The Residue Current of a Codimension Three Complete Intersection

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Abstract. Let $f_1, f_2,$ and $f_3$ be holomorphic functions on a complex manifold and assume that the common zero set of the $f_j$ has maximal codimension. We prove that the iterated Mellin transform of the residue integral has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^3$. We prove also that the natural regularization of the residue current converges unrestrictedly.

MSC: 32A27; 32C30

1. Introduction

Let $X$ be an $n$-dimensional complex manifold and $f = (f_1, \ldots, f_m)$ a holomorphic mapping $X \to \mathbb{C}^m$ such that the common zero set $V_f = \{f = 0\}$ has codimension $m$. In [8] Coleff and Herrera were able to associate a certain $(0, n)$-current to $f$, which has proven to be a good notion of a multi variable residue of $f$. They defined their current $R_f$, the Coleff-Herrera residue current, as follows. For a test form $\varphi \in \mathcal{D}_{n,n-m}(X)$, consider the residue integral

$$I_f^\varphi (\epsilon) = \int_{T_\epsilon} \frac{\varphi}{f_1 \cdots f_m},$$

where $T_\epsilon$ is the tube $\{|f_1|^2 = \epsilon_1, \ldots, |f_m|^2 = \epsilon_m\}$. Coleff and Herrera proved that the limit of the residue integral as $\epsilon$ tends to zero along a so-called admissible path exists and defines the action of a $(0, m)$-current on the test form $\varphi$. (The case $m = 1$ is due to Herrera and Lieberman, [11],) The limit along an admissible path here means that $\epsilon$ tends to zero along a path in the first orthant such that $\epsilon_j/\epsilon_{j+1}$ tends to zero for all $k \in \mathbb{N}$ and $j = 1, \ldots, m - 1$. The Coleff-Herrera residue current has many desirable properties. For instance, it is supported on $V_f$, it has the standard extension property, which more or less means that it has no mass concentrated on the singular parts of $V_f$, and it satisfies the duality property that a holomorphic function $h$ on X annihilates it if and only if $h$ belongs to the ideal generated by $f$. The duality property is due to Dickenstein-Sessa, [9], and Passare, [13], independently. It is natural to ask if the restriction to limits along admissible paths is necessary. It actually is and the first example showing this was found by Passare and Tsikh, [16]. Björk later realized that this
Indeed is the typical case, [6]; see also Pavlova, [18]. The Coleff-Herrera definition is in this sense quite unstable and one could try to look for more stable ones. One step in this direction was taken by Passare in [14] where he introduced the following regularized version of the residue integral. Let $\chi_j$ be smooth function on $[0, \infty]$ taking the value 0 at 0 and 1 at $\infty$. (Actually, Passare considered functions identically 0 close to 0 and identically 1 close to $\infty$.) The regularized residue integral is then the volume integral

$$\int \frac{\overline{\partial} \chi_1(|f_1|^2/\epsilon_1) \wedge \cdots \wedge \overline{\partial} \chi_m(|f_m|^2/\epsilon_m)}{f_1 \cdots f_m} \wedge \varphi.$$  

Note that in the (not allowed) case when all the $\chi_j$ are the characteristic function of $[1, \infty]$ we get back the residue integral. It follows from Coleff’s and Herrera’s result that the limit of (1) along admissible paths exists and equals the limit along admissible paths of the residue integral but Passare proves a more general result. In fact, he proves that for almost all parabolic paths, $\epsilon(\delta) = (\delta^{a_1}, \ldots, \delta^{a_m})$, the limit of (1) exists. Here "almost all" means that one has to impose finitely many linear conditions $\sum a_j b_j \neq 0$ in order to assure convergence. In the special case when $m = 2$ the author was able to prove that (1) actually depends Hölder continuously on $\epsilon$ in the closed first quarter, [19], [20]. In this paper we generalize this result to the codimension three case and we prove

**Theorem 1.** Let $f_1$, $f_2$, and $f_3$ be holomorphic functions on a complex manifold $X$ of dimension $n$ and assume that the common zero set of the $f_j$ has maximal codimension. Let also $\chi_1$, $\chi_2$, and $\chi_3$ be smooth functions on $[0, \infty]$ taking the value 0 at 0 and 1 at $\infty$ and denote $\chi_j(|f_j|^2/\epsilon_j)$ by $\chi_j^\epsilon$. Then, for any test form $\varphi \in \mathcal{D}_{n,n-2}(X)$, the integral

$$\int \frac{\chi_1^\epsilon \overline{\partial} \chi_2^\epsilon \wedge \overline{\partial} \chi_3^\epsilon}{f_1 \cdot f_2 \cdot f_3} \wedge \varphi$$

depends Hölder continuously on $\epsilon$ in the closed first octant.

The value at the origin is the current $PR^2[1/f]$ (acting on $\varphi$) in Passare’s notation from [14] and it is a $\overline{\partial}$-potential to the Coleff-Herrera residue current. That (1) converges unrestrictedly in the case $m = 3$ thus follows from Theorem 1 by applying it to $\overline{\partial}$-exact test forms.

Another approach to the Coleff-Herrera residue current based on analytic continuation of currents, a technique with roots in the works of Gelfand and Shilov, [10], and Atiyah, [4], has been considered by several authors, e.g., Yger, [22], Berenstein, Gay, and Yger, [5], and Passare and Tsikh, [15], [16]. Computing the Mellin transform of the residue integral one obtains

$$\int \frac{\overline{\partial} |f_1|^{2\lambda_1} \wedge \cdots \wedge \overline{\partial} |f_m|^{2\lambda_m}}{f_1 \cdots f_m} \wedge \varphi,$$

where $\lambda_1, \ldots, \lambda_m$ are complex parameters with large real parts, see e.g., [5] or [15]. It is proved in [22] and [5] that the restriction of (2) to any complex line of the form $\lambda(t_1, \ldots, t_m)$, $t_j \in \mathbb{R}_{>0}$, can be analytically continued to a neighborhood of the origin. Moreover, the value at the origin equals the Coleff-Herrera residue $R^\ell \cdot \varphi$. It is also proved in [5] that in the case when
$m = 2$, (2) can be analytically continued to a neighborhood of the origin as a function of two complex variables. It is generally believed, but not yet fully proved, that this holds for arbitrary $m$. Our second main result confirms this conjecture for $m = 3$.

