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Vienna, Preprint ESI 1836 (2006)

September 1, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

THE RESIDUE CURRENT OF A CODIMENSION THREE COMPLETE INTERSECTION

HÅKAN SAMUELSSON

Abstract. Let f_1, f_2, and f_3 be holomorphic functions on a complex manifold and assume that the common zero set of the f_j has maximal codimension. We prove that the iterated Mellin transform of the residue integral has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^3$. We prove also that the natural regularization of the residue current converges unrestrictedly.
MSC: 32A27; 32C30

1. Introduction

Let X be an n-dimensional complex manifold and $f = (f_1, \dots, f_m)$ a holomorphic mapping $X \to \mathbb{C}^m$ such that the common zero set $V_f = \{f = 0\}$ has codimension m. In [8] Coleff and Herrera were able to associate a certain $(0, n)$-current to f, which has proven to be a good notion of a multi variable residue of f. They defined their current R^f, the Coleff-Herrera residue current, as follows. For a test form $\varphi \in \mathscr{D}_{n,n-m}(X)$, consider the residue integral

$$I_f^{\varphi}(\epsilon) = \int_{T_\epsilon} \frac{\varphi}{f_1 \cdots f_m},$$

where T_ϵ is the tube $\{|f_1|^2 = \epsilon_1, \dots, |f_m|^2 = \epsilon_m\}$. Coleff and Herrera proved that the limit of the residue integral as ϵ tends to zero along a so called admissible path exists and defines the action of a $(0, m)$-current on the test form φ. (The case $m = 1$ is due to Herrera and Lieberman, [11].) The limit along an admissible path here means that ϵ tends to zero along a path in the first orthant such that $\epsilon_j / \epsilon_{j+1}^k$ tends to zero for all $k \in \mathbb{N}$ and $j = 1, \dots, m-1$. The Coleff-Herrera residue current has many desirable properties. For instance, it is supported on V_f, it has the standard extension property, which more or less means that it has no mass concentrated on the singular parts of V_f, and it satisfies the duality property that a holomorphic function h on X annihilates it if and only if h belongs to the ideal generated by f. The duality property is due to Dickenstein-Sessa, [9], and Passare, [13], independently. It is natural to ask if the restriction to limits along admissible paths is necessary. It actually is and the first example showing this was found by Passare and Tsikh, [16]. Björk later realized that this

Author supported by a Post Doctoral Fellowship from the Swedish Research Council. Part of the work was also done when the author was visiting the ESI in Vienna Oct-Dec 2005, then supported by a SVeFUM Post Doctoral Fellowship, and he wishes to thank the organizers of the ESI project 'Complex Analysis, Operator Theory and Applications to Mathematical Physics' for their hospitality.

indeed is the typical case, [6]; see also Pavlova, [18]. The Coleff-Herrera definition is in this sense quite unstable and one could try to look for more stable ones. One step in this direction was taken by Passare in [14] where he introduced the following regularized version of the residue integral. Let χ_j be smooth function on $[0,\infty]$ taking the value 0 at 0 and 1 at ∞. (Actually, Passare considered functions identically 0 close to 0 and identically 1 close to ∞.) The regularized residue integral is then the volume integral

$$\int \frac{\bar\partial\chi_1(|f_1|^2/\epsilon_1)\wedge\cdots\wedge\bar\partial\chi_m(|f_m|^2/\epsilon_m)}{f_1\cdots f_m}\wedge\varphi. \tag{1}$$

Note that in the (not allowed) case when all the χ_j are the characteristic function of $[1,\infty]$ we get back the residue integral. It follows from Coleff's and Herrera's result that the limit of (1) along admissible paths exists and equals the limit along admissible paths of the residue integral but Passare proves a more general result. In fact, he proves that for almost all parabolic paths, $\epsilon(\delta)=(\delta^{a_1},\ldots,\delta^{a_m})$, the limit of (1) exists. Here "almost all" means that one has to impose finitely many linear conditions $\sum a_jb_j\neq 0$ in order to asure convergence. In the special case when $m=2$ the author was able to prove that (1) actually depends Hölder continuously on ϵ in the closed first quarter, [19], [20]. I this paper we generalize this result to the codimension three case and we prove

Theorem 1. *Let f_1, f_2, and f_3 be holomorphic functions on a complex manifold X of dimension n and assume that the common zero set of the f_j has maximal codimension. Let also χ_1, χ_2, and χ_3 be smooth functions on $[0,\infty]$ taking the value 0 at 0 and 1 at ∞ and denote $\chi_j(|f_j|^2/\epsilon_j)$ by χ_j^ϵ. Then, for any test form $\varphi\in\mathscr{D}_{n,n-2}(X)$, the integral*

$$\int \frac{\chi_1^\epsilon\bar\partial\chi_2^\epsilon\wedge\bar\partial\chi_3^\epsilon}{f_1\cdot f_2\cdot f_3}\wedge\varphi$$

depends Hölder continuously on ϵ in the closed first octant.

The value at the origin is the current $PR^2[1/f]$ (acting on φ) in Passare's notation from [14] and it is a $\bar\partial$-potential to the Coleff-Herrera residue current. That (1) converges unrestrictedly in the case $m=3$ thus follows from Theorem 1 by applying it to $\bar\partial$-exact test forms.

Another approach to the Coleff-Herrera residue current based on analytic continuation of currents, a technique with roots in the works of Gelfand and Shilov, [10], and Atiyah, [4], has been considered by several authors, e.g., Yger, [22], Berenstein, Gay, and Yger, [5], and Passare and Tsikh, [15], [16]. Computing the Mellin transform of the residue integral one obtains

$$\int \frac{\bar\partial|f_1|^{2\lambda_1}\wedge\cdots\wedge\bar\partial|f_m|^{2\lambda_m}}{f_1\cdots f_m}\wedge\varphi, \tag{2}$$

where $\lambda_1\ldots,\lambda_m$ are complex parameters with large real parts, see e.g. [5] or [15]. It is proved in [22] and [5] that the restriction of (2) to any complex line of the form $\lambda(t_1,\ldots,t_m)$, $t_j\in\mathbb{R}_{>0}$, can be analytically continued to a neighborhood of the origin. Moreover, the value at the origin equals the Coleff-Herrera residue $R^f.\varphi$. It is also proved in [5] that in the case when

$m=2$, (2) can be analytically continued to a neighborhood of the origin as a function of two complex variables. It is generally believed, but not yet fully proved, that this holds for arbitrary m. Our second main result confirms this conjecture for $m=3$.

