Explicit Doubly–Hermitian Metrics

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EXPLICIT DOUBLY-HERMITIAN METRICS

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ABSTRACT. We construct explicit examples of 4-dimensional Riemannian metrics which admit precisely two independent hermitian structures with the same orientation. It is shown that metrics of this type exist on four-dimensional tori.

1. Introduction

If $M$ is an oriented Riemannian 4-manifold then it is well known that the (local) existence of a hermitian structure $I$ compatible with the orientation imposes constraints on the self-dual part $W_+$ of the Weyl tensor of $M$ (see for example [Sal81]). More precisely, such hermitian structures must necessarily be roots of $W_+$ thought of as a real quartic polynomial ($W_+$ lives in the spin bundle $S^4V_+$ while almost hermitian structures are sections of the twistor space $\mathbb{P}V_+$). In particular, if $M$ admits three independent hermitian structures with the same orientation then it is half conformally flat, that is, $W_+ = 0$. The same happens if there are two anti-commuting hermitian structures on $M$ since their composition is again hermitian. But $W_+ = 0$ implies that the twistor space $\mathbb{P}V_+$ is integrable [AHS78] so locally $M$ admits a whole family of hermitian structures parametrised by the germs of holomorphic functions in two variables (hermitian structures on $M$ are parametrised by complex submanifolds of the twistor space, transversal to the fibres). If, on the other hand, $W_+$ is non-trivial then there are at most two (modulo sign) candidates for hermitian structures compatible with the orientation on $M$. A natural question arises: does the existence of two compatible hermitian structures imply that $W_+ = 0$? In other words, we are asking whether there exist 4-dimensional metrics which admit locally precisely two independent compatible hermitian structures. The aim of this note is to present a simple proof of existence of such metrics. We shall exhibit metrics of this type on four-dimensional tori.

Conventions: For brevity, we shall say that two almost hermitian structures $I$, $J$ are independent if $I_x \neq \pm J_x$ for all $x \in M$; an almost hermitian structure is said to be compatible if it has the same orientation as $M$. Hermitian means ‘almost hermitian and integrable’. Riemannian 4-manifolds which admit precisely two compatible hermitian structures will be called doubly-hermitian. Finally, we emphasize again that questions dealt with in this paper are of local nature.

2. Deforming the flat metric

Let us consider the complex manifold $M = \mathbb{C}^2$ with complex coordinates $(z_1, z_2)$, orientation $-dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ and with the metric

$$g_f = dz_1 \circ d\bar{z}_1 + f dz_2 \circ d\bar{z}_2$$

where $f : \mathbb{C} \to \mathbb{R}_{>0}$ is a smooth strictly positive function.
We let \( I \) denote the standard complex structure on \( M \). If \( J \) is a compatible almost hermitian structure such that \( I \) and \( J \) are independent then there is a unique smooth complex valued function \( u : \mathbb{C}^2 \to \mathbb{C} \) such that the space of \((1,0)\)-forms with respect to \( J \) is spanned by

\[
\alpha_1 = \bar{u}dz_1 - d\bar{z}_2 \quad \text{and} \quad \alpha_2 = \bar{u}dz_2 + d\bar{z}_1.
\]

(The complex conjugate over \( u \) is introduced purely for convenience.) The reader familiar with the twistor language will guess that \( \bar{u} \) is in fact a section of the twistor space \( M \times \mathbb{C}P^1 \); the complex structure \( I \) corresponds to the section \( M \times \{\infty\} \) (see Appendix).

There is no need, however, to resort to twistor spaces here: in order to convince oneself that equations (2) do define an almost hermitian structure note that the annihilator of the \(1\)-forms \( \tilde{\alpha}_1 = u\bar{d}z_1 - dz_2 \) and \( \tilde{\alpha}_2 = u\bar{d}z_2 + dz_1 \) is spanned by the vectors

\[
Z_1 = -u\bar{d}\bar{\alpha}_1 + \bar{\alpha}_2, \quad Z_2 = u\bar{d}\bar{\alpha}_2 + \bar{\alpha}_1.
\]

where \( \bar{\alpha}_k = \bar{\alpha}_k \) and \( \tilde{\alpha}_k = \bar{\alpha}_k \). Since \( g_j(Z_i, Z_j) = 0 \) for \( i,j \in \{1,2\} \), the \(2\)-dimensional distribution \( \langle Z_1, Z_2 \rangle_{\mathbb{C}} \) (the complex linear span of \( Z_1 \) and \( Z_2 \)) is isotropic in \( T^C M \). Then