**Theorem 2.** Let $f_1$, $f_2$, and $f_3$ be holomorphic functions on a complex manifold $X$ of dimension $n$, and assume that the common zero set of the $f_j$ has maximal codimension. Then, for any test form $\varphi \in \mathcal{D}_{n,n-2}(X)$, the holomorphic function

$$
(\lambda_1, \lambda_2, \lambda_3) \mapsto \int \frac{|f_1|^{2\lambda_1} \partial \overline{\partial}|f_2|^{2\lambda_2} \wedge \partial \overline{\partial}|f_3|^{2\lambda_3}}{f_1 \cdot f_2 \cdot f_3} \wedge \varphi,
$$

originally defined when $\Re \lambda_j$, $j = 1, 2, 3$, is large enough, has a holomorphic continuation to a neighborhood of $\lambda = 0$ in $\mathbb{C}^3$ and the value at the origin equals $PR^2[1/f] \cdot \varphi$.

We will not be concerned with it in this paper but we also mention a third, and very successful way to gain stability in the definition of the Coleff-Herrera residue current introduced by Passare, Tsikh, and Yger in [17]. They use the Bochner-Martinelli kernel as a blueprint for the definition of the residue instead of the Cauchy kernel. One can view their Bochner-Martinelli type current as the limit of a certain average of the residue integral. The advantage is that this averaging process reduces the number of parameters to just one and it is then proved in [17] that the limit as this single parameter tends to zero exists. It is even true when $f$ does not define a complete intersection! However, it is non trivial to prove that the obtained current actually equals the Coleff-Herrera current. Quite recently, Andersson put the ideas in [17] into an algebraic framework and introduced more general currents of the Cauchy-Fantappiè-Leray type, which have been useful in applications, see e.g. [1], [2], [3].

The disposition of the paper is as follows. In the next section we settle the notations for some frequently appearing objects and we discuss the main elements of the proofs of Theorems 1 and 2. In section 3 we compute an example showing that the codimension three case is different from the codimension two case. Section 4 contains some technical results about the normal crossings case. At the end of the section we also prove a combinatorial algebra type result which will enable us to use the assumption that $f$ defines a complete intersection efficiently. In the last section, Section 5, we prove our main theorems. However, we only prove Theorem 2 in detail since the proofs are almost identical.

## 2. Notations and an overview of the proof

We will have to use Hironaka's theorem, [12], to resolve singularities locally. It gives us for any sufficiently small open set $U \subset X$ a complex manifold $X$ and a proper holomorphic map $\pi: X \to U$ with the properties that $Z_f := \{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3 = 0\}$ has normal crossings and $\pi$ restricted to $X \setminus Z_f$ is a biholomorphism. The varieties we will be most interested in are the varieties $V_f := \{f_1 = f_2 = f_3 = 0\}$ and $Z_f := \{f_1 f_2 f_3 = 0\}$ in $X$, and their total transforms, $\mathcal{V}_f := \{\pi^* f_1 = \pi^* f_2 = \pi^* f_3 = 0\}$ and
$Z_f = \{\pi^*f_1 \cdot \pi^*f_2 \cdot \pi^*f_3 = 0\}$. Varieties in calligraphic letters are always varieties in the resolution manifold $X$. Moreover, varieties denoted by $Z$ (or $\mathcal{Z}$) are always varieties of codimension 1 and a holomorphic function as a subscript means the zero variety of that holomorphic function, e.g., 
$Z_{f_1} := \{f_1 = 0\}$ ($Z_{f_1} := \{\pi^*f_1 = 0\}$). Varieties of higher codimensions are denoted by $V$ (or $\mathcal{V}$). Occasionally we will encounter varieties for which this nomenclature is not efficiently applicable and we will then use more ad hoc notations. That the variety $Z_f$ has normal crossings in $X$ means that locally on $X$ one can find holomorphic coordinates $z$ such that $\pi^*f_j = z^{a_j}\tilde{f}_j$, $j = 1, 2, 3$, where the $\tilde{f}_j$ are non vanishing holomorphic functions. We will call $z_k$ a simple factor if $z_k$ divides precisely one of the monomials $z^a$. The following analytic sheaves on $X$ will be referred to frequently; the sheaves of holomorphic $k$-forms, $\Omega^k$, and the subsheaves of them of holomorphic $k$-forms vanishing on a normal crossings divisor $Z$, $\mathcal{Z}^k_Z$. A holomorphic $k$-form, $\alpha$, vanishes on a normal crossings divisor if the pullback of $\alpha$ (under the inclusion map) to any irreducible component of $Z$ vanishes. If $z$ are local coordinates such that $Z$ is the zero set of a monomial, $z^a$, then $\alpha$ vanishes on $Z$ if and only if $(dz_j/z_j) \wedge \alpha$ is a holomorphic $k+1$-form for any $z_j$ dividing the monomial $z^a$. This, in turn, holds if and only if, for all $r \geq 1$, 
$(dz_{j_1}/z_{j_1}) \wedge \cdots \wedge (dz_{j_r}/z_{j_r}) \wedge \alpha$ is a holomorphic $k+r$-form for $z_{j_i}$ dividing $z^a$.

Now, some comments to the proofs of Theorems 1 and 2. After a partition of unity we may assume that our test form $\varphi$ has support in a neighborhood $U$ such that it exists a Hironaka resolution of singularities $\pi: X \to U$ as described above. We then pull our integral back to the resolution manifold $X$ and as explained above, we find local holomorphic coordinates $z$ on $X$ such that $\pi^*f_j = z^{a_j}\tilde{f}_j$, $j = 1, 2, 3$. After a partition of unity on $X$ one is then able to start computing. From a computational point of view it is of course easier if one could arrange so that the $\tilde{f}_j \equiv 1$. This is possible if the integer vectors $a_j$ are linearly independent, see e.g., [14]. If one restricts to limits along admissible paths, as in [8], or allowed parabolic paths, as in [14], one will encounter only charts on $X$ where the $a_j$ are linearly independent. However, in the general case one will encounter also charts where the $a_j$ are linearly dependent, so called charts of resonance. This is precisely what happens in the Passare-Tsikh example, [16]. Charts of resonance can therefore be seen as the reason for the discontinuity of the residue integral. In codimension two the author showed in [19] and [20] how the charts of resonance can be handled when one considers the regularized residue integral (1). The main tool is Proposition 11 in [20] and it can be generalized. The general version is Proposition 4 below but we have omitted the proof since it is a straightforward generalization of the proof of Proposition 11 in [20]. The presence of charts of resonance is therefore not a problem when we prove Theorems 1 and 2. The main difficulty is another problem that is harder to handle when $m \geq 3$ then when $m = 2$. On the resolution manifold $X$ the functions $\pi^*f_j$ almost never define a complete intersection and for arbitrary test forms in $\mathcal{D}_{n,n-m}(X)$ the corresponding residue integral will in general be discontinuous. One has to use that the test forms we consider are of
the special form $\pi^*\varphi$ because, in such test forms, the information that $f$
defines a complete intersection in $X$ is somehow coded. When $m = 2$ it
is quite easy to extract this information. Actually, it follows from a degree
argument, see e.g. [8], that $\pi^*\varphi$ vanishes on all components of codimension
1 of the variety $\mathcal{V}_f$, i.e., on the exceptional divisor. This vanishing is then
seen to be enough to get the results in codimension two. In codimension
three however, $\pi^*\varphi$ will in general not vanish on the exceptional divisor with
the consequence that it is not a local problem on the resolution manifold $\Lambda$
to prove Theorems 1 and 2. In the next section we give a simple example
showing this. So one has to work a little more to extract the information
hidden in $\pi^*\varphi$ when $m = 3$. The degree argument is still very useful though.
With the aid of the slightly technical Lemma 7 it enables us to locally modify
the test form $\pi^*\varphi$, without affecting the integral, so that the modified test
form has good enough vanishing properties on the exceptional divisor. The
process, however, produces also a global term which requires some additional
attention.

