots, n \} \) and \( L_{X_j} f_m = 0 \) for any \( j \in \{1, \ldots, n \} \) and for any eigenfunction. We use these eigenfunctions to define: \( Y^m_{a} = f_m X_a \), \( Y_0 = -iX_0 \). If for each eigenvalue there is only one eigenfunction satisfying previous requirements, we get

\[
[Y_0, Y^m_{a}] = m Y^m_{a},
\]

\[
[Y^m_{a}, Y^n_{b}] = Y^m_{a} Y^n_{b} + C_{ab}^k k
\]

If \( X_0 \) has a closed orbit \( \gamma \), we can define a central extension by setting
\[
\omega(Y^m_a, Y^n_b) = \frac{k}{2} \int \langle Y^m_a [X_0, Y^n_b] \rangle - \langle Y^n_b [X_0, Y^m_a] \rangle \cdot \theta,
\]
where \(\langle \ | \rangle\) is an invariant symmetric product on the algebra \(\mathfrak{g}\). An example of this situation arises for instance on \(T^*SU(2)\), where we consider \(X_0\) to be the canonical lift of a left invariant generator, while \(X_1, X_2, X_3\) are generators of canonical lifted left action. Eigenfunctions for \(X_0\), invariant under left action, are to be found among left invariant momenta, and an appropriate choice is to be made. If \(X_0\) has no closed orbits, we may use formulae similar to those of (1) and (2).

A typical situation arises from principal bundles on \(S^1\). We consider a \(G\)-bundle \(P\) with base space \(S^1\). The vector field \(X_0\) is any projectable vector field which commutes with the \(G\)-action on \(P\). Eigenfunctions of \(\pi_*X_0\) can be pulled back to \(P\) to construct vector fields \(Y^m_i\).

We give an example by considering again the carrier space \(\mathbb{R}^2 - \{0\}\). We use coordinates \((r, \varphi)\). The following vector fields

\[
X_0 = -i \frac{\partial}{\partial \varphi}, \quad Y_{-1} = \frac{\partial}{\partial r}, \quad Y_0 = r \frac{\partial}{\partial r}, \quad Y_1 = r^2 \frac{\partial}{\partial r},
\]

with eigenfunctions \(f_m = e^{im\varphi}\), satisfy all previous conditions. An extension of \(sl(2, \mathbb{R})\) is realized by the vector fields

\[
X_0 = -i \frac{\partial}{\partial \varphi}, \quad Y^m_{-1} = e^{im\varphi} \frac{\partial}{\partial r}, \quad Y^m_0 = e^{im\varphi} r \frac{\partial}{\partial r}, \quad Y^m_1 = e^{im\varphi} r^2 \frac{\partial}{\partial r}.
\]

We note that from any realization of current algebras in terms of vector fields on a manifold \(M\) we can construct a realization in terms of functions with the Lie algebra structure given by a Poisson bracket. We lift the vector fields to \(T^*M\), equipped with the canonical symplectic structure, and obtain in this way an Hamiltonian realization of the given algebra.

Another realization of affine Lie algebras in terms of Poisson brackets, which is not the cotangent lift of an action on the base manifold, is given in the following example. On \(\mathfrak{g}(M \times \mathfrak{g}^*)\) we define the following Poisson bracket

\[
\Lambda = P_k C^k_{ij} \frac{\partial}{\partial P_i} \wedge \frac{\partial}{\partial P_j} + \Lambda_M,
\]

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where $\Lambda_M$ is any Poisson bracket on $M$ and $P_i$ are a basis of linear functions on $g^*$. With $H_0 \in \mathfrak{g}(M)$ and $f \in \mathfrak{g}(M)$ such that
\[
\{H_0, f\} = f,
\]
we define $F_a^m = f^m P_a$ and we get
\[
\{H_0, F_a^m\} = mF_a^m,
\]
\[
\{F_a^m, F_b^n\} = C_{abc} F_c^{m+n}.
\]

A central extension can be constructed if there is an invariant symmetric product in $g$ (and therefore in $g^*$), we can set
\[
\omega_0(F, G) = \frac{k}{2} \int \langle F | L_{X_0} G \rangle - \langle G | L_{X_0} F \rangle \rangle \theta,
\]
where $F, G$ are assumed to be linear functions on $g^*$. Note that as in the case of the Virasoro algebra these central extensions cannot be realized in terms of vector fields

**Theorem 2** Affine Kac-Moody algebras cannot be realized neither in terms of vector fields nor in terms of Poisson brackets on finite-dimensional manifolds.

