Analysis of Quantum Semigroups with GKS–Lindblad Generators II. General

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Analysis of quantum semigroups with GKS–Lindblad generators II.
General

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Abstract

Semigroups describing the time evolution of open quantum systems in finite-dimensional spaces have generators of a special form, known as Lindblad generators. These generators and the corresponding processes of time evolution are analyzed, characterized as Decay, Dissipation and Dephasing. In relation to these processes the Hilbert space of the system is equipped with a special structure, a decomposition into a sum of mutually orthogonal subspaces. The complete set of all the stationary states and the asymptotic behavior of the evolutions are presented in detail. Some unusual special facts about invariant operators and symmetries are studied, examples are demonstrated. Perturbation theory for the structure and for the stationary states is discussed and performed in case studies.

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1 Introduction

Hundreds of papers dealing with “Lindblad equations” have been written since their fundamental importance in the theory of open quantum systems was shown (see [L76, GKS76]). They give the proper mathematical form to Markovian semigroups of completely positive norm continuous maps which are needed in this context [D76]. The early investigations (see, for example [S76, F78, S80]) mostly aimed at establishing theories for the approach to thermal equilibrium, and all the subsequent investigations pursued special physical questions; none of them, however, dealt with the general mathematical structure. And this is now the theme of this paper. We give an analysis of the mathematical properties. It is analogous to the functional analysis of unitary groups of evolution, which was presented in the early time of quantum mechanics. We continue the study on semigroups of completely positive maps, acting on states for systems with finite dimensional Hilbert spaces, which we began in [BNT08].

While Schrödinger equations are studied in Hilbert space spanned by the pure-state vectors, the Lindblad equations will be studied on two levels: One is the linear space of density matrices, the other is the underlying Hilbert space. A relation between these two levels is established. In this relation we find a clear distinction between different processes: Decay, Dissipation and Dephasing. Characterizing these processes involves a structuring of the underlying Hilbert space \( \mathcal{H} \). Each Lindblad generator is related to a special decomposition of \( \mathcal{H} \) into mutually orthogonal subspaces. This decomposition can be seen as a generalization of the spectral decomposition of Schrödinger operators. Orthogonality of the subspaces holds in spite of non-hermiticity of the defining operators.

We do not discuss “decoherence”. The meaning of this concept in physics is connected with the transition from quantumness to classicality, [J03, Z03a, Z03b], the mathematics of it requires the definition of a preferred basis. “Dephasing” is a similar concept, and we define what it means precisely in a mathematical fashion.

Our starting point, the connection with the earlier studies, is the result of [GKS76, L76]:

1 PROPOSITION. Generators of semigroups: Every generator of a semigroup of completely positive trace preserving maps \( T^t : \rho(s) \mapsto \rho(s + t) \) for \( t \geq 0 \) on the set of finite dimensional density matrices \( \rho \), can be written in the form

\[
\dot{\rho} = D(\rho) = -i[H, \rho] + \sum_\alpha D_{h_\alpha}(\rho)
\]

where \( H = H^\dagger \) is a hamiltonian. Transition operators \( h_\alpha \) define the irreversible parts, the simple generators:

\[
D_{h}(\rho) = hph^\dagger - \frac{1}{2}(h^\dagger h\rho + \rho h^\dagger h)
\]
This representation of $\mathcal{D}$ is known as “diagonal”. For our purposes it is optimal. All what is needed to characterize the processes can be extracted from the set of the operators $\{H, h_\alpha, h_\alpha^\dagger\}$, and from the algebra of operators which commute with each element of this set. (A remark on the notation: We use here $h_\alpha$ with lower index, instead of the use of an upper index, which has been employed in our first paper [BNT08].) It is well known, [AF01, BP02], that the division of $\mathcal{D}$ into a sum of several simple generators, attribution of $\{H, h_\alpha\}$ to $\mathcal{D}$, is not unique. Different sets of operators $h_\alpha$ can be attributed to a given generator $\mathcal{D}$, some actions can be shifted from the Hamiltonian $H$ to the transfer operators $h_\alpha$ or in the other way. There exist attempts to favor special ways, like demanding $\text{Tr}[h_\alpha] = 0$, but we make here, in this paper, no restriction.

The mathematical analysis is of course related to special questions concerning physics:

- Existence and characterization of stationary states.
- Geometry of the paths and characterizing Hilbert space subspaces.
- Aspects of symmetries.
- Perturbation.

Identifying stationary states is closely related to identifying special subspaces of the Hilbert space. The summary of our investigations on the first two of these themes can be stated as follows:

2 THEOREM. Structuring of the Hilbert space:

1. Decay: The Hilbert space can be represented in a unique way as a direct sum of two orthogonal subspaces, $\mathcal{H} = P_0\mathcal{H} \oplus P_0^\perp\mathcal{H}$, where $P_0^\perp\mathcal{H}$ is the maximal decaying subspace, i.e.

$$\forall \rho \in \mathcal{S} : \lim_{t \to \infty} P_0^\perp T^t(\rho)P_0^\perp = 0,$$

and $P_0\mathcal{H}$ contains no decaying sub-subspace.

$$\forall Q \leq P_0 \exists \rho \in \mathcal{S} : \limsup_{t \to \infty} \text{Tr}[Q T^t(\rho)Q] \neq 0.$$  

2. Dephasing: The collecting subspace $P_0\mathcal{H}$ can further be divided in a unique way by splitting $P_0 = \sum_k Q_{0,k}$, $Q_{0,k} \cdot Q_{0,\ell} = \delta_{k\ell} Q_{0,k}$, into minimal subspaces $Q_{0,k}\mathcal{H}$ with relative dephasing

$$\forall \rho(0), \forall k, \ell, \ k \neq \ell : \lim_{t \to \infty} Q_{0,k}\rho(t)Q_{0,\ell} = 0,$$

and the property, that the time evolution of each block $Q_{0,k}\rho(t)Q_{0,\ell}$ is independent of the other blocks.
3. Asymptotics: Each subspace $Q_{0,k}\mathcal{H}$ can be represented as $Q_{0,k}\mathcal{H} = \mathbb{C}^{n(k)} \otimes \mathcal{H}_{00,k}$, so that the time evolution at large times is described in block form, with Hamiltonians $H_{0,k} \otimes \mathbb{I}$ acting on $Q_{0,k}\mathcal{H}$ and with unique density matrices acting on $\rho_k$ on $\mathcal{H}_{00,k}$ inside each block:

$$\forall \rho(0) \exists \{\lambda_k, R_k\} : \lim_{t \to \infty} |\rho(t) - \bigoplus_k \lambda_k e^{-iH_{0,k}t} R_k e^{iH_{0,k}t} \otimes \rho_k| = 0,$$

where each $R_k$ is a positive matrix with trace one acting on $\mathbb{C}^{n(k)}$, and $0 \leq \lambda_k \leq 1$, $\sum_k \lambda_k = 1$.

The set of stationary states is given by the set of density matrices

$$\bigoplus_k \lambda_k R_k \otimes \rho_k,$$

where $[H_{0,k}, R_k] = 0$, and $\rho_k$ again unique for each $k$.

4. Dissipation: The limiting density matrices inside the minimal blocks are of maximal rank: $\text{rank}(\rho_k) = \text{dim}(\mathcal{H}_{00,k})$.

5. Cascades with basins: The decaying subspace $P_{0}^\perp \mathcal{H}$ can further be divided by splitting it as a cascade with “basins” $P_{k,\ell} \mathcal{H}$, $k \geq 1$, all mutually orthogonal, $P_{k,\ell} \cdot P_{j,m} = \delta_{kj} \delta_{m\ell} P_{k,\ell}$, and arranged in levels $P_{k} \mathcal{H} = \bigoplus_{\ell} P_{k,\ell} \mathcal{H}$. So $P_{0}^\perp = \sum_{k \geq 1} P_{k,\ell}$, and the time evolution is like a “flow”, out of each basin into the “lower” levels including collecting basins $P_{0,k} \mathcal{H} \in P_{0}^\perp \mathcal{H}$, where each $Q_{0,\ell} \mathcal{H}$ mentioned in the item above contains $n(\ell)$ basins:

$$\text{for } t > 0 : \quad P_{j} \cdot T^{t}(P_{k,\ell} \rho P_{k,\ell}) \cdot P_{j} \neq 0 \text{ if } j < k,$$

$$\text{and } P_{j} \cdot T^{t}(P_{k,\ell} \rho P_{k,\ell}) \cdot P_{j} = 0 \text{ if } j > k,$$

$$P_{k,m} \cdot T^{t}(P_{k,\ell} \rho P_{k,\ell}) \cdot P_{k,m} = 0 \text{ if } m \neq \ell.$$

In special cases there is a possibility of unitarily reshuffling some basins, defining other basins $\tilde{P}_{k,\ell} = U \cdot P_{k,\ell} \cdot U^\dagger$. The number of minimal basins is unique, as are their dimensions.