3. An example

We consider an example showing that proving analyticity of the Mellin
transform and continuity of the regularized residue integral are not local
problems on the resolution manifold. We will look at the integral

$$\int \frac{|x_1|^{2\lambda_1} |x_2|^{2\lambda_2} |x_3|^{2\lambda_3}}{x_1 x_2 x_3} \varphi(x) dx \wedge d\bar{x}$$

in $\mathbb{C}^3$, where $\varphi$ is a function defined as follows. Let $\phi$, $\varphi_2$ and $\varphi_3$ be smooth
functions on $\mathbb{C}$ with support close to the origin but non-vanishing there, and
put $\varphi_1 = \partial \phi / \partial \bar{z}$. We define $\varphi(x)$ to be the function $\varphi_1(x_1) \varphi_2(x_2) \varphi_3(x_3)$ in
$\mathbb{C}^3$. First of all, note that (3) is the Mellin transform of a residue integral by
the choice of $\varphi$; we can move the $\partial_1$ in front of $\phi$ to $|x_1|^{2\lambda_1}$ by an integration
by parts. Secondly, (3) equals

$$\int \frac{|x_1|^{2\lambda_1} |x_2|^{2\lambda_2} |x_3|^{2\lambda_3}}{x_1 x_2 x_3} \varphi_1 \frac{\partial \varphi_2}{\partial x_2} \frac{\partial \varphi_3}{\partial x_3} dx \wedge d\bar{x}$$

after two integrations by parts, from which we see that (3) is analytic at
$\lambda = 0$. Now we blow up $\mathbb{C}^3$ along the $x_1$-axis and study the pullback of (3)
to this manifold. Let $\pi: \mathbb{C} \times \mathcal{B}_0 \mathbb{C}^2 \rightarrow \mathbb{C}^3$ be the blow up map. In the natural
coordinates $z$ and $\zeta$ on $\mathbb{C} \times \mathcal{B}_0 \mathbb{C}^2$ it looks like

$$\pi(z_1, z_2, z_3) = (z_1, z_2, z_2 z_3),$$

$$\pi(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1, \zeta_2 \zeta_3, \zeta_2).$$

Since $\varphi$ has support close to the origin, $\pi^*\varphi$ has support close to $\pi^{-1}(0) =
\{z_1 = z_2 = 0\} \cup \{\zeta_1 = \zeta_2 = 0\} \cong \mathbb{C} P^1$. Note that $z_3$ and $\zeta_3$ are natural
coordinates on this $\mathbb{C} P^1$ and choose a partition of unity, $\{\rho_1, \rho_2\}$ on supp($\pi^*\varphi$)
such that supp($\rho_1$) $\subset \{|z_3| < 2\}$ and supp($\rho_2$) $\subset \{|\zeta_3| < 2\}$. The pullback of
(3) under $\pi$ now equals

$$\int \frac{|z_1|^{2\lambda_1} |z_2|^{2\lambda_2} |z_3|^{2\lambda_3}}{z_1 z_2 z_3} \rho_1(z) \varphi_1(z_1) \varphi_2(z_2) \varphi_3(z_2 z_3) dz_2 dz \wedge d\bar{z}_1$$
\[- \int \frac{|\zeta_1|^{2\lambda_1} \partial \bar{z}_1 \zeta_3^{2\lambda_2} \wedge \partial \bar{z}_3^{2\lambda_3}}{\zeta_1 \zeta_2 \zeta_3} \wedge \rho_2(\zeta) \varphi_1(\zeta_1) \varphi_2(\zeta_2) \varphi_3(\zeta_3) \zeta_2 d\zeta \wedge d\bar{\zeta}_1.\]

We know that this sum (difference) is analytic at \( \lambda = 0 \) but we will check that non of the terms are. We consider the first term. It is easily verified that it can be written as

\[\frac{\lambda_2}{\lambda_2 + \lambda_3} \int \frac{|z_1|^{2\lambda_1} \partial |z_2|^{2(\lambda_2 + \lambda_3)} \wedge \partial |z_3|^{2\lambda_3}}{z_1 z_2 z_3} \wedge \rho_1(z) \varphi_1(z_1) \varphi_2(z_2) \varphi_3(z_2 z_3) dz \wedge d\bar{z}_1.\]

We denote this integral, with the coefficient \( \lambda_2/(\lambda_2 + \lambda_3) \) removed, by \( I(\lambda) \). After two integrations by parts one sees that \( I(\lambda) \) is analytic at the origin, and so \( \lambda_2 I(\lambda)/(\lambda_2 + \lambda_3) \) is analytic at the origin if and only if \( I(\lambda) \) vanishes on the hyperplane \( \lambda_2 + \lambda_3 = 0 \). In particular we must have that \( I(0) = 0 \). But \( I(0) \) can be computed using Cauchy’s formula, and one obtains \( I(0) = -(2\pi i)^3 \varphi(0) \rho_2(0) \varphi_3(0) \neq 0 \). Hence, proving analyticity of the Mellin transform of the residue integral is not a local problem on the blown up manifold. The same example can also be used to see that continuity of the regularized residue integral is not a local property on the resolution manifold.