\[
T^C M = \langle Z_1, Z_2 \rangle_{\mathbb{C}} \oplus \langle \bar{Z}_1, \bar{Z}_2 \rangle_{\mathbb{C}}
\]

and \( J \) is determined by the property that the above is a decomposition into \( J \)-eigenspaces with eigenvalues \(+i\) and \(-i\). Moreover, \( I \) and \( J \) have the same orientation since \( J \) is a smooth deformation of \(-I\). This can be also verified by a direct calculation:

\[
\alpha_1 \land \bar{\alpha}_1 \land \alpha_2 \land \bar{\alpha}_2 = (1 + fu \bar{u}) dz_1 \land d\bar{z}_2 \land dz_2 \land d\bar{z}_2.
\]

With the vectors \( Z_1 \) and \( Z_2 \) at hand it is easy to establish when \( J \) is hermitian: this happens precisely when the distribution \( \langle Z_1, Z_2 \rangle_{\mathbb{C}} \) is integrable, that is,

\[
[Z_1, Z_2] \in \langle Z_1, Z_2 \rangle_{\mathbb{C}}.
\]

We have:

\[
[Z_1, Z_2] = \left[ -u\bar{d}\bar{\alpha}_1 + \bar{\alpha}_2, u\bar{d}\bar{\alpha}_2 + \bar{\alpha}_1 \right]
= (-u\bar{d}\bar{\alpha}_1 + \bar{\alpha}_2) \bar{\alpha}_2 + (u\bar{d}\bar{\alpha}_2 + \bar{\alpha}_1) \bar{\alpha}_1.
\]

The above vector field has no components in \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) so it must vanish, and the condition (4) is equivalent to the following system of equations:

\[
\begin{cases}
uf \bar{d}_1 u - \bar{\alpha}_2 u = 0, \\
u \bar{d}_2 (uf) + \bar{\alpha}_1 (uf) = 0.
\end{cases}
\]

Obviously, \( u = 0 \) is a solution of (5) for any \( f \). This corresponds to the complex structure \(-I\). The equations \( \ldots \) do not see \( I \) since it comes from the section at infinity. On the other hand if \( u \) (corresponding to \( J \)) is a solution of the equations then so is \(-\frac{1}{fu}\) as it gives rise to \(-J\). According to what was said in the introduction, we have:

**Lemma 1.** The metric \( g_I \) defined in (1) is doubly-hermitian if and only if (for fixed \( f \)) the equation (5) has precisely two nonzero solutions in \( u \).
3. Examples of doubly-hermitian metrics

Since we are only interested in explicit examples, we will solve the system (5) with the condition \( u = c = \text{const}, c \neq 0 \). In this case it reduces to a single equation

\[
(c \partial_b + \bar{\partial}_1) f = 0
\]

which has solutions of the form

\[
f(z_1, z_2) = h(z_2 - c z_1)
\]

where \( h : \mathbb{C} \to \mathbb{R}_{\geq 0} \) is an arbitrary smooth function. To simplify things further, set \( c = -1 \) (there is no loss of generality as one can get rid of nonzero \( c \) by redefining the coordinates \( z_i \) and the function \( f \). In this case

\[
\partial_2 f = \bar{\partial}_1 f,
\]

and

\[
f(z_1, z_2) = h(z_2 + z_1).
\]

Let us find now how many solutions system (5) can have. To do this note that it can be rewritten simply as \( [Z_1, Z_2] = 0, \) or

\[
\begin{cases}
Z_1 u = 0, \\
Z_2 (uf) = 0.
\end{cases}
\]

We will get more equations by differentiating the above:

\[ Z_1 Z_2 (uf) = 0. \]

Since \( [Z_1, Z_2] = 0 \), this is the same as

\[ Z_2 Z_1 (uf) = 0. \]

Now expand this, remembering that \( Z_1 u = 0 \):

\[ 0 = Z_2 Z_1 (uf) = Z_2 (u Z_1 f) = Z_2 u Z_1 f + u Z_2 Z_1 f. \]

The idea is to get rid of the partial derivatives of \( u \). From (7) we have \( Z_2 u = -\frac{1}{f} u Z_2 f \), so (8) gives

\[ u (f Z_2 Z_1 f - Z_2 f Z_1 f) = 0. \]