All projectors $P_k, P_{k,\ell}$ and $Q_{0,k}$ are orthogonal projectors.

Perpendicular to the structuring into levels there are dissections into “enclosures” with a division of density matrices into blocks with mutually independent evolutions.

3 THEOREM. Enclosures and blocks: If there exists a set of mutually orthogonal projectors $Q_m$, each commuting with $\mathcal{H}$ and every $h_\alpha$, then the basins $P_{k,\ell} \mathcal{H}$ can be chosen in such a way that each subspace $Q_m \mathcal{H}$ is a direct sum of basins. We call such a $Q_m \mathcal{H}$ an enclosure. The time evolution of any density matrix $\rho$ splits into mutually independent evolutions of blocks $Q_m \rho Q_\ell$:

$$T^{t}(Q_m \rho Q_\ell) = Q_m T^{t}(\rho) Q_\ell.$$
As an example for the structuring, already well known, may serve the Grotrian diagram for Helium. The dissection into enclosures is there the distinction between ortho- and para-helium. The lowest energy levels represent the collecting subspace. Energy levels with angular momentum zero are basins, those with higher angular momenta \( \vec{J} \) can further be split into basins, by diagonalizing a component of \( \vec{J} \). The choice of the axis of the component is not unique, but different kinds of splitting, with different basins, are unitarily equivalent.

The \( Q_m \) are projectors onto the “enclosures” – for details see Section 3.1. In the Heisenberg picture, with time evolution of the observables, they are invariant. This is a conventional symmetry. But there are cases when other symmetries turn up in non conventional way, without conserved observables; cases when conservation of observables appears, without an overall symmetry; and the conserved observables need not commute with all the \( \hbar \alpha \), and they need not form an algebra. “Degeneracy”, the existence of several stationary states, may be connected with occurrence of symmetries or not. The well known connection of symmetries with conservation laws, fundamental for Langrangian mechanics and quantum mechanics, is here no longer valid. All this is discussed in Section 4. It is also an important aspect when perturbations are studied, as is done in Section 6.

2 Basic properties of the superoperators

2.1 The Hilbert Schmidt space of matrices

Consider the set \( S \) of states as the set of \( n \times n \) density matrices \( \rho_{ij} = \langle i | \rho | j \rangle \), with \( n = \dim(\mathcal{H}) \), \( \{|i\rangle\} \) some basis of \( \mathcal{H} \), and consider it as embedded into the linear space of complex \( n \times n \) matrices. Now consider the general mathematical wisdom on linear differential equations with constant coefficients, acting in a finite dimensional space, which is here the space spanned by \( n \times n \) dimensional matrices:

To each superoperator \( \mathcal{D} \) there exist eigenmatrices \( \sigma \), proper or generalized:

\[
\mathcal{D}(\sigma) = \lambda \sigma \quad \text{or} \quad (\mathcal{D} - \lambda)^n(\sigma) = 0,
\]

So one has the special time evolutions: \( \sigma(t) = e^{\lambda t} \cdot Polynomial(t, \mathcal{D}^n(\sigma)) \). The general solution to \( \dot{\sigma} = \mathcal{D}(\sigma) \) is a linear combination of these special solutions. But the eigenmatrices to our \( \mathcal{D} \) may be not self-adjoint, although \( \mathcal{D} \) preserves self-adjointness. Instead, there may be pairs of eigenvalues \( \{\lambda, \lambda^*\} \) with pairs of eigenmatrices \( \{\sigma, \sigma^\dagger\} \). That’s the reason why we do not stay in the space of self-adjoint matrices.

We endow this linear space with the Hilbert Schmidt norm \( ||\sigma|| = (\text{Tr}[\sigma^\dagger \sigma])^{1/2} \) and the inner product \( \langle \langle \sigma | \rho \rangle \rangle = \text{Tr}[\sigma^\dagger \rho] \). The HS-superspace of operators enables the definition of the adjoint of a super-operator. The adjoint super-operator \( \mathcal{D}^\dagger \) generates the evolution of observables in the Heisenberg-picture, \( \dot{F} = \mathcal{D}^\dagger (F) \). It acts as
\[ \mathcal{D}^\dagger(F) = i[H, F] + \sum_{\alpha} \mathcal{D}^\dagger_{h_{\alpha}}(F), \]  
(12)

where
\[ \mathcal{D}^\dagger_{h}(F) = h^\dagger F h - \frac{1}{2}(h^\dagger h F + F h^\dagger h). \]  
(13)

We remark that self-adjointness of \( \mathcal{D} \) is rather exceptional. See Section 5 for examples.

### 2.2 Evolutions inside the set of states

\( T^t \) preserves self-adjointness, trace-norm and, for \( t \geq 0 \), positivity; it maps \( \mathbf{S} \mapsto \mathbf{S} \), the set of normed positive matrices into itself.

The dissipative character of the evolution is reflected in the general changes of eigenvalues \( r(t) \) of \( \rho(t) \). Zero eigenvalues may become positive, positive eigenvalues may increase or decrease. But the decrease is bounded from below:

#### 4 PROPOSITION. A differential inequality: If \( r(t) \) is a non-negative eigenvalue of \( \rho(t) \), its change in time is bounded from below as

\[ \dot{r}(t) \geq -\left( \sum_{\alpha} \|h_{\alpha}\|^2 \right) r(t). \]  
(14)

**Proof.** Differentiating the eigenvalue equation \( \rho(t)\psi(t) = r(t)\psi(t) \), and using \( \langle \psi|\psi \rangle = 0 \) gives

\[
\dot{r}(t) = \langle \psi|\dot{\rho}|\psi \rangle = \sum_{\alpha} \langle \psi|(h_{\alpha}\rho h_{\alpha}^\dagger - \frac{1}{2}(h_{\alpha}^\dagger h_{\alpha}\rho + \rho h_{\alpha}^\dagger h_{\alpha}))|\psi \rangle \\
\geq -\sum_{\alpha} \frac{1}{2} \langle \psi|(h_{\alpha}^\dagger h_{\alpha}\rho + \rho h_{\alpha}^\dagger h_{\alpha})|\psi \rangle \\
= -\sum_{\alpha} \langle \psi|h_{\alpha}^\dagger h_{\alpha}|\psi \rangle r(t). 
\]  
(15)

Here, the finite dimension of the Hilbert space is essential: it implies that the number of simple generators which are necessary to form some given \( \mathcal{D} \) is bounded by \( \dim^2(\mathcal{H}) - 1 \). All the sums over \( \alpha \) are finite.

An immediate consequence is that the positive eigenvalues \( r_j \) of the density operator obey the inequality

\[ r_j(t) \geq \exp \left( -\sum_{\alpha} \|h_{\alpha}\|^2 t \right) \cdot r_j(0) > 0. \]

The rank of \( \rho \) cannot decrease in finite time, purification can only occur in the limit \( t \to \infty \).

Moreover, we can affirm quite generally
5 **PROPOSITION. Existence of stationary states.** For each $D$ there exists at least one stationary state $\rho \in S$.

*Proof.* The density matrices in $B(H)$ form a compact convex set $S$. We know that the semigroup maps $S$ into itself, and we may consider the map $\rho(0) \mapsto \rho(\Delta t)$ for some fixed time interval $\Delta t$. By the Brouwer fixed point theorem, there exists at least one fixed point, $\rho(\Delta t) = \rho(0)$. So there exists the stationary state

$$\bar{\rho} = \frac{1}{\Delta t} \int_0^{\Delta t} \rho(t) dt,$$

satisfying $D(\bar{\rho}) = 0$. \qed

This abstract statement is needed in 3.3. In the following it is then elucidated by more concrete formulas.