Theorem 2. *Let f_1, f_2, and f_3 be holomorphic functions on a complex manifold X of dimension n, and assume that the common zero set of the f_j has maximal codimension. Then, for any test form $\varphi \in \mathscr{D}_{n,n-2}(X)$, the holomorphic function*

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto \int \frac{|f_1|^{2\lambda_1}\bar{\partial}|f_2|^{2\lambda_2} \wedge \bar{\partial}|f_3|^{2\lambda_3}}{f_1 \cdot f_2 \cdot f_3} \wedge \varphi,$$

originally defined when $\mathfrak{Re}\,\lambda_j$, $j=1,2,3$, is large enough, has a holomorphic continuation to a neighborhood of $\lambda = 0$ in $\mathbb{C}^3$ and the value at the origin equals $PR^2[1/f].\varphi$.

We will not be concerned with it in this paper but we also mention a third, and very successful way to gain stability in the definition of the Coleff-Herrera residue current introduced by Passare, Tsikh, and Yger in [17]. They use the Bochner-Martinelli kernel as a blueprint for the definition of the residue instead of the Cauchy kernel. One can view their Bochner-Martinelli type current as the limit of a certain average of the residue integral. The advatage is that this averaging process reduces the number of parameters to just one and it is then proved in [17] that the limit as this single parameter tends to zero exists. It is even true when f does not define a complete intersection! However, it is non trivial to prove that the obtained current actually equals the Coleff-Herrera current. Quite recently, Andersson put the ideas in [17] into an algebraic framework and introduced more general currents of the Cauchy-Fantappiè-Leray type, which have been useful in applications, see e.g. [1], [2], [3].

The disposition of the paper is as follows. In the next section we settle the notations for some frequently appearing objects and we discuss the main elements of the proofs of Theorems 1 and 2. In section 3 we compute an example showing that the codimension three case is different from the codimension two case. Section 4 contains some technical results about the normal crossings case. At the end of the section we also prove a combinatorial algebra type result which will enable us to use the assumption that f defines a complete intersection efficiently. In the last section, Section 5, we prove our main theorems. However, we only prove Theorem 2 in detail since the proofs are almost identical.

2. Notations and an overview of the proof

We will have to use Hironaka's theorem, [12], to resolve singularities locally. It gives us for any sufficiently small open set $U \subset X$ a complex manifold $\mathcal{X}$ and a proper holomorphic map $\pi\colon \mathcal{X} \to U$ with the properties that $\mathcal{Z}_f := \{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3 = 0\}$ has normal crossings and π restricted to $\mathcal{X} \setminus \mathcal{Z}_f$ is a biholomorphism. The varieties we will be most interested in are the varieties $V_f := \{f_1 = f_2 = f_3 = 0\}$ and $Z_f := \{f_1 f_2 f_3 = 0\}$ in X, and their total transforms, $\mathcal{V}_f := \{\pi^* f_1 = \pi^* f_2 = \pi^* f_3 = 0\}$ and

$\mathcal{Z}_f = \{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3 = 0\}$. Varieties in calligraphic letters are always varieties in the resolution manifold $\mathcal{X}$. Moreover, varieties denoted by Z (or $\mathcal{Z}$) are always varieties of codimension 1 and a holomorphic function as a subscript means the zero variety of that holomorphic function, e.g., $Z_{f_1} := \{f_1 = 0\}$ ($\mathcal{Z}_{f_1} := \{\pi^* f_1 = 0\}$). Varieties of higher codimensions are denoted by V (or $\mathcal{V}$). Occasionally we will encounter varieties for which this nomenclature is not efficiently applicable and we will then use more ad hoc notations. That the variety $\mathcal{Z}_f$ has normal crossings in $\mathcal{X}$ means that locally on $\mathcal{X}$ one can find holomorphic coordinates z such that $\pi^* f_j = z^{a_j} \tilde{f}_j$, $j = 1, 2, 3$, where the $\tilde{f}_j$ are non vanishing holomorphic functions. We will call z_k a simple factor if z_k divides precisely one of the monomials z^{a_j}. The following analytic sheaves on $\mathcal{X}$ will be referred to frequently; the sheaves of holomorphic k-forms, Ω^k, and the subsheaves of them of holomorphic k-forms vanishing on a normal crossings divisor $\mathcal{Z}$, $\mathscr{I}^k_{\mathcal{Z}}$. A holomorphic k-form, α, vanishes on a normal crossings divisor if the pullback of α (under the inclusion map) to any irreducible component of $\mathcal{Z}$ vanishes. If z are local coordinates such that $\mathcal{Z}$ is the zero set of a monomial, z^a, then α vanishes on $\mathcal{Z}$ if and only if $(dz_j/z_j) \wedge \alpha$ is a holomorphic $k+1$-form for any z_j dividing the monomial z^a. This, in turn, holds if and only if, for all $r \geq 1$, $(dz_{j_1}/z_{j_1}) \wedge \cdots \wedge (dz_{j_r}/z_{j_r}) \wedge \alpha$ is a holomorphic $k+r$-form for z_{j_l} dividing z^a.