**Remark 3.** This example could be a little confusing. The variable \( z_1 \) just appears as a "dummy variable" in the computations above, to which nothing interesting happens. This indicates that it is not a local problem on the resolution manifold to prove analytic continuation of (2) (or an unconditional limit of (1)) already in the case \( m = 2 \). Actually, if one pulls (2) back to a resolution manifold and then starts computing one will encounter global problems already for \( m = 2 \). But analytic continuation of

\[
(\lambda_1, \lambda_2) \mapsto \int \frac{|f_1|^{2\lambda_1} \partial |f_2|^{2\lambda_2}}{f_1 \cdot f_2} \wedge \varphi, \quad \varphi \in \mathcal{D}_{n,n-1}(X),
\]

implies analytic continuation of (2) for \( m = 2 \), and proving analytic continuation of (4) is a local problem on the resolution manifold. Thus, the analyticity problem can always be reduced to a local problem on the resolution manifold when \( m = 2 \) but, and this is the point, for \( m \geq 3 \) this is not always possible.

### 4. Preliminary Lemmas

The first proposition in this section is a straightforward generalization of Proposition 11 in [20] so we omit the proof.

**Proposition 4.** Let \( \psi_j, \ j = 1, \ldots, m, \) be strictly positive smooth functions on an open set \( \Omega \subset \mathbb{C}^d \) and let \( a_j = (a_{j1}, \ldots, a_{jm}), \ j = 1, \ldots, m, \) be multi-indices. Let also \( \chi_j \in \mathcal{C}^{\infty}(\Omega) \) be zero at zero. Then, for any test form \( \phi \in \mathcal{D}_{n,n}(\Omega), \) the integral

\[
\int \frac{\chi_1(\psi_1|z^{a_{11}}|^2/\epsilon_1) \cdots \chi_m(\psi_m|z^{a_{m1}}|^2/\epsilon_m)}{z^{a_{11}} \cdots z^{a_{m1}}} \wedge \phi
\]

depends Hölder continuously on \( \epsilon = (\epsilon_1, \ldots, \epsilon_m) \) in the closed first orthant.

The following two lemmas more or less reduce the proofs of Theorems 1 and 2 to a study of the pullback of the test form to the resolution manifold with the result that the two theorems can be treated almost identically.
Lemma 5. Let \( \tilde{f}_j, j = 1, \ldots, m, \) be non vanishing holomorphic functions on an open set \( \Omega \subset \mathbb{C}^n \) and let \( a_j = (a_{j1}, \ldots, a_{jn}) \), \( j = 1, \ldots, m, \) be multiindices. Assume that the test form \( \varphi \in \mathbb{D}_{n,n-r}(\Omega) \) has the property that \( (d\bar{z}_k/\bar{z}_k) \wedge \varphi \in \mathbb{D}_{n,n-r+1}(\Omega) \) for all non simple factors \( z_k \) dividing some monomial \( z^{a_j} \) with \( 1 \leq j \leq r. \) Then the integral

\[
\int \frac{\partial|z^{a_1} \tilde{f}_1|^{2\lambda_1} \cdots \partial|z^{a_m} \tilde{f}_m|^{2\lambda_m}}{z^{a_1} \tilde{f}_1 \cdots z^{a_m} \tilde{f}_m} \wedge \varphi
\]

has an analytic continuation to neighborhood of \( \lambda = 0 \in \mathbb{C}^m. \)

Lemma 6. Let \( \tilde{f}_j \) and \( a_j, j = 1, \ldots, m, \) and \( \varphi \) be as in Lemma 5 and let \( \chi_j \in \mathcal{C}^\infty([0, \infty)) \) be zero at zero. Then the integral

\[
\int \frac{\partial \chi_1^\epsilon \cdots \partial \chi_m^\epsilon \cdot \chi_{r+1}^\epsilon \cdots \chi_m^\epsilon}{z^{a_1} \tilde{f}_1 \cdots z^{a_m} \tilde{f}_m} \wedge \varphi,
\]

where \( \chi_j^\epsilon = \chi_j(|z^{a_j} \tilde{f}_j|^2/\epsilon_j) \), depends Hölder continuously on \( \epsilon = (\epsilon_1, \ldots, \epsilon_m) \) in the closed first orthant.

Proof of Lemmas 5 and 6. It is well known that integrals of the form

\[
\int \frac{|z^{a_1} \tilde{f}_1|^{2\lambda_1} \cdots |z^{a_m} \tilde{f}_m|^{2\lambda_m}}{z^{a_1} \tilde{f}_1 \cdots z^{a_m} \tilde{f}_m} \wedge \phi
\]

have an analytic continuation to a neighborhood of \( \lambda = 0 \in \mathbb{C}^m \) without any assumptions on the \((n, n)\)-test form \( \phi \), see e.g. [1]. Moreover, by Proposition 4 we have Hölder continuity in the first orthant for integrals like

\[
\int \frac{\chi_1^\epsilon \cdots \chi_m^\epsilon}{z^{a_1} \tilde{f}_1 \cdots z^{a_m} \tilde{f}_m} \wedge \phi
\]

for all \((n, n)\)-test forms \( \phi \). Using the assumption on our test form \( \varphi \) we will reduce the computations of the integrals in the Lemmas 5 and 6 to sums of integrals of the forms (5) and (6) respectively. This is done in more or less the same way in both cases. We start by writing every \( \partial \) as the sum

\[
\partial = \partial_1 + \cdots + \partial_n
\]

and splitting up the integrals into sums accordingly. We deal with an expression \( \partial_k|z^{a_j} \tilde{f}_j|^{2\lambda_j} \), respectively \( \partial_k \chi_j(|z^{a_j} \tilde{f}_j|^2/\epsilon_j) \), as follows. If \( z_k \) is a non simple factor dividing the monomial \( z^{a_j} \) we let \( \partial_k \) act, obtaining

\[
\lambda_j|z_{a_j} \tilde{f}_j|^{2\lambda_j} \left( a_{jk} \frac{d\bar{z}_k}{\bar{z}_k} + \frac{\partial_k \tilde{f}_j}{\tilde{f}_j} \right),
\]

respectively

\[
\tilde{\chi}_j(|z^{a_j} \tilde{f}_j|^2/\epsilon_j) \left( a_{jk} \frac{d\bar{z}_k}{\bar{z}_k} + \frac{\partial_k \tilde{f}_j}{\tilde{f}_j} \right),
\]

where \( \tilde{\chi}_j(t) = t \chi_j^\epsilon(t) \). Note that \( \tilde{\chi}_j \) is zero at zero and smooth on \([0, \infty]\), since \( \chi_j^\epsilon(t) \in \mathcal{O}(1/t^2) \) as \( t \to \infty \), and hence satisfies the properties required by Proposition 4. The assumption on our test form \( \varphi \) means that, for any expression \( d\bar{z}_k/\bar{z}_k \) arising in this way, \( (d\bar{z}_k/\bar{z}_k) \wedge \varphi \) is again a test form. More
generally, it is easy to see, e.g. by making a Taylor expansion à la Lemma 6 in [20] of the coefficients of \( \varphi \), that the assumption on \( \varphi \) implies that