**Proof.** It suffices to show that the cocycles in the class of $\omega_0$ are of infinite rank. But the cocycle
\[
\omega(F_a^m, F_b^n) = m\delta_{m,-n} \langle F_a, F_b \rangle + \phi([F_a, F_b]^{m+n}),
\]
with a given functional $\phi$, is of infinite rank even when reduced to $F_a, F_b$ from a Cartan subalgebra of $g$. $\square$

### 1.3 A Virasoro type algebra from a Kac-Moody algebra

Consider now the affine $A_1^1$ algebra being an extension of the loop $sl(2, \mathbb{R})$ algebra. Choosing for $sl(2, \mathbb{R})$ a basis $X^i$, $i = 0, \pm 1$, with
\[
[X^i, X^j] = [j - i] X^{i+j} \mathbb{R}.
\]
where \([n]_3 = 0, \pm 1\) if \(n = 0, \pm 1 (\text{mod}~ 3)\), we get the basis of the loop algebra being \(\{X^i_n : i = 0, \pm 1, \ n \in \mathbb{Z}\}\) with the commutation rules

\[
[X^i_n, X^j_m] = [j - i]_3 X^{[i+j]}_{n+m}.
\]

Changing the gradation to integers by putting

\[Y_{3n+i} = X^i_n,
\]

we get the commutation rules similar to those of the Witt algebra:

\[\left[Y_n, Y_m\right] = [m - n]_3 Y_{n+m}.
\]

It is in fact the only, except for the Witt, \(\mathbb{Z}\)-graded algebra \([Y_n, Y_m] = f(n, m) Y_{n+m}\) with the structure constants being translation invariant, i.e. satisfying \(f(n, m) = g(m - n)\). A version of the Virasoro cocycle

\[\omega_0(Y_n, Y_m) = [n^3]_3 \delta_{n,-m}
\]

is in the case of our Witt\(_3\) algebra trivial, since \([n^3]_3 = [n]_3\). We have, however, the nontrivial cocycle obtained from the loop algebra:

\[\omega(X^i_n, X^j_m) = n \delta_{n,-m} \delta_{i,-j},
\]

which gives us

\[\omega(Y_n, Y_m) = E(n/3) \delta_{n,-m},
\]

where \(E(x)\) is the integer closest to \(x\) (there is no ambiguity for \(x\) of the form \(n/3\)).

In the following sections we are going to consider a family of realizations of the infinite dimensional algebras, we have discussed, in terms of some interesting dynamical systems: the harmonic oscillator and the Euler top.

## 2 A realization in terms of the harmonic oscillator

1. We give now a family of realizations of the Virasoro algebra associated with a two-dimensional isotropic harmonic oscillator.
Here $M = \mathbb{C}^2 = \mathbb{R}^4$, we introduce coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$ to get

$$X_0 = i \left( z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right).$$

Obvious eigenfunctions for $X_0$ are provided by

$$f^r_{m-r} = z_1^{m-r}_1 z_2^{m-r}_2,$$
$$f^r_{-m-r} = \bar{z}_1^{m-r}_1 \bar{z}_2^{m-r}_2,$$

which along with

$$z_1^{r-p}_1 z_2^{q}_2 z_1^{s}_1 z_2^{t}_2,$$

$r + p - q - s = m$, belongs to the eigenvalue $m$.

By using

$$Y_0 = i \left( z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right)$$

and eigenfunctions $f_1 = z_1 z_2$, $f_m = (z_1 z_2)^m$, $m \in \mathbb{Z}$, we get $Y_m = f_m Y_0$ and $[Y_m, Y_n] = (n - m) Y_{m+n}$.