Now, some comments to the proofs of Theorems 1 and 2. After a partition of unity we may assume that our test form φ has support in a neighborhood U such that it exists a Hironaka resolution of singularities $\pi \colon \mathcal{X} \to U$ as described above. We then pull our integral back to the resolution manifold $\mathcal{X}$ and as explained above, we find local holomorphic coordinates z on $\mathcal{X}$ such that $\pi^* f_j = z^{a_j} \tilde{f}_j$, $j = 1, 2, 3$. After a partition of unity on $\mathcal{X}$ one is then able to start computing. From a computational point of view it is of course easier if one could arrange so that the $\tilde{f}_j \equiv 1$. This is possible if the integer vectors a_j are linearly independent, see e.g. [14]. If one restricts to limits along admissible paths, as in [8], or allowed parabolic paths, as in [14], one will encounter only charts on $\mathcal{X}$ where the a_j are linearly independent. However, in the general case one will encounter also charts where the a_j are linearly dependent, so called charts of resonance. This is precisely what happens in the Passare-Tsikh example, [16]. Charts of resonance can therefore be seen as the reason for the discontinuity of the residue integral. In codimension two the author showed in [19] and [20] how the charts of resonance can be handled when one considers the regularized residue integral (1). The main tool is Proposition 11 in [20] and it can be generalized. The general version is Proposition 4 below but we have omitted the proof since it is a straightforward generalization of the proof of Proposition 11 in [20]. The presence of charts of resonance is therefore not a problem when we prove Theorems 1 and 2. The main difficulty is another problem that is harder to handle when $m \geq 3$ then when $m = 2$. On the resolution manifold $\mathcal{X}$ the functions $\pi^* f_j$ almost never define a complete intersection and for arbitrary test forms in $\mathscr{D}_{n,n-m}(\mathcal{X})$ the corresponding residue integral will in general be discontinuous. One has to use that the test forms we consider are of the special form $\pi^*\varphi$ because, in such test forms, the information that f defines a complete intersection in X is somehow coded. When $m = 2$ it is quite easy to extract this information. Actually, it follows from a degree argument, see e.g. [8], that $\pi^*\varphi$ vanishes on all components of codimension 1 of the variety $\mathcal{V}_f$, i.e., on the exceptional divisor. This vanishing is then seen to be enough to get the results in codimension two. In codimension three however, $\pi^*\varphi$ will in general not vanish on the exceptional divisor with the consequence that it is not a local problem on the resolution manifold $\mathcal{X}$ to prove Theorems 1 and 2. In the next section we give a simple example showing this. So one has to work a little more to extract the information hidden in $\pi^*\varphi$ when $m = 3$. The degree argument is still very useful though. With the aid of the slightly technical Lemma 7 it enables us to locally modify the test form $\pi^*\varphi$, without affecting the integral, so that the modified test form has good enough vanishing properties on the exceptional divisor. The process, however, produces also a global term which requires some additional attention.

3. An example

We consider an example showing that proving analyticity of the Mellin transform and continuity of the regularized residue integral are not local problems on the resolution manifold. We will look at the integral

$$\int \frac{|x_1|^{2\lambda_1}\bar{\partial}|x_2|^{2\lambda_2} \wedge \bar{\partial}|x_3|^{2\lambda_3}}{x_1x_2x_3} \wedge \varphi(x)dx \wedge d\bar{x}_1 \tag{3}$$

in $\mathbb{C}^3$, where φ is a function defined as follows. Let ϕ, φ_2 and φ_3 be smooth functions on $\mathbb{C}$ with support close to the origin but non vanishing there, and put $\varphi_1 = \partial\phi/\partial\bar{z}$. We define $\varphi(x)$ to be the function $\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)$ in $\mathbb{C}^3$. First of all, note that (3) is the Mellin transform of a residue integral by the choice of φ; we can move the $\bar{\partial}_1$ in front of ϕ to $|x_1|^{2\lambda_1}$ by an integration by parts. Secondly, (3) equals

$$\int \frac{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}}{x_1x_2x_3}\varphi_1\frac{\partial\varphi_2}{\partial\bar{x}_2}\frac{\partial\varphi_3}{\partial\bar{x}_3}dx \wedge d\bar{x}$$

after two integrations by parts, from which we see that (3) is analytic at $\lambda = 0$. Now we blow up $\mathbb{C}^3$ along the x_1-axis and study the pullback of (3) to this manifold. Let $\pi\colon \mathbb{C} \times \mathcal{B}_0\mathbb{C}^2 \to \mathbb{C}^3$ be the blow up map. In the natural coordinates z and ζ on $\mathbb{C} \times \mathcal{B}_0\mathbb{C}^2$ it looks like

$$\pi(z_1, z_2, z_3) = (z_1, z_2, z_2z_3),$$

$$\pi(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1, \zeta_2\zeta_3, \zeta_2).$$

Since φ has support close to the origin, $\pi^*\varphi$ has support close to $\pi^{-1}(0) = \{z_1 = z_2 = 0\} \cup \{\zeta_1 = \zeta_2 = 0\} \cong \mathbb{CP}^1$. Note that z_3 and ζ_3 are natural coordinates on this $\mathbb{CP}^1$ and choose a partition of unity, $\{\rho_1, \rho_2\}$ on $\mathrm{supp}(\pi^*\varphi)$ such that $\mathrm{supp}(\rho_1) \subset \{|z_3| < 2\}$ and $\mathrm{supp}(\rho_2) \subset \{|\zeta_3| < 2\}$. The pullback of (3) under π now equals

$$\int \frac{|z_1|^{2\lambda_1}\bar{\partial}|z_2|^{2\lambda_2} \wedge \bar{\partial}|z_2z_3|^{2\lambda_3}}{z_1z_2^2z_3} \wedge \rho_1(z)\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_2z_3)z_2dz \wedge d\bar{z}_1$$

$$-\int \frac{|\zeta_1|^{2\lambda_1}\bar{\partial}|\zeta_2\zeta_3|^{2\lambda_2}\wedge\bar{\partial}|\zeta_2|^{2\lambda_3}}{\zeta_1\zeta_2^2\zeta_3}\wedge\rho_2(\zeta)\varphi_1(\zeta_1)\varphi_2(\zeta_2\zeta_3)\varphi_3(\zeta_2)\zeta_2 d\zeta\wedge d\bar{\zeta}_1.$$

We know that this sum (difference) is analytic at $\lambda = 0$ but we will check that non of the terms are. We consider the first term. It is easily verified that it can be written as

$$\frac{\lambda_2}{\lambda_2+\lambda_3}\int \frac{|z_1|^{2\lambda_1}\bar{\partial}|z_2|^{2(\lambda_2+\lambda_3)}\wedge\bar{\partial}|z_3|^{2\lambda_3}}{z_1z_2z_3}\wedge\rho_1(z)\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_2z_3)dz\wedge d\bar{z}_1.$$

We denote this integral, with the coefficient $\lambda_2/(\lambda_2+\lambda_3)$ removed, by $I(\lambda)$. After two integrations by parts one sees that $I(\lambda)$ is analytic at the origin, and so $\lambda_2 I(\lambda)/(\lambda_2+\lambda_3)$ is analytic at the origin if and only if $I(\lambda)$ vanishes on the hyperplane $\lambda_2+\lambda_3 = 0$. In particular we must have that $I(0) = 0$. But $I(0)$ can be computed using Cauchy's formula, and one obtains $I(0) = -(2\pi i)^3\phi(0)\varphi_2(0)\varphi_3(0) \neq 0$. Hence, proving analyticity of the Mellin transform of the residue integral is not a local problem on the blown up manifold. The same example can also be used to see that continuity of the regularized residue integral is not a local property on the resolution manifold.