\[
\frac{dz_{k_1}}{z_{k_1}} \wedge \cdots \wedge \frac{dz_{k_p}}{z_{k_p}} \wedge \varphi
\]

is a test form if the \( z_{k_i} \) are non simple factors each dividing some monomial \( z^a \) with \( 1 \leq j \leq r \). Hence, all singular forms \( \frac{dz_k}{z_k} \) arising from the action of \( \partial_k \), where \( k \) is such that \( z_k \) a non simple factor, can be incorporated in the test form. On the other hand, if \( z_k \) is a simple factor dividing the monomial \( z^a \) we do not let \( \partial_k \) act on \( |z_j f_j|^{2\lambda_j} \), respectively \( \chi_j(|z_j f_j|^{2\lambda_j}/\varepsilon_j) \). Instead we then integrate this \( \partial_k \) by parts. Since \( z_k \) is a simple factor it does not divide any monomial other then \( z^a \) and so, after the integration by parts, \( \partial_k \) will not encounter any monomial containing \( z_k \) as a factor and hence not produce the singular expression \( \frac{dz_k}{z_k} \). Hence, the integrals in Lemmas 5 and 6 can be written as sums of integrals of the form (5) and (6) respectively, concluding the proof.

The rest of this section is devoted to a proof of the following lemma. It will enable us to use the fact that we have a complete intersection on the original manifold in an efficient way when we do computations on the blown up one, where we in general do not have complete intersection.

**Lemma 7.** Consider the monomials \( \sigma = z_1^{a_1} \cdots z_{p-1}^{a_{p-1}} \) and \( \tau = z_r \cdots z_s \) and let \( \alpha \) be a holomorphic \( k \)-form such that \( d\sigma \wedge \alpha \in \mathcal{I}^{k+1}_Z \). Then there exists a holomorphic \( k \)-form \( \alpha' \) such that

1. \( d\sigma \wedge \alpha' = 0 \),
2. \( \alpha' \in \mathcal{I}^k_Z \), and
3. \( \alpha - \alpha' \in \mathcal{I}^k_Z \).

The lemma will follow from the next one, which says that property (i) implies property (ii), and Proposition 9, which should be compared to Lemma 6 in [20].

**Lemma 8.** Let \( \sigma \) be a monomial and \( \alpha \) a holomorphic \( k \)-form. Then \( (d\sigma/\sigma) \wedge \alpha \in \Omega^{k+1} \) if and only if \( \alpha \in \mathcal{I}^k_Z \).

**Proof.** We have by definition that \( \alpha \in \mathcal{I}^k_Z \) if and only if \( (dz_j/z_j) \wedge \alpha \in \Omega^{k+1} \) for all \( z_j \) dividing \( \sigma \) and the "if"-part of the lemma is clear. For the "only if"-part we will use induction on the number of coordinate functions \( z_j \) dividing \( \sigma \). (One could also use Proposition 9 but we choose to give a direct argument.) If just one coordinate function divides \( \sigma \) then we are done, again by definition. We therefore assume that we have proved the "only if"-direction for \( p - 1 \) coordinate functions dividing \( \sigma \). Now let \( z^a = z_1^{a_1} \cdots z_{p-1}^{a_{p-1}} \) and assume that \( (dz^a z_{ap}^a)/z^a z_{ap}^a) \wedge \alpha \in \Omega^{k+1} \). We then write \( \alpha = \alpha' + dz_p \wedge \alpha'' \), where \( \alpha' \) and \( \alpha'' \) do not contain any \( dz_p \). Then

\[
\frac{dz^a z_{ap}^a}{z^a z_{ap}^a} \wedge \alpha = \frac{dz^a}{z^a} \wedge \alpha' + dz_p \wedge \left( \frac{\alpha'}{z_p} - \frac{dz^a}{z^a} \wedge \alpha'' \right) \in \Omega^{k+1}.
\]

Since the first term on the right hand side does not contain any \( dz_p \) it follows that both terms on the right hand side are in \( \Omega^{k+1} \). Then by the induction
hypothesis, $\alpha' \in \mathcal{J}^k_{z_p=0}$. Moreover, $\alpha' / z_p - (d(z^a)/z^a) \wedge \alpha'' \in \Omega^k$ since $\alpha'$ and $\alpha''$ do not contain any $dz_p$. But then $\alpha'$ must be divisible with $z_p$ and $(d(z^a)/z^a) \wedge \alpha''$ must be in $\Omega^k$ (since $\alpha' / z_p$ is smooth in $z_1, \ldots, z_{p-1}$ and $(d(z^a)/z^a) \wedge \alpha''$ is smooth in $z_p$). Thus, $\alpha' \in \mathcal{J}^k_{z_p=0}$, and again by the induction hypothesis, $\alpha'' \in \mathcal{J}^k_{z_p=0}$. Hence, $\alpha = \alpha' + dz_p \wedge \alpha'' \in \mathcal{J}^k_{z_p=0}$, finishing the induction step.

**Proposition 9.** Consider the monomial $\tau = z_r \cdots z_s$ and the corresponding variety $Z_\tau$ in $\mathbb{C}^n$. Denote the index set $\{r, \ldots, s\}$ by $I$ and let $I(j)$ denote an arbitrary subset of $I$ with precisely $j$ elements fewer than $I$. Let also $V_I(j)$ and $Z_I(j)$ be the varieties $\cap_{i \in I(j)} \{z_i = 0\}$ and $\cup_{i \in I(j)} \{z_i = 0\}$ respectively. (In this notation $Z_\tau = Z_I(s-r+1)$.) For any holomorphic $k$-form $\omega$ we then let $\omega_I(j)$ denote the holomorphic $k$-form obtained from $\omega$ by first pulling $\omega$ back to $V_I(j)$ and then extending constantly to $\mathbb{C}^n$. Now, let $\alpha$ be a holomorphic $k$-form and put $\alpha^1 = \alpha - \alpha_I$ and recursively, $\alpha^{i+1} = \alpha^i - \sum_{I(i)} \alpha^{i}_I(I)$. Then

$$\alpha = \alpha_I + \sum_{I(1)} \alpha^{1}_{I(1)} + \cdots + \sum_{I(s-r)} \alpha^{s-r}_{I(s-r)} + \alpha^{s-r+1},$$

where $\alpha^{i}_I(I) \in \mathcal{J}^k_{Z_I(j)}$ and $\alpha^{s-r+1} \in \mathcal{J}^k_{Z_\tau}$.