We can compute the Gelfand-Fuks cocycle

$$\omega(Y_m, Y_n) = c(\gamma) \int (L_{Y_0} f_m) d(\bar{L}_{Y_0} f_n) - (\bar{L}_{Y_0} f_n) d(L_{Y_0} f_m))$$

$$= c(\gamma) \int_{\gamma} mn f_m f_n (n - m) \theta,$$

where $\theta = -i \frac{d(z_1 z_2)}{z_1 z_2}$. We find

$$\omega(Y_m, Y_n) = ic(\gamma) \int_{\gamma} mn (n - m) (z_1 z_2)^m (z_1 z_2)^n \frac{d z_1 z_2}{z_1 z_2}$$

which is zero, as $\gamma$ is a closed curve, unless $m + n = 0$. Therefore $\omega(Y_m, Y_n) = ic(\gamma) 4\pi m^3 \delta_{m+n,0}$. By making other appropriate choices of eigenfunctions we get different realizations.

2. We can also give a realization of the Witt algebra in terms of Poisson brackets on $\mathbb{C}^n$. As an example we use the one dimensional harmonic oscillator. From here it will be easier to go to creation and annihilation operators and provide a realization in the spirit of the Jordan-Schwinger map.
We consider $H_0 = -iz\bar{z}$, $\{z, \bar{z}\} = i$. We notice that a possible choice of eigenfunctions is provided by $f_m = \bar{z}^m$, $m \in \mathbb{Z}$:

$$\{H_0, f_m\} = m f_m.$$ 

A realization of Witt algebra obtains by using

$$t^m = f_m H_0$$

and we find

$$[t^m, t^n] = i \{t^m, t^n\} = (m - n)t^{m+n}.$$ 

An extension is obtained by using

$$\omega(t^m, t^n) = \int \{H_0, t^m\}d\{H_0, t^n\} - \{H_0, t^n\}d\{H_0, t^m\}.$$ 

In the next section we shall consider $X_0$ to represent the dynamics of the Euler top. We will find a realization of the Witt algebra in terms of elliptic functions. We will also compute the Gelfand-Fuks cocycle for a particular orbit.

3 Realization in terms of the Euler top

1. Let us begin with consideration of motion on intersection $C$ of two quadrics $Q_1$ and $Q_2$ in $\mathbb{R}^3$

$$Q_1 \equiv x_1^2 + x_2^2 + x_3^2 - 1 = 0,$$

$$Q_2 \equiv \frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} - 1 = 0.$$ 

As it is well known, the genus of $C$ is equal one and, hence, this curve may be parametrized by elliptic functions

$$x_1 = \sqrt{\alpha_1} \text{cn}(u, k), \quad x_2 = \sqrt{\alpha_2} \text{sn}(u, k), \quad x_3 = \sqrt{\alpha_3} \text{dn}(u, k).$$

Here $\text{cn}(k, u)$, $\text{sn}(u, k)$ and $\text{dn}(u, k)$ are standard elliptic functions (see for example [BE 1955], [WW 1962]) and