Remark 3. This example could be a little confusing. The variable z_1 just appears as a "dummy variable" in the computations above, to which nothing interesting happens. This indicates that it is not a local problem on the resolution manifold to prove analytic continuation of (2) (or an unconditional limit of (1)) already in the case $m = 2$. Actually, if one pulls (2) back to a resolution manifold and then starts computing one will encounter global problems already for $m = 2$. But analytic continuation of

$$(\lambda_1,\lambda_2)\mapsto\int\frac{|f_1|^{2\lambda_1}\bar{\partial}|f_2|^{2\lambda_2}}{f_1\cdot f_2}\wedge\varphi,\quad \varphi\in\mathscr{D}_{n,n-1}(X), \tag{4}$$

implies analytic continuation of (2) for $m = 2$, and proving analytic continuation of (4) is a local problem on the resolution manifold. Thus, the analyticity problem can always be reduced to a local problem on the resolution manifold when $m = 2$ but, and this is the point, for $m \geq 3$ this is not always possible.

4. Preliminary lemmas

The first proposition in this section is a straightforward generalization of Proposition 11 in [20] so we omit the proof.

Proposition 4. *Let ψ_j, $j = 1,\dots,m$, be strictly positive smooth functions on an open set $\Omega\subset\mathbb{C}^n$ and let $a_j = (a_{j1},\dots,a_{jn})$, $j = 1,\dots,m$, be multi-indices. Let also $\chi_j\in C^\infty([0,\infty])$ be zero at zero. Then, for any test form $\phi\in\mathscr{D}_{n,n}(\Omega)$, the integral*

$$\int\frac{\chi_1(\psi_1|z^{a_1}|^2/\epsilon_1)\cdots\chi_1(\psi_m|z^{a_m}|^2/\epsilon_m)}{z^{a_1}\cdots z^{a_m}}\wedge\phi$$

depends Hölder continuously on $\epsilon = (\epsilon_1,\dots,\epsilon_m)$ in the closed first orthant.

The following two lemmas more or less reduce the proofs of Theorems 1 and 2 to a study of the pullback of the test form to the resolution manifold with the result that the two theorems can be treated almost identically.

Lemma 5. *Let $\tilde{f}_j$, $j = 1, \dots, m$, be non vanishing holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$ and let $a_j = (a_{j1}, \dots, a_{jn})$, $j = 1, \dots, m$, be multiindices. Assume that the test form $\varphi \in \mathscr{D}_{n,n-r}(\Omega)$ has the property that $(d\bar{z}_k/\bar{z}_k)\wedge\varphi \in \mathscr{D}_{n,n-r+1}(\Omega)$ for all non simple factors z_k dividing some monomial z^{a_j} with $1 \le j \le r$. Then the integral*

$$\int \frac{\bar{\partial}|z^{a_1}\tilde{f}_1|^{2\lambda_1} \wedge \dots \wedge \bar{\partial}|z^{a_r}\tilde{f}_r|^{2\lambda_r} \cdot |z^{a_{r+1}}\tilde{f}_{r+1}|^{2\lambda_{r+1}} \cdots |z^{a_m}\tilde{f}_m|^{2\lambda_m}}{z^{a_1}\tilde{f}_1 \cdots z^{a_m}\tilde{f}_m} \wedge \varphi$$

has an analytic continuation to neighborhood of $\lambda = 0$ in $\mathbb{C}^m$.

Lemma 6. *Let $\tilde{f}_j$ and a_j, $j = 1, \dots, m$, and φ be as in Lemma 5 and let $\chi_j \in C^\infty([0, \infty])$ be zero at zero. Then the integral*

$$\int \frac{\bar{\partial}\chi_1^\epsilon \wedge \dots \wedge \bar{\partial}\chi_r^\epsilon \cdot \chi_{r+1}^\epsilon \cdots \chi_m^\epsilon}{z^{a_1}\tilde{f}_1 \cdots z^{a_m}\tilde{f}_m} \wedge \varphi,$$

where $\chi_j^\epsilon = \chi_j(|z^{a_j}\tilde{f}_j|^2/\epsilon_j)$, depends Hölder continuously on $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ in the closed first orthant.

Proof of Lemmas 5 and 6. It is well known that integrals of the form

$$\int \frac{|z^{a_1}\tilde{f}_1|^{2\lambda_1} \cdots |z^{a_m}\tilde{f}_m|^{2\lambda_m}}{z^{a_1}\tilde{f}_1 \cdots z^{a_m}\tilde{f}_m} \wedge \phi \tag{5}$$

have an analytic continuation to a neighborhood of $\lambda = 0$ in $\mathbb{C}^m$ without any assumptions on the (n, n)-test form ϕ, see e.g. [1]. Moreover, by Proposition 4 we have Hölder continuity in the first orthant for integrals like

$$\int \frac{\chi_1^\epsilon \cdots \chi_m^\epsilon}{z^{a_1}\tilde{f}_1 \cdots z^{a_m}\tilde{f}_m} \wedge \phi \tag{6}$$

for all (n, n)-test forms ϕ. Using the assumption on our test form φ we will reduce the computations of the integrals in the Lemmas 5 and 6 to sums of integrals of the forms (5) and (6) respectively. This is done in more or less the same way in both cases. We start by writing every $\bar{\partial}$ as the sum $\bar{\partial} = \bar{\partial}_1 + \dots + \bar{\partial}_n$ and splitting up the integrals into sums accordingly. We deal with an expression $\bar{\partial}_k|z^{a_j}\tilde{f}_j|^{2\lambda_j}$, respectively $\bar{\partial}_k\chi_j(|z^{a_j}\tilde{f}_j|^2/\epsilon_j)$, as follows. If z_k is a non simple factor dividing the monomial z^{a_j} we let $\bar{\partial}_k$ act, obtaining

$$\lambda_j|z_{a_j}\tilde{f}_j|^{2\lambda_j}\big(a_{jk}\frac{d\bar{z}_k}{\bar{z}_k} + \frac{\overline{\partial_k\tilde{f}_j}}{\bar{\tilde{f}}_j}\big),$$