According to formula (3)

\[ Z_1 f = -u f f_1 + f_2, \quad Z_2 f = u f_2 + f_1, \]

and, since \( Z_2 (uf) = 0 \),

\[
Z_2 Z_1 f = Z_2 (u f f_1 + f_2) = -u f Z_2 f_1 + Z_2 f_2
\]

\[ = -u^2 f f_{12} - u f f_{11} + u f_{22} + f_{12}. \]

The above can be used to expand (9); in order to make the result more compact, let \( p_{ij} \) denote the following determinant:

\[
p_{ij} := \det \begin{pmatrix} f & f_i \\ f_j & f_{ij} \end{pmatrix}
\]

where \( i, j \in \{1, 2, 1, 2\} \). Then (9) can be written as

\[ u (-u^2 f p_{12} + u (p_{22} - f p_{11}) + p_{12}) = 0. \]
Let us consider functions $f$ which satisfy (6). In this case $f_2 = f_1 = h_z$ and $f_3 = f_1 = h_z$, so all $p_{ij}$ which appear in (10) are equal to the determinant $hh_{z\bar{z}} - h_z h_{z\bar{z}}$. This can be written simply as $h^2 \partial \bar{\partial} \ln h$. As a result (10) gives

$$u(1 - uf)(1 + u) \partial \bar{\partial} \ln h = 0. \quad (11)$$

It is clear that, unless $\partial \bar{\partial} \ln h = 0$, the system (5) can only have solutions $u = 0$, $u = -1$, or $u = \frac{1}{f}$. We know that these actually are solutions of (5) with $f$ given by (6) so, from Lemma 1, we get the following simple

**Proposition 2.** Let $h : \mathbb{C} \to \mathbb{R}_{\geq 0}$ be such that it does not satisfy the equation

$$\partial \bar{\partial} \ln h = 0, \quad (12)$$

and let $f(z_1, z_2) = h(z_2 + z_1)$. Then the metric $g_f$ defined by (1) is doubly-hermitian.

Note that $\partial \bar{\partial} \ln h = 0$ (i.e. $\ln h$ is a harmonic function) iff $W_+ = 0$; for the metrics of type (1) this means that also $W_- = 0$ so the solutions of (12) give conformally flat metrics.

**Example 3.** Consider $h(z) = 2 + \cos(z + \bar{z})$. Then (12) evaluates to $-2 \cos(z + \bar{z}) - 1$ so the corresponding metric $g_f$ is doubly-hermitian. Moreover, both $g_f$ and the complex structures $u = 0$ and $u = -1$ are periodic so they define a doubly-hermitian structure on the 4-dimensional torus $\mathbb{C}^2/2\pi\mathbb{Z}^2$.

**Remark 4.** Rather than looking for the obstructions to integrability of (5) one could calculate the $W_+$ part of the Weyl tensor directly from the metric $g_f$. We chose a different approach in order to avoid tedious computations. In fact it is known that the obstruction to integrability of an almost hermitian structure on a 4-manifold does give $W_+$ (see [Nur93, Ch.3]), so we did compute $W_+$ after all. (The polynomial (10) is cubic rather than quartic since we chose the coordinates on the twistor space in such a way that one of the roots of $W_+$ lies at infinity.)

**Remark 5.** If $W_+$ has double roots then there is at most one (modulo sign) hermitian structure compatible with the orientation. One can get explicit examples of such ‘one-hermitian’ metrics by choosing the function $f$ so that $p_{12} = 0$ and the polynomial (10) does not vanish. For example set $f = z_1 + \bar{z}_1$. This gives a metric which admits one independent compatible hermitian structure in each orientation.

Another class of examples can be obtained from the following well-known result (cf. [Der83, Prop.2] and the references therein): For a Kählerian 4-manifold the tensor $W_+$ (thought of as an endomorphism of $\Lambda^2_+ T^* M$) has multiple eigenvalues (this means that $W_+ \in S^4 V_+$ has multiple roots). For example consider $\mathbb{CP}^2$ with the Fubini-Study metric and the orientation compatible with the standard Kähler structure. It follows that the standard complex structure on $\mathbb{CP}^2$ is the unique (up to sign) hermitian structure compatible with the Fubini-Study metric. (Of course the Fubini-Study metric, being antiselfdual, admits locally many hermitian structures with opposite orientation.)