$$\alpha_1 = 1 - \alpha_3, \quad \alpha_2 = 1 - (1 - k^2)\alpha_3.$$
Let us take the standard metric in $\mathbb{R}^3$:
\[
d s^2 = dx_1^2 + dx_2^2 + dx_3^2.
\]
Then it is easy to calculate induced metric on $C$:
\[
d s^2 = g(u) du^2,
\]
\[
g(u) = A - k^2 \text{sn}^2 u, \quad A = 1 - (1 - k^2) \alpha_3.
\]
So the function $g(u)$ is meromorphic double periodic function second order with periods $2K$ and $2iK'$ and $g(u)$ has two zeros, say $u = a$ and $u = -a$, and one pole of second order at the point $u = iK'$ in the fundamental domain
\[
0 \leq \Re u < 2K, \quad 0 \leq \Im u < 2K'.
\]
The position of zero $u = a$ depends on constant $A$ and when $A$ changes from $k^2$ to 1, $a$ changes from $K$ to $K + iK', a = K + ia$. The quantity $\alpha$ may be found from transcendental equation
\[
d n(\alpha, k') = k A^{-1/2}, \quad k' = \sqrt{1 - k^2},
\]
which follows from relation
\[
\text{sn}(K + ix, k) = [dn(x, k')]^{-1}.
\]
Now we may factorize the function $g(u)$
\[
g(u) = B f(u)f(\bar{u}),
\]
where
\[
f(u) = \frac{\sigma(u - a)}{\sigma(u + iK)} e^{\zeta(u) + \zeta(iK)u}, \quad \zeta(u) = \frac{\sigma'(u)}{\sigma(u)}
\]
is meromorphic one-periodic function
\[
f(u + 2K) = f(u)
\]
with one zero $u = a$ and one pole $u = iK'$ in fundamental domain. Here $\sigma(u)$ is standard Weierstrass function which entire function with zeros at the
points of the lattice $u_{m,n} = 2m\omega + 2n\omega'$.

We may define now a vector field on $C$:

$$X_0 = \frac{1}{f(u)} \frac{d}{du}.$$ 

Let us find eigenfunctions of this vector field

$$X_0\Psi_\lambda = \lambda \Psi_\lambda.$$ 

We have

$$\frac{d\Psi_\lambda}{\Psi_\lambda} = \lambda f(u) du,$$

so

$$\Psi_\lambda = C e^{\lambda F(u)}, \quad F(u) = \int f(u) du.$$ 

But $f(u)$ is periodic function of the variable $u$, so

$$f(u) = f_0 + \tilde{f}(u), \quad f_0 = \tilde{f} = \frac{1}{2\omega} \int_0^{2\omega} f(u) du, \quad \tilde{f} = 0$$

and $\Psi_\lambda(u)$ should be periodic in $u$. From this we obtain

$$\lambda f_0 2\omega = i 2\pi m, \quad \lambda_m = \frac{i\pi}{f_0 \omega} m,$$

and, after a redefinition of the vector field so that $f_0 = \frac{\pi}{\omega}$, we have

$$\Psi_{\lambda m} = C e^{i\lambda F(u)} = C (\Psi_1(u))^m = C \Psi_m(u), \quad \lambda_m = m$$

$$\Psi_1(u) = e^{iF(u)}.$$ 

So now we are ready to define the algebra of vector fields on the circle

$$X_m = \Psi_m(u) X_0 = \Psi_m(u) \frac{1}{f(u)} \frac{d}{du}.$$ 

Let us calculate the commutator

$$[X_m, X_n] = [\Psi_m(u) X_0, \Psi_n(u) X_0] = (n - m) X_{m+n},$$
i.e. it is a realization of the standard algebra of vector fields on the circle. But the concrete realization of this algebra is related to the function

\[ F(u) = \int f(v) \, dv = \int \frac{\sigma(v) - a}{\sigma(v)} e^{i v \omega} \, dv, \quad a = \omega + i \alpha, \quad 0 < \alpha \leq i \omega', \]

so depends essentially on parameters \( \alpha \). The integral under consideration may be calculated analytically, but give sufficiently complicated function, so here we consider only the simplest case \( \alpha = \omega' \). In this case

\[ f(u) = \text{dn}(u, k) \]

and we have

\[ F(u) = \int^u \text{dn}(v, k) \, dv = -i \ln \left( \text{cn}(u) + i \, \text{sn}(u) \right), \]

what may be checked directly. Finally we have

\[ \Psi_m(u) = \left( \text{cn}(u) + i \, \text{sn}(u) \right)^m \]

and

\[ X_m = -i \left( \text{cn}(u) + i \, \text{sn}(u) \right)^m \frac{1}{\text{dn}(u)} \frac{d}{du} \]

Note that when \( k \to 0 \) we obtain the standard expression

\[ X_m = -i e^{im \omega} \frac{d}{du}. \]

Let us consider another limiting case

\[ f(u) = \text{cn}(u, k). \]