respectively

$$\tilde{\chi}_j(|z^{a_j}\tilde{f}_j|^2/\epsilon_j)\big(a_{jk}\frac{d\bar{z}_k}{\bar{z}_k} + \frac{\overline{\partial_k\tilde{f}_j}}{\bar{\tilde{f}}_j}\big),$$

where $\tilde{\chi}_j(t) = t\chi_j'(t)$. Note that $\tilde{\chi}_j$ is zero at zero and smooth on $[0, \infty]$, since $\chi_j'(t) \in \mathcal{O}(1/t^2)$ as $t \to \infty$, and hence satisfies the properties required by Proposition 4. The assumption on our test form φ means that, for any expression $d\bar{z}_k/\bar{z}_k$ arising in this way, $(d\bar{z}_k/\bar{z}_k)\wedge\varphi$ is again a test form. More generally, it is easy to see, e.g. by making a Taylor expansion á la Lemma 6 in [20] of the cofficients of φ, that the assumption on φ implies that

$$\frac{d\bar{z}_{k_1}}{\bar{z}_{k_1}}\wedge\cdots\wedge\frac{d\bar{z}_{k_p}}{\bar{z}_{k_p}}\wedge\varphi$$

is a test form if the z_{k_l} are non simple factors each dividing some monomial z^{a_j} with $1\leq j\leq r$. Hence, all singular forms $d\bar{z}_k/\bar{z}_k$ arising from the action of $\bar{\partial}_k$, where k is such that z_k a non simple factor, can be incorporatad in the test form. On the other hand, if z_k is a simple factor dividing the monomial z^{a_j} we do not let $\bar{\partial}_k$ act on $|z^{a_j}\tilde{f}_j|^{2\lambda_j}$, respectively $\chi_j(|z^{a_j}\tilde{f}_j|^2/\epsilon_j)$. Instead we then integrate this $\bar{\partial}_k$ by parts. Since z_k is a simple factor it does not divide any monomial other then z^{a_j} and so, after the integration by parts, $\bar{\partial}_k$ will not encounter any monomial containing z_k as a factor and hence not produce the singular expression $d\bar{z}_k/\bar{z}_k$. Hence, the integrals in Lemmas 5 and 6 can be written as sums of integrals of the form (5) and (6) respectively, concluding the proof. □

The rest of this section is devoted to a proof of the following lemma. It will enable us to use the fact that we have a complete intersection on the original manifold in an efficient way when we do computations on the blown up one, where we in general do not have complete intersection.

Lemma 7. *Consider the monomials $\sigma=z_1^{a_1}\cdots z_{r-1}^{a_{r-1}}$ and $\tau=z_r\cdots z_s$ and let α be a holomorphic k-form such that $d\sigma\wedge\alpha\in\mathscr{I}^{k+1}_{Z_\tau}$. Then there exists a holomorphic k-form α' such that*

(i) $d\sigma\wedge\alpha'=0$,
(ii) $\alpha'\in\mathscr{I}^k_{Z_\sigma}$, *and*
(iii) $\alpha-\alpha'\in\mathscr{I}^k_{Z_\tau}$.

The lemma will follow from the next one, which says that property (i) implies property (ii), and Proposition 9, which should be compared to Lemma 6 in [20].

Lemma 8. *Let σ be a monomial and α a holomorphic k-form. Then $(d\sigma/\sigma)\wedge\alpha\in\Omega^{k+1}$ if and only if $\alpha\in\mathscr{I}^k_{Z_\sigma}$.*

Proof. We have by definition that $\alpha\in\mathscr{I}^k_{Z_\sigma}$ if and only if $(dz_j/z_j)\wedge\alpha\in\Omega^{k+1}$ for all z_j dividing σ and the "if"-part of the lemma is clear. For the "only if"-part we will use induction on the number of coordinate functions z_j dividing σ. (One could also use Proposition 9 but we choose to give a direct argument.) If just one coordinate function divides σ then we are done, again by definition. We therefore assume that we have proved the "only if"-direction for $p-1$ coordinate functions dividing σ. Now let $z^a=z_1^{a_1}\cdots z_{p-1}^{a_{p-1}}$ and assume that $(d(z^az_p^{a_p})/z^az_p^{a_p})\wedge\alpha\in\Omega^{k+1}$. We then write $\alpha=\alpha'+dz_p\wedge\alpha''$, where α' and α'' do not contain any dz_p. Then

$$\frac{d(z^az_p^{a_p})}{z^az_p^{a_p}}\wedge\alpha=\frac{d(z^a)}{z^a}\wedge\alpha'+dz_p\wedge(\frac{\alpha'}{z_p}-\frac{d(z^a)}{z^a}\wedge\alpha'')\in\Omega^{k+1}.$$

Since the first term on the right hand side does not contain any dz_p it follows that both terms on the right hand side are in Ω^{k+1}. Then by the induction hypothesis, $\alpha' \in \mathscr{I}^k_{z^a=0}$. Moreover, $\alpha'/z_p - (d(z^a)/z^a) \wedge \alpha'' \in \Omega^k$ since α' and α'' do not contain any dz_p. But then α' must be divisible with z_p and $(d(z^a)/z^a) \wedge \alpha''$ must be in Ω^k (since α'/z_p is smooth in $z_1, \ldots, z_{p-1}$ and $(d(z^a)/z^a) \wedge \alpha''$ is smooth in z_p). Thus, $\alpha' \in \mathscr{I}^k_{z^a z_p=0}$, and again by the induction hypothesis, $\alpha'' \in \mathscr{I}^k_{z^a=0}$. Hence, $\alpha = \alpha' + dz_p \wedge \alpha'' \in \mathscr{I}^k_{z^a z_p=0}$, finishing the induction step. $\square$

Proposition 9. *Consider the monomial $\tau = z_r \cdots z_s$ and the corresponding variety Z_τ in $\mathbb{C}^n$. Denote the index set $\{r, \ldots, s\}$ by I and let $I(j)$ denote an arbitrary subset of I with precisely j elements fewer than I. Let also $V_{I(j)}$ and $Z_{I(j)}$ be the varieties $\cap_{i \in I(j)}\{z_i = 0\}$ and $\cup_{i \in I \setminus I(j)}\{z_i = 0\}$ respectively. (In this notation $Z_\tau = Z_{I(s-r+1)}$.) For any holomorphic k-form ω we then let $\omega_{I(j)}$ denote the holomorphic k-form obtained from ω by first pulling ω back to $V_{I(j)}$ and then extending constantly to $\mathbb{C}^n$. Now, let α be a holomorphic k-form and put $\alpha^1 = \alpha - \alpha_I$ and recursively, $\alpha^{i+1} = \alpha^i - \sum_{I(i)} \alpha^i_{I(i)}$. Then*

$$\alpha = \alpha_I + \sum_{I(1)} \alpha^1_{I(1)} + \cdots + \sum_{I(s-r)} \alpha^{s-r}_{I(s-r)} + \alpha^{s-r+1}, \tag{7}$$

where $\alpha^i_{I(i)} \in \mathscr{I}^k_{Z_{I(i)}}$ and $\alpha^{s-r+1} \in \mathscr{I}^k_{Z_\tau}$.