3 **Processes and Structuring**

3.1 **Definitions**

**Enclosure** is not a standard concept, although it appears in Hamiltonian dynamics as a consequence of the conservation of energy. We define “enclosure” as a subspace $K$ with the property that for all $\rho(0)$ the expectation of the orthogonal projector onto this subspace, $Tr[PK\rho(t)]$, is constant in time. In the Schrödinger picture this means that the system can neither leave nor enter this subspace. In the Heisenberg picture it means invariance of the observable $PK$ in the course of time. Each $PK$ is one of the projectors $Q_m$ appearing in Theorem 3. So, enclosure is a case of “conservation”, but in the special way that it is an orthogonal projector which is the conserved observable. In Section 5 it is shown that there are cases where invariant observables exist, but no invariant projector, different to the evolutions under Hamiltonian dynamics.

Our analysis of general Lindblad equations $T^t$ starts with investigating an eventual enclosing of the system in the subspaces $K$ with conserved projectors $PK$. Then, *inside* the enclosures, decay and/or dissipation occurs. The phase-relations between the enclosures may eventually show a dephasing. Dephasing is a typical quantum-effect, but it is also related to *decoherence*, the transition from quantum to classical appearance, [Z03a, Z03b]. We introduce precise mathematical definitions of these physical events of irreversible processes:

6 **DEFINITION. Irreversible events:**

**Decay** occurs, if there exists a decaying subspace $K$; i.e.

$$\forall \rho(0) : \text{Tr}[PK\rho(t)] \to 0.$$  (17)
**Dissipation** occurs, if there exists a subspace $\mathcal{K}$ such that

$$\forall \rho(0) = P_\mathcal{K}\rho(0) : \text{rank}(\rho(t)) \to \text{dim}(\mathcal{K}).$$  \hspace{1cm} (18)

**Dephasing** occurs for phase relations, in the case of conservations of $P_\mathcal{K}$ and $P_\mathcal{L}$, with $\mathcal{K} \perp \mathcal{L}$, if

$$\forall \rho(0) : P_\mathcal{K}\rho(t) P_\mathcal{L} \to 0.$$  \hspace{1cm} (19)

All arrows are meant as limit of $t$ to infinity.

Enclosures, decaying and - their complement - collecting subspaces are the analogue to the energy eigenspaces in Hamiltonian dynamics.

### 3.2 Enclosures and conservation of projectors

In the Heisenberg picture “conservation” of $P$ means $\mathcal{D}(P) = 0$. In the Schrödinger picture this means that the state of the system can neither leave nor enter the subspace $P\mathcal{H}$.

**7 LEMMA.** Conservation of $P = P^\perp = P^2$ is possible if and only if this observable commutes both with the Hamiltonian $H$ and with all the transfer operators $h_\alpha$; in other words, iff it is an element of the von Neumann algebra $\{H, h_\alpha\}'$.

**Proof.** We consider $\rho = P\rho P$, systems in the subspace $P\mathcal{H}$. Leaving this subspace is possible for the system, if

$$P^\perp \mathcal{D}(\rho) P^\perp = \sum_\alpha P^\perp h_\alpha \rho h_\alpha^\dagger P^\perp \neq 0 \iff \exists \alpha : P^\perp h_\alpha P \neq 0.$$  

Entering the same subspace means leaving the orthogonal complement and is thus possible for the system if $\exists \alpha : P h_\alpha P^\perp \neq 0$. That neither leaving nor entering is possible implies therefore $\forall \alpha : P h_\alpha P = h_\alpha P$. Under this restriction one gets for the evolution of the phase relations, the off diagonal part, $P \rho(t) P^\perp$:

$$P \mathcal{D}(\rho) P^\perp = -iP[H, \rho] P^\perp = iP \rho P H P^\perp,$$

which vanishes for every $\rho = P\rho P$ iff $H$ commutes with $P$. And its vanishing is necessary, otherwise $P^\perp \mathcal{D}^2(\rho) P^\perp = P^\perp H P \rho PH P^\perp > 0$. \hfill $\square$

Assume that both $P_i$ and $P_j$ are conserved projectors. The commutations imply $\mathcal{D}(P_i \rho) = P_i \mathcal{D}(\rho)$ and $\mathcal{D}(\rho P_j) = \mathcal{D}(\rho) P_j$. So the evolution of each block $P_i \rho P_j$ is independent from the other parts of $\rho$. The set of conserved projectors generates the algebra $\mathcal{N} = \{H, h_\alpha\}'$. A maximal abelian subalgebra of the von Neumann algebra $\mathcal{N}$ gives a set of mutually orthogonal minimal conserved projectors $P_i$, a decomposition of $\mathcal{H}$, and a decomposition of $\rho$ as a block-matrix, with split evolutions of the blocks. Such a decomposition is unique if $\mathcal{N}$ is an
abelian algebra. If this algebra is not abelian, then different decompositions are possible. They are related by unitary transformations with $U \in \mathcal{N}$ which leave $\mathcal{D}$ invariant. This follows from the classification of finite dimensional von Neumann algebras, stated for example in [J03b].

Conserved projectors come along with an algebra of conserved observables, $\{H, h_{\alpha}\}'$, and, moreover, with a dynamical symmetry:

$$ T^t(U^\dagger \rho U) = U^\dagger \rho(t) U \quad \text{if} \quad U \in \{H, h_{\alpha}\}' . $$  \hfill (20)

But there are cases of conserved observables which do not form an algebra; and, on the other hand, conservation of projectors is not necessary for occurrence of a dynamical symmetry. This is discussed in Section 4.

A maximal set of mutually orthogonal minimal enclosing subspaces enables now the discussion of processes inside.

### 3.3 Collection into subspaces, dissipation inside

In studying the time evolution in [BNT08] we identified the condition “laziness”. A subspace $\mathcal{P} \mathcal{H}$ is “lazy” if there is no flow out of it in first order of time. This condition appears here as the first part for characterizing collecting spaces, the orthogonal complements of decaying spaces.

8 DEFINITION. Lazy subspaces: $\mathcal{P} \mathcal{H}$ is a lazy subspace if

$$ \forall \rho = P \rho P : \quad \frac{d}{dt} \text{Tr}[P \rho(t) P] = 0 \quad \text{at} \quad t = 0 . $$  \hfill (21)

9 LEMMA. $\mathcal{P} \mathcal{H}$ is a lazy subspace $\iff$

$$ \forall \alpha : \quad h_{\alpha} \cdot P = P \cdot h_{\alpha} \cdot P . $$  \hfill (22)

Proof. Since $T^t$ conserves the trace, one has $\text{Tr}[\mathcal{P} \dot{\rho}(t) \mathcal{P}] = -\text{Tr}[\mathcal{P}^\perp \dot{\rho}(t) \mathcal{P}^\perp]$. At time $t = 0$ this is to be expressed with the generating operator as

$$ -\text{Tr}[\mathcal{P}^\perp \mathcal{D}(\rho) \mathcal{P}^\perp] = - \sum_{\alpha} \text{Tr}[\mathcal{P}^\perp h_{\alpha} \rho h_{\alpha}^\dagger \mathcal{P}^\perp] = - \sum_{\alpha} \text{Tr}[(\mathcal{P}^\perp \cdot h_{\alpha} \cdot P) \rho (P \cdot h_{\alpha}^\dagger \cdot \mathcal{P}^\perp)] . $$

So, if $\mathcal{P}^\perp \cdot h_{\alpha} \cdot P$ does not vanish for each $\alpha$, there exists a state $\rho = P \rho P$, such that $\text{Tr}[\mathcal{P} \dot{\rho}(t) \mathcal{P}] \neq 0$. This inequality holds in particular for any $\rho = P \rho P$ which has $\text{rank}(\rho) = \text{Tr}[\mathcal{P}]$.  \hfill \Box

10 DEFINITION. Collecting subspaces

$\mathcal{P} \mathcal{H}$ is a collecting subspace, if

$$ \forall t > 0 \quad \forall \rho : \quad T^t(P \rho P) = PT^t(P \rho P) . $$  \hfill (23)
11 LEMMA. $P\mathcal{H}$ is a collecting subspace, iff it is a lazy subspace also satisfying

$$P(iH - \frac{1}{2} \sum_{\alpha} h_\alpha^\dagger h_\alpha)P^\perp = 0.$$  \hfill (24)