Then we have

\[ F(u) = \int^u \text{cn}(v, k) \, dv = -i \ln(\text{dn}(u) + i \, k \, \text{sn}(u)) \]

and

\[ X_m = -i (\text{dn}(u) + i \, k \, \text{sn}(u))^m \frac{1}{\text{cn}(u)} \frac{d}{du}. \]
Now:

\[ [X_m, X_n] = k(m - n)X_{m+n}. \]

It is easy to calculate the cocycle for this case and we have

\[ \omega_{\gamma}(X_m, X_n) = 2n^3T(\gamma)\delta_{m+n,0}, \quad T(\gamma) = 2\pi. \]

2. Note that the following three vector fields

\[
X_0 = \frac{1}{\frac{1}{\partial n(u)} \frac{d}{du}}, \\
X_1 = \frac{1}{\frac{1}{\partial n(u)} \frac{1}{(\partial n(u))} \frac{d}{du} \frac{+ i sn(u) \frac{d}{du} \frac{\partial n(u)}{u}}, \\
X_{-1} = \frac{1}{\frac{1}{\partial n(u)} \frac{1}{(\partial n(u))} \frac{d}{du} \frac{- i sn(u) \frac{d}{du} \frac{\partial n(u)}{u},}
\]

form the elliptic realization of the SU(1,1) algebra.

Let us give another elliptic realization of this algebra. Let us define

\[
X_1 = sn(u) \frac{d}{du}, \\
X_2 = cn(u) \frac{d}{du}, \\
X_3 = dn(u) \frac{d}{du}.
\]

Then we have

\[
[X_1, X_2] = -X_3, \\
[X_2, X_3] = (1 - k^2)X_1 = k'^2X_1, \\
[X_3, X_1] = X_2.
\]

Let us introduce

\[ X_0 = X_3, \quad X_{+1} = X_2 + ik'X_1, \quad X_{-1} = X_2 - ik'X_1, \]

then we obtain standard basis in SU(1,1). The Casimir operator for this case has the form

\[ C = \frac{1}{2}(k'^2X_1^2 + X_2^2 - X_3^2) \]

and, if we substitute the explicit expressions (6) for \( X_1, X_2, X_3 \), we obtain \( C \equiv 0 \).
Now we can consider a product bundle $\mathbb{R}^3 \times \mathbb{R}^3$ and double our construction. From the $X_0$ on the first $\mathbb{R}^3$ and the given $SU(2)$ algebra realization on the second $\mathbb{R}^3$ we can build up an extension of $SU(2)$-Kac-Moody as given in general terms in section 2. More explicitly, we set

\[
Y_0 = \frac{1}{\text{dn}(u)} \frac{d}{du},
Y^m_i = (\text{cn}(u) + i \text{sn}(u))^m \text{sn}(v) \frac{d}{dv}, \quad Y^m = (\text{cn}(u) + i \text{sn}(u))^m \text{cn}(v) \frac{d}{dv}, 
Y^m_3 = (\text{cn}(u) + i \text{sn}(u))^m \text{dn}(v) \frac{d}{dv},
\]

and we get an extension of the $SU(1,1)$-Kac-Moody algebra in terms of elliptic functions.

It is also possible to use the vector fields $X_0, X_1, X_{-1}$ in formula (5) to get a different realization of the same algebra.

4 Conclusions

In this paper we have shown how, by using an appropriate vector field $X_0$ on a generic manifold $M$, it is possible to select subalgebras of vector fields on $M$ which give rise via central extensions along trajectories of $X_0$ to parametric realizations of Virasoro or Kac-Moody algebras. We have also discussed their realizations in terms of Poisson brackets on $T^*M$. We hope that this approach may provide a more direct way to the construction of generalization of Virasoro algebra for Riemann surfaces of higher genus. [KN 1987].

We have also shown that the centrally extended algebras cannot be realized as algebras of vector fields on finite-dimensional manifolds.

In addition our realization in terms of oscillators opens the way to the construction of $q$-oscillator deformation of these algebras, different from the one by [RS 1990]. We shall come back to some of these issues in future papers.

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