Proof. Using induction over the number of coordinate functions dividing the monomial τ it is easy to see that (7) holds and so, what remains is to see that the $\alpha^i_{I(i)}$ have the correct vanishing properties. We fix r and s with $r \le s$ and we show that $\alpha^i_{I(i)} \in \mathscr{I}^k_{Z_{I(i)}}$ for $i = 1, \ldots, s-r+1$, again with induction. Note that $Z_\tau = Z_{I(s-r+1)}$ and that $\alpha^{s-r+1} = \alpha^{s-r+1}_{I(s-r+1)}$. First we put $i = 1$. We have $\alpha^1 = \alpha - \alpha_I$ and so $\alpha^1_I = \alpha_I - \alpha_I = 0$. Now, if $I(1) = I \setminus \{j\}$, then the pullback of $\alpha^1_{I(1)}$ to $\{z_j = 0\}$ equals $\alpha^1_I = 0$. Hence, $\alpha^1_{I(1)} \in \mathscr{I}^k_{Z_{I(1)}}$. For the induction step, assume that $\alpha^{p-1}_{I(p-1)} \in \mathscr{I}^k_{Z_{I(p-1)}}$. We have $\alpha^p = \alpha^{p-1} - \sum_{I(p-1)} \alpha^{p-1}_{I(p-1)}$ by definition. If I' is a fixed set of the type $I(p-1)$ we get that

$$\alpha^p_{I'} = \alpha^{p-1}_{I'} - \sum_{I(p-1)} (\alpha^{p-1}_{I(p-1)})_{I'} = \alpha^{p-1}_{I'} - \alpha^{p-1}_{I'} = 0.$$

The second equality follows from the induction hypothesis since if $I(p-1) \neq I'$ then I' contains at least one index j not in $I(p-1)$. Then, since $\{z_j = 0\} \subset Z_{I(p-1)}$, the induction hypothesis implies that $\alpha^{p-1}_{I(p-1)} \in \mathscr{I}^k_{z_j=0}$, which in turn gives that $(\alpha^{p-1}_{I(p-1)})_{I'} = 0$. Now, let I'' be a set of the type $I(p)$. We can write $I'' = I' \setminus \{j\}$ for non unique I' of the type $I(p-1)$ and j. Then the pullback of $\alpha^p_{I''}$ to $\{z_j = 0\}$ equals $\alpha^p_{I'} = 0$. Repeating this for all possible decompositions $I'' = I' \setminus \{j\}$ of I'' we get $\alpha^p_{I''} \in \mathscr{I}^k_{Z_{I''}}$ finising the induction step. $\square$

Proof of Lemma 7. Property (ii) follows from property (i) according to Lemma 8. We claim that

$$\alpha' = \alpha_I + \sum_{I(1)} \alpha^1_{I(1)} + \cdots + \sum_{I(s-r)} \alpha^{s-r}_{I(s-r)},$$

where we have used the notations from Proposition 9, has the properties (i) and (iii). That α' has the property (iii) is part of the statement of Proposition 9 so we only need to check that it has property (i). By assumption we have that

$$(8)\quad d\sigma\wedge\alpha' = d\sigma\wedge\alpha_I + d\sigma\wedge\sum_{I(1)}\alpha^1_{I(1)} + \cdots + d\sigma\wedge\sum_{I(s-r)}\alpha^{s-r}_{I(s-r)} \in \mathscr{I}^{k+1}_{Z_\tau}.$$

If we pull back $d\sigma\wedge\alpha'$ to $V_I = \{z_r = \cdots = z_s = 0\}$ we get by Proposition 9 that the pullback of $d\sigma\wedge\alpha_I$ is zero since all other terms on the right hand side of (8) vanish on this set by Proposition 9. But $d\sigma\wedge\alpha_I$ is independent of all z_j and dz_j with $j = r,\ldots,s$ and so $d\sigma\wedge\alpha_I = 0$ in $\mathbb{C}^n$. Next we pull $d\sigma\wedge\alpha'$ back to $V_{I\setminus\{r\}} = \{z_{r+1} = \cdots = z_s = 0\}$. Then, using that $d\sigma\wedge\alpha_I = 0$ and Proposition 9, we get that the pullback of $d\sigma\wedge\alpha^1_{I\setminus\{r\}}$ to this set is zero. But $d\sigma\wedge\alpha^1_{I\setminus\{r\}}$ is independent of all z_j and dz_j with $j = r+1,\ldots,s$ and thus vanishes in all of $\mathbb{C}^n$. Continuing in this way, running through the indices, then pulling back to varieties of dimension $+1$ and running through pairs of indices and so on, we eventually obtain that $d\sigma\wedge\alpha' = 0$ in $\mathbb{C}^n$. □

5. Proof(s) of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2. Apart from using Lemma 5 when proving Theorem 2 and Lemma 6 when proving Theorem 1 the proofs are almost identical and we choose to focus on Theorem 2. Any differences will be pointed out explicitly.