**Proof.** Using induction over the number of coordinate functions dividing the monomial $\tau$ it is easy to see that (7) holds and so, what remains is to see that the $\alpha^{i}_I(I)$ have the correct vanishing properties. We fix $r$ and $s$ with $r \leq s$ and we show that $\alpha^{i}_I(I) \in \mathcal{J}^k_{Z_I(j)}$ for $i = 1, \ldots, s-r+1$, again with induction. Note that $Z_\tau = Z_I(s-r+1)$ and that $\alpha^{s-r+1} = \alpha^{s-r+1}_I(I)$. First we put $i = 1$. We have $\alpha^1 = \alpha - \alpha_I$ and so $\alpha^1_I = \alpha - \alpha_I = 0$. Now, if $I(1) = I \setminus \{j\}$, then the pullback of $\alpha^1_I(I)$ to $\{z_j = 0\}$ equals $\alpha^1_j = 0$. Hence, $\alpha^1_I(I) \in \mathcal{J}^k_{Z_I(j)}$. For the induction step, assume that $\alpha^{p-1}_I(I(p-1)) \in \mathcal{J}^k_{Z_I(j)}$. We have $\alpha^p = \alpha^{p-1} - \sum_{I(p-1)} \alpha^{p-1}_I(I(p-1))$ by definition. If $I'$ is a fixed set of the type $I(p-1)$ we get that

$$\alpha^{p}_I = \alpha^{p-1}_I - \sum_{I(p-1)} (\alpha^{p-1}_{I(p-1)})_I = \alpha^{p-1}_I - \alpha^{p-1}_I = 0.$$ 

The second equality follows from the induction hypothesis since if $I(p-1) \neq I'$ then $I'$ contains at least one index $j$ not in $I(p-1)$. Then, since $\{z_j = 0\} \subset Z_I(p-1)$, the induction hypothesis implies that $\alpha^{p-1}_{I(p-1)} \in \mathcal{J}^k_{z_j=0}$, which in turn gives that $(\alpha^{p-1}_{I(p-1)})_I = 0$. Now, let $I''$ be a set of the type $I(p)$. We can write $I'' = I' \setminus \{j\}$ for non unique $I'$ of the type $I(p-1)$ and $j$. Then the pullback of $\alpha^p_I$ to $\{z_j = 0\}$ equals $\alpha^p_I = 0$. Repeating this for all possible decompositions $I'' = I' \setminus \{j\}$ of $I''$ we get $\alpha^p_I \in \mathcal{J}^k_{Z_I(j)}$, finishing the induction step.

**Proof of Lemma 7.** Property (ii) follows from property (i) according to Lemma 8. We claim that

$$\alpha' = \alpha_I + \sum_{I(1)} \alpha^1_{I(1)} + \cdots + \sum_{I(s-r)} \alpha^{s-r}_{I(s-r)},$$

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where we have used the notations from Proposition 9, has the properties (i) and (iii). That \( \alpha' \) has the property (iii) is part of the statement of Proposition 9 so we only need to check that it has property (i). By assumption we have that

\[
d\sigma \wedge \alpha' = d\sigma \wedge \alpha_I + d\sigma \wedge \sum_{l(1)} \alpha_I^1 + \cdots + d\sigma \wedge \sum_{l(s-r)} \alpha_I^{s-r} \in \mathcal{F}^{k+1}.
\]

If we pull back \( d\sigma \wedge \alpha' \) to \( V_I = \{z_r = \cdots = z_s = 0\} \) we get by Proposition 9 that the pullback of \( d\sigma \wedge \alpha_I \) is zero since all other terms on the right hand side of (8) vanish on this set by Proposition 9. But \( d\sigma \wedge \alpha_I \) is independent of all \( z_j \) and \( dz_j \) with \( j = r, \ldots, s \) and so \( d\sigma \wedge \alpha_I = 0 \) in \( \mathbb{C}^n \). Next we pull \( d\sigma \wedge \alpha' \) back to \( V_{I \setminus \{r\}} = \{z_r+1 = \cdots = z_s = 0\} \). Then, using that \( d\sigma \wedge \alpha_I = 0 \) and Proposition 9, we get that the pullback of \( d\sigma \wedge \alpha_I^{1} \) to this set is zero. But \( d\sigma \wedge \alpha_I^{1} \) is independent of all \( z_j \) and \( dz_j \) with \( j = r+1, \ldots, s \) and thus vanishes in all of \( \mathbb{C}^n \). Continuing in this way, running through the indices, then pulling back to varieties of dimension +1 and running through pairs of indices and so on, we eventually obtain that \( d\sigma \wedge \alpha' = 0 \) in \( \mathbb{C}^n \).

5. Proof(s) of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2. Apart from using Lemma 5 when proving Theorem 2 and Lemma 6 when proving Theorem 1 the proofs are almost identical and we choose to focus on Theorem 2. Any differences will be pointed out explicitly.

Proof of Theorems 1 and 2. After a preliminary partition of unity on \( X \) we may assume that the test form has as small support as we want. Moreover, from [5] we are done if the support of the test form does not intersect \( V_f = \{f_1 = f_2 = f_3 = 0\} \). (In the case of Theorem 1 this follows from [20],) Assume therefore that \( \varphi \) has support in a small neighborhood \( O \) of a point \( x \in V_f \). We may assume that \( \varphi \) has the form \( \varphi = \tilde{\varphi} \wedge \tilde{\sigma} \), where \( \tilde{\varphi} \) is a smooth \((n,0)\)-form with support close to \( x \) and \( \varphi \) is a holomorphic \((n-2)\)-form.

Hironaka’s theorem implies that there is a complex \( n \)-dimensional manifold \( \mathcal{X} \) and a proper holomorphic map \( \pi: \mathcal{X} \to O \) such that the variety \( \mathcal{Z}_f = \{\pi^*f_1 \cdot \pi^*f_2 \cdot \pi^*f_3 = 0\} \) has normal crossings in \( \mathcal{X} \) and \( \pi \) is biholomorphic outside \( \mathcal{Z}_f \). Since \( \pi \) is proper and \( \varphi \) has support close to \( x \in V_f \), the pullback, \( \pi^*\varphi \), has compact support close to \( \pi^{-1}(x) \subset V_f : = \{\pi^*f_1 = \pi^*f_2 = \pi^*f_3 = 0\} \). We also introduce the notation \( \mathcal{Z}_{12} \) for the variety consisting of the components of codimension one of \( \{\pi^*f_1 = \pi^*f_2 = 0\} \) on which \( \pi^*f_3 \) does not vanish identically. The varieties \( \mathcal{Z}_{23} \) and \( \mathcal{Z}_{13} \) are defined analogously.