Proof. Since the Lindblad equation is of first order in taking the time-derivative, the equation

$$\forall \rho : \quad D(P\rho P) = PD(P\rho P)P$$  \hfill (25)

is equivalent to (23). The laziness condition is equivalent to disabling, as is shown above, appearance of diagonal blocks $P^\perp \rho P^\perp$:

$$\forall \rho : \quad P^\perp D(P\rho P)P^\perp = 0,$$

and (24) is equivalent to disabling appearance of off-diagonal blocks of $\rho(t)$,

$$\forall \rho : \quad PD(P\rho P)P^\perp = 0.$$

This is seen just by inserting the Lindblad equation (1):

$$PD(P\rho P)P^\perp = P\rho P(iH - \frac{1}{2} \sum_{\alpha} h_\alpha^\dagger h_\alpha)P^\perp$$  \hfill (26)

For the system there is no way (back) out of $P\mathcal{H}$. But some part of $\rho$ may enter this subspace, if $P$ does not commute with the Hamiltonian $H$ and all the transition operators $h_\alpha$. In general $P \cdot \mathcal{T}(P^\perp \rho P^\perp) \cdot P \neq 0$. If $P$ does not commute with all the $h_\alpha$, this occurs for some $\rho$, for example $\rho = P^\perp / \text{Tr}[P^\perp]$, in first order in time, i.e. $P \cdot D(P^\perp \rho P^\perp) \cdot P \neq 0$. If $P$ does commute with all the $h_\alpha$, but does not commute with the Hamiltonian, entering the subspace occurs in second order: $P \cdot D^2(P^\perp \rho P^\perp) \cdot P \neq 0$; see the proof of Lemma 7.

If $P^\perp$ projects onto a decaying subspace, then $P\mathcal{H}$ is collecting. This can be seen in the structuring of $\mathcal{H}$, performed in Section 3.4. For the simple $\mathcal{D} = \mathcal{D}_h$ there was no reason for such a detailed investigation: A collecting subspace is there just a proper eigenspace for the eigenvalue zero of $h$. But here the structures are in general richer. There is still an evolution going on in the collecting subspace. It can be seen as an evolution for density matrices defined on the reduced Hilbert space $P\mathcal{H}$.

$$\rho = P\rho P \quad \Rightarrow \quad \dot{\rho} = \dot{\mathcal{D}}(\rho) \quad \text{defined with} \{ \ddot{H} = PHP, \ddot{h}_\alpha = P h_\alpha P \}.$$  \hfill (27)

If the collecting subspace is minimal, i.e. does not contain a smaller collecting subspace, the evolution inside is of the same type as in a minimal enclosure; it is dissipative, unless $\dim(P\mathcal{H}) = 1$.

The relations between characteristic subspaces and stationary states can now be analyzed, first for one direction of implications:
12 LEMMA. If \( \rho \) is a stationary state and \( P \) is the projector onto its range, then \( PH \) is a collecting subspace or an enclosure.

Proof. This proof is done by revisiting the proofs for Lemma 9 and Lemma 11, with the special extra situation \( \rho = P\rho P \) and \( \text{range}(\rho) = PH \).

The other relations state that inside a minimal enclosure, with no decay inside, there holds

13 THEOREM. Uniqueness of minimal stationary states. Let \( \mathcal{K} \) be a subspace which is a minimal enclosure or a minimal collecting subspace, containing no smaller enclosure or collecting subspace. Then there exists one and only one stationary state supported by \( \mathcal{K} \). Its density matrix has maximal rank, \( \text{rank}(\rho) = \dim(\mathcal{K}) \).

Proof. There has to exist a stationary state \( \rho \), see Theorem 5, applied to the restricted evolution inside of \( \mathcal{K} \). Linearity of the Lindblad equation implies that the stationary states, if more than one, form a line or a (hyper)plane. This would include elements in the boundary of \( \mathbf{S} \) with lower rank; but there are no invariant states at the boundary of \( \mathbf{S}(\mathcal{K}) \): Any invariant state has a collecting subspace as its range (Lemma 12), and it is assumed that \( \mathcal{K} \) contains no smaller collecting subspace.

3.4 Cascades of decay

In the cases where decay occurs it might be helpful or necessary to define a structure of the Hilbert space, analogous to the “energy levels” in Grotrian diagrams of atoms.

We define the lowest level of the Cascade, \( P_0\mathcal{H} \), as the smallest subspace of \( \mathcal{H} \) which contains all minimal collecting subspaces. The strategy of further procedure to give more structuring is as follows: Consider the complement of the lowest level, with the same evolution, except the flow out into the lowest level. This evolution, acting on \( P_0^\perp\mathcal{H} \), is generated by \( \mathcal{D}_1 \), formed with \( P_0^\perp HP_0^\perp \) and \( \{P_0^\perp h_\alpha P_0^\perp\} \). Then the first higher level \( P_1\mathcal{H} \) of the Cascade for \( \mathcal{D} \) is defined as the lowest level of the Cascade for \( \mathcal{D}_1 \). Iteration gives a series of levels \( P_0\mathcal{H}, P_1\mathcal{H}, \ldots P_n\mathcal{H}, \) until \( \bigoplus_i P_i\mathcal{H} = \mathcal{H} \). Basins \( P_{i,j}\mathcal{H} \) in the level \( P_i\mathcal{H} \) are defined as minimal collecting subspaces for the reduced evolutions with no flow out into the lower levels.

The precise details:

14 PROPOSITION. Decomposition into basins. Each level \( P_i\mathcal{H} \) can be decomposed into a direct sum of mutually orthogonal basins \( P_{i,j}\mathcal{H} \). This decomposition is either unique or unique up to some unitary equivalence, which reshuffles basins among a set of partners with equivalent dissipations inside.
Proof. Construct the decomposition inductively and begin with the lowest level. Consider a sum of minimal collecting mutually orthogonal subspaces $P_H := \bigoplus_j P_{0,j} \mathcal{H}$, and consider another minimal collecting subspace $P_\sigma \mathcal{H}$, not contained in $P_H$. $P_\sigma \mathcal{H}$ contains a unique stationary state $\sigma$ (Theorem 13). If $P_\sigma \mathcal{H}$ is orthogonal to $P_H$, define $P_{0,j+1} = P_\sigma$ and proceed inductively. If $P_\sigma \mathcal{H}$ is neither orthogonal to $P_H$ nor contained in it, consider the Lindblad equation acting on $\sigma$, with restriction to the subspace spanned by adding $P_H$ and $P_\sigma \mathcal{H}$, in the block-matrix representation as stated in the Appendix. There $R = P_\sigma P_{\perp} \neq 0$, $Q = P_\sigma P_{\perp} \neq 0$, since $P_\sigma \mathcal{H}$ is not subspace of $P_H$ but not orthogonal to it. Also $S = P_\sigma P_{\perp} \neq 0$, by positivity of $\sigma$, and $\text{rank}(S) = \text{rank}(P + P_\sigma) - \text{rank}(P)$, since $P_\sigma \mathcal{H}$ is minimal. The condition “laziness” on $P_H$, as stated in Lemma 9, is $C_\alpha = P_{\perp} h_\alpha P = 0$. Together with the condition stated in equation (24), which is $iPHP_{\perp} - \frac{1}{2} \sum_\alpha A^\dagger_\alpha B_\alpha = 0$ (here with the notations as defined in Appendix, Section 8), one gets for $\dot{\sigma} = 0$ the part

$$
\dot{S} = -\frac{1}{2} \sum_\alpha (B^\dagger_\alpha B_\alpha S + S B^\dagger_\alpha B_\alpha) = 0.
$$

This implies $\forall \alpha : B_\alpha = PH_\alpha P_{\perp} = 0$, where $h_\alpha$ is the restricted transfer operator, restricted to the subspace $P_H + P_\sigma \mathcal{H}$. Together with the “laziness” condition this means commutation of every restricted $h_\alpha$ with $P$, and this implies moreover, again using equation (24), that also the restricted $\dot{H}$ has to commute with $P$. It follows that $P_{0,j+1} \mathcal{H} := (P_{\perp} \mathcal{H} \oplus P_\sigma \mathcal{H}) \ominus (P_H \oplus P_\sigma \mathcal{H})$ is another collecting subspace, orthogonal to $P_H$.