Proof of Theorems 1 and 2. After a preliminary partition of unity on X we may assume that the test form has as small support as we want. Moreover, from [5] we are done if the support of the test form does not intersect $V_f = \{f_1 = f_2 = f_3 = 0\}$. (In the case of Theorem 1 this follows from [20].) Assume therefore that φ has support in a small neighborhood O of a point $x\in V_f$. We may assume that φ has the form $\varphi = \tilde{\phi}\wedge\overline{\phi}$, where $\tilde{\phi}$ is a smooth $(n,0)$-form with support close to x and ϕ is a holomorphic $(n-2)$-form. Hironaka's theorem implies that there is a complex n-dimensional manifold $\mathcal{X}$ and a proper holomorphic map $\pi\colon\mathcal{X}\to O$ such that the variety $\mathcal{Z}_f = \{\pi^*f_1\cdot\pi^*f_2\cdot\pi^*f_3 = 0\}$ has normal crossings in $\mathcal{X}$ and π is biholomorphic outside $\mathcal{Z}_f$. Since π is proper and φ has support close to $x\in V_f$, the pullback, $\pi^*\varphi$, has compact support close to $\pi^{-1}(x)\subset\mathcal{V}_f := \{\pi^*f_1 = \pi^*f_2 = \pi^*f_3 = 0\}$. We also introduce the notation $\mathcal{Z}_{12}$ for the variety consisting of the components of codimension one of $\{\pi^*f_1 = \pi^*f_2 = 0\}$ on which π^*f_3 does not vanish identically. The varieties $\mathcal{Z}_{23}$ and $\mathcal{Z}_{13}$ are defined analogously. Now, consider a point $p\in\mathcal{V}_f$. Since $\mathcal{Z}_f$ has normal crossings we may choose local holomorphic coordinates z close to p such that $z(p) = 0$, $\pi^*f_1 = z^a = z_1^{a_1}\cdots z_{r-1}^{a_{r-1}}$ and π^*f_j, $j = 2,3$, are monomials times non vanishing holomorphic functions. Generically, p does not lie on $\mathcal{Z}_{23}$, but if it does, then, after possibly renumbering the coordinates $z_r,\ldots,z_n$, we can write $\mathcal{Z}_{23} = \{z_r\cdots z_s = 0\}$. We can symbolically let $s<r$ denote the case that p does not lie on $\mathcal{Z}_{23}$. Since f_1, f_2, f_3 is a regular sequence, $df_1\wedge\phi$ vanishes on $\{f_2 = f_3 = 0\}$ for degree reasons, and so $d\pi^*f_1\wedge\pi^*\phi = d(z^a)\wedge\pi^*\phi\in\mathscr{I}^{n-1}_{\mathcal{Z}_{23}}$. By Lemma 7 we may therefore choose a holomorphic $n-2$-form, α_1, in

a neighborhood U of p such that $d\pi^* f_1 \wedge \alpha_1 = 0$, $\alpha_1 \in \mathscr{I}^{n-2}_{\mathcal{Z}_{f_1}}(U)$, and $\phi - \alpha_1 \in \mathscr{I}^{n-2}_{\mathcal{Z}_{23}}(U)$. In case p does not lie on $\mathcal{Z}_{23}$ we can take $\alpha_1 = 0$ close to p. In the same way, perhaps after shrinking U, we can find α_2 and α_3 such that $d\pi^* f_2 \wedge \alpha_2 = 0$, $\alpha_2 \in \mathscr{I}^{n-2}_{\mathcal{Z}_{f_2}}(U)$, and $\pi^*\phi - \alpha_2 \in \mathscr{I}^{n-2}_{\mathcal{Z}_{13}}(U)$ and similarily for α_3. In this way we get an open covering of $\mathcal{V}_f$ and we choose a locally finite subcovering, $\{U_j\}_1^\infty$. In each U_j we have holomorphic $n-2$-forms, α_1^j, α_2^j, and α_3^j with the properties described above. We may assume that $\pi^*\varphi$ has support in $\cup U_j$ and then, since $\pi^*\varphi$ has compact support, it suffices to take finitely many U_j to cover it. For convenience we denote this finite family by $\{U_j\}_1^q$. Subordinate to this family we choose a partition of unity, $\{\rho_j\}_1^q$, such that $\sum_1^q \rho_j = 1$ on the support of $\pi^*\varphi$. The integral we are interested in can now be written

$$\sum_1^q \int \frac{|\pi^* f_1|^{2\lambda_1} \bar\partial |\pi^* f_2|^{2\lambda_2} \wedge \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho_j \pi^* \tilde\phi \wedge \pi^* \bar\phi. \tag{9}$$

Since $d\pi^* f_i \wedge \alpha_i^j = 0$ in U_j we may replace $\pi^*\bar\phi$ with $\pi^*\bar\phi - \bar\alpha_2^j - \bar\alpha_3^j$ in (9) without affecting the integral. After this is done we integrate the $\bar\partial$ in front of $|\pi^* f_2|^{2\lambda_2}$ by parts, obtaining

$$\sum_1^q \int \frac{|\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho_j \bar\partial(\pi^* \tilde\phi \wedge (\pi^*\bar\phi - \bar\alpha_2^j - \bar\alpha_3^j)) \tag{10}$$

$$-\sum_1^q \int \frac{\bar\partial |\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \wedge \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho_j \pi^* \tilde\phi \wedge (\pi^*\bar\phi - \bar\alpha_2^j - \bar\alpha_3^j) \tag{11}$$

$$+\int \sum_1^q \frac{|\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \bar\partial\rho_j \wedge \pi^* \tilde\phi \wedge (\pi^*\bar\phi - \bar\alpha_2^j - \bar\alpha_3^j). \tag{12}$$

We first show that each term in the sum (10) has an analytic continuation to a neighborhood of $\lambda = 0$. To this end, we fix a j and write $\pi^* f_i = z^{a_i} \tilde f_i$, $i = 1, 2, 3$, as monomials times non vanishing functions in U_j. (The multiindices a_i and the functions $\tilde f_i$ of course also depend on j but we suppress this dependence to avoid to many subscripts.) By Lemma 5 it is sufficient to show that

$$\frac{d\bar z_k}{\bar z_k} \wedge \bar\partial(\pi^* \tilde\phi \wedge (\pi^*\bar\phi - \bar\alpha_2^j - \bar\alpha_3^j)) \tag{13}$$

is a smooth form for all non simple factors z_k dividing z^{a_3}. Assume first that z_k also divides z^{a_2}. Then we write (13) as

$$\frac{d\bar z_k}{\bar z_k} \wedge \pi^* \bar\partial\varphi - \frac{d\bar z_k}{\bar z_k} \wedge \bar\partial(\pi^* \tilde\phi \wedge (\bar\alpha_2^j + \bar\alpha_3^j)). \tag{14}$$