Now, consider a point \( p \in V_f \). Since \( \mathcal{Z}_f \) has normal crossings we may choose local holomorphic coordinates \( z \) close to \( p \) such that \( z(p) = 0, \pi^*f_1 = z^a = z_1^{a_1} \cdots z_r^{a_r-1} \) and \( \pi^*f_j, j = 2, 3 \), are monomials times non vanishing holomorphic functions. Generically, \( p \) does not lie on \( \mathcal{Z}_{23} \), but if it does, then, after possibly renumbering the coordinates \( z_r, \ldots, z_n \), we can write \( \mathcal{Z}_{23} = \{z_r = \cdots = z_s = 0\} \). We can symbolically let \( s < r \) denote the case that \( p \) does not lie on \( \mathcal{Z}_{23} \). Since \( f_1, f_2, f_3 \) is a regular sequence, \( df_1 \wedge \varphi \) vanishes on \( \{f_2 = f_3 = 0\} \) for degree reasons, and so \( d\pi^*f_1 \wedge \pi^*\varphi = d(z^a) \wedge \pi^*\varphi \in \mathcal{F}^{r-1}_{23} \).

By Lemma 7 we may therefore choose a holomorphic \( n - 2 \)-form, \( \alpha_1 \), in
a neighborhood $U$ of $p$ such that $d\pi^* f_1 \wedge \alpha_1 = 0$, $\alpha_1 \in [\mathcal{F}]_{\mathbb{Z}}^{n-2}(U)$, and $\phi - \alpha_1 \in [\mathcal{F}]_{\mathbb{Z}}^{n-2}(U)$. In case $p$ does not lie on $\mathbb{Z}_3$ we can take $\alpha_1 = 0$ close to $p$. In the same way, perhaps after shrinking $U$, we can find $\alpha_2$ and $\alpha_3$ such that $d\pi^* f_2 \wedge \alpha_2 = 0$, $\alpha_2 \in [\mathcal{F}]_{\mathbb{Z}}^{n-2}(U)$, and $\pi^* \phi - \alpha_2 \in [\mathcal{F}]_{\mathbb{Z}}^{n-2}(U)$ and similarly for $\alpha_3$. In this way we get an open covering of $\mathcal{V}_f$ and we choose a locally finite subcovering, $\{U_j\}_1^\infty$. In each $U_j$ we have holomorphic $n - 2$-forms, $\alpha_1^j$, $\alpha_2^j$, and $\alpha_3^j$ with the properties described above. We may assume that $\pi^* \phi$ has support in $\cup U_j$ and then, since $\pi^* \phi$ has compact support, it suffices to take finitely many $U_j$ to cover it. For convenience we denote this finite family by $\{U_j\}_1^q$. Subordinate to this family we choose a partition of unity, $\{\rho_j\}_1^q$, such that $\sum^q_1 \rho_j = 1$ on the support of $\pi^* \phi$. The integral we are interested in can now be written

$$\sum^q_1 \int \frac{|\pi^* f_1|^2 [2\lambda_1] |\pi^* f_2|^2 [2\lambda_2] \wedge |\pi^* f_3|^2 [2\lambda_3]}{|\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3|} \wedge \rho_j \pi^* \phi \wedge \pi^* \phi. \quad (9)$$

Since $d\pi^* f_i \wedge \alpha_i^j = 0$ in $U_j$ we may replace $\pi^* \phi$ with $\pi^* \phi - \bar{\alpha}_2^j - \bar{\alpha}_3^j$ in (9) without affecting the integral. After this is done we integrate the $\bar{\partial}$ in front of $|\pi^* f_2|^2 [2\lambda_2]$ by parts, obtaining

$$\sum^q_1 \int \frac{|\pi^* f_1|^2 [2\lambda_1] |\pi^* f_2|^2 [2\lambda_2] \wedge |\pi^* f_3|^2 [2\lambda_3]}{|\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3|} \wedge \rho_j \bar{\partial} (\pi^* \phi \wedge (\pi^* \phi - \bar{\alpha}_2^j - \bar{\alpha}_3^j)) \quad (10)$$

$$- \sum^q_1 \int \frac{\bar{\partial} |\pi^* f_1|^2 [2\lambda_1] |\pi^* f_2|^2 [2\lambda_2] \wedge |\pi^* f_3|^2 [2\lambda_3]}{|\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3|} \wedge \rho_j \pi^* \phi \wedge (\pi^* \phi - \bar{\alpha}_2^j - \bar{\alpha}_3^j) \quad (11)$$

$$+ \sum^q_1 \int \frac{|\pi^* f_1|^2 [2\lambda_1] |\pi^* f_2|^2 [2\lambda_2] \wedge |\pi^* f_3|^2 [2\lambda_3]}{|\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3|} \wedge \rho_j \pi^* \phi \wedge (\pi^* \phi - \bar{\alpha}_2^j - \bar{\alpha}_3^j) \quad (12)$$

We first show that each term in the sum (10) has an analytic continuation to a neighborhood of $\lambda = 0$. To this end, we fix a $j$ and write $\pi^* f_i = z^{\alpha_i} \hat{f}_i$, $i = 1, 2, 3$, as monomials times non-vanishing functions in $U_j$. (The multiindices $\alpha_i$ and the functions $\hat{f}_i$ of course also depend on $j$ but we suppress this dependence to avoid to many subscripts.) By Lemma 5 it is sufficient to show that

$$\frac{d\bar{z}_k}{\bar{z}_k} \wedge \bar{\partial} (\pi^* \phi \wedge (\pi^* \phi - \bar{\alpha}_2^j - \bar{\alpha}_3^j)) \quad (13)$$

is a smooth form for all non simple factors $z_k$ dividing $z^{\alpha_3}$. Assume first that $z_k$ also divides $z^{\alpha_2}$. Then we write (13) as

$$\frac{d\bar{z}_k}{\bar{z}_k} \wedge \pi^* \bar{\partial} \phi - \frac{d\bar{z}_k}{\bar{z}_k} \wedge \bar{\partial} (\pi^* \phi \wedge (\bar{\alpha}_2^j + \alpha_3^j)) \quad (14)$$

Since $\alpha_2^j$ and $\alpha_3^j$ vanish on $\{z_k = 0\}$ the last term is a smooth form. On the other hand, $\bar{\partial} \phi$ can be written as a sum of products of smooth $(n, 0)$-forms and anti-holomorphic $(0, n - 1)$-forms. These anti-holomorphic forms vanish on $\{f_2 = f_3 = 0\}$ for degree reasons, and hence, their pullback under $\pi$ vanish on $\{z_k = 0\}$. Thus, the first term in (14) is also a smooth form. It
remains to check the case when \( z_k \) divides \( z^{a_1} \) and \( z^{a_3} \) but not \( z^{a_2} \), i.e., that \( \{ z_k = 0 \} \) is (part of) an irreducible component of \( Z_{13} \). We then write (13) as

\[
\frac{\overline{d\bar{z}_k}}{\bar{z}_k} \wedge \partial(\pi^*\bar{\phi} \wedge (\pi^*\phi - \alpha^j_3)) - \frac{d\bar{z}_k}{\bar{z}_k} \wedge \overline{\partial(\pi^*\phi \wedge \alpha^j_3)},
\]

which is a smooth form since both \( \alpha^j_3 \) and \( \pi^*\bar{\phi} - \alpha^j_3 \) vanish on \( Z_{13} \).