Iterating this procedure, until no other collecting subspace, no other stationary state not already contained in $P_0 \mathcal{H} := P_H$ is left, gives a decomposition of the lowest level.

Now we investigate the restricted evolution of the complement of $P_0 \mathcal{H}$. Consider the subspace $P_0 \mathcal{H}$ and the processes generated by $\{P_0 \mathcal{H} P_0, P_0 h_\alpha P_0 \}$. The collecting subspaces of this evolution, with all outflow into the lowest level disabled, give the basins of the first higher level $P_1 \mathcal{H} = \bigoplus_j P_{1,j} \mathcal{H}$. Then one iterates the disabling of the outflow of the remaining subspace, constructing the higher levels with basins $P_{i,j} \mathcal{H}$ until nothing more of $\mathcal{H}$ remains.

The lowest level is the subspace spanned by all possible stationary states. So its definition is unique. By induction, the entire decomposition into levels is unique.

Now consider two different decompositions of a level, w.l.o.g. of the lowest level. This gives exactly the situation treated above, with at least one stationary state $\sigma$ with support in a basin $P_\sigma \mathcal{H}$ which is neither orthogonal to some basin $P_{i,j} \mathcal{H}$ nor contained in it. This is the case, iff there are stationary phase relations $P_{i,j} \sigma P_{k,l}$. The discussion of such cases is postponed to the following Section 3.5, using the Proposition 16.
The characterizations of basins involves a common Schur triangulation ([L69]) in block form of all the transfer operators. The blocks $P_{i,j}h_{\alpha}P_{k,\ell}$ with $i > k$ are zero. The change in time of a basin’s content $P_{i,j}$ consists of

- inflow from (several) $P_{k,\ell}\mathcal{H}$ with $k > i$, generated by $\{P_{i,j}HP_{k,\ell} + P_{k,\ell}HP_{i,j}, P_{i,j}h_{\alpha}P_{k,\ell}\}$
- outflow into (several) $P_{k,\ell}\mathcal{H}$ with $k < i$, generated by $\{P_{i,j}HP_{k,\ell} + P_{k,\ell}HP_{i,j}, P_{k,\ell}h_{\alpha}P_{i,j}\}$
- dissipation inside the basin, generated by $\{P_{i,j}HP_{i,j}, P_{i,j}h_{\alpha}P_{i,j}\}$.

If higher basins $P_{k,\ell}\mathcal{H}$ are empty, all contents of the basin $P_{i,j}\mathcal{H}$ will decay if $i \geq 1$. Dissipation leads to density matrices with full rank inside the basin; then, if $i \neq 0$, there is some outflow since $\sum_{k<i,\ell}P_{k,\ell} \cdot (i\mathcal{H} + \frac{1}{2} \sum_{\alpha} h_{\alpha} h_{\alpha}^\dagger) \cdot P_{i,j} \neq 0$.

**15 PROPOSITION. Emptying of higher levels.** Only the lowest level in the cascade carries stationary states, formed by combinations of unique states inside each single basin; eventually there may be stationary phase relations. The collection of the higher levels in the cascade,

$$K = \bigoplus_{i \geq 1, j} P_{i,j}\mathcal{H} = P_0^\perp\mathcal{H},$$

is the maximal decaying subspace. It is completely emptied in the course of the evolution.

**Proof.** There are no stationary states in $K$, by construction. So the diagonal blocks $P_{i,j}\rho P_{i,j}$ with $i \geq 1$ vanish. By preservation of positivity, the off-diagonal blocks $P_{i,j}\rho P_{k,\ell}$ and $P_{k,\ell}\rho P_{i,j}$ have to vanish also. \qed

### 3.5 Dephasing and the geometry of paths

Here we study, as $t \to \infty$, the phase relations between basins. Knowing that basins in the upper levels of the Cascade get empty, and phase relations involving one or two of the decaying basins have to vanish because of preservation of positivity, it remains to study phase relations between minimal collecting subspaces. We may restrict the system and consider only the lowest level, $P_0\mathcal{H}$, which is collecting. In this level the time evolution is identical to the evolution defined in equation (27), generated by the restricted operators. Every collecting subspace is there, in this restriction, an enclosure. So we simplify the discussion and consider a system with can be decomposed into minimal enclosures $P_i\mathcal{H}$ without decay. The time evolution of each block $P_i\rho P_j$ is independent of all the other blocks. When considering the phase relations between diagonal blocks we may therefore simplify further, and restrict the system to a space with just two basins $\mathcal{H} = P_i\mathcal{H} \oplus P_j\mathcal{H}$. 
16 PROPOSITION. **Stationary phase relations.** A stationary phase relation \( P_i \rho P_j \) exists if and only if there exists a unitary operator \( U \) commuting with \( H \) and each \( h_\alpha \) which intertwines between the two enclosures.

\[
UP_j = P_i U, \quad U^2 = \quad (28)
\]

The stationary phase relation is unique up to a constant factor.

**Proof.** Assume that such an intertwiner \( U \) exists. It creates a dynamical symmetry, \( D(U \rho U^\dagger) = U D(\rho) U^\dagger \), and it transforms the stationary density matrices of the enclosures into each other,

\[
U \rho_j U^\dagger = \rho_i. \quad (29)
\]

The commutation with the generating operators implies stationarity of phase relation blocks \( P_i \rho P_j = r_{i,j} \rho_j \) and \( P_j \rho P_i = r_{j,i} \rho_j U \).

On the other hand, assume that some stationary block \( P_i \rho P_j \) exists. Then also \( P_j \rho P_i = (P_i \rho P_j)^\dagger \) is stationary. Since the matrices for \( \rho \) in diagonal blocks are of full rank (Theorem 13), the density matrices

\[
\sigma_\lambda = \frac{1}{2}(\rho_i + \lambda P_i \rho P_j + \lambda^* P_j \rho P_i + \rho_j)
\]

are positive for \( |\lambda| \) small. Now at some critical value of \( |\lambda| \) the state \( \sigma_\lambda \) is at the boundary of \( S \), and \( \text{rank}(\sigma_\lambda) < \text{rank}(P_i) + \text{rank}(P_j) \). This implies that \( P_\lambda \mathcal{H} \), the range of \( \sigma_\lambda \), is an enclosure (Lemma 12), and that \( P_\lambda \in \{ H, h_\alpha \}' \). Since \( P_i \mathcal{H} \) and \( P_j \mathcal{H} \) are undecomposable and \( P_\lambda \) does neither commute with \( P_i \) nor with \( P_j \), the only possibility for such a situation is, that the von Neumann algebra \( \{ H, h_\alpha \}' \) is not abelian, and

\[
\{ H, h_\alpha \}' \cong \mathbb{C}^2 \otimes P_i \mathcal{H},
\]

including a unitary intertwiner \( U \) acting as in equation (29).

**Proof. Uniqueness of decomposition into basins.** Proposition 16 gives the completion for proving Proposition 14. The decomposition into basins is not unique, iff there exist stationary phase relations. And such stationary phase relations exist, iff there is a unitary equivalence as stated in Proposition 16.

The geometry of paths \( \{ \rho(t) \} \subset S \) is related to the eigenvalues of the superoperator \( \mathcal{D} \). Each path can be decomposed into a sum of at least one stationary state and paths of self-adjoint matrices which are eigenmatrices or pairwise sums of eigenmatrices of \( \mathcal{D} \).

**Special paths for self-adjoint matrices:**

- eigenvalue zero \( \Leftrightarrow \) stationary state
- imaginary eigenvalue \( \Leftrightarrow \) circular path
• negative eigenvalue $\iff$ path leading straight to zero
• complex pairs of eigenvalues with negative real part $\iff$ paths formed as
  $\gamma \sigma(t) + \gamma^* \sigma^\dagger(t)$, spiraling in to zero

General paths arise as superpositions of special paths. Geometric considerations give some implications for the eigenvalues of $D$: Decay processes lead straight to the boundary of $S$, so negative eigenvalues are involved. Eigenmatrices whose range contains some part of $P_0^\perp \mathcal{H}$ belong to eigenvalues with negative real part. The eigenmatrices whose support is in $P_0 \mathcal{H}$ can be chosen such that their supports are in single collecting basins or in blocks giving phase relations between pairs of basins. In this way one gets a complete set of eigenmatrices spanning the space of all matrices with support in $P_0 \mathcal{H}$. Since no path can leave $S$, there are no eigenvalues with positive real part. Moreover we can state

17 LEMMA. To the eigenvalue zero of $D$ there exist only proper eigenmatrices. The corresponding eigenspace is spanned by positive density matrices.