Since α_2^j and α_3^j vanish on $\{z_k = 0\}$ the last term is a smooth form. On the other hand, $\bar\partial\varphi$ can be written as a sum of products of smooth $(n,0)$-forms and anti-holomorphic $(0, n-1)$-forms. These anti-holomorphic forms vanish on $\{f_2 = f_3 = 0\}$ for degree reasons, and hence, their pullback under π vanish on $\{z_k = 0\}$. Thus, the first term in (14) is also a smooth form. It remains to check the case when z_k divides z^{a_1} and z^{a_3} but not z^{a_2}, i.e., that $\{z_k = 0\}$ is (part of) an irreducible component of $\mathcal{Z}_{13}$. We then write (13) as

$$\frac{d\bar{z}_k}{\bar{z}_k} \wedge \bar{\partial}(\pi^*\tilde{\phi} \wedge (\pi^*\bar{\phi} - \alpha_2^j)) - \frac{d\bar{z}_k}{\bar{z}_k} \wedge \bar{\partial}(\pi^*\tilde{\phi} \wedge \alpha_3^j),$$

which is a smooth form since both α_3^j and $\pi^*\bar{\phi} - \alpha_2^j$ vanish on $\mathcal{Z}_{13}$.

Next we show that each term in the sum (11) has an analytic continuation to a neighborhood of $\lambda = 0$. Now we have a $\bar{\partial}$ in front of $|\pi^* f_1|^{2\lambda_1}$ and so we may subtract also α_1^j in the test form. By Lemma 5 it will thus be sufficient to show that

$$\frac{d\bar{z}_k}{\bar{z}_k} \wedge \pi^*\tilde{\phi} \wedge (\pi^*\bar{\phi} - \bar{\alpha}_1^j - \bar{\alpha}_2^j - \bar{\alpha}_3^j) \tag{15}$$

is a smooth form for any non simple factor z_k dividing at least one of z^{a_1} and z^{a_3}. If z_k divides all three monomials z^{a_1}, z^{a_2}, and z^{a_3} then α_1^j, α_2^j, and α_3^j all vanish on $\{z_k = 0\}$. But then $\pi^*\phi$ does also, because the holomorphic $n-2$-form ϕ vanishes on V_f for degree reasons and hence $\pi^*\phi$ vanishes on $\mathcal{V}_f \supset \{z_k = 0\}$. In this case, (15) is therefore a smooth form. If instead z_k divides precisely two of the monomials, say for simplicity z^{a_1} and z^{a_2}, then α_1^j, α_2^j, and $\pi^*\phi - \alpha_3^j$ vanish on $\{z_k = 0\}$ and we see again that (15) is a smooth form.

Finally we prove that (12) has an analytic continuation to a neighborhood of $\lambda = 0$. If one could choose the α_i^j to agree on overlaps, i.e., to be independent of j, then this would be easy because in this case, the only thing depending on j would be ρ_j, and since the ρ_j sum up to 1, the $\bar{\partial}\rho_j$ sum up to 0, and (12) would be identically 0. Maybe it is possible to choose the α_i^j in such a good way but even if the differences $\alpha_i^j - \alpha_i^k$ are not 0 on overlaps they have sufficiently good properties for our purposes. Actually, $(dz_l/z_l) \wedge (\alpha_i^j - \alpha_i^k)$ is a holomorphic form, where it is defined, for any non simple factor z_l. We verify this for $i = 2$. If z_l divides $\pi^* f_2$ it is clear. On the other hand, if z_l does not divide $\pi^* f_2$, but divides $\pi^* f_1$ and $\pi^* f_3$, i.e., is (part of) a component of $\mathcal{Z}_{13}$, then it follows from the fact that $\alpha_2^j - \alpha_2^k = \pi^*\phi - \alpha_2^k - (\pi^*\phi - \alpha_2^j) \in \mathscr{I}_{\mathcal{Z}_{13}}^{n-2}$. We will show that (12) is analytic by showing that each point in $\mathrm{supp}(\pi^*\varphi)$ has a neighborhood such that if ρ is a smooth compactly supported function in this neighborhood then the integral of ρ multiplied with the sum in (12) has an analytic continuation to a neighborhood of $\lambda = 0$. It will then follow, after another partition of unity, that (12) also has. Now, consider a point $p \in \mathrm{supp}(\pi^*\varphi)$ and let $U_{j_1}, \ldots, U_{j_r}$ be those sets from our cover which contains p. Let U be a neighborhood of p contained in $\cap_i U_{j_i}$ and such that the ρ_j sum up to 1 in U and let ρ be a smooth function with support in U. The differences $\beta_1^{kl} = \alpha_1^{j_k} - \alpha_1^{j_l}$ and $\beta_2^{kl} = \alpha_2^{j_k} - \alpha_2^{j_l}$ with $1 \le k, l \le r$ are all defined in U. If we multiply the sum in (12) by ρ and integrate we obtain

$$\int \sum_{k=1}^{r} \frac{|\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \bar{\partial} |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho \bar{\partial}\rho_{j_k} \wedge \pi^*\tilde{\phi} \wedge (\pi^*\bar{\phi} - \bar{\alpha}_2^{j_k} - \bar{\alpha}_3^{j_k}) =$$

$$(16)\quad \int \sum_{k=1}^{r} \frac{|\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho \bar\partial \rho_{j_k} \wedge \pi^* \tilde\phi \wedge (\pi^* \bar\phi - \bar\alpha_2^{j_1} - \bar\alpha_3^{j_1})$$

$$(17)\quad - \sum_{k=1}^{r} \int \frac{|\pi^* f_1|^{2\lambda_1} |\pi^* f_2|^{2\lambda_2} \bar\partial |\pi^* f_3|^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho \bar\partial \rho_{j_k} \wedge \pi^* \tilde\phi \wedge (\bar\beta_2^{1k} + \bar\beta_3^{1k}).$$

The only thing in the sum in (16) that depends on k is ρ_{j_k} and since the ρ_{j_k} sum up to 1 in U the $\bar\partial \rho_{j_k}$ sum up to 0, and so (16) is identically 0. But $(dz_l/z_l) \wedge \beta_i^{1k}$ is a holomorphic form for any non simple factor z_l by the discussion above, and so by Lemma 5, each term in the sum (17) has an analytic continuation to a neighborhood of $\lambda = 0$. The proof is complete. $\square$

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