Next we show that each term in the sum (11) has an analytic continuation to a neighborhood of \( \lambda = 0 \). Now we have a \( \partial \) in front of \( |\pi^*f_1|^{2\lambda_1} \) and so we may subtract also \( \alpha^j_1 \) in the test form. By Lemma 5 it will thus be sufficient to show that

(15) \[
\frac{\overline{d\bar{z}_k}}{\bar{z}_k} \wedge \pi^*\bar{\phi} \wedge (\pi^*\phi - \alpha^j_1 - \alpha^k_2 - \bar{\alpha}^j_3)
\]

is a smooth form for any non simple factor \( z_k \) dividing at least one of \( z^{a_1} \) and \( z^{a_3} \). If \( z_k \) divides all three monomials \( z^{a_1}, z^{a_2}, \) and \( z^{a_3} \) then \( \alpha^j_1, \alpha^j_2, \) and \( \alpha^j_3 \) all vanish on \( \{ z_k = 0 \} \). But then \( \pi^*\phi \) does also, because the holomorphic \( n - 2 \)-form \( \phi \) vanishes on \( V_f \) for degree reasons and hence \( \pi^*\phi \) vanishes on \( V_f \supset \{ z_k = 0 \} \). In this case, (15) is therefore a smooth form. If instead \( z_k \) divides precisely two of the monomials, say for simplicity \( z^{a_1} \) and \( z^{a_2} \), then \( \alpha^j_1, \alpha^j_2, \) and \( \pi^*\phi - \alpha^j_3 \) vanish on \( \{ z_k = 0 \} \) and we see again that (15) is a smooth form.

Finally we prove that (12) has an analytic continuation to a neighborhood of \( \lambda = 0 \). If one could choose the \( \alpha^j_1 \) to agree on overlaps, i.e., to be independent of \( j \), then this would be easy because in this case, the only thing depending on \( j \) would be \( \rho_j \), and since the \( \rho_j \) sum up to 1, the \( \partial \rho_j \) sum up to 0, and (12) would be identically 0. Maybe it is possible to choose the \( \alpha^j_1 \) in such a good way but even if the differences \( \alpha^j_1 - \alpha^k_1 \) are not 0 on overlaps they have sufficiently good properties for our purposes. Actually, \( (dz_i/z_i) \wedge (\alpha^j_1 - \alpha^k_1) \) is a holomorphic form, where it is defined, for any non simple factor \( z_i \). We verify this for \( i = 2 \). If \( z_i \) divides \( \pi^*f_2 \) it is clear. On the other hand, if \( z_i \) does not divide \( \pi^*f_2 \), but divides \( \pi^*f_1 \) and \( \pi^*f_3 \), i.e., is (part of) a component of \( Z_{13} \), then it follows from the fact that \( \alpha^j_2 - \alpha^k_2 = \pi^*\phi - \alpha^k_2 \), where \( \phi \) is a smooth function in this neighborhood such that if \( \rho \) is a smooth compactly supported function in this neighborhood then the integral of \( \rho \) multiplied with the sum in (12) has an analytic continuation to a neighborhood of \( \lambda = 0 \). It will then follow, after another partition of unity, that (12) also has. Now, consider a point \( p \in supp(\pi^*\phi) \) and let \( U_1, \ldots, U_j \) be those sets from our cover which contains \( p \). Let \( U \) be a neighborhood of \( p \) contained in \( \cap \{ U_j \} \) and such that the \( \rho_j \) sum up to 1 in \( U \) and let \( \rho \) be a smooth function with support in \( U \). The differences \( \beta^{jk}_1 = \alpha^{jk}_1 - \alpha^{ji}_1 \) and \( \beta^{kl}_2 = \alpha^{jk}_2 - \alpha^{ji}_2 \) with \( 1 \leq k, l \leq r \) are all defined in \( U \). If we multiply the sum in (12) by \( \rho \) and integrate we obtain

\[
\int \sum_{k=1}^{r} \frac{|\pi^*f_1|^{2\lambda_1}|\pi^*f_2|^{2\lambda_2}|\partial(\pi^*f_3)|^{2\lambda_3}}{\pi^*f_1 \cdot \pi^*f_2 \cdot \pi^*f_3} \wedge \rho \partial \rho_j \wedge \pi^*\phi \wedge (\pi^*\phi - \alpha^{jk}_2 - \alpha^{ji}_3) =
\]
\begin{equation}
(16) \int \sum_{k=1}^{r} \frac{\pi^* f_1 |^{2\lambda_1} \pi^* f_2 |^{2\lambda_2} \bar{\partial} |^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho \bar{\partial} \rho_{jk} \wedge \pi^* \phi \wedge (\pi^* \phi - \tilde{\alpha}_2^{j} - \tilde{\alpha}_3^{j})
\end{equation}

\begin{equation}
(17) - \sum_{k=1}^{r} \int \frac{\pi^* f_1 |^{2\lambda_1} \pi^* f_2 |^{2\lambda_2} \bar{\partial} |^{2\lambda_3}}{\pi^* f_1 \cdot \pi^* f_2 \cdot \pi^* f_3} \wedge \rho \bar{\partial} \rho_{jk} \wedge \pi^* \phi \wedge (\tilde{\beta}_2^{jk} + \tilde{\beta}_3^{jk}).
\end{equation}

The only thing in the sum in (16) that depends on \( k \) is \( \rho_{jk} \) and since the \( \rho_{jk} \) sum up to 1 in \( U \) the \( \bar{\partial} \rho_{jk} \) sum up to 0, and so (16) is identically 0. But \( (dz_l/z_l) \wedge \beta_2^{jk} \) is a holomorphic form for any non simple factor \( z_l \) by the discussion above, and so by Lemma 5, each term in the sum (17) has an analytic continuation to a neighborhood of \( \lambda = 0 \). The proof is complete. \( \square \)

References


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