For recent results concerning existence of complex structures on symmetric spaces see [BGMR93] and [Gau94].

The condition $\# \text{spec} \Lambda_+ \leq 2$ (i.e. $W_+$ has multiple roots) was extensively studied in [Der83]. For example it is shown that if $M$ is Einstein (or, more generally, the divergence $\delta W_+$ vanishes) then the condition is equivalent to $M$ being locally

conformally Kählerian. Interestingly, for Einstein manifolds the existence of a compatible hermitian structure already implies that $W_+$ has multiple roots, see [PB83] and [Nur93, Ch.3].

4. Appendix: parametrizing hermitian structures

Consider the vector space $W = \mathbb{R}^4$ with the standard metric $dx_1^2 + \ldots + dx_4^2$ and with orientation $dx_1 \wedge \ldots \wedge dx_4$ and denote by $Z(W)$ the set of all hermitian structures on $W$ compatible with the orientation. There are many equivalent descriptions of $Z(W)$, for example one can identify $W$ with quaternions $\mathbb{H}$ and then $Z(W)$ is the 2-sphere of unit quaternions. Another well known way is to use the spinor formalism. In what follows we use the notation from [Sal91]. Let us write $(W^*)^C = V_+ \oplus V_-$ where $V_+$ are spin bundles with real symplectic forms $\eta_{\pm}$ and quaternionic structures $j_{\pm}$ (so $j_{\pm}$ are anti-$\mathbb{C}$-linear, and $(j_{\pm})^2 = -1$). The metric $g$ can be expressed in terms of $\eta_{\pm}$ while the complex conjugation on $(W^*)^C$ is equal to $j_+ \oplus j_-$. Choosing a hermitian structure $I$ on $W$ is equivalent to a choice of a spinor $v \in V_+$ as this gives the decomposition

$$(W^*)^C = (v)_C \oplus V_- + (\bar{v})_C \oplus V_- = \Lambda^{(1,0)}_v \oplus \Lambda^{(0,1)}_v$$

into $(1,0)$ and $(0,1)$-forms. Here $(v)_C$ denotes the complex vector space spanned by $v$ and $\bar{v}$ stands for $j_+ v$ (similarly, for $w \in V_-$ we will write $\bar{w}$ to denote $j_-(w)$). To be more explicit, we can write (with a choice of sections $v$, $w$ such that $\eta_+(v, \bar{v}) = \eta_-(w, \bar{w}) = 1$):

$$\Lambda^{(1,0)}_v = \langle dz_1, dz_2 \rangle_C, \quad \Lambda^{(0,1)}_v = \langle d\bar{z}_1, d\bar{z}_2 \rangle_C$$

with

$$dz_1 = v \otimes w, \quad dz_2 = \bar{v} \otimes \bar{w}, \quad d\bar{z}_1 = \bar{v} \otimes \bar{w}, \quad d\bar{z}_2 = -v \otimes w.$$ 

Since $Z(W) = \mathbb{P}(V_+)$ we can introduce an affine coordinate $c$ on $Z(W)$ in such a way that $I$ is the point at infinity. Then every hermitian structure $J \neq I$ compatible with the orientation corresponds to the spinor

$$v' = [c : 1] = cv + \bar{v}$$

with unique $c \in \mathbb{C}$. Note that $c = 0$ corresponds to the complex structure $-I$. In general, $-J$ corresponds to $j_+(v')$ so it is given by the complex number $-\frac{1}{c}$. Finally, the $(1,0)$-forms for $J$ can be written as

$$\zeta_1 = (cv + \bar{v}) \otimes w = cdz_1 - d\bar{z}_2,$$

$$\zeta_2 = (cv + \bar{v}) \otimes \bar{w} = cdz_2 + d\bar{z}_1.$$ 

The above can be applied to the metric (1), we just have to normalize $dz_2$ and $d\bar{z}_2$ — this involves rescaling them by $f^+$. We see that the $(1,0)$ forms for $J$ are spanned by

$$f^+ \zeta_1 = c f^+ dz_1 - d\bar{z}_2, \quad \zeta_2 = c f^+ dz_2 + d\bar{z}_1.$$ 

and one gets (2) simply by changing the notation (take $\bar{u} = cf^+ \bar{v}$).
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References


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