Proof. Consider the stationary eigenmatrix $\sigma$. If it is not self adjoint, then also $\sigma^\dagger$ is an eigenmatrix, as are the selfadjoint $\sigma + \sigma^\dagger$ and $i \sigma - i \sigma^\dagger$. If $\sigma$ is selfadjoint but not positive, consider it split as $\sigma = \rho_+ - \rho_-$, both parts being positive. Since $T^t$ is positivity preserving, both parts separately must be stationary. Now assume the existence of a generalized eigenmatrix $\tau_0$, with $D(\tau_0) = \sigma$, $D(\sigma) = 0$. Again one can conclude that the analog equations should hold for the adjoint matrices and for their linear combinations. One can therefore assume $\tau_0 = \tau_0^\dagger$. Integrating the evolution equation, assuming $\tau(0) = \tau_0$, gives $\tau(t) = \tau_0 + t \sigma$. Multiplying with some small $\varepsilon$ and adding some positive $\rho$ with full rank would give a path starting inside $S$ but leaving it as $t$ gets large. This is a contradiction to the preservation of positivity, so no such $\tau_0$ can exist.

The eigenvalues of $D$ lying on the imaginary axis correspond to circular paths. Such cases can appear for phase relations, for off-diagonal blocks of $\rho$.

18 THEOREM. Dephasing and non-dephasing; eigenvalues of $D$

1. For a minimal block at the diagonal, belonging to a collecting basin, $\{\sigma = P_{0,j} \sigma P_{0,j}\}$, there exists exactly one eigenmatrix to the eigenvalue zero. All other eigenvalues have negative real part.

2. For an off-diagonal block $\{\sigma = P_{0,j} \sigma P_{0,k}\}$ where there exists an intertwiner $U \rho_j = P_j U$ with $U \in \{P_0 \mathcal{H} P_0, P_0 h \rho_0 P_0\}'$, there exists exactly one eigenmatrix to the eigenvalue zero. The eigenmatrix is $U \rho_j$, where $\rho_j$ is the stationary eigenmatrix with support in $P_{0,j} \mathcal{H}$. All other eigenvalues have negative real part.
3. For an off-diagonal block \( \{\sigma = P_{0,j}\sigma P_{0,k}\} \) where there exists an intertwiner \( U \in \{P_0 H P_0 - (E_j P_{0,j} + E_j P_{0,k})^\dagger, P_0 h_a^\dagger P_0\} \), there exists exactly one eigenmatrix to an eigenvalue on the axis of imaginary numbers. The eigenvalue is \( \pm i (E_j - E_k) \), the eigenmatrix is \( U \rho_j \), where \( \rho_j \) is the stationary eigenmatrix with support in \( P_{0,j} \mathcal{H} \). All other eigenvalues have negative real part.

4. For an off-diagonal block \( \{\sigma = P_{0,j}\sigma P_{0,k}\} \) there is no intertwiner as in item 2 or 3 there exist only eigenvalues with negative real part.

Proof. The existence and uniqueness of an eigenmatrix to the eigenvalue zero in cases (1), (2), and the nonexistence in case (4) are stated and then proven in Theorem 13 and in Proposition 16; then Lemma 17 states that there are no generalized eigenspaces to this eigenvalue.

It remains to examine the existence or non-existence of other eigenvalues on the imaginary axis. The method is the same as in the proof of Theorem 17 “No circular paths” in [BNT08]. We switch between the Schrödinger and the Heisenberg picture. \( \mathcal{D}^\dagger \) has the same spectrum as \( \mathcal{D} \). Assume the existence of an eigenvalue \( \lambda = ir \) with eigenoperator \( F = P_{0,\ell} F P_{0,j} \), where \( r \in \mathbb{R} \), so \( \exp(t\mathcal{D}) F = e^{ir t} F \), and use the Kadison inequality. No further details for the time evolution are needed to deduce equation (30). We refer to [BNT08] for description of how to conclude that \( V := F/\sqrt{||F||} \) is a local isometry between \( P_{0,j} \mathcal{H} \) and \( P_{0,\ell} \mathcal{H} \) or a local unitary if \( j = \ell \):

\[
V^\dagger \cdot V = P_{0,j}, \quad V \cdot V^\dagger = P_{0,\ell}. \tag{30}
\]

Now we use the evolution equation (12) for \( V \), multiply from the left by \( V^\dagger \), and get

\[
V^\dagger \mathcal{D}^\dagger (V) = i(V^\dagger H V - P_{0,j} H) + \sum_a (V^\dagger h_a^\dagger h_a - \frac{1}{2} P_{0,j} h_a^\dagger h_a - \frac{i}{2} V^\dagger h_a^\dagger h_a V) = i r P_{0,j}. \tag{31}
\]

Since, by definition of the collecting basins, \( \{P_{0,j}, P_{0,\ell}\} \subset \{P_0 H P_0, P_0 h_a P_0\} \), we may define

\[
\hat{h}_a = P_{0,j} h_a = h_a P_{0,j}, \quad \hat{h}_a = V^\dagger h_a V, \quad \hat{H} = P_{0,j} H, \quad \hat{H} = V^\dagger \hat{H} V,
\]

and write the trace of (31) as

\[
ir \text{Tr}[\hat{H} - \hat{H}] + \sum_a (\text{Tr}[\hat{h}_a^\dagger \hat{h}_a] - \frac{1}{2} \text{Tr}[\hat{h}_a^\dagger \hat{h}_a] - \frac{i}{2} \text{Tr}[\hat{h}_a^\dagger \hat{h}_a]) = ir \dim (P_{0,j} H). \tag{32}
\]

The Cauchy-Schwarz inequality, applied to inner product \( \text{Tr}[\cdot, \cdot] \) in the H.S. space of matrices, and the inequality between geometric and arithmetic mean imply that the real part of the l.h.s. of (32) is zero iff \( \forall \alpha : \hat{h}_a = \hat{h}_a \), i.e. \( V^\dagger h_a V = P_{0,j} h_a \), \( h_a V = V h_a \). Using again (31) this implies \( H \cdot V = V \cdot (H + r) \).

There are no such imaginary eigenvalues, no circular paths, in cases 1, 2, 4. They exist only in case 3, with \( U = V + V^\dagger + \bigoplus_{k \neq j, k \neq \ell} P_{0,k} \). \( \square \)
### 3.6 Stationary states, collection of results

A collection of results stated above in this Section gives now the proofs of the main theorems.

**Proof.** For **Theorem 2 on structuring of the Hilbert space:** In Proposition 14 on decomposition into basins the decomposition of $\mathcal{H}$ into $P_0^\dagger \mathcal{H}$, defined as the lowest level of the cascade of decay, and its complement $P_0^\dagger \mathcal{H}$ is performed. This gives part (1) of the Theorem.

The same Proposition 14 gives also the further splitting of $P_0$ into basins. These are minimal collecting subspaces. If there exist stationary phase relations between two basins, and only then, can this splitting be varied, using another pair of basins which are related to the former two and among themselves by unitary transformations. This is stated and then proven in Proposition 16 on stationary phase relations. Their existence comes with a form of equivalence between basins, given by the unitary transformation, which has to commute with every $P_0 h_\alpha P_0$ and with $P_0^\dagger H P_0$.

There may exist a generalized form of equivalence between basins, where the unitary transformation commutes again with every $P_0 h_\alpha P_0$ but then with $P_0^\dagger H P_0 - E_j P_0^\dagger + E_\ell P_0$ instead of $P_0^\dagger H P_0$. Such a generalized form of equivalence implies again equivalence of the unique stationary states located in the basins; and it enables the occurrence of undamped oscillating phase relations. Collecting equivalent and generalized equivalent basins into subspaces $Q_{0,k}^\dagger \mathcal{H}$ gives the larger, unique, part of splitting $P_0 = \sum_k Q_{0,k}$. There are neither stationary nor undamped oscillating phase relations $Q_{0,k}^\dagger \mathcal{H} Q_{0,\ell}$. This is stated and then proven in Theorem 18 on dephasing and non-dephasing.

The recollection of basins can be seen as $Q_{0,k}^\dagger \mathcal{H} = C^n(k) \otimes \mathcal{H}_{00,k}$. Uniqueness of the stationary states $\rho_k$ in the minimal collecting subspaces, everyone equivalent to $\mathcal{H}_{00,k}$, is stated in Theorem 13. So parts (2) (3) and (4) of the Theorem are proven.

The splitting of $P_0^\dagger \mathcal{H}$ into higher levels and basins is stated in Proposition 16, proving the last part of the Theorem.

A subspace $\mathcal{K}$ is the minimal support of a minimal stationary state iff it fulfills the conditions

- “Laziness” – the projector $P_\mathcal{K}$ fulfills equation (22),
- “No creation of off-diagonal elements” – $P_\mathcal{K}$ fulfills equation (24),
- “Minimality” – $\mathcal{K}$ contains no smaller subspace fulfilling the first two conditions.

The characteristic equations have been found also by B. Kraus et al., [K08a, K08b], as determining “dark states”, which are pure stationary states.

A collection of results gives also the
Proof. For Theorem 3 on enclosures and blocks: The relations of projectors $Q_m \in \{\mathcal{H}, h_\alpha\}$ to the time evolution and to “enclosure” are analyzed in Lemma 7 and in the further discussions in Section 3.2 on “Conservation of projectors and enclosures”. Because of the enclosure, all further decompositions can be performed for the restricted evolution acting on the set of density matrices $\rho$ with support in $Q_m \mathcal{H}$. \qed

4 Invariance (conservation) and symmetry

4.1 Invariant observables

The stationary states span a subspace of the HS-space of matrices, the space of eigenmatrices to the eigenvalue zero of the superoperator $\mathcal{D}$. Its adjoint, $\mathcal{D}^\dagger$, has the same set of eigenvalues, so there is a linear set of invariant operators. It has the same dimension as the set of stationary states, and contains only proper “eigenmatrices”, in duality to Lemma 17.

Let us start a construction of an invariant observable with a nucleus, located in a collecting basin, say $P_{0,k} \mathcal{H}$. The defining condition (23) “nothing goes out” is equivalent to the dual condition “no observable comes in”

$$P_{0,k} T^t(F) P_{0,k} = P_{0,k} T^t(P_{0,k} F P_{0,k}) P_{0,k}. \quad (33)$$

Inside the basin is full dissipation with only one stationary state. There is therefore only one invariant observable inside the basin (up to constant factors). It is the projector $P_{0,k}$.

$$P_{0,k} T^t(P_{0,k}) P_{0,k} = P_{0,k}. \quad (34)$$

The Heisenberg evolution goes backward. It lets, applied to the projector $P_{0,k}$, the diagonal block $P_{0,k}$ unchanged, it creates an extension into the decaying subspace $P_{0}^\perp$, and also phase relations between this subspace and the basin where it started. It lets the whole block $P_{0} \mathcal{H}$ unchanged, since this collecting subspace is spanned by collecting basins, each one showing the “no observable comes in” condition (33). The evolved observable therefore stays HS-orthogonal to eventually existing undamped oscillating phase relations, and we can define

$$A_{0,k} := \lim_{t \to \infty} T^t(P_{0,k}) \quad (35)$$

as an invariant observable. It is positive, since preserving positivity goes over from $T^t$ to $T^\dagger$ by duality.

For a maximal set of mutually orthogonal collecting basins one obtains

$$\sum_k P_{0,k} = P_0 \quad \iff \quad \sum_k A_{0,k} = . \quad (36)$$
There may be a still larger set of linearly independent invariant observables. Let us represent subspaces with \( n \) equivalent collecting basins \( P_{0,\ell} \) allowing for stationary phase relations, but not undamped oscillating ones, as

\[
\bigoplus_{\ell} P_{0,\ell} = \mathbb{C}^n \otimes \mathcal{H}_{00}.
\]

For any \( n \times n \) matrix \( M \) the observable \( M \otimes \) can be extended, analogously to the procedure (35), to an invariant observable. \( n^2 \) of these observables can be linearly independent, \( n \) of them are as constructed in (35).

An algebraic analysis of the set of stationary states, of invariant observables and their relations to the generating operators relies on first cutting off the decaying subspace \( P_0^\perp \mathcal{H} \). The restricted time evolution, generated by \( \mathcal{D} \) defined with Hamiltonian and transfer operators \( \{P_0 HP_0, P_0 h_\alpha P_0\} \), is identical to the full one for \( \rho \) with support on \( P_0 \mathcal{H} \). For this restricted system, on \( P_0 \mathcal{H} \), the concepts of “basin” and “enclosure” are identical. Projectors onto basins are elements of the von Neumann algebra \( \mathcal{N}_0 = \{P_0 HP_0, P_0 h_\alpha P_0\}' \), which is the set of invariant observables for the restricted time evolution. The set of extended invariant observables in the large system is thus related to the commuting von Neumann algebra of the restricted system; restricted to the non-decaying level \( P_0 \mathcal{H} \).

\[
\exists F : \mathcal{D}^\dagger(F) = \dot{F} = 0 \iff P_0 FP_0 \in \mathcal{N}_0 \quad (37)
\]

So, in systems without decay, the invariant observables do form an algebra. But, in systems with decay, the extension (35) does in general destroy this property. See examples in Section 5.

### 4.2 Symmetries

A dynamical symmetry is defined by existence of unitary or anti-unitary operators \( V \), such that

\[
\forall t, \forall \rho \quad T^\dagger(V\rho V^{-1}) = V\rho(t)V^{-1}.
\]

This is equivalent to

\[
\forall \rho \quad \mathcal{D}(V\rho V^{-1}) = V\mathcal{D}(\rho)V^{-1}.
\]

Sometimes an appearance of a dynamical symmetry is in connection with existence of an algebra of conserved observables, a connection well known in Hamiltonian dynamics. If \( U \in \{H, h_\alpha\}' \) then \( T^\dagger(U\rho U^\dagger) = U\rho(t)U^\dagger \). Another way how a symmetry may be guaranteed is, that \( VHV^{-1} = H \) and the set \( \{Vh_\alpha V^{-1}\} \) equals the set \( \{h_\alpha\} \). But it may also be hidden, not immediately to be observed in the \( h_\alpha \). See examples in Section 5. For continuous groups, and for \( \mathcal{D} \) with a finite number of transfer operators \( h_\alpha \), the symmetry is necessarily not completely represented by invariance of the set of generators.
The maximal symmetry is invariance under all unitary and anti-unitary transformations. There is only one ray of generators compatible with this symmetry, in the center of the cone of generators, see [BNT08]. It consists of \( \{ D | D(\rho) = \lambda \cdot (\omega - \rho) \} \), where \( \omega = \sqrt{\dim(H)} \) is the completely mixed state. There are several ways of choosing the set \( \{ h_\alpha \} \) to form such a special \( D \). Examples are again in Section 5.

While, in Hamiltonian dynamics, the appearances of

- dynamical symmetry
- invariant (conserved) observables
- algebra formed by the invariant observables

must come together, these relations are not strict in irreversible dynamics. Here we observe cases of

- dynamical symmetry without invariant observables,
- Invariance of observables without a symmetry,
- Invariant observables which do not form an algebra.

For each of these cases we present examples.

There remains a relation between a dynamical symmetry and the set of stationary states. If \( V \) is a symmetry operator as in (38), then the set of stationary states obeys the symmetry for stationarity

\[
\{ \rho | \text{stationary} \} = \{ V \rho V^{-1} | \rho \text{ stationary} \}.
\]

But this is, in general, a one-way relation. The symmetry for stationarity (40) may be valid, without (38) being true. This comes, again, because of the restriction of the set of invariant states to the subspace \( P_0 H \).

5 Examples

The density matrices are representations of states in some basis of the usual type, employing a complete orthogonal set of basis vectors. Mostly we use \( D \) given as a sum of two simple generators, with transition operators \( \{ h_+, h_- \} \). Zeroes as matrix elements are represented with dots. Lower indices on \( C \) indicate the role of a subspace as a level or as a basin. Matrix elements of the density operator are denoted as \( r_{i,j} \). Representations using a tensor product are in accordance to its use in part (3) of Theorem 2. “No invariant observable” means, precisely: Only the constants are invariant.
5.0.1 Dissipation

Hilbert space $\mathcal{H} = \mathbb{C}^2$

$$h_+ = \begin{pmatrix} 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & 1 \end{pmatrix} \quad h_- = \begin{pmatrix} 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & 1 \end{pmatrix}$$

There is a unique stationary state. It is $\omega = \frac{\pi}{2}$, a fact which cannot occur with simple generators. To represent the dynamics we use Pauli matrices, so $h_\pm = \sigma_\pm = (\sigma_x + \sigma_y)/2$, and $\mathcal{D}(\sigma_\pm) = -\sigma_\pm$, $\mathcal{D}(\sigma_z) = -2\sigma_z$. The dynamics is symmetric under rotation around the z-axis, under reflection $\sigma_z \leftrightarrow -\sigma_z$ and under complex conjugation $C$. The symmetry for stationarity is maximal, but there is no invariant observable.

5.0.2 Decay, two collecting basins, no stationary phase relations

$\mathcal{H} = \mathbb{C}^3 = \mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2} \oplus \mathbb{C}_{1}$

$$h_+ = \begin{pmatrix} 1 & 1 \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix} \quad h_- = \begin{pmatrix} 1 & -1 \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix}$$

Extremal stationary states are $1 \oplus 0 \oplus 0$ and $0 \oplus 1 \oplus 0$. The symmetry for stationarity includes exchange of these extremal states and complex conjugation. The whole system is only one enclosure. Invariant observables are diagonal matrices with components $(1, 0, 2/3)$, or $(0, 1, 1/3)$, and their linear combinations. They do not form an algebra. No dynamical symmetry (but $C$).

5.0.3 Decay, two collecting basins with stationary phase relations

$\mathcal{H} = \mathbb{C}^3 = \mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2} \oplus \mathbb{C}_{1}$

$$h_+ = \begin{pmatrix} 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & 1 \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix} \quad h_- = \begin{pmatrix} 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & 1 \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix}$$

Stationary states: Any $2 \times 2$ density matrix with support on $\mathbb{C}_{0,1} \oplus \mathbb{C}_{0,2}$. Symmetry for stationarity: $U \oplus$ with any unitary $U$, and $C$. No dynamical symmetry but $C$. Invariant observables: Linear combinations of $A_1 \ldots A_4$,

$$A_1 = \begin{pmatrix} 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & 1/3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdot \cdot \cdot & 1 \\ \cdot \cdot \cdot & 1/3 \end{pmatrix}, \quad A_3 = A_4^\dagger = \begin{pmatrix} \cdot \cdot \cdot & 1 \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{pmatrix}.$$

They do not form an algebra.
5.0.4 Decay, basins with dissipation inside

\[ H = C^4 = C_0^2 \oplus C_1^2 \]

\[ h_\pm = \begin{pmatrix}
\cdots & \pm 1 & 1 & \\
1 & \cdots & \\
\cdots & \cdots & \pm 1 & \\
\cdots & 1 & \\
\end{pmatrix} \]

The restricted evolution on \( C_0^2 \) is almost the same as in 5.0.1, only with twice the speed. One stationary state, \( \omega \oplus 0 \), no invariant observables. Symmetry for stationarity: all the unitary and anti unitary transformations acting on \( C_0^2 \). No invariant observables.

5.0.5 Dephasing of two enclosures

\[ H = C^4 = C_{0,1}^2 \oplus C_{0,2}^2 \]

\[ h_+ = \begin{pmatrix}
\cdots & 1 & \\
\cdots & \cdots & \\
\cdots & \cdots & 1 & \\
\cdots & \cdots & \\
\end{pmatrix} \quad h_- = \begin{pmatrix}
\cdots & \cdots & 1 & \\
1 & \cdots & \\
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix} \]

The restricted evolutions on each \( C_{0,j}^2 \) is the same as in 5.0.1. Acting on the off diagonal blocks, these evolutions are “out of phase”, destroying every phase relation. Such an off diagonal block evolves according to

\[ \frac{d}{dt} \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} = \begin{pmatrix} r_{2,4} - r_{1,3} & -r_{1,4} \\ -r_{2,3} & -r_{1,3} - r_{2,4} \end{pmatrix} \]

Two extremal stationary states \( \omega \oplus 0 \) and \( 0 \oplus \omega \), two invariant observables, \( \oplus 0 \) and \( 0 \oplus \), very rich symmetry.

5.0.6 Undamped oscillating phase relation

\[ H = C^4 = C_{0,1}^2 \oplus C_{0,2}^2 \cong C^2 \otimes C_{0,0}^2 \]

\[ h_+ = \begin{pmatrix}
\cdots & 1 & \\
\cdots & \cdots & \\
\cdots & \cdots & 1 & \\
\cdots & \cdots & \\
\end{pmatrix} \quad h_- = \begin{pmatrix}
\cdots & \cdots & 1 & \\
1 & \cdots & \\
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix} \]

The evolutions of states of the enclosures \( C_{0,j}^2 \) are again as in 5.0.1. Here they are in phase when acting on the off diagonal blocks, leaving a special phase relation undamped. But the Hamiltonian creates an oscillation.

\[ \frac{d}{dt} \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} = -i \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} + \begin{pmatrix} r_{2,4} - r_{1,3} & -r_{1,4} \\ -r_{2,3} & r_{1,3} - r_{2,4} \end{pmatrix}. \]
So \( T^t \begin{pmatrix} r_{1,3} & r_{1,4} \\ r_{2,3} & r_{2,4} \end{pmatrix} \sim_{t \to \infty} e^{-it} \frac{r_{1,3} + r_{2,4}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

The equation for full dynamics can be written as
\[
\frac{d}{dt}(M \otimes \rho) = -i[H, M] \otimes \rho + M \otimes D_{0,0}(\rho),
\]
with \( D_{0,0} \) as in 5.0.1. Stationary states are the same as in 5.0.5. They can be represented as tensor products of diagonal \( 2 \times 2 \) matrices \( M \) with \( \omega \). Symmetry of stationary states includes reflection, exchanging \( \rho_1 \) and \( \rho_2 \). Invariant observables are given by all linear combinations of the two projectors onto \( \mathbb{C}^2_0 \). They form an algebra.

5.0.7 Stationary phase relation

The transition operators \( h_\pm \) are as above in 5.0.6, but \( D \) is given without the Hamiltonian. Very rich symmetry, much more symmetry operations than in 5.0.5. There are more stationary states than in 5.0.5, they can be represented as tensor products \( M \otimes \omega \) with any \( 2 \times 2 \) matrix \( M > 0, \operatorname{Tr}[M] = 1 \). Invariant observables are \( A \otimes \); they form an algebra.

5.0.8 Cascade of decay

\( \mathcal{H} = \mathbb{C}^4 = \mathbb{C}_0 \oplus \mathbb{C}_{1,1} \oplus \mathbb{C}_{1,2} \oplus \mathbb{C}_2 \)

\[
h_\pm = \begin{pmatrix}
\cdots & 1 & \pm 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 1 
\end{pmatrix}
\]

There are two lines of flow: \( |1, 1\rangle \langle 1, 1| \to |0\rangle \langle 0| \) and \( |2\rangle \langle 2| \to |1, 2\rangle \langle 1, 2| \to |0\rangle \langle 0|. \) Both lines have the same end. The differential equations are \( \dot{r}_{1,1} = r_{2,2} + 2r_{3,3} \), \( \dot{r}_{2,2} = -r_{2,2}, \dot{r}_{3,3} = -2r_{3,3} + r_{4,4}, \dot{r}_{4,4} = -r_{4,4} \) for the diagonal matrix elements. The off-diagonals are just exponentially decaying, with one accompanying part of the flow, \( \dot{r}_{1,3} = -2r_{1,3} + r_{2,4} \) and the same for the adjoint. No invariant observables.

5.0.9 Maximal symmetric evolution

\( \mathcal{H} = \mathbb{C}^n; \) The evolution is

\[
\dot{\rho} = \dim(\mathcal{H}) \cdot (\omega - \rho).
\]

\( D \) can be defined with \( H = 0, \{ h_{i,j} = |i\rangle \langle j| \} \) for some basis \( \{|i\rangle\} \). There is a unique stationary state, it is \( \omega = / \dim(\mathcal{H}) \). Symmetry under every unitary